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The Functoriality of Khovanov Homology and the Monodromy of Knots

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Candidato
Carlo Collari

Relatore
Prof. Paolo Lisca
Università di Pisa

Controrelatore
Dott. Bruno Martelli
Università di Pisa

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Introduction

The aim of this thesis is to present a proof of the functorial properties of the Khovanov homology and to study the monodromy group of knots. In this introduction we provide an overview of the context we are going to work in and explain the underlying motivations.

The matematical context

Knots and the Jones polynomial. Knot theory is the best starting point to outline the mathematical context of this thesis. Classical knot theory is the study of the possible ways to smoothly embed a collection of circles into $\mathbb{R}^3$, up to ambient isotopy of the space. A *knot* is the embedding of a single circle. A *link* is a finite collection of knots, called *components*, with disjoint image. The *trivial knot* is the embedding of a circle into a plane. The *unlink* is a collection of disjoint circles in a plane.

An immediate question is: how can one tell if a given knot is, up to ambient isotopy, trivial? Or more generally, given two knots, are they equivalent? And if we consider links instead of knots? There are various approaches to these problems. One of the most effective ones is the diagrammatic approach.

A *link diagram* is the projection of a link into a plane satisfying certain conditions: this projection is one-to-one except for a finite number of points. The points where the projection fails to be injective are double points and at each double point is specified which arc undercrosses and which one overcrosses.

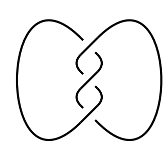


Figure 1. A knot diagram.

In 1920 Alexander proved that two link diagrams are related by a finite sequence of oriented Reidemeister moves – see Figure 5 at page 7 – and planar isotopies if and only if they represent equivalent links – see Theorem 1 at page 6. This theorem allowed the introduction of combinatorial invariants for knots and links.

In 1984 Vaughan Jones, see [**VFRJones1**], introduced a polynomial link invariant $\mathcal{J}$ satisfying the following skein relation

$$(\spadesuit) \qquad q^{-1} \cdot \mathcal{J}_{\mathcal{L}_1}(q) - q \cdot \mathcal{J}_{\mathcal{L}}(q) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) \cdot \mathcal{J}_{\mathcal{L}_0}(q),$$

where $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}$ are three oriented link diagrams which differ in a small region where they look like, respectively, , and .

This literally caused a revolution. The words of Cromwell – in [**Cromwell**] page 217 – may give an idea of the effect caused by the definition of this new invariant.

> *"This discovery had a tremendous impact, and not only on knot theory. Once it was known that the Alexander polynomial invariant was not the only polynomial link invariant, people started to search for more – some using combinatorics and others following the algebraic route used by Jones. Close connections with physics generated a lot of interdisciplinary research, and polynomial were defined via physical methods related to statistical mechanics, were the Yang-Baxter equations provided an analogue of the third braid relation, and quantum groups."*

Fifteen years later, in 1999, another major event in knot theory took place: Khovanov's categorification of the Jones polynomial – [**KhovCat**].

Categorification. The term *categorification*, coined by Louis Crane in [**Crane**], describes the process of replacing sets by categories, functions with functors, and equations by natural isomorphisms of functors satisfying additional properties called *coherence laws*. The result of this process may reveal deep insights on the categorified object.

An example of categorification that may help the reader to understand the basic idea behind this concept is given by the de Rham cohomology. Let us consider the collection of all smooth surfaces, say S, for each element of this collection we can compute the Euler-Poincaré characteristic χ. We can categorify χ by replacing S with the category $\mathscr{S}$, whose objects are smooth surfaces and whose arrows are differentiable maps between them, $\mathbb{Z}$ with the category $\mathscr{M}od_{\mathbb{Z}}$ of the graded $\mathbb{Z}$-modules and χ – which is a function between the set S and $\mathbb{Z}$ – with the de Rham cohomology – which is a functor between $\mathscr{S}$ and $\mathscr{M}od_{\mathbb{Z}}$. Moreover we can recover the categorified object from the categorification: χ is the result of the alternating sum

$$\sum_{i\in\mathbb{Z}}(-1)^i rank(H^i),$$

where H^i is the i-th de Rham cohomology group.

Within the last few years, also thanks to Khovanov categorification of the Jones polynomial, the importance of categorification raised, involving various mathematical areas and leading to lots of interesting results. Categorification by itself goes behind the scope of our work, any reader who wishes to know more on the subject can consult [**Crane**] or also [**Mazorchuk**].

Khovanov homology. Khovanov's idea, see [**KhovCat**], was to define for each oriented link diagram $\mathcal{D}$, a complex $KH^{\bullet,\bullet}(\mathcal{D})$ whose *graded Euler chatacteristic*, i.e. the Laurent polynomial defined by the sum

$$\sum_{i,j\in\mathbb{Z}}(-1)^i(q)^j rank(KH^{i,j}(\mathcal{D}))$$

is the Jones polynomial of $\mathcal{D}$. The starting point for Khovanov's construction is Kauffman's state model of the Jones polynomial.

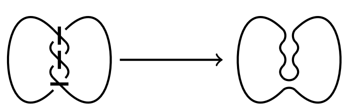
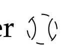
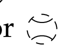


Figure 2. The resolution of a state

Given a link diagram $\mathcal{D}$, a resolution of one of its crossings is the replacement of a crossing  with either , a 0-resolution, or , a 1-resolution. A *state* is the choice of a possible resolution for each crossing. The *resolution* of a diagram with respect to a state is the diagram obtained by resolving all the crossings according to the state.

In [**Kauff2**] Kauffman, using the skein relation (♠), proved that the Jones polynomial could be written as a $\mathbb{Z}[q,q^{-1}]$-linear combination of the Jones polynomials of the resolutions, whose coefficients are of the form $(-q)^j$ and j depends on both the type of the resolution performed at each crossing and the orientation of the diagram.

The Khovanov chain complex is defined from the *cube of resolutions*, also called *cube of smoothings*. This is an n-dimensional cube whose vertices are identified with the resolutions of $\mathcal{D}$ and whose edges are identified with cobordisms between resolutions.

If we replace each resolution of $\mathcal{D}$ with a graded module such that its graded dimension is exactly the Jones polynomial of the associated resolution and the cobordisms representing the edges with maps in such way that the cube is skew commutative, it is possible to obtain the *Khovanov chain complex*. The way to obtain a complex from a skew-commutative cube is a general procedure which works in every abelian category, and its described in [**KhovCat**] or in Chapter 1 Section 1.3.

TQFT. The way to replace a resolution with modules and cobordisms with maps comes from the late '80s. In his article [**MAtiyah**] Atiyah suggested a set of axioms for topological quantum field theory which was inspired by the axioms for conformal field theory given by Segal, see [**Segal**], and the geometric meaning of supersymmetry given by Witten, [**Witten**].

A n-dimensional TQFT is a monoidal functor between the category of $(n+1)$-dimensional closed smooth manifolds and n-dimensional cobordisms between them and the category $\mathscr{M}od_{\mathbb{Z}}$. In simpler words, a $(1+1)$-TQFT is just a way to replace a circle with a $\mathbb{Z}$-module, a cobordism with a map and the empty set with the ring $\mathbb{Z}$, in such way that the glueing of two cobordisms along the boundary corresponds to the composition of the associated maps and the disjoint union correspond to the tensor product.

The use of any $(1+1)$-TQFT allows us to obtain new chain complexes from the cube of resolutions. To define to a *link homology theory*, i.e. to define a chain complex whose homology is a link invariant, in this way it a few more hypotheses are necessary – see [**KhovUniv**]. In particular, it is possible to obtain Khovanov homology theory from the cube of resolutions by applying a particular TQFT.

Functoriality. Khovanov homology defines a functor from the category of oriented link diagrams and oriented link cobordisms to the category of bi-graded $\mathbb{Z}$-modules and morphisms.

This statement could be refined: Khovanov homology gives a functor between the category of link diagrams and boundary fixing isotopy classes of link cobordisms to the *projectivization* of the category, i.e. the category with the same objects but whose morphisms are considered only up to sign, of bi-graded $\mathbb{Z}$-modules and morphisms.

At this point a natural question arises: what happens if we drop the condition on the fixed boundary? Is the functoriality preserved? The answer to this question is known to be negative. Jacobsson, see [**Jacobss**], proved that there are cobordisms ambient isotopic to cylinders which induce automorphisms in Khovanov homology that are neither the identity map nor its opposite. These cobordisms are described by movies that begin and end with the same diagram and involve only planar isotopies and

Reidemeister moves. The automorphisms associated to these cobordisms form a group with respect to composition and this group is precisely the *monodromy group*.

Outline of the paper

In Chapter 0 of this paper we introduce the basic concepts of knot theory such as diagrams, oriented diagrams and sign conventions on the crossings. In this chapter also the Jones polynomial is defined.

In Chapter 1 we summarize different constructions, as well as some of their properties. The fundamental constructions we present are: the cube of smoothings, Khovanov bracket, Khovanov chain complex and the formal Khovanov chain complex. The approach we adopt is the same one described in [**BarNatan**]. In this chapter is also sketched a proof of the invariance of the formal Khovanov chain complex.

In Chapter 2 we prove the functoriality of the Khovanov homology. We prove the result not only in the case of link diagrams but also in the more general case of tangle diagrams. Section 1 provides an introduction to the possible representations of knotted surfaces in 4-dimensional space. In Section 2 the new categorical setting of our theory and the fundamental tools that are necessary for the proof of the main result are presented. Section 3 is devoted to prove of the main theorem: up to sign, the map induced by two surfaces ambient isotopic relative to the boundary are homotopy equivalent. A final section is dedicated to suggestions of further readings.

Finally, Chapter 3 is devoted to the study of the monodromy group. In Section 1 we prove the invariance, up to isomorphisms, of the monodromy group. In Section 2 we describe two techniques that can be applied to the explicit computation of the monodromy group. Section 3 is devoted to the proof of the Rasmussen-Tanaka theorem. This result will be used for the computations performed in Section 4. In the last section we describe the limits of our approach and suggest a few possible ways to improve our methods.

CHAPTER 0

Knot Theory

> Whatever the twists and turns of a system of threads in space, one can always obtain an expression for the calculation of its dimensions, but this expression will be of little use in practice. The craftsman who fashions a braid, a net, or some knots will be concerned, not with questions of measurement, but with those of position: what he sees there is the manner in which the theads are interlaced.
>
> A. T. Vandermonde

The quote above is the first paragraph of [**Vanderm**]. This book, written in 1771, deals with the position problem. This problem consists of understanding in how many different ways we can place an object, for example a system of treads or a necklace, in space. A way to state formally the mathematical problem of position, in its most general version, is the following.

Problem (Position problem). *Given a topological space $\mathcal{X}$ and two subspaces, say $\mathcal{A}$ and $\mathcal{B}$, homeomorphic to a topological space $\mathcal{Y}$. The subspaces $\mathcal{A}$ and $\mathcal{B}$ are said to be of the same* type *in $\mathcal{X}$ if and only if there exists an isotopy H of $\mathcal{X}$, such that $H(\mathcal{A}, 1) = \mathcal{B}$. Classify all the types of spaces homeomorphic to $\mathcal{Y}$.*

Knot theory deals with probably the simplest non-trivial instance of this problem: when the space $\mathcal{X}$ is the usual 3-dimensional space $\mathbb{R}^3$ – or its Alexandroff compactification $\mathbb{S}^3$ – and the subspaces are disjoint union of circles – the so called *links*, or *knots* if the subspaces are homeomorphic to a single circle.

A function that goes from the set of all possible subspaces of $\mathcal{X}$ that are homeomorphic to $\mathcal{Y}$, to another set – that might be a collection of vector spaces, as well as a numeric set or, in general, the class of the objects of a given category – such that two subspaces of the same type have the same image is called an *invariant*. The aim of knot theory is to find easily computable *complete invariants*, i.e. invariants such that if two links have the same image then they are of the same type.

Up to now a few complete invariants are known, such as the fundamental quandle or the peripheral system, and they are not easily computable even for the simplest knots. So, it is generally preferable to find incomplete invariants that are easier to compute, rather than hard-to-compute complete ones.

Most of these incomplete invariants are in practice more than enough to discern if two knots are or are not of the same type. Combinatorial invariants are based on the study of knots and links diagrams which are, roughly speaking, projections of knots and links onto a plane by applying combinatorial techniques. Combinatorial invariants are easy to compute: most of them can be calculated using a computer, and are very powerful. Khovanov homology can be seen as a particular combinatorial invariant.

In this chapter we describe the basic definition and techniques of combinatorial knot theory. We will proceed as follows: in the first section we formally define knots and links; in the second section we will add the orientation to the picture. Finally, in the third section we will give an example of a combinatorial invariant: the Jones polynomial. The latter is deeply related with Khovanov homology.

Any reader who is interested in knot theory, even after they read this chapter, can consult the fourth and last section of this chapter where we will give some references.

1. Knots and links

1.1. Definitions. A knot can be thought of as a thin piece of rope, knotted and with its ends glued together. A link is a collection of knots that could be tangled and linked together. The formal definitions are given below.

Definition 0.1. A *knot* $\mathfrak{K}$ is a topological embedding of $\mathbb{S}^1$ in $\mathbb{R}^3$ or $\mathbb{S}^3$. A *link* is a finite collection $\mathfrak{K}_1, ..., \mathfrak{K}_m$ of knots, called *components*, with disjoint images.

Two knots, say $\mathfrak{K}$ and $\mathfrak{L}$, are *equivalent* – or of the *same type* – if exists a continuous map

$$H : \mathbb{S}^3 \times [0,1] \longrightarrow \mathbb{S}^3,$$

such that $h_t(\cdot) = H(\cdot, t)$ is an homeomorphism, for each $t \in [0,1]$, h_0 is the identity map of $\mathbb{S}^3$ and $h_1(\mathfrak{K}(\mathbb{S}^1)) = \mathfrak{L}(\mathbb{S}^1)$; such a map is called *ambient isotopy*. The previous definition formalizes the intuitive idea of a continuous deformation of $\mathfrak{K}(\mathbb{S}^1)$ *which takes place in* $\mathbb{R}^3$ into $\mathfrak{L}(\mathbb{S}^1)$; as time t passes the points of our knot as well as its surrounding, move – continuously – in $\mathbb{R}^3$ until, at the final instant, they overlap the points of the second knot.

This is obviously an equivalence relation, so it partitions the set of knots into classes called *types*, and indicated $[\mathfrak{K}]$.

Two links $\mathfrak{K}_1, ..., \mathfrak{K}_m$ and $\mathfrak{L}_1, ..., \mathfrak{L}_m$ are equivalent if and only if there exists an ambient isotopy H and a permutation $\sigma \in \mathfrak{S}_m$, such that:

$$h_1(\mathfrak{K}_j(\mathbb{S}^1)) = \mathfrak{L}_{\sigma(j)}(\mathbb{S}^1), \text{ for all } j \in \{1, ..., m\}.$$

The *unknot* is a circle that lies in a plane; any knot which is equivalent to the unknot is said to be unknotted. Similarly, the *unlink* with m components is a collection of m disjoint circles in a plane.

Remark. The equivalence between links is more than the equivalence of the single components; we require the components to be deformed together and without intersecting each other. An example of two non-equivalent links with components of the same type is given by the unlink with two components and the Hopf link shown in Figure 8 at page 86.

1.2. Tame and wild knots. Any mathematician with a little background of topology knows that dealing with continuous maps, without any further regularity hypotesis, leads to a variety of pathological cases. Knot theory makes no exception: as things are now, there are knots far from our intuitive concept and from the physical entities we are modelling in knot theory; an example of such a knot is shown in the figure below.

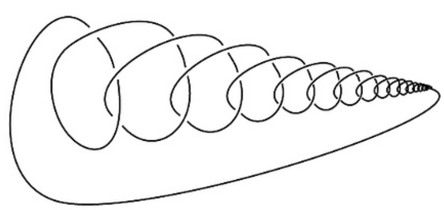


FIGURE 1. A wild knot.

Knots like the one shown in figure are called "wild" – the one depicted is actually mild wild, there are even worse examples. These knots, apart from being far from our intuitive idea of a knot, have also bad combinatorial properties – which are essential in this work – and are difficult to study.

There are different ways to rule wild knots out; the most common apporaches are: requiring knots to be "sufficently regular", i.e. C^1 or smooth, or forcing our knots to be "finite" in some sense. These approaches are more or less equivalent, but the second one avoids technical difficulties and needs less pre-requisites.

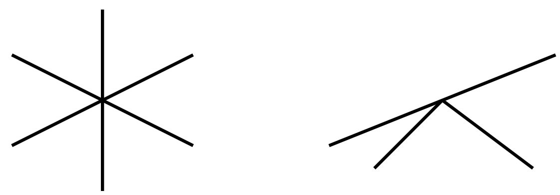


FIGURE 2. Local picture of non-regular projections.

Definition 0.2. A *polygonal knot* is a knot whose image is given by a finite number of points, called *vertices*, joined by a finite number of line segments, called *edges*, such that: two edges intersects at most in a vertex and each vertex is shared by exactly two edges. A *polygonal link* is a link whose components are polygonal knots.

Any link, or knot, equivalent to a polygonal one is called *tame*, the others are called *wild*. This distinction is important: most of the known techniques, in particular those used in this paper, cannot be applied to wild knots or links. From now on, all knots and links are supposed to be tame. All knots which are sufficently regular are tame, in the sense of our definition.

Proposition 1. *Any knot of class C^1, up to reparametrization, is tame.*

This fact is basically due to the rectificability of C^1 curves parametrized by arc lenght; the proof of this proposition could be found in the appendix I of **[FoxCrom]**.

1.3. Diagrams. Knots in $\mathbb{R}^3$ or $\mathbb{S}^3$ are generally difficult to describe; for this reason usually a knot is presented by a projection called diagram. This diagrams allow us to use combinatorial techniques to study knots, and also provide a way to compute effectively lots of invariants.

Just any projection would not work, so we must make some assumptions.

Definition 0.3. Given a polygonal link $\mathfrak{K}_1, \ldots, \mathfrak{K}_m$, we say that it is in *regular position with respect to a plane* Π if the projection of $\bigcup_j \mathfrak{K}_j(\mathbb{S}^1)$ is one to one except for a finite number of double points, none of wich is the image of a vertex. Any projection wich satisfies the properties just stated is called *regular projection*.

Proposition 2. *Given a knot $\mathfrak{K}$ and a plane Π exists an arbitrarily small rotation ρ of $\mathbb{R}^3$ such that $\rho(\mathfrak{K})$ is in regular position with respect to Π.*

This theorem and its proof could be found in the first chapter of [**FoxCrom**]. As a consequence, every tame knot is equivalent to a polygonal knot in regular position with respect to a fixed plane. Up to ambient isotopy we can smooth out the vertices of our polygonal knots, as well as those of the projections, so usually we will draw them without corners.

Definition 0.4. A *link diagram* is the image of a regular projection, together with the information of which arc undercrosses at each double point. The *components* of a link diagram are the images of the components of the link.

A way to codify this information is to break the undercrossing arc as shown in figure; this type of link diagram is also called *broken diagram*.

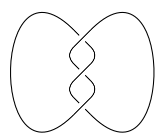


FIGURE 3. A diagram for the trefoil knot.

The image of a regular projection is a compact set, so, up to applying ambient isotopy, this image can be taken to be contained in $\mathbb{D}^2$, i.e. the unit disc of $\mathbb{R}^2$. This fact will be implicitly assumed in the rest of the paper.

1.4. Tangles an local moves. Sometimes one has to deal with links that have the same diagram,except in a small region of the plane, and it may be useful to work only in that region "forgetting" the rest of the diagram. To work in such a local context it is necessary to use tangles.

Definition 0.5. A *tangle* is the intersection of a link with a 3-disc $\mathbb{D}^3$. The *boundary set* of a tangle is the – finite – set of points that lies in $\partial\mathbb{D}^2$.

A tangle $\mathfrak{T}$ is said to be in *regular position* with respect to a plane Π if and only if the original link is in regular position with respect to Π and there are no crossings in the projection of the boundary set of $\mathfrak{T}$. The projection of a tangle in regular position, together with the information of which arc undercrosses at each crossing is called *tangle diagram*.

The tangle diagrams are contained in $\mathbb{D}^2$, which is the projection of $\mathbb{D}^3$ onto a plane, and their boundary set, i.e. the itersection of the tangle diagram with $\partial\mathbb{D}^2$, will be always considered linaerly ordered.

A *local move* is the replacement of a tangle with another tangle having the same boundary set. There are two different types of local moves: one that changes the link type and one that does not. An example of moves of the second type is given in Figure 4. These moves give us a way to relate diagrams of equivalent knots.

Theorem 1 (Alexander 1920). *Two links are equivalent if and only if there exists a diagram $\mathcal{H}$ of the first one, and a diagram $\mathcal{K}$ of the second one, that are related by a finite sequence of planar isotopies and Reidemeister moves.*

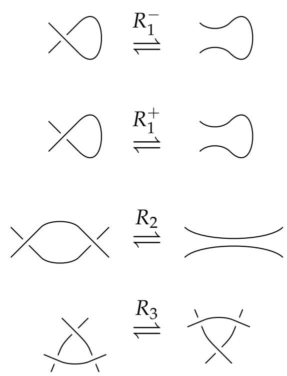


FIGURE 4. The unoriented Reidemeister moves.

2. Orientations

Knots, links and tangles are, technically speaking, topological 1-manifolds – with boundary in the case of tangles – and, as a consequence, they admit an orientation. Orientations are, roughly speaking, just the choice of a preferred direction of travel along each component.

Definition 0.6. An *oriented link* is a collection of knots $\mathfrak{K}_1, ..., \mathfrak{K}_m$, together with a fixed orientation on their image.

Considering our definition of knot equivalence, it turns out that two copies of the same knot with different orientations may not be equivalent. Any oriented knot that is equivalent to itself with the opposite orientation is called *invertible*. Knots that admit a diagram with less than nine crossings are all invertible.

A choice of a orientation for the a link $\mathfrak{L}$ induces an orientation on each diagram of $\mathfrak{L}$. This orientation can be indicated with an arrow on each component.

Also Reidemeister moves have an oriented version: there is more than one oriented version for each unoriented move. Luckily, we can obtain all the possible oriented Reidemeister moves from the four moves depicted in Figure 5 – see [**Polyak**] for the proof. In order to check that something – e.g. a group, a chain complex, a polynomial ect. – is an *oriented link invariant*, i.e. does not change if two oriented links are of the same type, it is necessary, and also sufficent, to check the invariance under oriented Reidemeister moves.

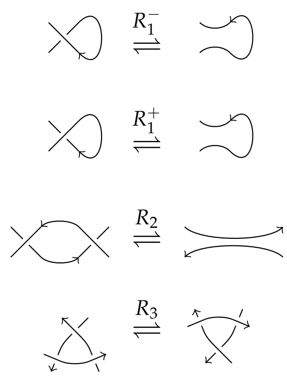


FIGURE 5. The oriented Reidemeister moves.

Along with the concept of orientation on a diagram we have the concept of positive and negative crossings – these are conventionally defined as in Figure 6. The *sign of a crossing* is $+1$ if the crossing is positive and -1 if the crossing is negative.

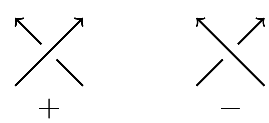


FIGURE 6. Positive and negative crossings.

Now we can define one of the simplest invariant for links with two components. Given an oriented link diagram $\mathcal{L}$ with two components, say $\mathcal{A}$ and $\mathcal{B}$, the *linking number* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $lk(\mathcal{A}, \mathcal{B})$, is half the sum of the signs of the crossings that are shared by $\mathcal{A}$ and $\mathcal{B}$. It is an easy exercise that the linking number is an oriented link invariant.

Definition 0.7. The *writhe* of an oriented diagram $\mathcal{D}$, indicated with $w(\mathcal{D})$, is the sum of the signs of the crossings of the diagram $\mathcal{D}$.

The writhe is neither a link nor a knot invariant because the second and third oriented Reidemeister moves do preserve the writhe, while the first move either raises – R_1^+ – or decreases – R_1^- – the writhe.

When something – e.g. a group, a chain complex, a polynomial ect. – is preserved by the second and the third Reidemeister moves, but not necessarely by any version of the first move, we will call it *invariant under regular isotopies* for links.

3. The Kauffman bracket and the Jones polynomial

In this section we define the Kauffman bracket polynomial and the Jones Polynomial. Moreover, we will see how these two polynomials are related.

Definition 0.8. The *trivariate Kauffman bracket polynomial* of a – unoriented – link diagram $\mathcal{L}$ is the polynomial $\langle \mathcal{L} \rangle$ in the variables A, B, d, defined recursively by the relations

(a) $\langle \bigcirc \rangle = 1$, where $\bigcirc$ denotes a single circle in the plane;
(b) $\langle \bigcirc \sqcup \mathcal{L}' \rangle = d \langle \mathcal{L}' \rangle$, for every link diagram $\mathcal{L}'$;
(c) $\langle \mathcal{L} \rangle = A \langle \mathcal{L}_0 \rangle + B \langle \mathcal{L}_1 \rangle$, where the diagrams $\mathcal{L}_0$ and $\mathcal{L}_1$ are obtained from $\mathcal{L}$ by replacing a crossing with, respectively, and .

If we consider a link diagram $\mathcal{L}$, in the local picture near a crossing there are four regions. Two of these local regions form a *pair* if they meet only at the vertex. A pair is called *positive* if is the first pair swept by the overcrossing arc under counterclockwise rotation. The non-positive pair is called *negative*.

Figure 7. The pairing of regions: the "+"s indicate the positive pair and the "-" the negative one.

Given a link diagram $\mathcal{L}$ the *universe* associated to $\mathcal{L}$ is the 4-valent planar graph obtained by placing a vertex at each crossing of $\mathcal{L}$ and considering the arcs of $\mathcal{L}$ as the arcs of the graph. A *state* on a universe is the choice of a pair of regions for each vertex. A *resolution*, or *smoothing*, of a crossing in a diagram, or of a vertex in a universe, is one of the local moves that replaces the crossing, or the vertex, with either or .

For each crossing a smoothing is called positive if the local regions of the positive pair are merged together. Otherwise, the smoothing is called negative.

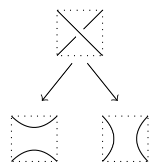


FIGURE 8. The two possible resolutions of a crossing: on the left the positive one and on the right the negative one.

Let $\mathcal{L}$ a link diagram and s a state of the associated universe. We will denote with $i(s)$ the number of positive pairs in s, $j(s)$ the number of negative pairs and $|s|$ the number of circles we obtain by smoothing each crossing according to the pair chosen, the collection of these circles will be called *smoothing* relative to s.

Definition 0.9. Given a link diagram $\mathcal{L}$ and a state s, the *weight* of s in $\mathcal{L}$ is the monomial defined as

$$\langle \mathcal{L} | s \rangle = A^{i(s)} B^{j(s)}.$$

Since the smoothings relative to the states are in one-to-one correspondence with the summands in the expansion of the bracket, and since the coefficent of the trivariate Kauffman bracket of the smooting relative to a state s is exactly $A^{i(s)} B^{j(s)} = \langle \mathcal{L} | s \rangle$, we have that

$$\langle \mathcal{L} \rangle = \sum_{s \text{ state}} \langle \mathcal{L} | s \rangle \cdot d^{|s|-1}. \tag{1}$$

The expression in (1) is called *expansion of the bracket as a state summation*.

The bracket is not invariant under regular isotopies for links. A few computations – see **[Kauff]**, pages 216-220 – show that necessary and sufficent conditions for the invariance under regular isotopies of the Kauffman bracket are

$$B = A^{-1}, \quad d = -(A + A^{-1}).$$

The *Kauffman bracket* of a link diagram is the Laurent polynomial in the variable A obtained from the trivariate Kauffman bracket by means of the equations above.

Nonetheless, we do not have the invariance of the Kauffman bracket, because the first move changes the bracket by multiplying it by either $-A^3$ or $-A^{-3}$, depending on the version of the first move we are considering. To obtain an invariant of oriented links we set

$$f_{\mathcal{L}}(A) = (-A)^{-3w(\mathcal{L})} \langle \mathcal{L} \rangle,$$

where $\mathcal{L}$ is an oriented link diagram and $w(\cdot)$ is the writhe. The Laurent polynomial $f_{\mathcal{L}}$ is called the *Kauffman polynomial* of $\mathcal{L}$.

From the Kauffman polynomial we can define the *Jones polynomial* $\mathcal{J}$ of an oriented diagram as

$$\mathcal{J}_{\mathcal{L}}(q) = f_{\mathcal{L}}(q^{-\frac{1}{4}});$$

As a consequence of the state summation formula in (1), we get the following result.

Proposition 3. *For each oriented link diagram $\mathcal{L}$ the following state summation formula holds*

$$\mathcal{J}_{\mathcal{L}}(q) = \sum_{s \text{ state}} (-1)^{3w(\mathcal{L})} q^{\frac{i(s)-j(s)-3w(\mathcal{L})}{4}} (-q^{\frac{1}{4}} - q^{-\frac{1}{4}})^{|s|-1},$$

where $i(s)$ is the number of positive smoothings in s, $j(s)$ is the number of negative smoothings in s and $|s|$ is the number of circles in the resolution associated with the state s.

Remark. The Jones polynomial is usually defined by the following identities

(a) $\mathcal{J}_{\bigcirc} = 1$;
(b) if $\mathcal{L}$ is an oriented link diagram, $\mathcal{L}_1$ is obtained from $\mathcal{L}$ by replacing a negative crossing c, , with a positive one, , and $\mathcal{L}_0$ is obtained by replacing c with the smoothing , we have

$$q^{-1}\mathcal{J}_{\mathcal{L}_1} - q\mathcal{J}_{\mathcal{L}} = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)\mathcal{J}_{\mathcal{L}_0}.$$

For a proof that the Jones polynomial just defined is the same as the one previously defined, see [**Kauff2**].

4. Further reading

A very good book to get started with knot thory is [**FoxCrom**], where a beautiful introduction to knots and links is given. This book treats only a few arguments: the knot group, the Alexander polynomial and colorations, but it is self contained and clear.

A book that gives just the ideas behind most of the constructions in knot theory without being too technical is [**Living1**]. The strong point of this book is the simplicity; the prerequisites are just a little background in topology and the knowledge of some linear algebra.

One of the most complete books about knot theory is Dale Rolfsen's "Knots and Links", [**Rolfsen**]. Finally, if one wants to know more about the Kauffman bracket or combinatorial knot theory, can consult [**Kauff**].

CHAPTER 1

Khovanov Homology

In this chapter different constructions are summarized as well as some of their properties; these are the fundamental constructions for the Khovanov homology theory. We start with the definition of the smoothing cube, a combinatorial object employed to define the Khovanov chain complex. Following [**BarNatan**], after a discussion on cubes – inspired by [**KhovCat**] – and a bit of abstract nonsense, we start the formal construction of the so called Khovanov bracket, a complex over a suitable category.

Before proving the invariance – up to chain homotopy and in the right category – of the bracket $[\![\cdot]\!]$, a little aside on the costruction of the Khovanov complex is made. At first, we just say how the chain complex could be defined from the bracket; afterwards, we investigate more deeply the relations between the formal complex and the chain complex.

In the third section we provide a proof of the invariance of the Khovanov bracket, using the already cited Bar-Natan's approach; we deal with the problem in a formal way by using planar algebras and local arguments in order to conclude our proofs. In the same section, to be precise in the concluding subsection, the grading and a graded version of the main theorems are dealt with.

Finally, in the conclusive section, we describe – without too much detail – different appoaches to the invariance of the Khovanov homology, as well as other definitions, generalizations, and alternative constructions.

1. Khovanov bracket

The aim of this section is to introduce the Khovanov bracket; this is a "formal complex", in a sense described in the second subsection, based on the combinatorial structure called smoothings cube.

1.1. The Cube. The cube of smoothings, as its name reveals, is a structure based on the possible resolutions of all the crossings in a tangle diagram; so, in order to study this structure, one needs to make a few considerations about the smoothings.

Let c be a crossing of a tangle diagram $\mathcal{T}$. In the local picture of c there are four regions; a couple of them is called are a *pair* if they meet only at the vertex. A pair is called *positive* if is the pair swept by the overcrossing arc, under counterclockwise rotation, until the undercrossing arc; otherwise, the pair is called *negative*.

FIGURE 1. The pairing of regions: the "A"s indicate the positive pair and the "B"s the negative one.

Any resolution of c is called accordingly to which pair of regions it fuses: if the positive pair is unified, then the resolution is positive, otherwise negative.

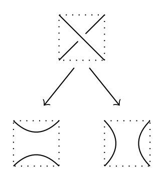


FIGURE 2. The two possible resolutions of a crossing: on the left the positive one and on the right the negative one.

Let $c_1, ..., c_n$ be an order for the crossings of $\mathcal{T}$; until the end of the chapter we suppose this order fixed – unless otherwise stated. The example we will use the most will be the trefoil diagram depicted in figure 3, in this case the order will be descending – *c'est-à-dire*, the first crossing is on the top, while the third crossing is on the bottom of the diagram.

We can assign to each smoothing a vector of ones and zeros, called *splitting* or *smoothing vector*, whose i-th entry is 0 if c_i is resolved in a positive way and 1 otherwise. Viceversa, given a vector v – with n components – of zeros and ones, we can associate to it a smoothing by splitting the i-th crossing according to the i-th entry of v.

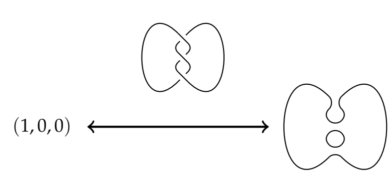


FIGURE 3. The smoothing corresponding to the vector $(1, 0, 0)$.

The bijective correspondence between smoothings and smoothing vectors induces an identification of the resolutions of $\mathcal{T}$ with the vertices of the *n-dimensional standard cube* in $\mathbb{R}^n$ – the set of all the points with coordinates in $\{0,1\}$, called *vertices*, together with the segments of straight lines joining two vertices that differ by a single coordinate, called *edges*.

Definition 1.1. Let v be a vector in $\mathbb{R}^n$; its *lenght*, denoted $|v|$, is the sum of the absolute value of its coordinates. The length of a smoothing is the lenght of the associated vector.

To obtain the smoothings cube we arrange the resolutions in columns according to their length – i.e. two smoothings are in the same column if and only if their lengths are the same – and place this columns in such a way that the module increases from left to right. In this case "a picture is worth a thousand words", to quote Bar-Natan – **[BarNatan]** –, so a picture of the smoothings cube of the trefoil is provided.

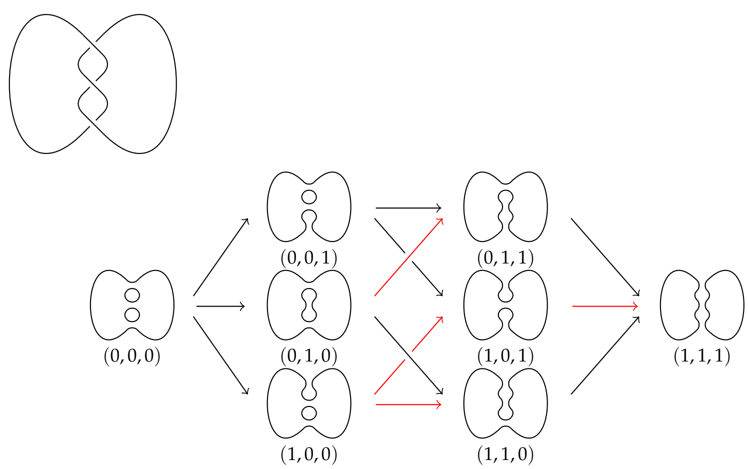


FIGURE 4. The cube of smoothings of a trefoil diagram.

The cube depicted in the previous figure has, between certain smoothings, arrows. These arrows are the projection of the cube edges – so they join smoothings that differ by a single coordinate – oriented in such a way that they are directed towards the smoothing with higest module in the couple.

Remark. The costruction just done works for any cube in a category, see next section, and we call its result the *standard projection*.

There is also a standard way to label arrows: to each one of them we associate a vector, identical to its tail smoothing vector, except in the coordinate that changes, where it has a $\star$. An example clarifies more than just a description, so below is shown the labelling of an arrow in the trefoil's cube.

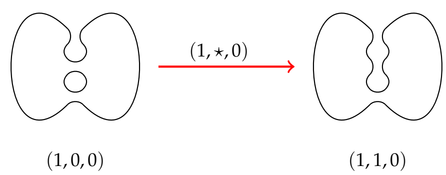


FIGURE 5. Labeling of states and arrow in the trefoil's cube.

In figure 4 there are two types of arrows: red ones and black ones; those in red are the arrows whose - not yet defined - associated morphism will carry a minus sign. In general, an arrow in a cube of smoothings will be colored red if the star in its label is preceded by an odd number of ones. This detail will be essential when defining the differential in the Khovanov bracket: this choice of the sign ensure the skew-commutativity of our cube and ththe fact that the composition of the differentials is zero.

Let $v_\star$ be an arrow in the smoothings cube associated to $\mathcal{T}$, with $\mathcal{T}$ a link, from the resolution associated to v_0 to the one associated to v_1; an example is shown below. The smoothings corresponding to v_0 and v_1 are identical except in a small area near the crossing corresponding to the $\star$ in $v_\star$, where they are smoothed differently. A neighbourhood of this crossing, which does not intersect other arcs or crossings, is called *changing region* or *changing disc*.

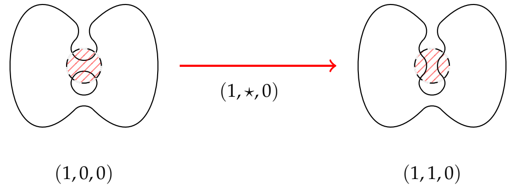
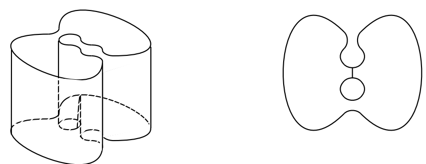


FIGURE 6. The area in red represent the changing disc.

The replacement of by – or viceversa, depending on the smoothed crossing – can be seen as the passage through a saddle point in a cobordism. The latter can be built as follows: take the cylinder over the resolution associated with v_0, remove the one over the changing disc and plug in a saddle cobordism , – or , depending on the replacement – which will be indicated with the symbol - resp. .



$(1, \star, 0)$

FIGURE 7. Two ways to represent an arrow: with the notation described and as a cobordism.

Everything said until now works just fine for knot and link diagrams, but cobordisms between smoothing of tangle diagrams are a little trickier to define.

Definition 1.2. A *cobordism between tangle smoothings* is a smooth surface S, properly embedded in $\mathbb{D}^2 \times [0,1]$, with the following properties:

(a) $S \cap \partial\left(\mathbb{D}^2 \times [0,1]\right) = \partial S$.
(b) $S \cap \mathbb{D}^2 \times \{i\} = \mathcal{T}_i$, for $i \in \{0,1\}$, is a tangle smoothing.
(c) T_0 and T_1 have boundary set B.
(d) $S \cap \partial\mathbb{D}^2 \times [0,1] = B \times [0,1]$.

$\mathcal{T}_0$ is the *upper boundary*, $\mathcal{T}_1$ is the *lower boundary* and $\partial S \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$ is called *vertical boundary*.

If $B = \emptyset$, our cobordism turns out to be a cobordism between link resolutions. If $\mathcal{T}$ is a tangle diagram, given two resolution of $\mathcal{T}$, say v_0 and v_1, connected by an arrow, $v_\star$, we can repeat the above construction and obtain a cobordism between tangle diagrams smoothings.

1.2. Cubes in categories. At the begining of this chapter we said that, to formalize our construction, some abstract nonsense would be needed.

Remark. For the rest of the paper, if not explicitly stated otherwise, the word "category" will mean "small category" – i.e. the objects and, as a consequence, the morphisms form two sets instead of being proper classes. Any reader that wishes to investigate more on categories may refer to: [**AdHerStr**] or [**MacLane**].

Definition 1.3. A *cube* in a category $\mathscr{C}$ is a collection of objects $V_v \in \mathscr{O}bj(\mathscr{C})$, called *vertices* and indexed over the vertices of the n-cube, and morphisms ξ_a, called *edges* and defined for each edge a of the cube, such that: if a is the edge between $v,\ u$ and $|v|+1=|u|$, then:

$$\xi_a : V_v \longrightarrow V_u.$$

A cube is said to be *commutative* if for each square $v_{00},\ v_{01},\ v_{10},\ v_{11}$ – i.e. four vertices such that, for all $i \in \{0,1\}$, there exist edges $a_{i\star},\ a_{\star i}$ such that: $a_{i\star}$ joins v_{i0} and v_{i1}, and $a_{\star i}$ joins v_{0i} and v_{1i} – the following diagram commutes

$$\begin{tikzcd}
V_{v_{00}} \arrow[r, "\xi_{a_{0\star}}"] \arrow[d, "\xi_{a_{\star 0}}"'] & V_{v_{01}} \arrow[d, "\xi_{a_{\star 1}}"] \\
V_{v_{10}} \arrow[r, "\xi_{a_{1\star}}"'] & V_{v_{11}}
\end{tikzcd}$$

In this case we will say that *the square* $v_{00},\ v_{01},\ v_{10},\ v_{11}$ *commutes*.

Our cube of smoothings is not yet a cube in the abstract (non)sense just defined: we need a category where it could fit in; we will worry about this in the next section, for now let us continue with categories and cubes. Given a cube we will use the labelling for the arrows explained in the previous subsection.

Definition 1.4. A *pre-additive* category is a category $\mathscr{A}$ together with a family of operations $\{+_{A,B}\}_{A,B\in\mathscr{O}bj(\mathscr{A})}$ such that:

(a) $(\mathscr{A}r(A,B), +_{A,B})$ is an abelian group, for all $A,\ B \in \mathscr{O}bj(\mathscr{A})$;
(b) the composition $\circ_{\mathscr{A}}$ is bilinear.

Given an arbitrary category $\mathscr{C}$ its *pre-additive closure* is the category $\mathscr{C}_{pa}$, which is: $\mathscr{C}$ itself, if it is pre-additive to begin with; otherwise, $\mathscr{C}_{pa}$ has the same objects as $\mathscr{C}$, but the morphisms are given by the free abelian group over $\mathscr{A}r(\mathscr{C})$ and the composition is the bilinear extension of the original composition.

Remark. Given a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$, F extends to their pre-additive closure in a natural way: by linear extension.

A cube in a pre-additive category is *skew-commutative* if, for each square $v_{00},\ v_{01},\ v_{10},\ v_{11}$, holds:

$$\xi_{a_{1\star}} \circ \xi_{a_{\star 0}} + \xi_{a_{\star 1}} \circ \xi_{a_{0\star}} = 0,$$

where 0 is the neutral element of $\mathscr{A}r(V_{v_{00}}, V_{v_{11}})$.

Lemma 1. *Let C be a commutative cube in the pre-additve category $\mathscr{C}$. The cube $\overline{C}$, obtained from C by defining*

$$\overline{\xi}_{v_\star} = sign(v_\star)\xi_{v_\star},$$

where $sign(v_\star)$ is $(-1)^k$ and k is the number of ones preceding the $\star$ in the label associated with $v_\star$, is skew-commutative.

Proof. Let v_{00}, v_{01}, v_{10}, v_{11} be a square. To show the skew-commutativity of this square we must take into account the position of the changing coordinates: there are four possible cases, depending on the oddity of the number of ones before the first changing coordinate and beween the two changing coordinates; it is easy to see that the following relations hold

$$sign(a_{0\star}) = -sign(a_{1\star}), \qquad sign(a_{\star 1}) = sign(a_{\star 0}). \tag{2}$$

By definition, we have the following equality:

$$\overline{\xi}_{a_{1\star}} \circ \overline{\xi}_{a_{\star 0}} = sign(a_{1\star}) \cdot sign(a_{\star 0})\xi_{a_{1\star}} \circ \xi_{a_{\star 0}} =,$$

from the commutativity of C it follows

$$= sign(a_{1\star}) \cdot sign(a_{\star 0})\xi_{a_{\star 1}} \circ \xi_{a_{0\star}} =,$$

finally, from (2), we obtain

$$= -sign(a_{\star 1}) \cdot sign(a_{0\star})\xi_{a_{\star 1}} \circ \xi_{a_{0\star}} = -\overline{\xi}_{a_{\star 1}} \circ \overline{\xi}_{a_{0\star}}$$

Q.E.D.

Let $\mathscr{C}$, $\mathscr{D}$ be two pre-additive categories. A functor F from $\mathscr{C}$ to $\mathscr{D}$ is *pre-additive* if

$$F(n \cdot f + m \cdot g) = n \cdot F(f) + m \cdot F(g),$$

for every $f,\ g \in \mathscr{A}r(\mathscr{C})$ and $m,\ n \in \mathbb{Z}$.

Proposition 4. *Let C be a cube in $\mathscr{C}$ and F a pre-additive covariant functor from $\mathscr{C}$ to $\mathscr{D}$. The cube $F(C)$, with vertices $W_v = F(V_v)$ and edges $\zeta_a = F(\xi_a)$ – where V_v and ξ_a are vertices and edges of C – is (skew) commutative if C is (skew) commutative.*

Proof. The assertion is immediate from the definition of covariant and pre-additive functors.

Q.E.D.

In proposition 4, the covariance is necessary; otherwise we would have needed to re-label the vertices and the arrows by switching ones and zeros to obtain a true cube. Modulo this re-labelling of vertices and edges, the proposition holds also for contravariant pre-additive functors.

1.3. Complexes. Let $\mathscr{C}$ be a pre-additive category, a (*cochain*) *complex* over $\mathscr{C}$ is a family of objects $(\Omega_i)_{i\in\mathbb{Z}}$, together with a family of morphisms $(d^i)_{i\in\mathbb{Z}}$ such that: $d^i : \Omega_i \to \Omega_{i+1}$ and $d^i \circ d^{i+1} = 0$, for all $i \in \mathbb{Z}$. The morphisms d^i are called *differentials* while the Ω_is are called (*co*)*chain spaces*.

A *morphism between two complexes*, say (Ω_i, d^i) and (Γ_j, δ^j), is a family of morphisms $f_i : \Omega_i \to \Gamma_{i+k}$ – with $k \in \mathbb{Z}$, called *degree* of f, indipendent of i – such that:

$$\delta^{i+k} \circ f_i = f_{i+1} \circ d^i,$$

or, equivalently, the following diagram commutes

$$\begin{tikzcd} \Omega_i \arrow[r, "d^i"] \arrow[d, "f_i"'] & \Omega_{i+1} \arrow[d, "f_{i+1}"] \\ \Gamma_{i+k} \arrow[r, "\delta^{i+k}"'] & \Gamma_{i+k+1} \end{tikzcd}$$

for each $i \in \mathbb{Z}$.

The composition of two morphisms $f = (f_i)_{i\in\mathbb{Z}}$ and $g = (g_j)_{j\in\mathbb{Z}}$ is the morphism defined by

$$(f \circ g)_i = f_{i+deg(g)} \circ g_i;$$

and the neutral element for composition is given by the identity $id_\Omega = (id_{\Omega_i})_i$.

Remark. Notice that the degree is additive with respect to the composition. In particular, if we compose two degree-0 morphisms we obtain a degree-0 morphism.

Finite complexes and morphisms over a pre-additive category $\mathscr{C}$ form themself a category, indicated as $\mathscr{K}om(\mathscr{C})$. Sometimes a sub-category of $\mathscr{K}om$ will be used; this sub-category, denoted $\mathscr{K}om_0$, has the same objects as $\mathscr{K}om$ and its morphisms are the degree-0 morphisms in $\mathscr{K}om$.

The aim of the chapter is to prove the invariance of the Khovanov homology but, if we are not in an abelian category, one cannot properly define the homology of a formal complex. Needless to say, the category we are going to work with is not abelian, but, luckily, there is a condition for chain complexes which implies having isomorphic homologies: the chain equivalence.

Definition 1.5. Given two degree-0 morphisms $F, G : \Omega \to \Gamma$, with Ω, Γ complexes, F and G are *chain homotopic* if there exists a degree -1 morphism P between Ω and Γ, such that:

$$F_i - G_i = \delta^{i-1} \circ P_i \pm P_{i+1} \circ d^i.$$

The morphism P is called *prism map*, or *chain homotopy map*. Two complexes Ω and Γ are said to be *chain equivalent*, or *homotopy equivalent*, if there exist two morphisms

$$K : \Omega \to \Gamma, \qquad H : \Gamma \to \Omega,$$

such that $K \circ H$ and $H \circ K$ are degree-0 and chain homotopic to Id_Ω, and Id_Γ, respectively.

Being chain equivalent is an equivalence relation in the category of complexes and all the morphisms descend to the quotient – because they commute with the differentials; so we can define the category of *complexes modulo homotopy* $\mathscr{K}om_{/h}$ – also $\mathscr{K}om_{0/h}$ – as the category whose objects and morphisms are the equivalence classes of the objects in $\mathscr{K}om$ – resp. $\mathscr{K}om_0$ – with respect to chain equivalence.

1.4. The formal complex. Let B the boundary set of some tangle diagram. The category $\mathscr{C}ob^3(B)$ is the pre-additive closure of the category $\mathscr{C}_2(B)$; the latter is defined as follows: its objects are smoothings of tangle diagrams having B as a boundary set , and the arrows are cobordisms between tangle smoothings, considered up to boundary fixing isotopies. The domain of a morphisms in $\mathscr{C}_2(B)$ is its upper boundary, while the codomain is given by the lower boundary.

Remark. According to the definitions of tangle and tangle diagram we have given, also the empty set can be seen both as a tangle and as a tangle diagram. The unique smoothing of this tangle diagram is the *empty smoothing*, and this will be included as object in $\mathscr{C}_2$. Moreover, a cobordism between two empty smoothings can be built in different ways, e.g. a sphere, a torus or the *empty cobordism*, i.e. the empty set viewed as a morphism in $\mathscr{C}_2$.

Consider $S,\ S'$, two cobordisms between tangle smoothings, their composition is possible if the lower boundary of S, is equal to the upper boundary of S' - or viceversa; in this case, $S' \circ S$ is - up to boundary fixing isotopies - the surface obtained by gluing together a tubular neighbourhood of the lower boundary of S with a tubular neighbourhood of the upper boundary of S'.

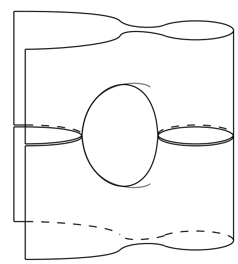


FIGURE 8. Composition of two morphisms in $\mathscr{C}_2$.

Definition 1.6. Given a pre-additive category $\mathscr{C}$, its *additive closure* – or *matrix category* – denoted $\mathscr{M}at(\mathscr{C})$, is the category defined by the following properties:

(a) $\mathscr{O}bj\,(\mathscr{M}at(\mathscr{C}))$ are formal finite – or empty – direct sums of objects of $\mathcal{C}$.
(b) Given $C = \bigoplus_i^n C_i,\ D = \bigoplus_j^m D_j \in \mathscr{O}bj\,(\mathscr{M}at(\mathscr{C}))$, a morphism between them is a $m \times n$ matrix $\left(F_{j,i}\right)$, with $F_{j,i} \in \mathscr{A}r(C_i, D_j)$.
(c) If C, D are object in $\mathscr{M}at(\mathscr{C})$, $\mathscr{A}r(C, D)$ has a natural structure of abelian group given by matrix addition.
(d) The composition of two morphisms $\left(F_{j,i}\right)$, $\left(G_{k,j}\right)$ is given by the "matrix multiplication" rule:

$$\left(\left(G_{k,j}\right) \circ \left(F_{j,i}\right)\right)_{s,t} = \sum_j G_{s,j} \circ F_{j,t}.$$

Remark. Any functor between pre-additive categories extends naturally – not in the technical sense – to their additive closure.

Remark. The empty sum, also denoted 0, is an initial and also a final object in $\mathscr{M}at(\mathscr{C})$. The unique morphism from an object O to 0 is given by the empty matrix, i.e. the matrix without entries, and this is also the unique morphism with source 0 and target any other object; so $\mathscr{A}r(0, O)$, as well as $\mathscr{A}r(O, 0)$, can be given the trivial group structure and its unique element will be, with an abuse of notation, called 0.

Remark. Sometimes we will need to take the formal direct sum of objects in $\mathscr{M}at(\mathscr{C})$, and the result should be an object in $\mathscr{M}at(\mathscr{C})$. In the present setup this is not true: we cannot add an object $\bigoplus A_i$ to 0, because 0 is not an object in the original category. It is not difficult to solve this problem: every time we add 0, the object we are adding it to remains untouched.

Now everything is set up; given an oriented tangle diagram $\mathcal{T}$ - with boundary set B - we can consider any of its smoothings as an object in $\mathscr{M}at(\mathscr{C}ob^3_{pa}(B))$ and define:

$$[\![\mathcal{T}]\!]^i = \bigoplus_{|S|=i+n_-} S,$$

where $|S|$ is the module of S. Moreover, to each arrow $v_\star$ is associated a morphism, denoted $d_{v_\star}$, in $\mathscr{M}at(\mathscr{C}ob^3_{pa}(B))$, so we can define:

$$d^i = \sum_{|v_0|=i+n_-} sgn(v_\star) d_{v_\star},$$

with v_0 the tail of $v_\star$, i.e. the source smoothing, and $sgn(v_\star)$ is $+1$, if there is an even nuber of ones before the $\star$, in the notation described in the first subsection, or -1, if that number is odd.

Remark. Most of the constructions described until now are indipendent of the orientation of the tangle diagram; but, the previous definition depends on the orientation: when we take the – direct – sum along the columns we shift the complex by a n_- on the right; whether a crossing is positive or negative, depends on the chosen orientation. This shift is indeed necessary to prove invariance.

Proposition 5. *Let $\mathcal{T}$ be a tangle; $d^i \circ d^{i+1}$ is zero.*

Proof. The proof of this proposition consists of showing that without signs the cube of smoothings, which is a cube in $\mathscr{C}ob^3(B)$, commutes and the assertion will follow from proposition 1. Given a square v_{00}, v_{10}, v_{01}, v_{11}, the two compositions $d_{v_{\star 0}} \circ d_{v_{1\star}}$ and $d_{v_{0\star}} \circ d_{v_{\star 1}}$ are cylinders except in two changing areas; in each changing area the cylinder over the changing disc is replaced by a saddle. Because the compositions both start from v_{00} and arrive at v_{11}, the two cobordisms must have the same "type" of saddles in the same changing areas;they only differ in the height of the two saddle points. By standard Morse theory, the two saddles can be "height re-ordered" by a boundary fixing isotopy.

Q.E.D.

The complex $[\![\mathcal{T}]\!] = ([\![\mathcal{T}]\!]^i, d^i)_{i\in\mathbb{Z}}$ is called the *Khovanov bracket* of $\mathcal{T}$.

The construction just described is pretty general. Given a skew-commutative cube C in a pre-additive category $\mathscr{C}$, we can see it as a cube in $\mathscr{M}at(\mathscr{C})$; considering its standard projection and taking the direct sum along columns of the vertices and the sum along columns of the edges, as done for the Khovanov bracket, one can define a formal complex $[\![C]\!]$ which will be called the *bracket of C*.

Moreover, if we have a functor F between two pre-additive categories, once extended the functor to their additive closure and also to complexes, we have $[\![F(C)]\!] = F\left([\![C]\!]\right)$ for every (skew) commutative cube C. In particular, if the Khovanov bracket is invariant under Reidemeister moves and planar isotopies, that is to say is a tangle invariant, also its functorial image is a tangle invariant.

2. Khovanov Complex

Now we want to define a chain complex of $\mathbb{Z}$-modules from our formal complex, so that we can compute homology. For some reasons, which will be explained at the end of the chapter, in this section we will consider only knots or links, and it is important to consider them oriented – at least in the case of links.

2.1. Graded modules and quantum dimension. Before starting the construction it is necessary to recall some algebraic definitions.

Definition 1.7. A ($\mathbb{Z}$-)*graded module* (over a commutative ring R) is a module M together with a decomposition:

$$M = \bigoplus_{n \in \mathbb{Z}} M_n,$$

where M_n is a (possibly trivial) submodule of M, for each n. An element $x \in M$ is called *homogeneous* of degree k if $x \in M_k \setminus \{0\}$.

Remark. Every ring, if unless stated otherwise, from now on will be trivially graded – i.e. all its elements have degree 0.

Given two graded modules, say M and N, their direct sum, as well as their tensor product, inherits the structure of a graded module naturally:

$$(M \oplus N)_n = M_n \oplus N_n,$$

$$(M \otimes N)_n = \bigoplus_{i+j=n} M_i \otimes N_j.$$

Let M be a finitely generated graded module, say over $\mathbb{Z}$ – this is the only case we shall consider – each M_n is a sub-module of M so its rank – i.e. the dimension of the free part – is defined.

The *graded* (or *quantum*) *dimension* of M is defined as:

$$qdim(M) = \sum_{n \in \mathbb{Z}} rank(M_n) T^n;$$

while the *graded Euler characteristic* of M is

$$\chi_g(M) = \sum_{n \in \mathbb{Z}} (-1)^n rank(M_n) T^n.$$

Notice that both the quantum dimension and the graded Euler characteristic are Laurent polynomial in T; furthermore, the following equalities hold

$$\chi(M) = \chi_g(M)[1], \quad \chi_g(M)[-T] = qdim(M)[T].$$

The basic properties of the graded dimension, which can be easily verified, are listed below:

(1) $qdim(M \oplus N) = qdim(M) + qdim(N)$;
(2) $qdim(M \otimes N) = qdim(M) \cdot qdim(N)$;
(3) $qdim(\mathcal{S}_k(M)) = T^k \cdot qdim(M)$;

where $S_k(M)$ is the k-degree shift of M, that is to say:

$$\mathcal{S}_k(M) = \bigoplus_{n \in \mathbb{Z}} M_{n-k}.$$

2.2. From the bracket to the complex. Now we turn to the definition of the Khovanov complex; this is based on the Khovanov bracket: to each smoothing will be assigned a module and to each arrow a map. The sum – direct sum in the case of modules – over a column will give us the chain group, if we sum modules, and the differential when we sum arrows, in accordance of what was previously done with the bracket.

Let V be the free $\mathbb{Z}$-module generated by x_-, x_+, with grading induced by:

$$deg(x_+) = 1, \qquad deg(x_-) = -1.$$

Given a resolution S to each circle we associate the module V and then tensor over all the circles in S; so, if k_S is the number of circles in S, we obtain $V^{\otimes k_S}$, then we apply a grade shift:

$$V_S = \mathcal{S}_{r_S}\left(V^{\otimes k_S}\right),$$

where r_S is the sum $|S| + n_+ - 2n_-$.

Definition 1.8. The i-th *Khovanov chain group* is the $\mathbb{Z}$-module:

$$C^{i,*}(\mathcal{D}) = \bigoplus_{|S| = i + n_-} V_S.$$

The integer i is called *homological degree*.

Remark. It is an easy exercise to see that the graded Euler characteristic of the complex $C^{*,*}(\mathcal{D}) = \bigoplus_{i \in \mathbb{Z}} C^{i,*}(\mathcal{D})$ is the unnormalized Jones polynomial $\hat{\mathcal{J}}(\mathcal{D})$. The latter is defined as

$$\hat{\mathcal{J}}_{\mathcal{L}}(q) = (q - q^{-1}) \cdot \mathcal{J}_{\mathcal{L}}(q),$$

where $\mathcal{J}$ is the Jones polynomial defined in Chapter 0. [Hint: use the form in state summation of $\mathcal{J}$.]

Every Khovanov chain group has, thanks to the grading of V, a natural graded structure:

$$C^{i,*}(\mathcal{D}) = \bigoplus_{j\in\mathbb{Z}} C^{i,j}(\mathcal{D}),$$

where j is called the *quantum degree* and is defined by

$$qdeg(x) = deg(x) + i + n_+ - n_-;$$

where: x is an homogeneous element of V_S – with $|S| = i + n_-$ – and deg is its degree in V_S.

Now take a smoothing vector v_0, and an arrow $v_\star$ pointing from v_0 to v_1. As observed in the previous section, there are two possible changes from the tail to the head, depending on how many circles intersect the changing region: the fusion of two circles or the splitting of a circle in two, as shown below.

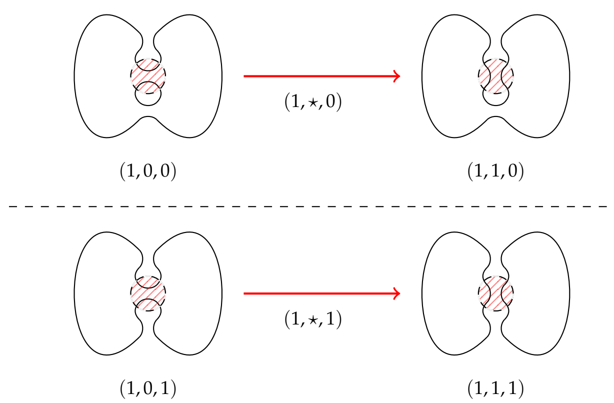


FIGURE 9. A fusion – on the top – and a splitting – on the bottom – of circles.

In the Khovanov bracket, to the arrow $v_\star$ is associated a cobordism $d_{v_\star}$ given by a number of cylinders and a pair-of-pants surface. This cobordism $d_{v_\star}$ is the identity cobordism – a cylinder – for all the circles except for the circle(s) involved in the changing disc; here is the pair-of-pants that takes care of the splitting or the fusion, depending on how it is directed: upward or downward.

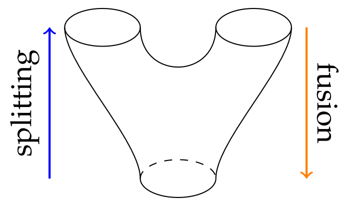


FIGURE 10. A pair-of-pants surface.

Remark. The cobordisms drawn across the paper are always directed downward, unless otherwise specified.

The map $\delta_{v_\star}$, that we want to associate to the arrow $v_\star$, goes from V_{v_0} to V_{v_1}. Over each V corresponding to a circle that doesn't intersect the changing disc, it must be the identity map; while, on the V(s) corresponding to the remaining circle(s) it must split V in $V \otimes V$ (resp. fuse $V \otimes V$ in V); as a consequence of everything said until now, $\delta_{v_\star}$ is the tensor of $k_{v_0} - 1$ identity maps with a fusion – or splitting map.

The two latter maps, denoted m and Δ, respectively, are defined as follows:

$$m : V \otimes V \to V,$$

is such that:

$$m(x_+ \otimes x_+) = x_+, \quad m(x_+ \otimes x_-) = m(x_- \otimes x_+) = x_-, \quad m(x_- \otimes x_-) = 0;$$

while

$$\Delta : V \to V \otimes V,$$

is defined by

$$\Delta(x_+) = x_- \otimes x_+ + x_+ \otimes x_-, \quad \Delta(x_-) = x_- \otimes x_-.$$

The *i-th differential in the Khovanov complex* is defined in a similar way to what was done for the bracket:

$$\delta_i = \sum_{|v_0| = i + n_-} sgn(v_\star)\delta_{v_\star}.$$

We can verify that the map defined just now is a differential and, consequently, that the Khovanov complex is a complex, but we postpone this matter until the next section.

2.3. TQFT. The construction of the Khovanov complex just given, is incomplete; what we have done is sufficent to define a chain complex but does not represent a fuctor yet: we do not know which morphism is associated to, for example, a cap. In this condition even proving the invariance – up to chain homotopy – could be challenging.

The underlaying functor is an example of TQFT (acronym for *Topological Quantum Field Theory*), which is a monoidal functor from the category $\mathscr{C}_n$, of $(n-1)$-dimensional closed smooth manifolds and n-dimensional cobordisms between them, to the category $\mathscr{M}od(\mathbb{Z})$; the word "monoidal" means that it is a functor between categories "with tensor product" – called monoidal categories – that respects this structure – i.e. commutes with $\otimes$ and sends unity object/morphism to unity object/morphism. The study of generic TQFTs, as well as monoidal functors, goes beyond the scope of this work; the interested reader may refer to [**MAtiyah**], [**JohnPMay**] or [**JacLurie**]. In our case the definition turns out to be:

Remark. The category $\mathscr{C}ob^3(\emptyset)$ is in fact the pre-abelianized of $\mathscr{C}_2$, so any functor defined over $\mathscr{C}_2$ extends by linearity to $\mathscr{C}ob^3$. Moreover, both the mentioned categories can be given a monoidal structure by using as tensor product the disjoint union and as unit object/morphism the empty set.

Definition 1.9. A $(1+1)$-*TQFT* is a functor F from $\mathscr{C}ob^3(\emptyset)$ to $\mathscr{M}od(\mathbb{Z})$, which satisfies the following properties:

(a) $F(\emptyset) = \mathbb{Z}$, $F(\emptyset_{cob}) = id_{\mathbb{Z}}$;
(b) $F(S \sqcup S') = F(S) \otimes_{\mathbb{Z}} F(S')$, for all $S,\ S' \in \mathscr{O}bj(\mathscr{C}ob^3(\emptyset))$;
(c) $F(c \sqcup c') = F(c) \otimes_{\mathbb{Z}} F(c')$, for all $c,\ c' \in \mathscr{A}r(\mathscr{C}ob^3(\emptyset))$;

where $\emptyset_{cob}$ is the empty set seen as 2-dimensional cobordism between $\emptyset$ and $\emptyset$.

A $(1+1)$-TQFT defines, and is defined by, a particular algebraic structure: a Frobenius algebra. If the interested reader wants to know more about this structure then he – or she – can refer to: [**LAbrams**].

Definition 1.10. Let R be a domain. A (commutative finite dimensional) *Frobenius algebra* over R is a finitely generated projective R-module M together with two linear maps:

$$m : M \otimes_R M \to M, \qquad \varepsilon : M \to R,$$

called, respectively, *multiplication* and *trace* – or *co-unit*–, such that:

(a) $m(u \otimes v) = m(v \otimes u)$.
(b) There exists an element $e \in M$ for which $m(v \otimes e) = v$, for each $v \in M$.
(c) $\prec \cdot, \cdot \succ : M \times M \to R : (v, u) \mapsto \varepsilon(m(v \otimes u))$ is non degenerate.

Given a Frobenius algebra M we can define, using the fact that $\prec \cdot, \cdot \succ$ is non-degenerate, another map:

$$\Delta : M \to M \otimes_R M,$$

where $\Delta(v) = \sum_i u_i \otimes_R v_i$ is the unique element such that:

$$m(v \otimes y) = \sum_i \prec v_i, y \succ u_i,$$

with v_i and u_i two bases of M. The map just defined, Δ, is called *co-multiplication*. Finally, we can define also

$$\iota : R \to M : 1_R \mapsto e;$$

which is called *unity* or *unit map*.

Remark. The structure $(M, m, \Delta, \iota, \varepsilon)$ is <u>not</u> an Hopf Algebra.

Example 1. *Let L be the $\mathbb{Q}$-module generated by x_+ and x_-; L is given a graded structure by grading x_+ and x_- as follows:*

$$deg(x_\pm) = \pm 1.$$

Take as multiplication on L the map m defined by

$$m(x_+ \otimes x_+) = x_+, \quad m(x_+ \otimes x_-) = m(x_- \otimes x_+) = x_-,$$

$$m(x_- \otimes x_-) = x_+;$$

and as a co-unit the map

$$\varepsilon(x_+) = 0, \quad \varepsilon(x_-) = 1.$$

One could verify that $\prec \cdot, \cdot \succ$ is non degenerate, which means that the maps above define a Frobenius algebra structure on L. Carrying on the computations we find out that the co-multiplication map Δ is defined by

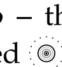
$$\Delta(x_+) = x_+ \otimes x_+ + x_- \otimes x_+ - x_+ \otimes x_+,$$

$$\Delta(x_-) = x_- \otimes x_- + x_+ \otimes x_+,$$

and the unit is

$$\iota(1) = x_+.$$

Notice that neither Δ nor m are degree homogeneous.

A TQFT is related with a Frobenius algebra in the following way: the pair-of-pants surface correspond to the multiplication or to the co-multiplication – depending if it is regarded as a fusion or as a splitting cobordism. The map associated to a cap – the disc seen as cobordism between the empty set and a circle, denoted  – is the unit, while the co-unit is associated to the disc but regarded as cup – i.e. cobordism between the circle and the empty set, denoted  .

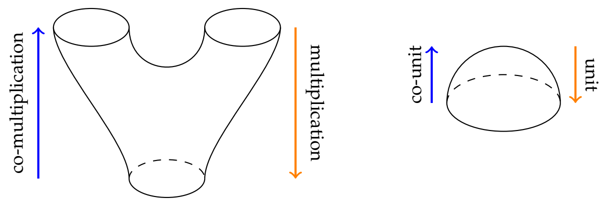


Figure 11. Relation between Frobenius algebras and TQFTs.

So our splitting and fusion maps are the Frobenius algebra's maps related to the applied TQFT. The unit and the co-unit, which were not defined, can be obtained from the multiplication an co-multiplication; for example, from:

$$\Delta(x_+) = x_- \otimes x_+ + x_+ \otimes x_-,$$

, we obtain:

$$x_+ = m(x_+ \otimes x_+) = \varepsilon(m(x_+ \otimes x_+))x_- + \varepsilon(m(x_- \otimes x_+))x_+ =$$

$$= \varepsilon(x_+)x_- + \varepsilon(x_-)x_+,$$

by the indipendence of x_+ and x_-, we must have:

$$\varepsilon(x_+) = 0, \qquad \varepsilon(x_-) = 1.$$

While, the unity is easily determined:

$$\iota(1) = x_+.$$

The fact that we are applying a TQFT, for what we have said about brackets of cubes and functors, implies that the Khovanov complex is the bracket of the cube obtained from the cube of smoothings via a covariant pre-additive functor; in particular, it is a complex.

3. Invariance

Let $\mathcal{T}$ an oriented tangle diagram. We supposed that an order of the crossings of $\mathcal{T}$ was fixed in order to obtain the correspondence smoothing-smoothing vectors. Our construction is based on this correspondence. A change of order has the effect of changing the level of the smoothings within a column at the level of cube, and of change in the order of the sum in $\mathscr{K}ob$. These changes produce an isomorphic complex. So the invariance under the change of ordering is easily proven. The invariance we want is the invariance for tangle type: if we perform a Reidemeister move on a tangle the complexes before and after the move must be equivalent.

3.1. Local moves. If we proved the invariance – up to chain homotopy – of the Khovanov bracket as things are now, we would prove that the Khovanov homology is invariant, and that would also mean every TQFT generates a link homology theory. This is false: the fact our theory is a link invariant depends on the TQFT applied. For example, if our TQFT associates to a circle the graded module A, the fact that $[\![\cdot]\!]$ is a link invariant implies that the characteristic of the complex is indipendent of the chosen diagram; in particular, the complexes associated to the two representations of the unknot shown below must have the same Euler characteristic.

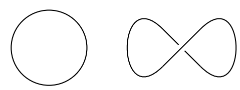


FIGURE 12. Two diagrams of the unknot.

The associated chian groups would be respectively:

$$0 \longrightarrow A \longrightarrow 0, \qquad 0 \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0,$$

so we must have:

$$rank(A) = -rank(A) + rank(A \otimes A) = rank(A)^2 - rank(A),$$

which implies

$$rank(A) = 2.$$

The rank of A is not the only property our theory must satisfy. From abstract considerations, see **[KhovUniv]**, it turns out that any TQFT that gives rise to a link homology theory must satisfy:

$$\varepsilon(\iota(1)) = 0,$$

and also

$$\varepsilon(m(\Delta(\iota(1)))) = 2;$$

these are called *S-relation* and *T-relation*, respectively, and their geometric counterpart is drawn below.

FIGURE 13. S and T local relations.

A TQFT trasforms the disjoint union in tensor products so, as a consequence, the S and T relations imply that any cobordism containing a sphere – or a torus – as connected component is the 0 morphism – resp. 2 times the cobordism without the torus.

A third geometric relation, which can be algebraically interpreted in different ways, is the 4-*Tu* relation shown in the figure below.

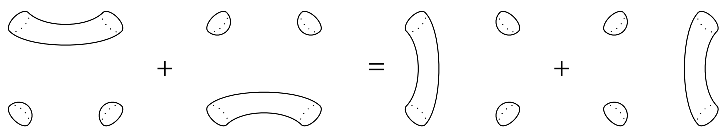


FIGURE 14. The 4-Tu relation.

The picture above is meant to be read in the following way: if we intersect a surface with a 3-disc, in such way that the boundary of the intersection are 4 circles, the sum of the surface obtained by replacing the inner part of the disc with the first configuration in figure 14, and the same surface with the inner part of the disc replaced by the second configuration is equal to the sum of the surfaced obtained, with the same technique, from the third and the fourth configurations. A complete proof that at least our TQFT satisfies this relation is given in [**Tamburr**]; this proof is divided into cases, depending on how many circles "come from" the upper boundary of the cobordism, and consist on the computation of the maps generated by discs and tubes in each configuration.

The category $\mathscr{C}ob^3(B)$ modulo the local relations S, T and 4-*Tu*, is denoted $\mathscr{C}ob^3_\ell(B)$; from the fact these relations hold we have that our TQFT descends to a functor from $\mathscr{C}ob^3_\ell$ to $\mathscr{M}od(\mathbb{Z})$.

3.2. Planar algebras. Now we want to introduce the "main tool" for demonstrating the invariance: the planar algebras. These will allow us to "patch up" the "local" proofs of the invariance, i.e. the proofs for the tangle diagrams involved in the Reidemeister moves – to obtain the invariance for all possible tangles.

A *k-tangle*, with $k \in \mathbb{N}$ positive, is a tangle diagram whose boundary set is the set of the $2k$-th roots of unity – a 0-tangle is, by definition, a link. The set of k-tangles is denoted by $\mathscr{T}(k)$; while the set of the oriented k-tangles is $\mathscr{T}(\underline{k})$, where $\underline{k}$ is a vector whose entries are in $\{\pm 1\}$: the i-th coordinate is 1 if an arc begins there, and -1 if it ends there.

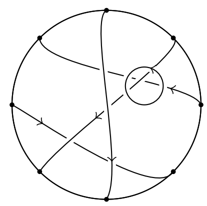


FIGURE 15. An oriented 4-tangle in $\mathscr{T}$(-1,-1,-1,+1,-1,+1,+1,+1).

A *planar arc diagram* is a k-tangle without crossing with a number of open balls removed – called *holes*. These holes are usually ordered and have marked points on their boundary, corresponding to the intersection of the strands with the boundary of the hole. These marked points are called *inner gates*, while the marked points on the boundary of the tangle are called *outer gates*. In the oriented version we can give a sign to the inner gates as well: if a strand begins in an inner gate then the sign will be -1, otherwise $+1$.

Remark. The sign convention for the inner gates is opposite to the sign convention for the outer gates.

The set of all the planar arc diagrams – p.a.d. – is denoted by $\mathscr{D}$, or $\mathscr{D}(k; k_1, ..., k_h)$ if we want to emphasize the number of outer gates – k – and the number of inner gates in the i-th circle – k_i, for $i \in \{1, ..., h\}$ with h the number of holes. The oriented version is similar but each k_i is replaced by $\underline{k}_i$, where the latter is a vector carrying the signs of the inner gates in the boundary of the i-th hole.

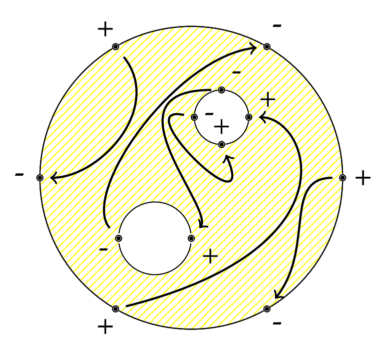


FIGURE 16. An oriented planar arc diagram.

A special class of planar arc diagrams is given by the so-called *identity diagrams* – an example of which is shown below. The identity diagram with $2k$ strands – or, equivalently, with $2k$ outer gates – is denoted by I_k.

The oriented version is almost identical, but with the strands oriented; the notation is the same with k replaced by $\underline{k}$, a sign vector for the outer gates.

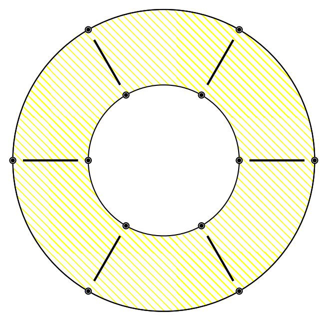


FIGURE 17. The I_3 diagram.

Before giving the definition of planar algebra, it is necessary to introduce the most important example in the subject: the *planar arc algebra* – p.a.a. This is given by: the collection of all the tangle spaces $\{\mathscr{T}(k)\}_{k\in\mathbb{N}}$, the collection of all the planar arc diagrams $\mathscr{D}$, an action of the planar arc diagrams over the tangle spaces – which is described below – and a composition rule for diagrams with an associative property.

The action of the planar arc diagrams over the tangle spaces is, as one could expect, just given by filling the holes with tangles.

Let $D \in \mathscr{D}(k; k_1, ..., k_n)$ be a planar arc diagram; D defines a map

$$D : \mathscr{T}(k_1) \times \cdots \times \mathscr{T}(k_n) \longrightarrow \mathscr{T}(k),$$

such that $D(t_1, ..., t_k)$ is the tangle obtained by shrinking – or enlarging – t_i to the dimension of the i-th hole and, paying attention to match the order of the i-inner gates with the boundary set of t_i, fill the i-th hole with t_i.

The action of the oriented diagrams over the oriented tangles is defined just in the same way, with the additional condition that the signs of the gates must match.

Moreover, we can compose the diagrams using the same technique, and the following associativity property is almost tautological:

$$D \odot C(t_1, ..., t_{i-1}, \bar{t}, t_{i+1}, ..., t_h) = D(t_1, ..., t_{i-1}, C(\bar{t}), t_{i+1}, ..., t_h), \tag{3}$$

where $t_1, ..., \hat{t}_i, ..., t_h$ are tangles with the correct boundary and $\bar{t}$ is a tangle vector such that (3) makes sense. The above equation just means that composing the diagrams and letting the result act over tangles is the same as composing the actions. A *planar arc sub-algebra* – p.a.s.-a. – is a subset of the diagrams set $\mathscr{D}'$, which acts over a family of subsets $\mathscr{S}(k) \subseteq \mathscr{T}(k)$, so that each $\mathscr{S}(k)$ is closed under the action of $\mathscr{D}'$; moreover, the latter must be closed under the composition $\odot$ and must contain the identity diagrams for every non-empty tangle set.

Example 1.1. There are few examples of p.a.s.-a. that must be kept in mind:

(a) a first example is given by the *smoothings planar algebra*; this is defined as the sub-algebra of the p.a.a. with diagram set all the planar arc diagrams, so $\mathscr{D}' = \mathscr{D}$, with the same action and composition; what changes is the tangle set: we consider as $\mathscr{S}(k)$ the set of all crossingless k-tangles.

(b) The *k-tangle algebra* has as tangle sets $\mathscr{S}(h)$: $\mathscr{T}(k)$ if $h = k$, otherwise $\mathscr{S}(h)$ is the empty set. The diagram set is given by all the diagrams that have k outer gates and, for each hole, k inner gates.

(c) The *k-smoothing algebra* has as tangle sets – resp. as diagram set – the intersection of the tangle sets – resp. diagram sets – of the previous two p.a.s.-a.

Definition 1.11. An *unoriented planar algebra* – p.a. – is a collection of sets, $\{\mathscr{P}_k\}_{k\in\mathbb{N}}$, together with a family of operators $\{\mathscr{O}_D\}$, indexed over the diagram set of a planar arc sub-algrebra, and a composition product $\odot$, such that:

(a) if $D : T_{k_1} \times \cdots \times T_{k_h} \to T_k$ then $\mathscr{O}_D : \mathscr{P}_{k_1} \times \cdots \times \mathscr{P}_{k_h} \to \mathscr{P}_k$;
(b) $\mathscr{O}_{D\odot(D_1\times\ldots\times D_h)} = \mathscr{O}_D \odot (\mathscr{O}_{D_1} \times \cdots \times \mathscr{O}_{D_h})$;
(c) $\mathscr{O}_{I_k} = Id_{\mathscr{P}_k}$;
(d) if $D \neq D'$ then $\mathscr{O}_D \neq \mathscr{O}_{D'}$.

If we take the oriented diagrams, instead of the unoriented ones, as indexing set for the planar operators and we replace $\mathbb{N}$ with sign vectors, i.e. k-uples of ± 1, the result will be an *oriented planar algebra*.

In fewer words: a planar algebra is given by a set of *planar operators* – i.e. the family $\{\mathscr{O}_D\}$ – that acts over a collection of *tangle spaces* – the family $\{\mathscr{P}_k\}_{k\in\mathbb{N}}$ – such that the composition of the operators and their action are compatible with the identifications of $\mathscr{O}_D$ with D, and of $\mathscr{P}_k$ with a subset of the set $\mathscr{T}_k$.

Remark. In a more formal language, a planar algebra P is a colored operad which is isomorphic to the operad of planar arc sub-algebra. An enthusiastic reader who wants to know more on the subject can consult the original paper by Vaughan Jones [**VFRJones**]; a faster, and perhaps clearer, introduction on the subject can be found in [**BWebster**].

Given two planar algebras, say

$$(\{\mathscr{O}_D\}_{D\in\mathscr{D}'}, \{\mathscr{P}_k\}_k, \odot), \qquad (\{\mathscr{Q}_{D'}\}_{D\in\mathscr{D}''}, \{\mathscr{R}_k\}_k, \odot'),$$

a *planar algebras morphism* Ψ between them is given by a family of maps:

$$\psi_k : \mathscr{P}_k \to \mathscr{R}_k,$$

plus a map:

$$\psi : \mathscr{D}^{'} \longrightarrow \mathscr{D}^{''},$$

such that:

$$\psi_k(\mathscr{O}_D(T_1, ..., T_h)) = \mathscr{Q}_{\psi(D)}(\psi_{k_1}(T_1), ..., \psi_{k_h}(T_h)). \tag{4}$$

For our purposes we need a somewhat stronger definiton of planar algebra: the one we just gave is a "set-theoretic" notion, but we need to work also over complexes. This requires the $\mathscr{P}_k$ to be (free) $\mathbb{Z}-$modules and the operators $\mathscr{O}_D$ to be multilinear maps – also p.a. morphisms will be required to be multilinear.

This extension does not pose a real problem: we can define our algebra over the basis of our (free) $\mathbb{Z}$-modules, and take the multilinear extension of the operators. The reader could verify that all the properties of the definition remain satisfied.

A non-trivial example of planar arc algebra is given by the morphisms of $\mathscr{C}ob^3_\ell$; this is defined as follows: its planar operators are the cylinders over the planar arc diagrams, the tangle spaces are the family $\{\mathscr{A}r\left(\mathscr{C}ob^3_\ell(k)\right)\}_{k\in\mathbb{N}}$, the action is given by filling the holes of the cylinders $D \times [0,1]$, where $D \in \mathscr{D}$ is a planar arc diagram, with tangle cobordisms, and the composition of two operators is given by the cylinder over the composition of the associated planar arc diagrams. An example of the action is shown in the figure below, for further examples the reader may refer to **[BarNatan]** page 1465.

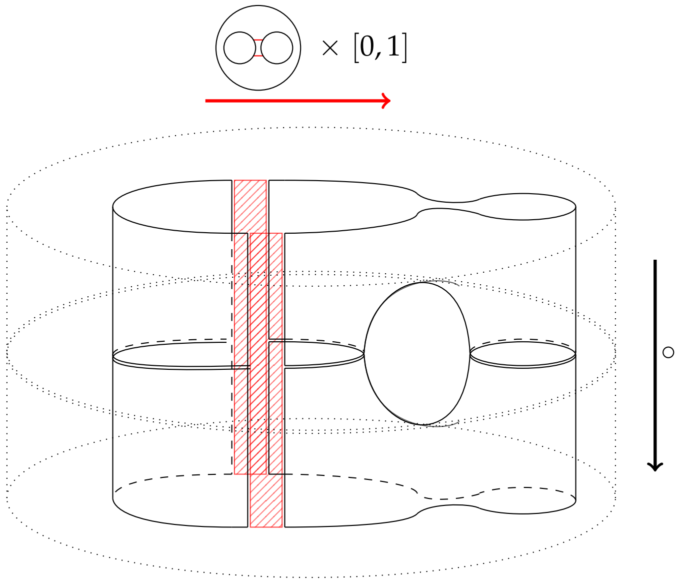


FIGURE 18. Commutativity between composition and the planar arc operator action in the planar algebra of cobordisms.

As shown in figure, the following relation holds

$$D(c_1 \circ c_2, d_1 \circ d_2) = D(c_1, d_1) \circ D(c_2, d_2),$$

for every $c_1,\ c_2,\ d_1,\ d_2 \in \mathscr{A}r(\mathscr{C}ob^3)$ and $D \in \mathscr{D}$ any two-holed diagram; this only means that plugging in two cobordisms in $D \times [0,1]$, taking another copy of $D \times [0,1]$ and plugging in other two cobordisms, composable with the first two cobordisms, and finally put the results one atop the other, is the same thing as composing the cobordisms and plug the result in the "holes" of $D \times [0,1]$.

Definition 1.12. The category $\mathscr{K}ob(k)$ is defined as the sub-category of $\mathscr{K}om\left(\mathscr{M}at(\mathscr{C}ob^3_\ell(k))\right)$ given by all the finite complexes, i.e. all the complexes with at most a finite number of non-zero chain spaces.

Theorem 2. *The following results hold:*

(a) $\mathscr{K}ob$ *has a natural planar algebra structure;*
(b) $[\![\cdot]\!]$ *descends to a planar algebra morphism between the p.a.a. and* $\mathscr{K}ob$*;*
(c) $[\![\cdot]\!]$ *descends to a planar algebra morphism between the p.a.a. and* $\mathscr{K}ob_{/h}$*, i.e.* $\mathscr{K}om$ *modulo chain equivalences.*

Proof. This proof, only skecthed in **[BarNatan]**, will be detailed here. The first step will be to extend the planar algebra structure of both $\mathscr{O}bj(\mathscr{C}ob^3_\ell)$ and $\mathscr{A}r(\mathscr{C}ob^3_\ell)$ to, respectively, $\mathscr{O}bj(\mathscr{M}at(\mathscr{C}ob^3_\ell))$ and $\mathscr{A}r(\mathscr{M}at(\mathscr{C}ob^3_\ell))$.

Any object of $\mathscr{M}at(\mathscr{C}ob^3_\ell(k))$ is a – possibly empty – formal direct sum like

$$O^j = S^1_j \oplus ... \oplus S^{r_j}_j, \quad S^i_j \in \mathscr{O}bj(\mathscr{C}ob^3_\ell(k));$$

given a diagram $D \in \mathscr{D}(k; k_1, ..., k_n)$ we define the corresponding planar operator, denoted D as well, by

$$D(O^1, ..., O^n) = \bigoplus_{i_1=0}^{r_1} \cdots \bigoplus_{i_n=0}^{r_n} D(S^{i_1}_1, ..., S^{i_n}_n),$$

which is nothing more than the multilinear extension of the operator D defined over $\mathscr{O}bj(\mathscr{C}ob^3_\ell)$; in particular, $D(0, ..., 0) = 0$.

The composition of the operators, defined as the operator induced by the composition, satisfies the required associativity property almost trivially. In the same way we extend over the morphisms of $\mathscr{M}at(\mathscr{C}ob^3_\ell(k))$ the planar algebra structure of $\mathscr{A}r(\mathscr{C}ob^3_\ell(k))$.

Now, take Ω_i complexes in $\mathscr{K}om(k_i)$, $i \in \{1, ..., n\}$, and D as above; also in this case we must define how D acts over the spaces $\mathscr{K}om(k_i)$. The idea is to define D as "tensor product" of complexes; so the operator $\mathscr{O}_D$ is defined as

$$(\mathscr{O}_D(\Omega_1, ..., \Omega_n))^r = \bigoplus_{i_1+...+i_n=r} D(\Omega^{i_1}_1, ..., \Omega^{i_n}_n),$$

over the chain spaces, and as

$$d_{i_1,...,i_n} = \sum_{j=1}^{n} (-1)^{\sum_1^j i_t} D(Id_{\Omega^{i_1}_1}, ..., d^{i_j}_{\Omega_j}, ..., Id_{\Omega^{i_n}_n}),$$

$$\mathscr{O}_D(d)^r = \bigoplus_{i_1+...+i_n=r} d_{i_1,...,i_n},$$

for the differentials. We need to check that:

$$\mathscr{O}_D(\Omega_1, ..., \Omega_n) = ((\mathscr{O}_D(\Omega_1, ..., \Omega_n))^r, \mathscr{O}_D(d)^r)_{r\in\mathbb{Z}}$$

is indeed a complex; this is just a routine verification, and could be done by mimicking the proof that the tensor product of cochain complexes is a cochain complex.

Remark. The finiteness of the complexes in $\mathscr{K}ob$ was used in order to obtain complexes in $\mathscr{K}ob$: the sum

$$\bigoplus_{i_1+...+i_n=r} D(\Omega^{i_1}_1, ..., \Omega^{i_n}_n),$$

is *a priori* infinite, because the indices range over $\mathbb{Z}$, and so, in general, is not an object in $\mathscr{M}at(\mathscr{C}ob^3_\ell)$. But, because the complexes in $\mathscr{K}ob$ are finite, only finitely many $\Omega_j^{i_j}$ are non zero and, consequently, the sum is finite.

Is a bit tedious, and does not provide any insight, to check that all the properties of planar algebra; for this reason we will check only the associative property (3), emphasizing where the finiteness of the complexes in the definition of $\mathscr{K}ob$ is used.

Let $D,\ D'$ two planar arc diagrams with D as above and

$$D' \in \mathscr{D}(k_1; h_1, ..., h_m),$$

which implies

$$D \odot D' \in \mathscr{D}(k; h_1, ..., h_m, k_2, ..., k_n);$$

given $\Theta_j \in \mathscr{K}ob(h_j)$ and $\Omega_i \in \mathscr{K}ob(k_i)$, for $j \in \{1, ..., m\}$ and $i \in \{2, ...n\}$, we can compute

$$\left(\mathscr{Q}_D \left(\mathscr{Q}_{D'}\left(\Theta_1, ..., \Theta_m\right), \Omega_2, ..., \Omega_n\right)\right)^r =$$

$$= \bigoplus_{i_1+...+i_n=r} D((\mathscr{Q}_{D'}\left(\Theta_1, ..., \Theta_m\right))^{i_1}, \Omega_2^{i_2}, ..., \Omega_n^{i_n}) =$$

$$= \bigoplus_{i_1+...+i_n=r} D\left(\bigoplus_{j_1+...+j_m=i_1} D'\left(\Theta_1^{j_1}, ..., \Theta_m^{j_m}\right), \Omega_2^{i_2}, ..., \Omega_n^{i_n}\right) =$$

by the definition of the planar algebra extension to the matrix category, we have

$$= \bigoplus_{i_1+...+i_n=r}\ \bigoplus_{j_1+...+j_m=i_1} D\left(D'\left(\Theta_1^{j_1}, ..., \Theta_m^{j_m}\right), \Omega_2^{i_2}, ..., \Omega_n^{i_n}\right) =$$

$$= \bigoplus_{j_1+...+j_m+i_2+...+i_n=r} D\left(D'\left(\Theta_1^{j_1}, ..., \Theta_m^{j_m}\right), \Omega_2^{i_2}, ..., \Omega_n^{i_n}\right) =$$

the property (3) for the smoothings planar algebra implies

$$= \bigoplus_{j_1+...+j_m+i_1+...+i_n=r} D \odot D'\left(\Theta_1^{j_1}, ..., \Theta_m^{j_m}, \Omega_2^{i_2}, ..., \Omega_n^{i_n}\right) =$$

$$= \left(\mathscr{Q}_{D\odot D'}\left(\Theta_1, ..., \Theta_m, \Omega_2, ..., \Omega_n\right)\right)^r.$$

In a similar way we can proceed with the differentials. Obviously, the identity diagram behaves as the identity of a complex.

Now that we have defined a planar algebra structure over $\mathscr{K}ob$, we have to show that the bracket descends to a planar algebra morphisms between $\mathscr{K}ob$ and the planar arc algebra; to do so we must verify that:

$$[\![\cdot]\!]_k : \mathscr{T}(k) \to \mathscr{K}ob(k),$$

satisfies (4). Because every planar arc diagram can be seen as the composition of two-holed diagrams, and because (3) holds, is sufficient to prove the assertion for this type of diagrams. Take $D \in \mathscr{D}(k; k_1, k_2)$, $\Omega_i \in \mathscr{K}ob(k_i)$ and $\mathcal{T}_i \in \mathscr{T}(k)$ – with $i \in \{1,2\}$ – such that:

$$\Omega_i = [\![\mathcal{T}_i]\!].$$

We want to show that

$$D(\Omega_1, \Omega_2) = [\![D(\mathcal{T}_1, \mathcal{T}_2)]\!];$$

let us verify it only for the chain modules, the proof for the differentials is almost identical. Now, we can suppose $D(\mathcal{T}_1, \mathcal{T}_2)$ to have the crossing ordered in such way that the crossings of $\mathcal{T}_1$ come before than the crossings of $\mathcal{T}_2$; so, any splitting vector v can be seen as (v_1, v_2), with v_i splitting vector, see page 12 of this paper, for $\mathcal{T}_i$, such that:

$$|v| = |v_1| + |v_2|,$$

and, viceversa, every pair of splitting vectors defines a splitting vector for $D(\mathcal{T}_1, \mathcal{T}_2)$; it is also clear that the splitting S_v, associated to the vector v, can be also obtained by using D to compose the splittings $S^1_{v_1}$, $S^2_{v_2}$ of $\mathcal{T}_1$, $\mathcal{T}_2$, given by, respectively, v_1, v_2. So we have

$$[\![D(\mathcal{T}_1, \mathcal{T}_2)]\!]^r = \bigoplus_{|v|=r+n_-} S_v = \bigoplus_{|v|=r+n_-} D(S^1_{v_1}, S^2_{v_2}) =$$

if we denote n^i_- the negative crossings of $\mathcal{T}_i$, we also have:

$$= \bigoplus_{(|v_1|-n^1_-)+(|v_2|-n^2_-)=r} D(S^1_{v_1}, S^2_{v_2}) =$$

by definition of the bracket,

$$= \bigoplus_{(|v_1|-n^1_-)+(|v_2|-n^2_-)=r} D(\Omega_1^{v_1-n^1_-}, \Omega_2^{|v_2|-n^2_-}) =$$

by a simple change of index

$$= \bigoplus_{i+j=r} D(\Omega_1^i, \Omega_2^j) = D(\Omega_1, \Omega_2)^r.$$

As for the last point of the theorem, it is sufficent to verify it for two-holed diagrams; in this case, the assertion is implied by the fact that diagram of two equivalent complexes are equivalent, and the chain equivalences are given by the "diagram of the chain equivalences" defined as for the differential in the diagram of complexes. This is also a routine verification that we leave to the reader. We provide a hint: is necessary to use the fact, stated before the definition of $\mathscr{K}ob$ at page 35, that the action of the planar operators and the composition of cobordisms commute.

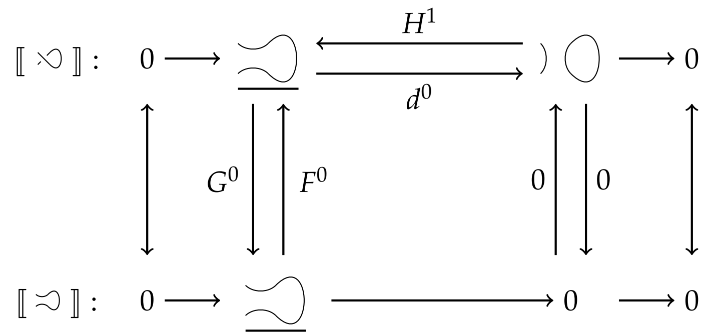


FIGURE 19. Invariance under $R1$.

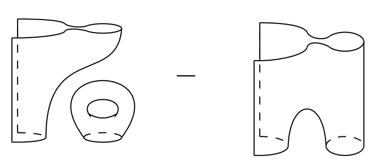
Q.E.D.

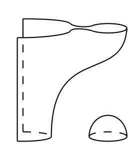
3.3. Invariance of $[\![\cdot]\!]$. Finally, to conclude our proof, we must show that for the diagrams involved in the Reidemeister moves we have invariance. All the proofs of these facts are well known, so we will not carry on all the computations; instead, we will sketch the proofs, stressing the details that will be used afterwards, and redirect the reader to [**BarNatan**] for the complete versions.

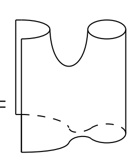
Proposition 6. *The two complexes $[\![\ ]\!]$ and $[\![\ ]\!]$, shown in figure 19, are homotopy equivalent.*

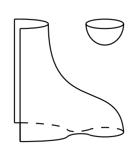
Proof. The proof of this theorem is pretty standard and direct. One defines the maps directly and verifies that they do the trick; in figure 19 is drawn a diagram containing the two complexes $[\![\ ]\!]$ and $[\![\ ]\!]$, the underlined smoothings are those with homological grading 0.

The maps in figure 19 are defined as:

$F^0 =$ 

$H^1 =$ 

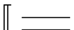
$d^0 =$ 

$G^0 =$ 

and the maps $d^i,\ G^i,\ F^i,\ H^{i+1}$ are defined to be 0 for each $i \neq 0$. The commutativity of F and G with the differential is almost immediate, so we turn directly to the verification that F and G are homotopy equivalences. It follows immediately from the T relation that the composition $G \circ F$ is the identity for $[\![$  $]\!]$.

The remaining composition is a bit more difficult: we must verify two cases $F^1 \circ G^1 = Id^1 + d^0 \circ H^1$ and $F^0 \circ G^0 = Id^0 - H^1 \circ d^0$; the first is immediate, while the second verification uses the 4-*Tu* relation depicted in figure 14.

Q.E.D.

Proposition 7. *The two complexes* $[\![$  $]\!]$ *and* $[\![$  $]\!]$ *are homotopy equivalent.*

Proof. This proof is similar to the previous one, so we refer the reader directly to [**BarNatan**] – Theorem 1, Invariance under the Reidemeister $R2$, pages 1458-1459.

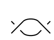
Q.E.D.

A little bit more machinery is needed to prove the invariance under the third move. This proof is from [**BarNatan**], and is inspired by the original proof of the invariance for regular isotopies of the Kauffmann bracket.

Definition 1.13. Let Ω, Θ be two complexes. A morphism $F : \Omega \to \Theta$ is a *strong deformation retract* if there exist $G : \Theta \to \Omega$ and a prism map $K : \Omega \to \Omega$, see page 18, such:

(a) $F \circ K = 0$ and $K \circ G = 0$.
(b) $G \circ F = Id$
(c) $F \circ G = K \circ d + d \circ K + Id$

In this case G is called *inclusion in a deformation retract*.

Remark. Actually, does exists a strong defomation retract G between the complexes $[\![$  $]\!]$ and $[\![$ $]\!]$, a picture of this morphism is shown in figure 6, page 1459, [**BarNatan**]. A modified version of this morphism will be used to prove the invariance under the third Reidemeister move.

Definition 1.14. Let Ω, Θ be a two complexes and $F : \Omega \to \Theta$ a morphism. The *cone* over F is the complex $\Gamma(F)$, defined by:

$$\Gamma(F)_i = \Omega_{i+1} \oplus \Theta_i, \quad d^i_{\Gamma(F)} = \begin{pmatrix} d^{i+1}_\Omega & 0 \\ F_i & d^i_\Theta \end{pmatrix}$$

The cone is well defined becuse of the commutativity of F with the differential. Now the proof of the invariance under the third move rest upon two lemmas, which we are not going to demonstrate. This lemmas give us a way to relate the third and the second move, in such way that the first will follow from the latter.

Lemma 2. *The following relations hold*

(a) $[\![\ \ ]\!] = \Gamma(\ [\![\ \ ]\!]\ )$.
(b) $[\![\ \ ]\!] = \Gamma(\ [\![\ \ ]\!]\ )[-1]$.

where $\cdot[\cdot]$ *is the shift of complexes defined by:* $\Omega[s]_i = \Omega^{i+s}$, $[\![\ \ ]\!]$ *is the saddle morphism between the complex* $[\![\ \ ]\!]$ *and the complex* $[\![\ \ ]\!]$, *and* $d^i_{[s]} = d^{i+s}$.

Lemma 3. *Given four complexes* Ω_0, Ω_1, Θ_0, Θ_1, *and morphisms* F_0, F_1, G_0, G_1, Ψ *as in the following diagram*

$$\begin{tikzcd}
\Omega_1 \arrow[r, bend left, "F_1"] \arrow[d, "\Psi"] & \Theta_1 \arrow[l, bend left, "G_1"] \\
\Omega_0 \arrow[r, bend left, "F_0"] & \Theta_0 \arrow[l, bend left, "G_0"]
\end{tikzcd}$$

then the following statements hold

(a) if F_1 *is a strong deformation retract with inclusion* G_1 *then: the cones* $\Gamma(\Psi)$ *and* $\Gamma(\Psi \circ G_1)$ *are homotopy equivalent;*
(b) if G_0 *is a strong deformation retract with inclusion* F_0 *then: the cones* $\Gamma(\Psi)$ *and* $\Gamma(F_0 \circ \Psi)$ *are homotopy equivalent.*

The proofs of the lemmas can be found in [**BarNatan**] – Lemma 4.4 and Lemma 4.5 at pages 1460-1462, respectively. Now we can conclude the proof of the invariance.

Proposition 8. *The two complexes* $[\![\ \ ]\!]$ *and* $[\![\ \ ]\!]$ *are homotopy equivalent.*

Proof. The complex $[\![\ \ ]\!]$ can be seen as the cone over the morphism $\Psi_R = [\![\ \ ]\!]$, shown in the figure below. This fact can be proved directly by computing the chain groups and differentials of the cone complex explicitly, and comparing the result with the complex $[\![\ \ ]\!]$.

Notation. With the symbol $[\![\ \ ]\!]$ we denote the map between the complexes $[\![\ \ ]\!]$ and $[\![\ \ ]\!]$, which is everywhere the identity, except from the crossing indicated with where it behaves as the saddle map.

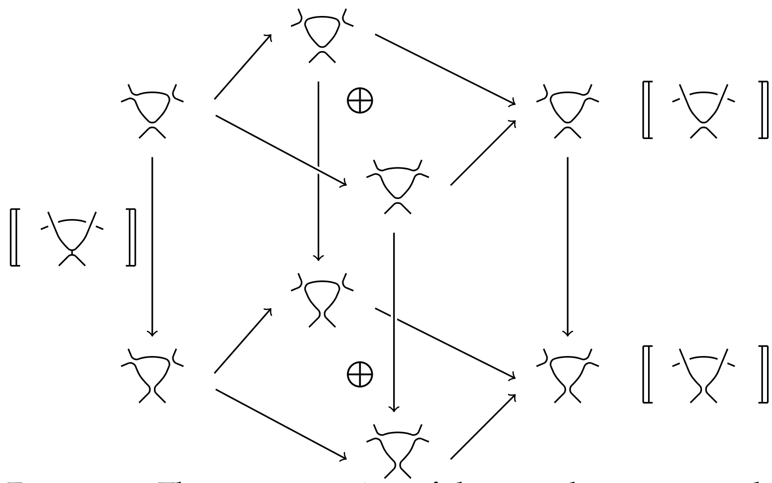


FIGURE 20. The representation of the complex corresponding to the right side of the third move as a cone.

By direct computation, one can see that the complex $[\![\;]\!]$ is the cone over $[\![\;]\!]$, in accordance with the previous case.

The top layer of the cube in figure 20 is the complex $[\![\;]\!]$, which deformation retracts over $[\![\;]\!]$ via the inclusion in a defomation retract generated by the second Reidemeister move – see the remark at page 40. This defines an inclusion in a deformation retract Φ_R, depicted in figure 21, from $[\![\;]\!]$ to the bottom layer of the cube in figure 20.

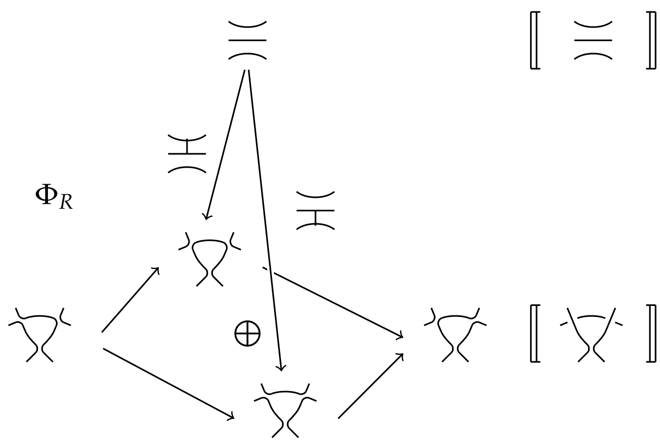


FIGURE 21. The morphism Φ_R.

Thanks to lemma 8, we can say that our original complex is the cone over the morphisms Φ_R. Everything just said can be restated for $[\![\;]\!]$, but exchanging Φ_R with Φ_L, illustrated in figure 22. We get that $[\![\;]\!]$ is the cone over the morphism Φ_L; but this two maps are obiouvsly homotopic and, thanks to lemma 2, this concludes the proof.

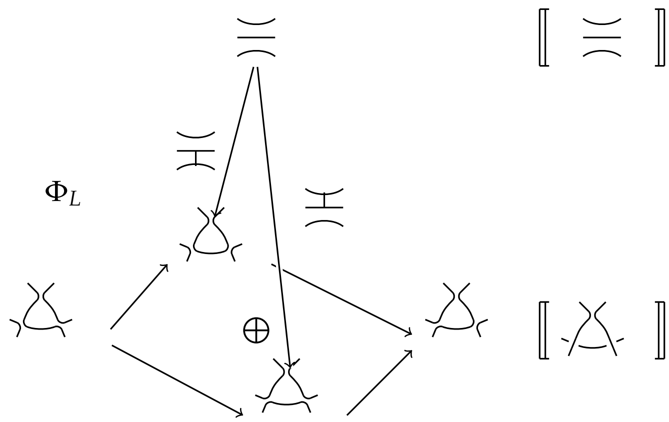


FIGURE 22. The morphism Φ_L.

Q.E.D.

3.4. Grading the bracket. Now the last thing to do is introduce the grading into our construction. The Khovanov complex is, in fact, bi-graded while the bracket has only the homological grading; by simply applying the TQFT without a grade shifting we lose the information on the quantum grading. We can, with a little help from abstract nonsense, define the formal grading and create a new graded complex from which we can immediatly obtain the Khovanov chain complex with the quantum grading and all.

Definition 1.15. A *graded category* is a pre-additive category with a $\mathbb{Z}$ action over objects and morphisms, called *shift*, and a notion of *degree* for the latters, such that:

(a) $deg(Id_O) = 0$, for each object O.
(b) $deg(f \circ g) = deg(f) + deg(g)$, for every pair of morphisms $f,\ g$ for which the composition makes sense.
(c) Given two objects, $O_1,\ O_2$, and denoted $\mathcal{S}_k$ the k-degree shift, then

$$\mathscr{A}r(\mathcal{S}_k(O_1), \mathcal{S}_h(O_2)) = \mathscr{A}r(O_1, O_2),$$

and, if $\mathcal{S}_k^h(f) \in \mathscr{A}r(\mathcal{S}_k(O_1), \mathcal{S}_h(O_2))$ is the k-source h-target shift of f, the following holds:

$$deg\left(\mathcal{S}_k^h(f)\right) = deg(f) + h - k.$$

Each pre-additive category admits a *graded closure* – with respect to a grading of the morphisms group, the latter seen as a $\mathbb{Z}$-module. As in the previous constructions we leave the category untouched if it is already graded; otherwise, we take as objects the pairs (m, O), for each $m \in \mathbb{Z}$ and O object, and we can define a grading on the morphisms by taking them as a graded group, and shifting in such a way that the property (c) of the grading is satisfied.

Once we have a graded category we can extend the graded structure to its matrix category by giving the degree d to a matrix whose entries are all morphisms of degree d; any matrix can be seen as a linear combination of "homogeneous matrices" so this gives the structure of graded module to the morphisms. The k-shift of a vector of objects V is the vector containing the k-shifted entries of V. In a similar way we can extend our grading to the category of complexes.

To define a graded structure over $\mathscr{K}ob(k)$ it is sufficent to give a grading to the morphisms of $\mathscr{C}ob^3(k)$, verify that the local relations are grade homogeneous – so that the graded structure descens to a graded structure over $\mathscr{C}ob^3_{\ell}(k)$ – and, finally, additivity under "horizontal" and "vertical" composition of cobordisms.

Definition 1.16. Let $C \in \mathscr{A}r(\mathscr{C}ob^3(k))$ be a cobordisms between tangle smoothings. The *degree* of C is defined as

$$deg(C) = \chi(C) - k,$$

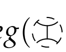
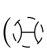
where $\chi(\cdot)$ is the Euler characteristic. Remind that k is half of the vertical boundary components of C.

Remark. We have to grade also the empty set, as it represent a morphism in $\mathscr{C}ob^3(\emptyset)$, so we define

$$deg(\emptyset) = 0;$$

this choice is not arbitrary, but descends from the request of the additivity with respect to planar algebra operations.

Simple computations show that

$deg($$) = deg($$) = -1, \quad deg($$) = deg($$) = +1,$

so, once we verify both "vertical " and "horizontal" additivity of the degree, we can conclude that the S relation and T relation are grade preserving. More tedious, but not difficult, is to check that the 4-Tu is grade homogeneous; we will treat only a case, that is when two discs lay in the lower boundary and the other two in the upper boundary. In this case we have a cylinder plus a cap and a cup this rises the degree by two; on the other side of the = there are two cups/caps plus an horizontal tube. The horizontal tube is given by compositon of a (closed) saddle plus a cup/cap, so there is a 2-degree shift.

The additivity property mentioned above can be verified, for example by triangulating the cobordisms. Finally we can define a graded complex, called *Khovanov formal complex*, and denoted $Kh^{i,j}(\mathcal{T})$, as:

$$Kh^j(\mathcal{T}) = \mathcal{S}_{j+n_+-n_-}([\![\mathcal{T}]\!]^j)$$

now the Khovanov (algebraic) chain complex can be seen as direct application of the TQFT to the formal Khovanov complex.

All the results obtained for the bracket can be extended and adapted for the complex just defined; these results are collected in the theorem below.

Theorem 3. *Given a tangle diagram $\mathcal{T}$, the following hold:*

(a) The differential of Kh is of degree 0*;*
(b) The TQFT introduced in the previous section is a degree 0 *functor between* $\mathscr{C}ob^3_\ell$ *graded and* $\mathscr{M}od_{gr}(\mathbb{Z})$*;*
(c) Kh is a tangle invariant up to degree 0 *homotopy equivalence;*
(d) Kh defines a degree 0 *planar algebra morphism between* $\mathscr{T}(k)$ *and* $\mathscr{K}om(k)$*, for each* $k \in \mathbb{N}$*.*

Proof. The theorem follows from 2 by simple degree computations and from the fact that the number of crossings, as well as the number of positive and negative crossings, is additive under the action of planar arc diagrams.

Q.E.D.

4. Alternative definitions and generalizations

So far we defined the Khovanov bracket and the Khovanov chain complex, and sketched a proof of their invariance under Reidemeister moves. The construction of the Khovanov complex we have given is the standard one. Another definition, more combinatorial, is given by Jacobsson in **[Jacobss]**, where he proves the invariance and the yet-to-be-defined functoriality. The advantage of Jacobsson's definition is that it is easier to handle in actual computations.

The invariance can be proven, even using the standard definition, in different ways: for knot and links different proofs of the invariance can be found in **[KhovCat]** – which uses cubes – and in **[EunSLee]** – a direct proof. To be precise, Khovanov introduces his homology using as a base ring $\mathbb{Z}[c]$, and demonstrates the invariance in this case; the construction of Bar-Natan, even the bracket, can be adapted to reproduce the original construction – see [**BarNatan**] page 1483.

A generalization of our construction – i.e. the algebraic chain complex – for tangles can be found in [**KhovTan1**]; this construction is fundamentally different from the one we have described for links: Khovanov associates to (m, n)-tangles – which are tangles in a box whose boundary lays in the union of two edges: an "input" edge, where there are m points of the boundary, and an "output" edge, with the remaining n boundary points – an (H^n, H^m)-modules bi-graded chain complex, with $\{H^n\}_n$ a family of rings, which coincide with the standard construction for $(0, 0)$-tangles.

A different homology, based on the Khovanov complex, was defined by Lee in [**EunSLee**]. Starting form the Khovanov chain complex she defines a $(1, 4)$-bidegree map Φ; the sum of this map with the Khovanov differential is a differential for the chain complex and gives rise to a link homology theory – with the loss of the quantum grade. This theory still comes from a TQFT which respects the S, T, 4-Tu relations, hence our proof of the invariance works for Lee's homology as well. To be precise, Lee's theory comes from the TQFT associated to the Frobenius algebra described in example 1 at page 27; the loss of the quantum grade can be deduced from the fact that both the multiplication and the co-multiplication are not degree homogeneous. It turns out, see [**PTurner**] for further references, that Khovanov and Lee's theories are the only relevant link homology theories that could be defined from the bracket via TQFT.

CHAPTER 2

Functoriality

The Jones polynomial represent a powerful and easily computable invariant for links, but Khovanov homology is at least as powerful as the Jones polynomial: we can obtain the Jones polynomial as the graded Euler characteristic of the Khovanov homology – see page 23. Moreover, there are knots with the same Jones polynomial but different Khovanov homology; hence Khovanov Homology is a strictly stronger invariant than the Jones polynomial.

Being a better invariant is not the main advantage of the Khovanov homology over the Jones polynomial. What makes Khovanov homology interesting are its functorial properties: if we consider two oriented link diagrams and a cobordism – embedded in a certain 4-dimensional space – between them, the latter induces a morphism between the Khovanov homologies of the two diagrams. Moreover, up to sign, the morphism is totally determined by the ambient isotopy class relative to the boundary of the chosen cobordism.

The aim of this chapter is to prove the above-mentioned functoriality of Khovanov homology; to be precise we will prove the result for Khovanov formal complexes of tangle diagrams. The first section of this chapter provides an introduction to knotted surfaces in a 4-dimensional space and their representation.

Afterwards, in the second section, we will introduce the new categorical setting of our theory: the categories $\mathscr{C}ob^4$ and $\mathscr{M}ov$. Always in the second section, we define canopoleis, a tool that we will use to reduce our proofs to local ones, and describe how to associate to a cobordism between tangles a map between the Khovanov formal complexes of its boundary.

Finally, the third section is devoted to the proof of the main theorem, that is to say: we will prove that, up to sign, the map induced by two ambient isotopic, relative to the boundary, surfaces with the same boundary are homotopy equivalent.

A fourth section, at the end of the chapter, is devoted to generalizations, alternative proofs of the main statement and further constructions.

1. Cobordisms

The source categories we are going to define are strictly related with cobordisms between links or, more generally, tangles. So, to give a proper description of those categories and to have some tools to verify the functoriality of Kh, we need to describe the cobordisms between tangles and their representations. The outline of this section is the following one: after a brief description of the cobordisms between links and tangles, we introduce a way to represent them through "movies". Further on we describe how these cobordisms, and their movies, are related with Khovanov homology. Finally, we describe a set of "movie moves", i.e. equivalences between two movies of cobordisms related by isotopies, that will be essential to prove the functoriality of Khovanov homology.

1.1. Generic cobordisms. Let $\mathcal{L}$ and $\mathcal{L}'$ be two oriented link diagrams, each of which contained in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon)$. A *link cobordism* between them is a smooth, oriented, compact surface Σ, neatly[1] embedded in $(\mathbb{D}^2 \times (-\varepsilon, \varepsilon)) \times [0,1]$ such that:

(a) $\mathcal{L} = \partial\Sigma \cap (\mathbb{D}^2 \times (-\varepsilon, \varepsilon)) \times \{0\}$, as sets.
(b) $\mathcal{L}' = \partial\Sigma \cap (\mathbb{D}^2 \times (-\varepsilon, \varepsilon)) \times \{1\}$, as sets.
(c) $\partial\Sigma = \mathcal{L} \cup \overline{\mathcal{L}'}$.

Where the overline means that the orientation is inverted. We will refer to $\mathcal{L}$ as the *source link*, or *starting link*, for Σ; while $\mathcal{L}'$ will be called *target link*, or *ending link*.

The definition in the case of tangles is similar; the only thing we must take into account is the boundary of the tangle.

Definition 2.1. A *tangle cobordism* Σ is a smooth oriented surface neatly embedded in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$, that satisfies the following properties:

(a) $\mathcal{T}_i = \Sigma \cap \mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times \{i\}$, for $i \in \{0,1\}$, is an oriented tangle with boundary $B \times \{0\} \times \{i\}$;
(b) $\partial\Sigma = \mathcal{T}_0 \sqcup \mathcal{T}_1 \sqcup B \times \{0\} \times (0,1)$;

$\mathcal{T}_0$ is the *source*, or *starting*, *tangle* while $\mathcal{T}_1$ is the *target*, or *ending*, *tangle*.

A surface Σ, embedded in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$, is in *generic position* if and only if the following conditions are satisfied:

(a) Σ is neatly embedded;
(b) its boundary is transversal to the boundary of $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$;
(c) the singular points in the image of the projection

$$p : \mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1] \to \mathbb{D}^2 \times \{0\} \times [0,1],$$

[1] An embedding $\iota : \mathcal{M} \hookrightarrow \mathcal{N}$ is *neat* if and only if $\iota(\partial\mathcal{M}) = \partial\mathcal{N} \cap \iota(\mathcal{M})$.

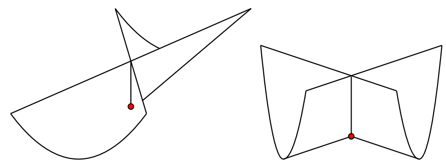


FIGURE 1. A Whitney umbrella point, indicated in red, seen from two different points of view.

are only double points, triple points and Whitney umbrella points;

(d) the singular points listed above appear only in the interior of the surface;
(e) double and triple points of self-intersection are trasversal;
(f) Whitney umbrella points appear only as isolated boundary points of double point sets;
(g) triple points appear as trasverse intersections of double point loci.

Without loss of generality, up to small perturbations by ambient isotopies, in the image of p we can suppose triple and Whitney umbrella points, as well as local maxima or minima of the double point loci with respect to the projection π described below, to "happen" at different "time levels", i.e. there is at most one of the listed singular points in $\mathbb{D}^2 \times \{0\} \times \{t\}$, for each t .

Any surface in generic position could be represented by a *surface diagram*; this is an analogue of a link diagram: after projecting our surface to $\mathbb{D}^2 \times [0,1]$, one introduces the information of which "surface strand overcrosses" along double point loci, or near triple points, by means of broken surface diagrams, see [**CaSaRie**] or also [**CaSa1**].

In this work we are not interested in broken surface diagrams; nonetheless, there it is an important fact about diagrams that needs to be recalled: two surfaces in generic position are isotopic if their diagrams are connected by a finite number of Roseman moves and ambient isotopies of the diagrams, see [**Rosem**].

Definition 2.2. A surface immersed in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$ is said to be *time generic* if and only if the "time projection"

$$\pi : \mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1] \to [0,1],$$

is a Morse function with distinct critical values for distinct critical points.

Let Σ be a *generic tangle cobordism* – possibly a link cobordism – in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$, i.e. both time generic and in generic position. The counter image of a regular time value t, with respect to $\pi_{|\Sigma}$, is an embedded smooth compact orientable, possibly disconnected, 1-manifold whose boundary B lies in $\mathbb{D}^2 \times \{0\} \times \{t\}$; that is to say a tangle with boundary set B.

Moreover, the projection of this tangle on $\mathbb{D}^2 \times \{0\} \times \{t\}$, contains only isolated double points of traverse self intersection; otherwise, in the surface diagram there would be either non-isolated triple points or points with multiplicity higher than three, which is absurd because the surface Σ is in generic position.

If t is a critical value then, as a result of the intersection of Σ with $\pi^{-1}(t)$, we obtain either a tangle with a single point of trasverse self-intersection or a tangle union a single point, depending on the index of the critical point.

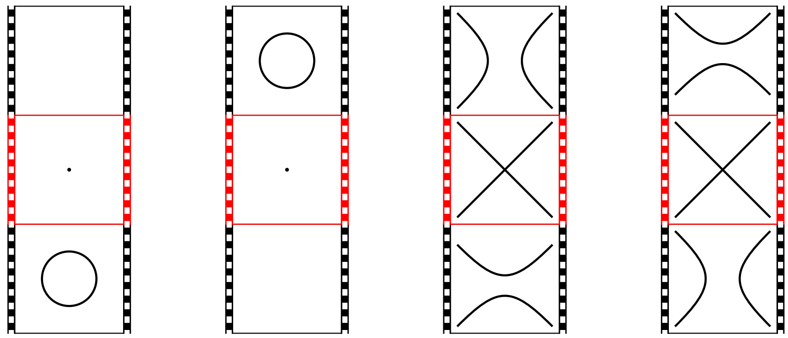
Definition 2.3. Let Σ be a generic link, or tangle, cobordism; the Σ-*still*, or simply *still*, at the instant t is the oriented link diagram, possibly with singularities if t is a critical value for π, obtained by projecting $\pi^{-1}(t) \cap \Sigma$ onto $\mathbb{D}^2 \times \{0\} \times \{t\}$. The still at instant t, when t is a critical time value, will be called by us *scenery change*.

Remark. With the exception of the starting still, we will suppose the stills to be oriented in the opposite way respect to the orientation induced by the cobordism. With this convention, a cylinder is represented by a sequence of identical stills; otherwise, all the stills would have had the opposite orientation respect to the starting still.

1.2. Movies. Given a generic surface Σ, the set of all Σ-stills provides a complete description of Σ. Such a description is not easy to handle because it is composed by infinitely many stills. We can reduce considerably the amount of stills needed without loosing any topological information; the result of this reduction will be a collection of finitely many stills that provides a good representation of Σ, this collection will be called *movie*.

Let us begin by reducing the stills near a critical time value t. At the instant t we have a scenery change, and for a sufficiently small interval of time I all we can see is a tangle diagram that undergoes a single transformation: in few small areas that do not involve crossing, called *changing areas*, the local picture changes by a single *Morse modification*, or *Morse move*; these are the local moves shown in Figure 2. So we can summarize all the stills relative to I by three stills: one before the Morse modifications, one relative to the scenery change – the red stills in the figure below – and a still representing the diagram after all the Morse moves. Sometimes we will omit the stills relative to the scenery changes.

Now we have to reduce the number of stills between two scenery changes to finitely many; basic Morse theory tells us that in this interval all we can see is a diagram that undergoes through a finite sequence of



Birth move Death move Fusion move I Fusion move II

FIGURE 2. Movie representation of the – unoriented – Morse moves.

planar isotopies and Reidemeister moves; each one of these modifications can be condensed in two stills one "before" and one "after".

As already said at page 48, up to small perturbations, far from the boundary of the surface, we can suppose that the Reidemeister moves and the planar isotopies "happen" to a diagram at different time levels. With this hypothesis two consecutive stills of our movies are related by exactly one of this local moves:

(a) a Morse move;
(b) a Reidemeister move;
(c) the rotation of one or more closed components of the tangle;
(d) the movement of an arc;
(e) the permutation of two closed components of the tangle.

Each one of the moves listed above is represented, in the movie, as a couple of stills: a "before" still and an "after" still. The portion of the embedded surface between two such stills can be seen as a morphism from the Khovanov formal complex of the "before" still to the Khovanov formal complex of the "after" still; these morphisms are, in the order, the following ones:

(a) The Birth move is associated to the cap

$$[\![\ \ ]\!] \ :\ Kh(\emptyset) \to Kh(\bigcirc),$$

to the Death move is associated a cup

$$[\![\ \ ]\!] \ :\ Kh(\bigcirc) \to Kh(\emptyset);$$

while to the fusion moves are associated the saddles

$$[\![\ \ ]\!] \ :\ Kh(\ \ ) \to Kh(\ \ )\,, \quad [\![\ \ ]\!] \ :\ Kh(\ \ ) \to Kh(\ \ )\,;$$

(b) to the Reidemeister moves are associated the morphisms used to prove the invariance of the Khovanov homology. For the first and second move the reader could refer to the previous chapter, third section, while for the third move one could see [**BarNatan**], p. 1463, where the morphism between the two formal complexes is detailed.
(c) the rotation of one or more closed components of the tangle has no effects, unless the diagram is symmetric with respect to that rotation, in which case permutes the circles in the diagram.
(d) the movement of an arc has no effect;
(e) the permutation of two closed components of the tangle, has the effect of permuting the circles in each smoothing.

Some of the moves described above seem to be trivial, like the rotation of closed components or the motion of an arcs, but, even if their effect at level of Khovanov homology is trivial, the presence or the absence of one of those moves may change the ambient isotopy class of the surface we are representing. For example, at page 5 of [**CaKaSa**] – Fig. 1.2 – is shown the movie of an unknotted torus where a trefoil component is rotated by $\pi/3$ radians; the same movie without the rotation represent a knotted torus instead of an unknotted one.

1.3. Movie moves. In [**CaSaRie**], Carter, Saito and Rieger introduced a full set of Reidemeister-type moves for movies, the so called *movie moves*. These moves include all the movie version of the Roseman moves, plus another set of moves that do not affect the topology of the surface diagram.

To the set of moves displayed in the Figures 3, 4 and 5 we can add other movie moves obtained from the ones given by one, or more, of the following operations:

(1) reading the move from bottom to top, or from right to left;
(2) reflecting all the stills of a move with respect to the x-axis;
(3) reflecting all the stills of a move with respect to the y-axis;
(4) changing all the crossings.

Our set of movie moves is not complete, also counting the moves obtained by performing the operations listed above; nonetheless, the moves provided are fit for our purposes. Any reader eager for details about movie moves can consult the article [**CaSaRie**], where the full set movie moves is carefully explained, or can read either of the – beautiful – books [**CaKaSa**] or [**CaSa1**], where are described different ways to represent a surface embedded in a 4-space, as well as various techniques to study them. In particular, in the first chapter of both the above-mentioned books, is provided a detailed description of the movies.

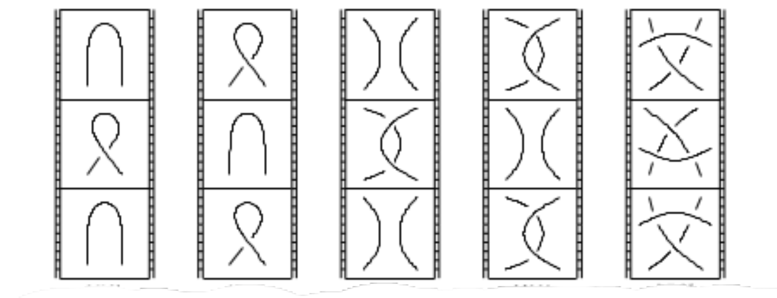


FIGURE 3. First group of movie moves. These are known as the Elementary String Interactions, and they consist in a Reidemeister move and its inverse. We label them, from left to right, as Ia, Ib, IIa, IIb and III.

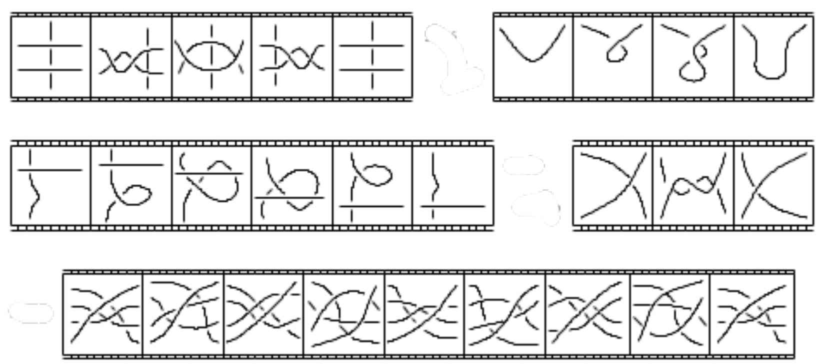


FIGURE 4. Second group of movie move, also known as circular movie clips. We label them, from left to right, as IV, V, VI, VII and VIII.

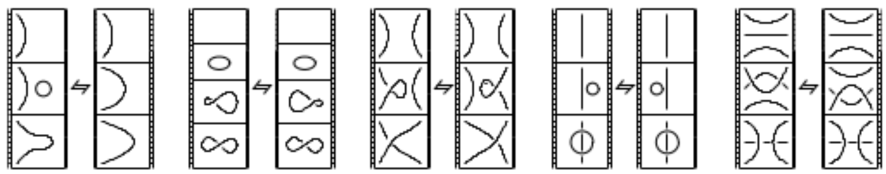


FIGURE 5. Third, and last, group of movie move. We label them, from left to right, as IX, X, XI, XII and XIII.

The movie moves shown, graphically speaking, can be divided in two main types: movie moves that involve only one movie and movie moves that involve two movies; this second type of movie moves should be read as follows: if M is a movie involving a sequence represented in one of the two sides of a move, than the surface Σ' represented by the movie obtained from M by the replacement of the mentioned sequence with the sequence on the other side of the move is ambient isotopic, relative to the boundary, to the surface represented by M.

The moves that involve only one movie should be interpreted similarly: in this case the "other side" of the move is represented by a sequence of identical copies of the first still of the movie shown.

The following result is a modified version of a result of Carter and Saito – see **[Jacobss]**, page 1235 – and will be fundamental to prove the functoriality of the Khovanov formal chain complex.

Theorem 4. *Let M and N be two movies representing generic tangle cobordisms, say Σ and Γ respectively, embedded in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$. Then Σ and Γ are ambient isotopic relative to the boundary if and only if M and N are related by a finite sequence of movie moves, or their modifications through operations from (1) to (4), and interchange of distant critical points. In this case M and N are said to be* equivalent*.*

We should spend few words on the meaning of "interchange of distant critical points". Let us consider a movie M whose first still is a diagram $\mathcal{D}$, then from a still to its subsequent a single small area that undergoes through a change. One can consider the corresponding area in the first still, this area will be called *t-changing area in $\mathcal{D}$*, where t is the instant where the change happens; given two instants, say t_0 and t_1, the t_0-changing area in $\mathcal{D}$ and the t_1-changing area in $\mathcal{D}$ are said *to intefere*, if there is sequence of t-changing areas in $\mathcal{D}$ that connects them, and that cannot be shrunk to be non intersecting.

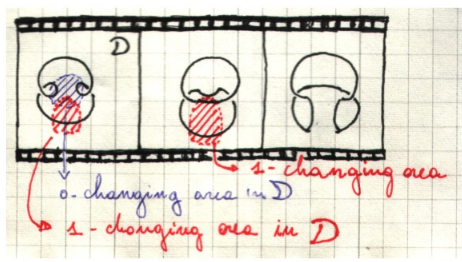


FIGURE 6. Two t-changing areas that do interfere.

Given two non-interfering t-changing areas in $\mathcal{D}$, say A and B, such that the changes in A happen before the changes in B, then the movie M' obtained from M by making the changes in B happen before the changes in A describes a surface ambient isotopic, relative to the boundary, to the surface described by M. In this case we sya that M' is obtained from M *by interchange of distant critical points*.

As for most of the constructions in this work, any description by words is less clear than a picture; so in Figure 7 is shown an example of two movies obtained one from the other by interchange of distant critical points.

2. Categories of tangles and canopoleis

In this section we will introduce the main tools needed to state and prove the theorem about the functoriality of Khovanov homology. This section will develop as follows: in the first subsection we introduce the

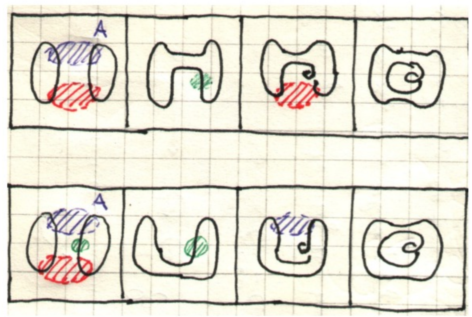


FIGURE 7. Two movies related by interchange of distant critical points.

notion of canopolis which is a kind of generalization of the concept of planar algebra; later on, in the second subsection, we define the new geometric categories that will play the role of source category for the functor Kh_0, defined in the third and last subsection.

2.1. Canopoleis. A canopolis, roughly speaking, is a planar algebra of categories with some "functorial properties". Canopoleis were introduced first in [**BarNatan**] to prove the functoriality of Khovanov homology, and represent an essential tool to study "geometric complexes" like Kh or $[\![\cdot]\!]$.

Canopoleis will play the same role played by planar algebras in the previous chapter, and they represent a good way to avoid cumbersome abstract nonsensical constructions. Many proofs in this section are similar, if not identical, in spirit to the ones in the section regarding planar algebras; for this reason some of them will be omitted and the reader will be referred to the corresponding theorems for planar algebras.

Let $\mathscr{P} = \{\mathscr{P}_k\}_{k\in K}$ be a planar algebra; we have suppressed the indication of the operators $\{\mathscr{O}_{\mathcal{D}}\}_{\mathcal{D}\in\mathscr{D}'}$, because further on we will be working with different planar algebra structures at the same time and, to avoid cumbersome notation, we will denote the operator associated to the planar diagram $\mathcal{D}$ with the same symbol. So we will invite the reader to pay attention to which operator is used and when.

Definition 2.4. A canopolis – over $\mathscr{P}$ – is a collection of categories $\{\mathscr{C}(k)\}_{k\in K}$, together with a planar algebra structure over $\{\mathscr{O}bj(\mathscr{C}(k))\}_{k\in K}$ and one over $\{\mathscr{A}r(\mathscr{C}(k))\}_{k\in K}$, such that:

(a) the two structures of planar algebras over $\{\mathscr{O}bj(\mathscr{C}(k))\}_{k\in K}$ and over $\{\mathscr{A}r(\mathscr{C}(k))\}_{k\in K}$ are isomorphic to $\mathscr{P}$;

(b) if $f_{i_j} : A_{i_j} \to B_{i_j}$, with $j \in \{1, ..., m\}$, and
$$\mathcal{D} : \mathscr{T}_{i_1} \times ... \times \mathscr{T}_{i_m} \to \mathscr{T}_i,$$
then
$$\mathcal{D}(f_{i_1}, ..., f_{i_m}) : \mathcal{D}(A_{i_1}, ..., A_{i_m}) \to \mathcal{D}(B_{i_1}, ..., B_{i_m});$$

(c) planar algebras operations commute with the composition
$$\mathcal{D}(f_{i_1} \circ g_{i_1}, ..., f_{i_m} \circ g_{i_m}) = \mathcal{D}(f_{i_1}, ..., f_{i_m}) \circ \mathcal{D}(g_{i_1}, ..., g_{i_m});$$

The first and the second properties, we call the latter *compatibility*, allow us to visualize morphisms as cans, whose top is the source and whose bottom is the target, and to see diagrams as containers where the cans can be inserted to fill the holes.

The third property ensures the *functoriality* of planar algebra operations; in terms of cans we can say that piling up cans – that corresponds to the composition in the category – and then putting them in a container – which visually represents the planar algebra composition – is the same thing as piling them up in the container.

Both the properties and the visual image just given should sound familiar to the reader; we encountered two examples of this structure in the previous chapter: the families $\{\mathscr{C}ob^3(k)\}_{k\in K}$ and $\{\mathscr{C}ob^3_{/\ell}(k)\}_{k\in K}$ are examples of canopoleis over the planar algebra of planar arc diagrams.

Some result of the previous chapter can be restated in this, more general, contest.

Theorem 5. *Given a canopolis $\{\mathscr{C}(k)\}_{k\in K}$ of additive categories, then the family $\{\mathscr{K}om(\mathscr{C}(k))\}_{k\in K}$ has a natural structure of planar algebra. In addition, this structure descends to a planar algebra structure over $\{\mathscr{K}om_{/h}(\mathscr{C}(k))\}_{k\in K}$.*

This theorem has been proven in the previous chapter: the proof is exactly the same of that the first point of theorem 2. The following result regarding the category of complexes over a canopolis will be useful later on.

Proposition 9. *Let $\{\mathscr{C}_k\}_k$ be a canopolis of additive categories. Let us consider*
$$C,\ D \in \mathscr{O}bj(\mathscr{K}om(\mathscr{C}_{k_1})),\ E \in \mathscr{O}bj(\mathscr{K}om(\mathscr{C}_{k_2})),$$

and a two-holed diagram $\mathcal{D} \in \mathscr{D}(k_1, k_2; k)$; given two homotopic morphisms

$$F,\ G : C \to D,$$

the morphism $\mathcal{D}(F, Id_E)$ and $\mathcal{D}(G, Id_E)$ are homotopic.

Proof. Remember that we extend the planar algebras structures over additive categories by making the diagrams multilinear. So, if H is the prism map for F and G then

$$\mathcal{D}(F, Id_E) - \mathcal{D}(G, Id_E) - d_{\mathcal{D}(D,E)} \circ \mathcal{D}(H, Id_E) - \mathcal{D}(H, Id_E) \circ d_{\mathcal{D}(C,E)} =$$

$$= \mathcal{D}(F - G, Id_E) - \mathcal{D}(d_D, Id_E) \circ \mathcal{D}(H, Id_E) - \varepsilon \cdot \mathcal{D}(Id_D, d_E) \circ \mathcal{D}(H, Id_E) +$$
$$- \mathcal{D}(H, Id_E) \circ \mathcal{D}(d_C, Id_E) - (-\varepsilon) \cdot \mathcal{D}(H, Id_E) \circ \mathcal{D}(Id_C, d_E) =$$

where ε is ± 1, depending on which chain group we are considering, see the previous chapter for details,

$$= \mathcal{D}(F - G - d_D \circ H - H \circ d_C, Id_E) = \mathcal{D}(Id_C, Id_E) = Id_{\mathcal{D}(C,E)}.$$

Q.E.D.

Definition 2.5. A *canopolis morphism*, between two canopoleis $\{\mathscr{C}(k)\}_k$ and $\{\mathscr{C}'(k)\}_k$ over the same planar algebra, is a collection of functors

$$F_k : \mathscr{C}(k) \to \mathscr{C}'(k),$$

that respects planar algebra operations; that is to say:

$$\mathcal{D}(F_{i_1}(f_1), ..., F_{i_m}(f_m)) = F_i(\mathcal{D}(f_1, ..., f_m)),$$

for all the diagrams, indices and all the maps for which the expression above makes sense.

2.2. New geometric categories. We cannot define a functor without telling its source and target category; hence we need to define the proper categorical setting to our theory. The aim of this section is to define two equivalent "geometric" category, non in a technical sense, whose the objects are tangle and such that we can define a new functor from these categories to $\mathscr{K}ob$, that associate to each tangle its Khovanov formal chain complex.

The first category we are going to define is called $\mathscr{C}ob^4(B)$. This category is the category whose objects are tangles in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon)$ with boundary set B and whose morphisms are generic tangle cobordisms in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0, 1]$. To avoid any ambiguity later on, we suppose each component of the tangles to have a region near a crossing marked.

The composition of two morphisms in $\mathscr{C}ob^4(B)$ is given by the cobordism obtained by gluing the two cobordisms together, similarly to what was done for $\mathscr{C}ob^3$ in the previous chapter, paying attention to match both the boundary set and the marked regions. The latter condition is necessary to rule out the symmetries in the diagram so that the composition can be unambiguous. It is easy verification that $\mathscr{C}ob^4(B)$ is indeed a category.

We can endow the set of morphisms in $\mathscr{C}ob^4$, i.e. the disjoint union of $\mathscr{C}ob^4(B)$ for all possibles boundary sets B, with the structure of planar algebra in a natural way. Let D be a planar arc diagram in $\mathscr{D}(B_1, ..., B_k; B)$, and $c_i \in \mathscr{A}r(\mathscr{C}ob^4(B_i))$, for $i \in \{1, ..., k\}$, we define $D(c_1, ..., c_k)$ to be the generic tangle cobordism obtained by placing each cobordism in the holes of the cylinder $D \times \{0\} \times [0,1]$. The composition of two planar operators is defined as the cylinder over the composition of the corresponding planar arc diagrams.

It is easy to verify that the planar algebra structure just defined, together with the natural planar algebra structure of the tangles, define a canopolis structure over the planar arc algebra of the planar arc diagrams on $\mathscr{C}ob^4$. Moreover, this structure descends to a canopolis structure on $\mathscr{C}ob^4_{/i}(B)$, which has the same objects as $\mathscr{C}ob^4$ but the morphisms are considered up to boundary fixing ambient isotopies of $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$.

Remark. When we write $\mathscr{C}ob^4(k)$ we mean the category $\mathscr{C}ob^4(B)$ where B is the set of the $2k$-th roots of unity; while $\mathscr{C}ob^4$ will mean, from now on, the disjoint unipon of $\mathscr{C}ob^4(k)$, for all $k \in \mathbb{N}$.

A combinatorial version of $\mathscr{C}ob^4_{/i}(k)$ can be obtained by considering, as objects, the – oriented – k-tangles, and, as morphisms, movies – with oriented stills – of generic k-tangles cobordisms. Here a structure of planar algebra over the morphisms can be obtained by defining the operator $\mathcal{D}$ as the one that takes two movies, say M and N, of the same length – we can always extend a movie to the desired length: it is sufficient to add more copies of a single still – to a movie whose still at the time t corresponds to the composition of the time t stills of M and N via $\mathcal{D}$ – se the picture below.

This structure of planar algebra, together with the natural structure of planar algebra of the tangle diagrams, defines a canopolis structure on the category just defined; the latter will be denoted by $\mathscr{M}ov(B)$. One can consider the category $\mathscr{M}ov_{/m}(B)$, obtained from $\mathscr{M}ov$ by considering its morphisms up to movie moves and exchange of distant critical points. As a consequence of Theorem 4 the following result, whose easy proof is omitted, holds.

Proposition 10. *There is a natural isomorphism of canopoleis between $\mathscr{M}ov_{/m}$ and $\mathscr{C}ob^4_{/i}$; this is given by associating to a cobordism a movie presentation and to a movie the rebuilt surface.*

2.3. The functor and grading. Now we will define our functor: Kh_0 is a functor defined from $\mathscr{C}ob^4$, or $\mathscr{M}ov$, to $\mathscr{K}ob$, and associates to each tangle $\mathcal{T}$ its Khovanov chain complex $Kh(\mathcal{T})$, and to each generic cobordism – or to a movie – the associated morphism in Khovanov homology.

It is not difficult to see this is a functor; our claim, which is also the main result of this chapter, is that this functor descends to a – degree preserving – canopolis morphism from $\mathscr{C}ob^4_{/i}$ to $\mathscr{K}ob_{/\pm h}$, i.e. the projectivization of $\mathscr{K}ob_h$. We must consider the projectivization because the two sides of some movie moves induces the same morphism only up to sign; we will discuss the sign problem in the next chapter.

Grading will be intensively used in the follow up, so we need a definition of canopolis that includes the grading.

Definition 2.6. A *graded canopolis* is a canopolis whose "cans", i.e. the elements of the morphism sets belonging to the categories that compose the considered canopolis, are graded and such that the grading is additive with respect to both composition and planar algebra operations.

We can grade $\mathscr{C}ob^4$, and also $\mathscr{M}ov$, by giving to a generic cobordism Σ the grade:

$$deg(\Sigma) = \chi(\Sigma) - k,$$

where k is half the number of connected components of the vertical boundary.

Remark. Each Morse move either increases – birth/death move – or decreases – fusion move – the degree by one.

It is a simple verification that the degree is additive for both types of composition, and thus $\mathscr{C}ob^4$ and $\mathscr{M}ov$ are given the structure of graded canopoleis.

3. The main theorem

In this section we give the proof of the following statement:

Claim. *The functor Kh_0 descends to a degree preserving canopolis morphisms between $\mathscr{C}ob^4_{/i}$, or its combinatorial counterpart $\mathscr{M}ov_{/m}$, and $\mathscr{K}ob_{/\pm h}$, the category $\mathscr{K}ob_{/h}$ whose morphisms are considered only up to sign.*

This statement is also known as *functoriality of Khovanov homology*; to be precise this is a geometric version involving formal complexes but, of course, it becomes the corresponding statement for Khovanov homology once a TQFT – with corners, in the case of tangles – is applied.

This section is divided in three parts: the first one contains a few preliminary results needed by the other two parts; the second part is dedicated to prove that the first and second group of movie moves are in the "kernel" of our functor, that is to say both sides of the movie moves induce the same morphism – up to homotopy and sign – at the chain level. Finally, in the third, and last, part we verify the statement for the third group of moves.

3.1. Some preliminary results. Following Bar-Natan – [**BarNatan**] – we begin with the definition of "having no automorphism" for tangles.

Definition 2.7. A tangle diagram is called *Kh-simple* if every degree 0 automorphism of $Kh(\mathcal{T})$ is homotopic to $\pm Id_{Kh(\mathcal{T})}$.

Lemma 4. *The empty set, seen as a the empty knot, is Kh-simple.*

Proof. The only non-zero degree 0 automorphisms of $Kh(\emptyset)$ are, up to sign, the torus and the empty cobordism. The first one corresponds to multiplication by 2, and so it is not an isomorphisms over $\mathbb{Z}$, which leaves $\emptyset$ as the sole morphisms that is non null; as the empty set represent the identity for the diagram $\emptyset$, this concludes the proof.

Q.E.D.

Now we want to show that a particular class of tangles, the *pairings*, are *Kh*-simple; a pairing is a tangle diagram without crossings and without closed connected components.

Lemma 5. *Any pairing is Kh-simple.*

Proof. Any degree 0 morphism is a $\mathbb{Z}$-linear combination of degree 0 cobordisms, so we begin by classifying the latters.

If Σ is a degree 0 cobordism, by definition, its Euler characteristic must be equal to half the number of its vertical boundary components; any handle attached to the vertical curtains decreases the degree by two, this implies that, to maintain degree 0 we must add two spherical components for each handle, but, for the S relation, then our cobordism represent the 0 morphism.

So we must have only vertical curtains with possibly disjoint closed connected components. Each component with genus higher than one – so with negative Euler characteristic – must be balanced against a number of spherical components so, as before, our cobordism is 0. The only remaining possibility is to have curtains and a number of disjoint tori, whose Euler characteristic is 0; by the T relation, we can substitute each torus with a multiplying factor of 2. But, as we are using $\mathbb{Z}$ as base ring, the tori cannot be allowed if we want Σ to represent an invertible surface; this concludes the proof of the lemma.

Q.E.D.

As one could expect, being Kh-simple does not depend on the diagram chosen.

Lemma 6. *If $\mathcal{T}$ and $\mathcal{T}'$ are two diagrams of the same tangle then $\mathcal{T}$ is Kh-simple if and only if $\mathcal{T}'$ is Kh-simple.*

Proof. By the invariance of Khovanov homology, the two diagrams have homotopy equivalent Khovanov complexes. Let F be the homotopy, from $Kh(\mathcal{T})$ to $Kh(\mathcal{T}')$, induced by a sequence of Reidemeister moves that transforms $\mathcal{T}$ in to $\mathcal{T}'$, and let G be its up-to-homotopy inverse.

If $\alpha : Kh(\mathcal{T}) \to Kh(\mathcal{T})$ is a degree 0 automorphism and $\mathcal{T}'$ is Kh-simple, then $F \circ \alpha \circ G =: \alpha'$, which is a degree 0 automorphism of $Kh(\mathcal{T}')$, must be, up to homotopy, $\pm Id_{Kh(\mathcal{T}')}$; thus we have

$$\alpha \sim G \circ F \circ \alpha \circ G \circ F = G \circ \alpha' \circ F \sim \pm G \circ F \sim \pm Id_{Kh(\mathcal{T})}.$$

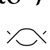
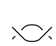
Q.E.D.

Now we must find a way to reduce our tangles to pairings; the trick is to show that if we remove a crossing from a simple diagram then the result will be simple. Then we can prove that if our diagram is simple adjoining a crossing gives a simple diagram.

We will say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ *by the addition of a crossing* if $\mathcal{T}' = \mathcal{D}(\mathcal{T}, \mathcal{X})$, where $\mathcal{X}$ is the diagram of a crossing and $\mathcal{D}$ is the planar arc diagram in Figure 8. Similarly, we will say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ *by the removal of a crossing* if $\mathcal{T}$ is obtained from $\mathcal{T}'$ by the addition of a crossing.

Proposition 11. *Given a Kh-simple tangle diagram $\mathcal{T}$, any tangle diagram obtained from $\mathcal{T}$ by the addition of a crossing is Kh-simple.*

Proof. Let $\mathcal{X}$ be the diagram of a crossing. If $\alpha : Kh(\mathcal{T}) \to Kh(\mathcal{T})$ is a degree 0 automorphism then $\alpha' = \mathcal{D}(\alpha, Id_{\mathcal{X}})$, where $\mathcal{D}$ is the diagram shown in Figure 8, is a degree 0 automorphism of $Kh(\mathcal{T}')$, hence homotopic to $\pm Id$.

Let X^{-1} be the mirror image of X; then the diagram $\mathcal{D}(\mathcal{D}(\mathcal{T}, X), X^{-1})$ is equivalent to $\mathcal{T}$ via a second Reidemeister move. The map that converts $\mathcal{D}(X, X^{-1}) =$  – or  , depending on which type of crossing is X – in induces an homotopy equivalence Ψ between $Kh(\mathcal{D}(\mathcal{T}, \mathcal{D}(X, X^{-1}))) = Kh(\mathcal{D}(\mathcal{D}(\mathcal{T}, X), X^{-1}))$ and $Kh(\mathcal{T})$. We have

$$\alpha \sim \Psi \circ \mathcal{D}(\mathcal{D}(\alpha, Id_X), Id_{X^{-1}}) \circ \Psi^{-1},$$

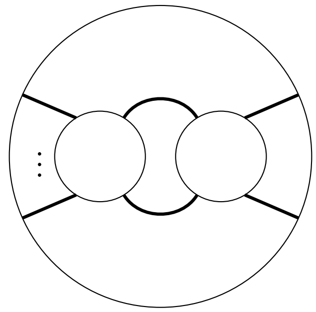


FIGURE 8. The diagram $\mathcal{D}$, also called *connected sum diagram*.

thanks to Proposition 9, we have

$$\Psi \circ \mathcal{D}(\mathcal{D}(\alpha, Id_X), Id_{X^{-1}}) \circ \Psi^{-1} \sim \Psi \circ \mathcal{D}(\pm Id_{\mathcal{D}(\mathcal{T},X)}, Id_{X^{-1}}) \circ \Psi^{-1},$$

hence

$$\alpha \sim \Psi \circ \mathcal{D}(\pm Id_{\mathcal{D}(\mathcal{T},X)}, Id_{X^{-1}}) \circ \Psi^{-1} \sim \pm\Psi \circ Id_{\mathcal{D}(\mathcal{T}(\mathcal{T},X),X^{-1})} \circ \Psi^{-1} \sim \pm Id_{\mathcal{T}}.$$

Q.E.D.

By putting together the two previous results, it is immediate the following corollary.

Corollary 1. *A tangle diagram $\mathcal{T}$ is Kh-simple if and only if any tangle diagram $\mathcal{T}'$, obtained from $\mathcal{T}$ by the addition of a crossing, is Kh-simple.*

3.2. First and second group of movie moves. The first group of movie moves are those corresponding to the Reidemeister moves, and it can be easily dealt with.

Theorem 6. *The movies Ia, Ib, IIa, IIb and III correspond, via Kh_0, to the identity map.*

Proof. The movies listed in the statement correspond to doing a Reidemeister move and its inverse, which induce maps that are homotopic to the identity – this is the invariance of *Kh* under the Reidemeister moves.

Q.E.D.

The second group of movie moves is composed by the so called *circular movie clips*; these are given by movies whose starting still is identical to the ending still, and no morse moves are perfomed.

Theorem 7. *The movies IV, V, VI, VII and VIII induce, via Kh_0, the identity map.*

Proof.

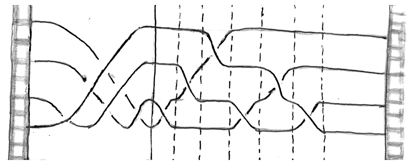


FIGURE 9. A way to transform, by the addition of crossings, the starting still of move VIII in to a *Kh*-simple diagram.

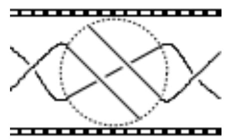


FIGURE 10. A way to transform, by the addition of crossings, the starting still of move IV in a *Kh*-simple diagram.

To prove the statement it is sufficient to show that the first still of each movie comes from a *Kh*-simple tangle diagram by addition or the removal of crossings. Figure 9 shows how the first still of move VIII is related to a tangle diagram equivalent to a pairing, by addition of crossings, and hence is *Kh*-simple by Lemmas 5 and 6 and Corollary 1.

Remark. With the same technique used for move VIII, one can prove that any braid is *Kh*-simple.

In a similar way, the starting still of move IV can be related by addition of crossings to a *Kh*-simple diagram, as shown in figure 10.

The other moves begin with either with a pairing or a crossing, so the starting still represents a *Kh*-simple diagram. The fact that the starting still, which is the same as the ending still, is *Kh*-simple implies that the automorphism of Khovanov homology induced by each circular movie clip is $\pm Id$; the latter are also the morphisms induced by the identity movie and its formal opposite – remember that we are considering pre-additive categories.

Q.E.D.

Remark. The same argument used for the moves from IV to VIII works also for the first group of movie moves.

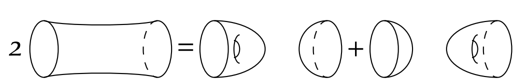
Remark. The ninth move can be seen as a circular movie clip, and so the argument used above works also for this move. Moreover, there is no need to add or remove crossings, because the starting tangle of both sides is a pairing, hence *Kh*-simple. the reason why move IX is in the third group instead of the second one, is because it involves Morse moves (a birth move and a fusion I move, hence its grade is zero).

3.3. The third group of movie moves. The last group of movie moves is composed of movie moves that involves Morse moves. This implies that there is a grade change during the reproduction of the movies involved. Move IX can easily be dealt with by the same argument used for moves from IV to VIII, as we said in the second remark at page 65. The remaining moves induce maps that are of degree ± 1 and involve non *Kh*-simple tangles – those that involve a loop, in general, are not *Kh*-simple, as we will see in the next chapter. So it is not possible to apply the same argument used for the second group of movie moves to moves from X to XIV.

To prove the statement for moves XI and XIII, we need a new geometric relation which is a consequence of the 4-*Tu* relation, depicted at page 30.

Remark. The relation in the Proposition 12 is called *neck cutting*, or *NC*, relation. This relation should be interpreted in the same way we interpreted the 4-*Tu* in the previous chapter. Let us consider a morphism in $\mathscr{C}ob^3$, which is a certain surface Σ embedded in $\mathbb{D}^2 \times [0,1]$. If we consider the intersection of a 3-disk D with our surface Σ, such that $\partial D \cap \Sigma$ is the disjoint union of two circles; then two times the surface $\Sigma \setminus Int(D) \cup c$, where c is a cylinder between the two circles in $\partial D \cap \Sigma$, is equal to the sum of the surface obtained from $\Sigma \setminus Int(D)$ by gluing the fist cobordism on the right side of the equality in *NC* along the circles in $\partial D \cap \Sigma$, with the cobordism obtained, with the same technique, from the second cobordism on the right side of the equality in *NC*.

Proposition 12. *If the 4-Tu holds, then also the relation depicted below holds in $\mathscr{C}ob^3$.*



Proof. The proof of this proposition is straightforward: just apply the 4-*Tu* relation to the cobordism drawn below.

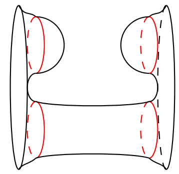


FIGURE 11. To obtain the NC relation is sufficient to apply the 4-Tu to the cobordism in figure.

Q.E.D.

Remark. If we require the morphism sets of $\mathscr{C}ob^3$ to be the $\mathcal{R}$ modules, where $\mathcal{R}$ is a commutative ring where 2 is invertible, instead of abelian groups and the matrices in $\mathscr{M}at$ to be $\mathcal{R}$ linear, then the NC relation will be equivalent to the 4-Tu.

Theorem 8. *The right and the left sides of movie moves X and XI induce, up to sign and homotopy, the same morphism via Kh_0.*

Proof. One can simply compute the maps induced by the left and right side of X, and of XI. In both cases, the difference – or the sum, depending on whether the move changes the sign or not – between the map induced by the left side and the map induced by the right side is just a version of the neck cutting relation proved above.

Q.E.D.

Theorem 9. *Both sides of move XII induce the same morphism via Kh_0, up to sign and homotopy.*

Proof. If one consider the chain group $Kh^0\left(\quad\right)$, which is composed by the direct sum of a circle on the right union a curtain, plus a circle on the left union a curtain. One can easily compute the maps induced between the Khovanov complexes. Figure 12 shows the explicit computation of the map for the left side of this move, the other map can be calculated similarly. In the end both maps consist in the sum of a curtain union a cap on the left and a curtain union a cap on the right, this implies that the two induced maps are the same.

Q.E.D.

Theorem 10. *The left side of move XIII induces, via Kh_0, the same morphism as the right side of the same move, up to sign and homotopy.*

Proof. If one compute explicitly the maps induced by the left and the right sides, one will obtain none other than the maps Φ_R and Φ_L, see page

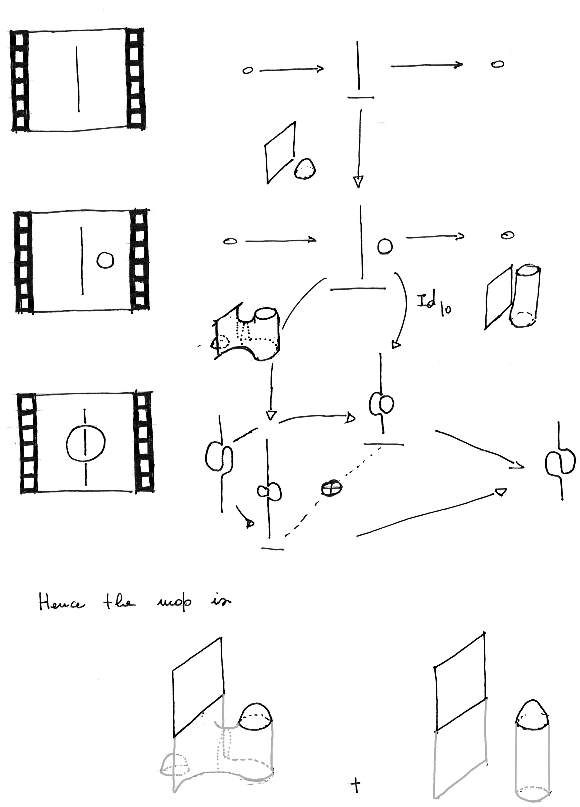


FIGURE 12. Computation of the map induced by the left side of move XII

42 and page 43, used to prove the invariance under the third Reidemeister move of the Khovanov homology; as already noticed, the two maps Φ_R and Φ_L are homotopic.

Q.E.D.

Finally we can complete the proof of our statement.

Theorem 11 (Functoriality of the formal Khovanov complex). *The functor Kh_0 descends to a degree preserving canopolis morphisms between $\mathscr{C}ob^4_{/i}$, or* *its combinatorial counterpart $\mathscr{M}ov_{/m}$, and $\mathscr{K}ob_{/\pm h}$, i.e. the category $\mathscr{K}ob_{/h}$ whose morphisms are considered only up to sign.*

Proof. That Kh_0 is a degree preserving canopolis morphism between $\mathscr{C}ob^4$, or its combinatorial counterpart $\mathscr{M}ov$, and $\mathscr{K}ob_{/h}$ is a consequence of Theorem 3 point (d). The fact that when we mod out by the isotopies – or movie moves – the grade is preserved, is due to the fact that the morphisms induced by either sides of the movie moves are degree homogeneous. Finally, Theorems from 6 to 10, as well as the second remark at page 65, almost complete the proof of our claim. The only remark left is that both side of the mirror image, the reflected version with respect either the x or y axis, as well as the *rewind*, i.e. the move read from bottom to top or from right to left, of the movie moves we have shown induce the same morphism via Kh_0; this could be done by using the same arguments used for the corresponding moves from Ia to XIII.

Q.E.D.

4. Foreword

In this last section we will describe results related to Theorem 11. An alternative proof of that statement can be found in [**Jacobss**]. This proof is done by a straightforward argument: Jacobsson computes, for each move, the morphisms induced by both sides and compares them. The notation used by Jacobsson allows him to carry out the computations without two much sweat. Moreover, he is able to find which moves preserve the sign and which moves change the sign; a small and incomplete list is given below, for further reference see [**Jacobss**].

To "fix" this sign problem Scott Morrison, David Carter and Kevin Walker in [**ClaMorWa**] defined another invariant, similar to the formal Khovanov chain complex described in chapter 1, using disoriented surfaces instead of oriented ones. Their construction depends on a parameter ω, which is a fourth root of unity, and is such that for the value $\omega = i$ we obtain a new theory which is completely functorial; while, for $\omega = 1$, we can recover the original theory.

In his paper [**KhovTan2**], Khovanov proves that his invariant for $(m, n)-$tangles, obtained via a TQFT with corners, is functorial. His proof is similar to the one described in this chapter – which is the [**BarNatan**] approach – at least in spirit. On the other hand, Khovanov does not use the canopolis formalism and make use of heavy machinery derived from abstract non-sense such as 2-categories.

Our proof of the functoriality, as it is given at geometrical level, adapts also to Lee theory – see [**BarNatan**], page 1483.

Move number	as shown	mirror	rewind
Ia, Ib, IIa, IIb, III	same	same	same
IV	opp	–	opp
V	same	same	opp
VI pos curl	–	opp	opp
VI neg curl	–	same	same
VII	opp	opp	same
VIII	same	–	same
IX	same	–	same
X	opp	same	same
XI	opp	same	same
XII	same	–	opp
XIII	opp	–	same

TABLE 1. Table of signs for the movie moves.

CHAPTER 3

Monodromy

In the previous chapter we defined a functor Kh_0, from the category $\mathscr{C}ob^4$ to the category $\mathscr{K}ob$, and we have shown that this functor is also well defined on the quotient $\mathscr{C}ob^4_{/i}$; in fact we proved a stronger statement: Kh_0 descends to a graded canopolis morphism between $\mathscr{C}ob^4_{/i}$ and $\mathscr{K}ob$.

In this chapter we are going to analyze the obstructions to define Kh_0 on the quotient $\mathscr{C}ob^4_{/I}$, which is the category with the same objects as $\mathscr{C}ob^4$ and whose morphisms are isotopy classes of generic tangle cobordisms; these obstructions come from the existence of surfaces, whose boundary is given by two copies of the same tangle, that are ambient isotopic to a cylinder – of course, not relatively to the boundary – and whose induced maps in Khovanov homology are neither the identity nor its opposite.

Given a link diagram $\mathcal{D}$, all the cobordisms with boundary two copies of $\mathcal{D}$ and ambient isotopic to a cylinder, form a subgroup of the endomorphisms group of $\mathcal{D}$ in $\mathscr{C}ob^4$; this group was called by Jacobsson, see [**Jacobss**], the *monodromy group*. Since its introduction the monodromy group has been largely ignored: to the author's knowledge, no one has invstigated this group.

The aim of this chapter is to pursue the study of the monodromy group. In the first section we will formally define the monodromy group of a link diagram and show that the isomorphism class of this group does not depend on the chosen diagram, hence it defines a link invariant.

In the second section we present some results that will help us to explicitly calculate the monodromy groups of some links. In the third section we prove the Rasmussen Tanaka theorem, which will be fundamental to our computations.

The fourth section contains the computation of the monodromy groups for the unknot, the unlink with two components and the unlik with three components.

The concluding section is dedicated to the description of the partial computations of the monodromy group of the Hopf link; here we describe the limits of our techniques and some possible ways to overcome them.

1. Definiton and invariance

Let $\mathcal{D}$ be an oriented link diagram, seen as an object of $\mathscr{M}ov$. A *circular movie starring* $\mathcal{D}$ is a movie whose first and last still is $\mathcal{D}$ and that does not involve any Morse move. Circular movies with starting and ending diagram equal to $\mathcal{D}$ represent unknotted cylinders in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$. Examples of circular movies were given in the previous chapter: the second group of movie moves – the so called circular movie clips – are circular movies. Now we can formally introduce the object of our study.

Definition 3.1. The *algebraic Khovanov monodromy group* of an oriented link diagram $\mathcal{D}$ is the set of all the automorphisms of the Khovanov homology $KH^{\bullet,\bullet}(\mathcal{D})$, associated to circular movies via Kh_0 and the Khovanov TQFT[1]. This group will be denoted by $\mathcal{M}on(\mathcal{D})$.

Remark. One could also consider the *geometric monodromy group* of an oriented link diagram $\mathcal{D}$. This group is given by all the equivalence classes, modulo movie moves, of circular movies starring $\mathcal{D}$ with the composition of movies – seen as morphisms in $\mathscr{M}ov$ – as operation. This is indeed a group and, *a priori*, gives more information than the algebraic Khovanov monodromy group. The study of the geometric group goes beyond the scope of this work.

The name Khovanov is not written in vain: the use of Lee's TQFT[2], instead of the Khovanov one, would have led to different results. From now on, we will drop both the name "Khovanov" and the adjective "algebraic" and refer to $\mathcal{M}on(\mathcal{D})$ simply as *monodromy group*.

As the circular movies do not contain any Morse move, they involve only Reidemeister moves and planar isotopies. Both Reidemeister moves and planar isotopies are associated, via Kh_0, to morphisms that are homogeneous of degree 0 with respect to both the homological and the quantum

[1]This is the TQFT used, in Chapter 1, to define Khovanov homology. Just to be clear, this TQFT associates, to a single circle, the free $\mathbb{Z}$-module V generated by x_+ and x_-. V is given a grading by the formula

$$deg(x_\pm) = \pm 1.$$

The multiplication m is defined by:

$$m(x_+ \otimes x_+) = x_+, \quad m(x_+ \otimes x_-) = m(x_- \otimes x_+) = x_-,$$
$$m(x_- \otimes x_-) = 0;$$

and the co-multiplication map Δ is such that

$$\Delta(x_+) = x_+ \otimes x_+ + x_- \otimes x_+,$$
$$\Delta(x_-) = x_- \otimes x_-.$$

Finally, the co-unit is defined by $\varepsilon(x_-) = 1$ and $\varepsilon(x_+) = 0$.

[2]The TQFT defined in Chapter 1, Example 1 at page 27.

degrees. Because the Khovanov TQFT preserves the gradings – both homological and quantum – we have that all the elements of $\mathcal{M}on(\mathcal{D})$ are of degree 0 with respect to both degrees. An obvious consequence of these considerations is the following lemma.

Lemma 7. *A Kh-simple – see page 61 for the definition – diagram has trivial monodromy.*

The inverse implication may not be true. Let $\mathcal{D}$ be an oriented link diagram, $\mathcal{A}ut_0(\mathcal{D})$ is the set of all the degree 0 automorphisms of $KH^{\bullet,\bullet}(\mathcal{D})$, obtained from movies whose first and last stills are copies of $\mathcal{D}$. The set $\mathcal{A}ut_0$ together with the composition is a group, and the monodromy group is a subgroup of $\mathcal{A}ut_0$.

Geometrically speaking, $\mathcal{A}ut_0$ consists of the morphisms associated to possibly knotted cylinders[3] with boundaries given by two copies of the same diagram. It is personal impression of the author that the maps induced by Kh_0 and Khovanov TQFT, do not "perceive" the knotting of the surfaces; in particular, the following statement is believed to be true.

Conjecture. *Given an oriented link diagram $\mathcal{D}$, it is true that*

$$\mathcal{M}on(\mathcal{D}) = \mathcal{A}ut_0(\mathcal{D}).$$

We can use both the groups $\mathcal{A}ut_0$ and $\mathcal{M}on$ to study knots and links.

Theorem 12. *The isomorphism classes of $\mathcal{M}on(\mathcal{D})$ and $\mathcal{A}ut_0(\mathcal{D})$ are a link invariants.*

Proof. Let $\mathcal{D}$ and $\mathcal{D}'$ two diagrams of the same link. By Reidemeister theorem there is a finite sequence of planar isotopies and Reidemeister moves that trasforms $\mathcal{D}$ into $\mathcal{D}'$; let Φ the map induced by this sequence in Khovanov homology. Given an element $\varphi \in \mathcal{M}on(\mathcal{D})$, we have that $\Phi \circ \varphi \circ \Phi^{-1}$ is the map associated to a circular movie starring $\mathcal{D}'$, obtained by playing the movie associated to Φ^{-1}, which is the rewind of the movie associated to Φ, then the movie associated to φ, which is a circular movie starring $\mathcal{D}$, and, finally, the movie associated to Φ.

In this way we found a group homomorphism between $\mathcal{M}on(\mathcal{D})$ to $\mathcal{M}on(\mathcal{D}')$, which is given by the conjugation by Φ. This homomorphism is clearly invertible, its inverse being the conjugation by Φ^{-1}, hence a group isomorphism. The same proof works for $\mathcal{A}ut_0$.

Q.E.D.

[3]These are surfaces with boundary $\mathcal{D} \sqcup \mathcal{C}$, whose Euler characteristic – that is also the degree of the associated map – is 0. Moreover, it exists a surface with boundary $\mathcal{C} \sqcup \mathcal{D}$ such that if we glue the two surfaces along $\mathcal{C}$, then the result is a cylinder over $\mathcal{D}$. THe only possibility is that these surfaces are disjoint union of cylinders.

2. Calculation techniques

This section is dedicated to the description of the techniques that will be used to compute monodromy groups in the next section. The techniques are fundamentally two: an algebraic technique, based on the fact that the maps considered are degree homogeneous, and a geometric tecnique, based on the knowledge of which maps are associated to closed surfaces.

Let $\mathcal{M}$ be a finitely generated bi-graded $\mathbb{Z}$-module and let $\mathcal{M}^{i,j}$ be the submodule of $\mathcal{M}$ composed by its elements of bi-degree (i,j). The fact that $\mathcal{M}$ is finitely generated implies that only a finite number of $\mathcal{M}^{i,j}$ is non-zero and that each $\mathcal{M}^{i,j}$ is finitely generated.

Let us fix a set of generators for $\mathcal{M}^{i,j}$ $\{\alpha^k_{i,j}\}_k$, with $k \in \{1, ..., m\}$ and $m = m(i,j)$; we may suppose m to be as little as possible. This gives us a set of generators for $\mathcal{M}$, given by the union of all these sets of generators. Moreover, we can fix an ordering on this set of generators by using the degree i first, and then the degree j.For elements that have the same degrees i and j, just choose an arbitrary order. Now we can express every $\mathbb{Z}$-linear map φ from $\mathcal{M}$ to itself by an integer matrix M_φ.

Remark. The matrix M_φ may not be unique because of the presence of torsion. As an example, consider $\mathcal{M} = \mathbb{Z}/5\mathbb{Z}$ and $\varphi = Id_{\mathbb{Z}/5\mathbb{Z}}$. In this case, the matrix associated to φ is just an integer number. All the numbers of the form $5h+1$, with $h \in \mathbb{N}$, describe the same application φ. To limit the ambiguity in the choice of the matrix it is sufficient to choose as entry the integer which has the least absolute value, and if there are two, then we will take the positive one.

Let φ be a *bi-homogeneous*, i.e. homogeneous with respect to both degrees, endomorphism of $\mathcal{M}$. Then a matrix M_φ associated to φ is of the form:

$$\left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & A_s \end{array}\right)$$

where each A_i is a matrix representing the effect of the map on the i-th module. The reason this matrix should be block diagonal lies in the homogeneity of φ with respect to the i-grading. On the other hand, we can repeat the same reasoning on each block A_i, using the j-grading, hence obtaining that also the A_i's should be block diagonal.

In this way, taking as $\mathcal{M}$ the Khovanov homology of a certain diagram, one can find a matrix group that admits $\mathcal{A}ut_0$, and hence $\mathcal{M}on$, as a subgroup.

The second technique is more "geometric" and is based on the following result from Rasmussen, see [**Rasmus**], and Tanaka, see [**Tanaka**].

Theorem (Rasmussen-Tanaka). *Given a closed connected surface Σ embedded in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$, we have either $KJ(\Sigma) = \pm 2$, and this happens if and only if Σ is a torus, or $KJ(\Sigma) = 0$, for all the other surfaces.*

The $KJ(\Sigma)$ in the previous proposition is the *Khovanov-Jacobsson number* of Σ and is defined as the number that represents the map induced by the closed surface Σ, via Kh_0 and Khovanov TQFT, from $\mathbb{Z}$ to $\mathbb{Z}$.

The result itself will be proven in the next section. For now let us assume this result as given and proceed to describe how it is related to monodromy.

Our technique consists of creating a movie of a closed surface that involves the given diagram to compute the monordomy. Then we replace this diagram with a circular movie starring it, we compute the induced map and, by the Rasmussen-Tanaka theorem, this map should be either ± 2 or 0. In this way we obtain conditions on the possible morphisms in $\mathcal{A}ut_0$ or $\mathcal{M}on$.

3. The Rasmussen-Tanaka theorem

One of the main tools at our disposal is the Rasmussen-Tanaka theorem for closed surfaces. This theorem asserts that the maps induced in Khovanov homology, via Kh_0 and the Khovanov TQFT, by a closed connected surface are either the 0 map or the multiplication by a factor of ± 2. This result has been proven independently, in 2005, by Tanaka, see [**Tanaka**], and Rasmussen, see [**Rasmus**]. Our proof is the one given by Tanaka, which uses the Bar-Natan construction. The proof given by Rasmussen is based on Lee Theory.

Let us begin by defining a Frobenius algebra over $\mathbb{Z}[T]$. Let B be the free $\mathbb{Z}[T]$-module of rank 2 – this condition is forced if we want to obtain a link homology theory, see Chapter 1 – generated by v_+ and v_-, and graded with the following conventions

$$deg(T) = -4, \quad deg(x_+) = 1, \quad deg(x_-) = -1.$$

To obtain a Frobenius algebra we must define the multiplication m_1, the co-multiplication Δ_1, the unit ι_1 and the co-unit ε_1. These maps are defined by

$$\Delta_1(x_+) = x_+ \otimes x_- + x_- \otimes x_+, \qquad \Delta_1(x_-) = x_- \otimes x_- + T \cdot x_+ \otimes x_+,$$

$$m_1(x_+ \otimes x_+) = x_+, \qquad m_1(x_- \otimes x_+) = m_1(x_+ \otimes x_-) = x_-,$$
$$m_1(x_- \otimes x_-) = T \cdot x_+;$$

$$\varepsilon_1(x_-) = 1, \qquad \varepsilon_1(x_+) = 0;$$

$$\iota_1(1) = x_+.$$

As a consequence of the definitions just given, we have that:

$$m_1 \circ \Delta_1(x_+) = 2x_-, \quad m_1 \circ \Delta_1(x_-) = 2T \cdot x_+. \tag{5}$$

This TQFT was defined by Bar-Natan – see [**BarNatan**], Section 9.2 – and also by Khovanov, the $\mathcal{F}_3$ in the paper [**KhovUniv**]. We will call it *Bar-Natan TQFT* and denote it by $\mathscr{F}_3$.

Notice that all the maps are grade homogeneous, hence the resulting cohomology, called *Bar-Natan cohomology* and denoted by $BN^{\bullet,\bullet}(\cdot)$, is bigraded.

Definition 3.2. The *Bar-Natan number*, denoted $BNn(\cdot)$, of a closed surface embedded in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$ is the number associated to the surface via Kh_0 and the Bar-Natan TQFT.

We can simply recover Khovanov theory from Bar-Natan theory by setting $T = 0$; in particular, the following equality holds

$$BNn(\Sigma)_{|T=0} = KJ(\Sigma),$$

where $KJ(\Sigma)$ indicates the Khovanov Jacobsson number of Σ.

Let us denote by ψ^{BN}_{Σ} the map induced by the link cobordism Σ, via Kh_0 and Bar-Natan TQFT, between the Bar-Natan cohomologies of its source and ending tangles. Clearly, we have

$$\psi^{BN}_{\Sigma}(1) = BNn(\Sigma),$$

for every closed surface Σ.

Given Σ, a generic closed surface in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$, one can consider a point p on Σ and a unknotted disc small neighbourhood U of p. By removing this small neighbourhood we can view $\Sigma \setminus U$ as a cobordism between the empty set and the trivial knot, or from the trivial knot to the empty set. Let

$$\varphi^{\Sigma}_1 : \; BN^{\bullet,\bullet}(\emptyset) \to BN^{\bullet,\bullet}(\bigcirc), \quad \varphi^{\Sigma}_2 : \; BN^{\bullet,\bullet}(\bigcirc) \to BN^{\bullet,\bullet}(\emptyset),$$

be the maps obtained by removing U from Σ, and applying Kh_0 and Bar-Natan TQFT, then we have

$$\psi^{BN}_{\Sigma} = \varphi^{\Sigma}_2 \circ \iota_1 = \varepsilon_1 \circ \varphi^{\Sigma}_1.$$

More generally, for every couple of closed surfaces Σ_1 and Σ_2, the following holds:

$$\psi^{BN}_{\Sigma_1\sharp\Sigma_2} = \varphi_2^{\Sigma_1} \circ \varphi_1^{\Sigma_2}.$$

Definition 3.3. A surface in $\mathbb{D}^2 \times (-\varepsilon, \varepsilon) \times [0,1]$ is said to be *unknotted* if and only if it bounds a solid torus of genus g.

From the Equation (5) the following lemma follows immediately:

Lemma 8. *For any unknotted surface Σ, of genus $2m+1 \geq 0$, we have that*

$$BNn(\Sigma) = 2 \cdot (4T)^m;$$

while, if the genus of Σ is $2m$, with $m \geq 0$, it holds

$$\varphi_2^{\Sigma}(x_-) = \pm\alpha \cdot T^m;$$

To prove the Rassmussen-Tanaka theorem we will need the two facts about knotted surfaces. The first one is the following.

Fact (Unknotting theorem). *Any knotted surface may be unknotted by attaching a finite number of 1-handles.*

The proof of this result can be found in both [**Kamada**] and [**HosKa**]. The minimum number of 1-handles needed to unknot a surface is called *unknotting number*, see [**HosKa**] for more.

A move on surface diagram that we need is the *ribbon move* shown in Figure 1. The movie version of the left side of the ribbon move is given by playing the movie on the left side of Movie Move XII and then the movie on its right side; while the other side of the ribbon move is the movie obtained by playing the right side of Move XII and then its left side. By the functoriality of Kh_0 we obtain that the map associated to each side of the ribbon move is the same, up to sign. The second fact we are going to use is the following.

Fact. *Any 1-handle on a surface-knot is ribbon move equivalent to a trivial 1-handle.*

This fact can be found in [**CaSaSa**], proof of Theorem 1, page 2780.

Now we can finally state and give Tanaka's proof of the following theorem on the Bar-Natan number.

Theorem 13 (Tanaka). *For any surface Σ of genus $g \geq 0$, we have the following:*

(1) if g is even, then $BNn(\Sigma) = 0$;
(2) if g is odd, then $BNn(\Sigma) = \pm 2^g T^{\frac{g-1}{2}}$.

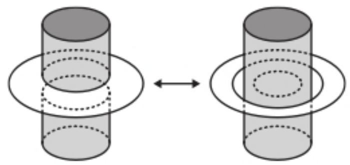


FIGURE 1. Ribbon move.

Proof. The map φ_Σ^{BN} has degree $\chi(\Sigma)$ and T is graded -4. If g is even, Lemma 8 tells us that φ_2^Σ sends x_- to an integer multiple of $T^{g/2}$. Then, since it holds

$$\psi_\Sigma^{BN} = \varphi_2^\Sigma \circ \iota,$$

hence the degree of ψ_Σ^{BN} is $-4^{g/2}+1+1$, which is different from 0 or any power of -4. Because the Bar-Natan cohomology of the empty set has only elements of degree 0 and powers of -4, we must have $\psi_\Sigma^{BN}=0$, which implies $BNn(\Sigma)=0$.

Let us suppose g to be odd. In this case, the following equality holds:

$$BNn(\Sigma) = \pm\alpha T^{(g-1)/2},$$

for a certain $\alpha \in \mathbb{N}$. The above equation implies

$$\varphi_2^\Sigma(x_+) = \pm\alpha T^{(g-1)/2}.$$

To prove the statement it is sufficient to show that $\pm\alpha = 2^g$. Let us consider an unknotted surface Γ_k of genus k. We have that

$$\psi_{\Sigma\sharp\Gamma_k}^{BN}(1) = \varphi_2^\Sigma \circ \varphi_1^{\Gamma_k}(1) = \pm\alpha T^{(g-1)/2}\cdot(4T)^{(k-1)/2};$$

for k even and greater than the unknotting number of Σ, we have that $\Sigma\sharp\Gamma_k$ is equivalent, up to a finite sequence of ribbon moves, to Γ_{g+k}. Since the ribbon moves do not change the map induced in Bar-Natan cohomology by our surface, we have the following equation

$$\alpha T^{(g-1)/2}\cdot(4T)^{(k-1)/2} = \pm 2(4T)^{\frac{(g+k)-1}{2}},$$

that leads us to the equality $\pm\alpha = 2^g$.

Q.E.D.

From the Tanaka theorem and the equality $BNn(\Sigma)_{|T=0} = KJ(\Sigma)$, we immediately have as a corollary the Rasmussen-Tanaka theorem.

Corollary 2 (Rasmussen-Tanaka theorem). *Given a closed connected surface Σ embedded in $\mathbb{D}^2\times(-\varepsilon,\varepsilon)\times[0,1]$, we have either $KJ(\Sigma)=\pm 2$, which happens if and only if Σ is a torus, or $KJ(\Sigma)=0$, for all the other surfaces.*

4. Monodromy groups

In this section we compute the monodromy groups in the case when our link is an unlink. Let us begin with the simplest case of unlink: the unknot. In this case, it is not difficult to compute the group $\mathcal{A}ut_0$, and show that it is equal to the monodromy group.

Proposition 13. *The group $\mathcal{A}ut_0$ of the unknot is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Proof. Let us consider the trivial diagram of the unknot $\mathcal{U}_1$. The Khovanov chain complex of this diagram is composed by a single non-trivial group: $C^{0,\bullet}(\mathcal{U})$; hence, its homology complex coincides with the chain complex.

The generators of $C^{0,\bullet}(\mathcal{U}) = KH^{0,\bullet}(\mathcal{U})$ are x_+ and x_-. Any linear map from $KH^{0,\bullet}(\mathcal{U}) = KH^{\bullet,\bullet}(\mathcal{U})$ to itself can be represented, with respect to the basis $x_+,\ x_-$, as a 2×2 integer matrix, say

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

As A represents a degree 0 automorphism, by the consideration made in Section 2, A should be diagonal, equivalently $c = b = 0$. Because the map associated to A is invertible and we have no torsion, A should be unimodular. So we get $a,\ d = \pm 1$.

To prove the statement it is sufficient to show that the only matrices A we can obtain are $\pm I_2$, where I_n denotes the $n \times n$ identity matrix.

We know that the matrices $\pm I_2$ are realizable by circular movies. All that is left is to rule out the other two possibilities. It is sufficient to prove that we cannot change the sign of only one of the two generators. Either generator will do, because if A is the matrix associated to an automorphism that changes the sign of the second generator, i.e. x_-, then $-A = -I_2 \cdot A$ is also an element of $\mathcal{A}ut_0$; hence, if we can change the sign of a generator, leaving the other untouched, then we can also obtain the automorphism that changes the sign of the other one.

Let us suppose that there is an oriented link cobordism between $\mathcal{U}_0$ and itself that induces a map φ that changes the sign of x_+ while leaving x_- untouched. Let us call M a circular movie that induces φ via Kh_0 and Khovanov TQFT.

Let us consider the movie N shown in Figure 2. This movie starts with a birth move, then a Fusion I, which actually splits, move is performed on the circle just born. The Fusion I move splits the newly born circle in two circles; our movie goes on by leaving one of these circles untouched, while the other one takes part in the movie M. At the end of this performance, we have again a couple of circles. Now a fusion moves between these

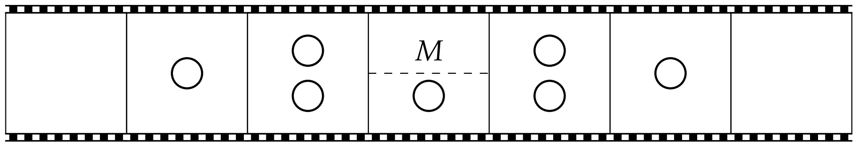


Figure 2. A movie representing a torus that involves the movie M.

circles takes place, and the two circles became a single circle. Finally, a death move is performed on the only circle left.

This movie describes a torus; hence, by the Rassmussen-Tanaka theorem, the map it induces should be multiplication by ± 2. But if we compute the map directly we obtain:

$$1 \stackrel{\iota}{\mapsto} x_+ \stackrel{\Delta}{\mapsto} x_+ \otimes x_- + x_- \otimes x_+,$$

then φ sends $x_+ \otimes x_- + x_- \otimes x_+$ to

$$x_+ \otimes x_- - x_- \otimes x_+ \stackrel{m}{\mapsto} 0 \stackrel{\varepsilon}{\mapsto} 0;$$

which is absurd.

Q.E.D.

As the monodromy group contains $\mathbb{Z}/2\mathbb{Z}$ as a subgroup, because the identity and minus the identity are always realizable, and $\mathcal{M}on$ is a subgroup of $\mathcal{A}ut_0$, the following corollary is immediate.

Corollary 3. *The monodromy group of the unknot is trivial.*

The following statement holds – see [**Jacobss**], pages 1231-1234, for the proof.

Proposition. *The knot* 8_{18} *has non-trivial monodromy.*

As a consequence, we obtain that the monodromy is a non-trivial invariant. Let us raise the difficulty level a bit by adding more components.

Let us fix an ordering of the basis elements of the Khovanov homology complex associated to any trivial unlink diagram $\mathcal{U}_n$, i.e. the diagram of the unlink with only disjoint ◯ components, all of them with the same orientation, whose centres lie on the same line and the components are ordered from left to right.

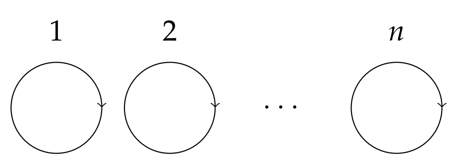


Figure 3. The diagram $\mathcal{U}_n$ with an orientation on the components.

Because there are no differentials, the Khovanov homology of such a diagram is the same as its Khovanov chain complex. The latter consist on a single non-zero group which is the free $\mathbb{Z}$-module generated by the 2^n elements of the form

$$v_{i_1 \dots i_n} := x_{i_1} \otimes \cdots \otimes x_{i_n}, \quad i_1, \dots, i_n \in \{+, -\};$$

the quantum grade of such an element is $n - 2k$, where k is the number of minus signs among $i_1, \dots, i_n$. For each quantum degree $d = n - 2k$, we have $\binom{n}{k}$ elements of that degree. Hence, to represent the elements of $\mathcal{A}ut_0(KH(\mathcal{U}_n))$ by matrices we have to fix an ordering for each quantum grade.

So we choose the ordering $+ > -$, on the set $\{+, -\}$; and, in case $v_{i_1 \dots i_n}$ and $v_{i_1 \dots i_n}$ have the same quantum grade, set

$$v_{i_1 \dots i_n} > v_{j_1 \dots j_n} \iff (i_1, \dots, i_n) >_{lex} (j_1, \dots, j_n),$$

where $>_{lex}$ indicates the lexicographic order.

Now we can proceed with the computation of the monodromy of the unlink with two components.

Proposition 14. *The monodromy group of the unlink with two components is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^2$.

Proof. If we consider $\mathcal{U}_2$ as diagram for the unlink with two components, i.e. the diagram depicted in Figure 3 with $n = 2$, and the ordering on the generators of $KH^{\bullet,\bullet}(\mathcal{U}_2)$ described at page 81; by the grade arguments illustrated in Section 2 we have that all morphisms in $\mathcal{A}ut_0(\mathcal{U}_2)$ are represented by matrices of the form

$$B = \left(\begin{array}{c|cc|c} \alpha & 0 & 0 & 0 \\ \hline 0 & a_{1,1} & a_{1,2} & 0 \\ 0 & a_{2,1} & a_{2,2} & 0 \\ \hline 0 & 0 & 0 & \beta \end{array} \right).$$

By computing the determinant of B we get:

$$\alpha\beta \cdot Det(A) = Det(B), \quad \text{with } A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

Since B is unimodular, because it represents an automorphism, and we are working over the integer ring we must have: $\alpha, \beta = \pm 1$ and $A \in \mathcal{G}l(2, \mathbb{Z})$. We want to show that the only matrices $A \in \mathcal{G}l(2, \mathbb{Z})$ that may represent elements of $\mathcal{M}on$ are those of the form

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Let M_A the movie of a cobordism whose associated map φ_A is represented by the matrix

$$B = \left(\begin{array}{c|c|c} \pm 1 & 0 & 0 \\ \hline 0 & A & 0 \\ \hline 0 & 0 & \pm 1 \end{array}\right), \quad \text{with } A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

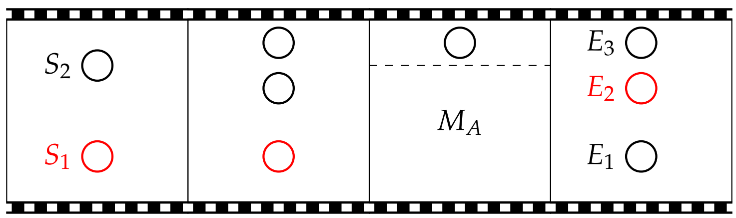


FIGURE 4. The movie N_1.

Consider the movie N_1 in Figure 4. This movie defines a map

$$\varphi_1 : V \otimes V \longrightarrow V \otimes V \otimes V,$$

given by:

$$\begin{aligned} \varphi_1(v_{++}) &= \alpha v_{++-} + a_{1,1} v_{+-+} + a_{2,1} v_{-++}, \\ \varphi_1(v_{+-}) &= a_{1,1} v_{+--} + a_{2,1} v_{-+-}, \\ \varphi_1(v_{-+}) &= a_{1,2} v_{+--} + a_{2,2} v_{-+-} + \beta v_{---}, \\ \varphi_1(v_{--}) &= \beta v_{---}; \end{aligned}$$

where α and β are, respectively, the $(1,1)$-entry and the $(4,4)$-entry of B.

To get a system of equation we have to complete N_1 to the movie of a closed connected surface. In order to do so, is necessary to understand the position of the red circle, i.e. the one that comes from S_1, at the end of N_1: this circle could be in either positions E_1 or E_2 depending on whether or not M_A permutes the two components of $\mathcal{U}_2$. Let $\sigma \in \mathfrak{S}_2 \simeq \mathbb{Z}/2\mathbb{Z}$ be the permutation such that the circle in position S_1 is in position $E_{\sigma(1)}$ at the end of N_1.

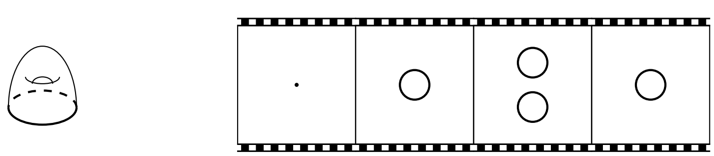


FIGURE 5. A torus cap and its movie version T_a.

Let N_2 be the movie obtained from N_1 by capping S_1 with a cap and S_2 with a torus cap. Its associated map is

$$\begin{aligned} \varphi_2 :\ \mathbb{Z} &\to V \otimes V \otimes V, \\ 1 &\mapsto 2 \cdot \varphi_1(v_{+-}). \end{aligned}$$

In order to get a closed surface from N_2, we can either fuse E_1 and E_2 and put a cup on E_3 or fuse E_3 with $E_{\sigma(1)}$ and put a cap on $E_{\sigma(2)}$. The surface obtained has a boundary component, if we put a cup on it the result is in both cases a torus. Thus we get the system of equations:

$$\begin{cases} 2 \cdot (a_{1,1} + a_{2,1}) & = & \pm 2 \\ \quad 2 \cdot a_{\sigma(1),1} & = & \pm 2 \end{cases}.$$

The solution of this system is

$$a_{1,1} = \pm\delta^1_{\sigma(1)}, \qquad a_{2,1} = \pm\delta^2_{\sigma(1)};$$

where δ^i_j is the *Kronecker delta*, i.e. the function which is 1 if $i = j$ and 0 otherwise. Consider the movie N_3 obtained from N_1 by capping S_2, instead of S_1, with a cap and S_1 with a torus cap. Its associated map is

$$\begin{aligned} \varphi_3 :\ \mathbb{Z} &\to V \otimes V \otimes V, \\ 1 &\mapsto 2 \cdot \varphi_1(v_{-+}). \end{aligned}$$

We can get a closed surface from N_3 by proceeding in the same way as in the case of N_2. We can either fuse E_1 and E_2 on the result and put a cup on E_3 or fuse E_3 with $E_{\sigma(1)}$ and put a cap on $E_{\sigma(2)}$. The surface obtained has a boundary component, if we put a cup on it the result is in both cases a torus. Thus we get the system of equations:

$$\begin{cases} 2 \cdot (a_{1,2} + a_{2,2}) & = & \pm 2 \\ \quad 2 \cdot a_{\sigma(2),2} & = & \pm 2 \end{cases}.$$

Remark. The term β vanishes because it is the coefficient of v_{---} that becomes 0 when we fuse a pair of circles.

the solution of this system is

$$a_{1,2} = \pm\delta^1_{\sigma(2)}, \qquad a_{2,2} = \pm\delta^2_{\sigma(2)}.$$

This leaves us with determining exactly which are the sign combination we can realize. By using the movie in Figure 2 with the third still replaced by M_A and Rasmussen-Tanaka theorem, one can rule out the cases of mixed signs in A, i.e. all the non-zero entries have the same sign.

In other words, the possibilities for the matrix A are:

$$\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

If we suppose that the element α of matrix B and the non-zero entries of the matrix A have different signs, then the movie in Figure 6, which represent a torus, is associated to the 0 morphism – which is absurd by the Rasmussen-Tanaka theorem. The same argument works for β: if we suppose β and the non-zero entries of A to have different sign, then the movie shown in Figure 7 represents a torus with Khovanov-Jacobsson number 0.

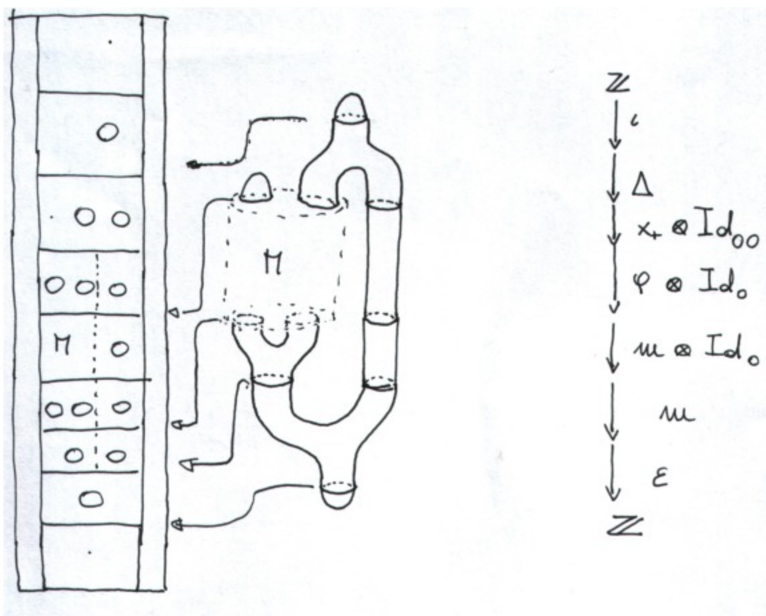


FIGURE 6. The movie of a torus that involves the circular movie M.

So we have obtained that all the elements of $\mathcal{M}on(\mathcal{U}_2)$ have matrices of the form

$$B(\sigma) = \pm \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & A_\sigma & 0 \\ \hline 0 & 0 & 1 \end{array} \right), \quad \text{with } A_\sigma = \begin{pmatrix} \delta^1_{\sigma(1)} & \delta^1_{\sigma(2)} \\ \delta^2_{\sigma(1)} & \delta^2_{\sigma(12)} \end{pmatrix},$$

where $\sigma \in \mathfrak{S}_2$. The matrices of this form with the matrix multiplication are a group which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathfrak{S}_2 \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Moreover, we can realize all these matrices by exchanging the components of $\mathcal{U}_2$ and/or using a movie whose associated map is $-Id_{KH^{\bullet,\bullet}(\mathcal{U}_2)}$. Hence, $\mathcal{M}on(\mathcal{U}_2) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

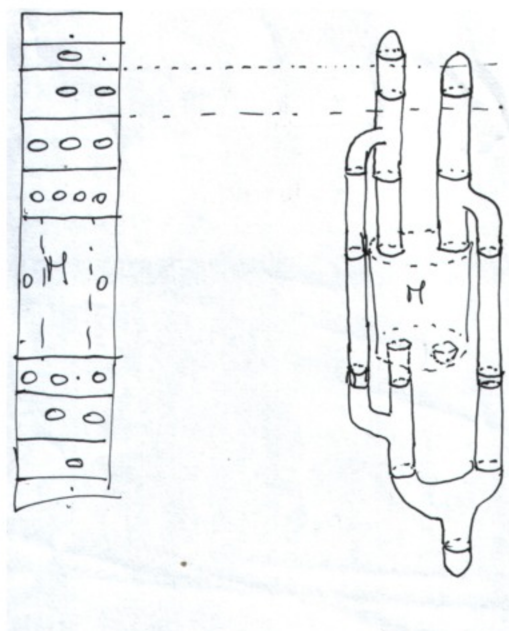


FIGURE 7. The movie of a torus that involves the circular movie M.

Q.E.D.

It is our opinion that the proof of the previous proposition can be adapted to prove the following.

Conjecture. *The monodromy group of the unlink with n-components is* $\mathbb{Z}/2\mathbb{Z} \oplus \mathfrak{S}_n$*, where* $\mathfrak{S}_n$ *is the group of permutations of n elements.*

5. Conclusions and further possibilities.

Our computations are limited to the unlinks. This is due to the fact that, performing the technique based on the Rasmussen-Tanaka theorem on more complicated knots or links could be quite challenging. For the Hopf link, for example, this tecnique proved itself to be useless. Let $\mathcal{H}$ be the oriented diagram of the Hopf link shown below.

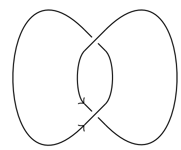
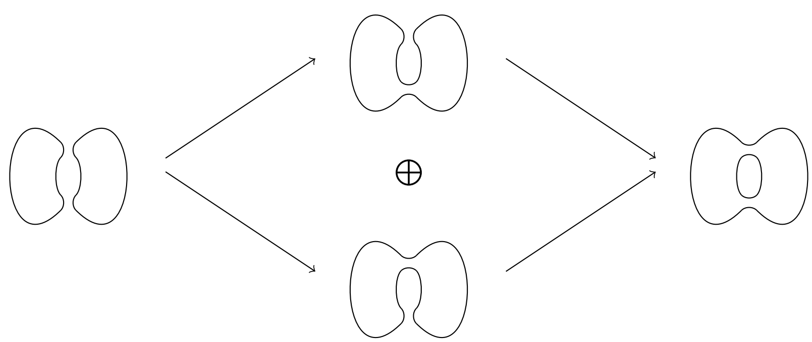


FIGURE 8. The Hopf link.

The formal Khovanov complex associated to the diagram $\mathcal{H}$ is



$$C^{-2} \longrightarrow C^{-1} \longrightarrow C^{0}$$

FIGURE 9. The formal Khovanov chain complex of $\mathcal{H}$.

If one considers the generators of the Khovanov homology of $\mathcal{H}$, then it is possible to describe the possible elemetns of the group $\mathcal{A}ut_0$ by matrices. Considered that the integral Khovanov homology of the Hopf link has non-trivial groups on bi-degrees $(-2,-2)$, $(-2,0)$, $(0,-2)$ and $(0,0)$, where the homology groups are $\mathbb{Z}$. Hence, by the grade argument described in Section 2, the possible matrices associated to the morphisms in $\mathcal{A}ut_0$ are those of the form

$$B = \left(\begin{array}{cc|cc} \pm1 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & 0 \\ \hline 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & \pm1 \end{array} \right)$$

Now we should understand which are the possible combinations of signs, but at least we found out that $\mathcal{A}ut_0(0_2)$, and hence $\mathcal{M}on(0_2)$, is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^4$. One can consider the planar isotopy given by the π-rotation of the diagram in Figure 8, this rotation exchanges the components. A few computations show that its effect, up to sign, in Khovanov homology is described by the matrix

$$A = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

and this tells us that $\mathcal{M}on(\mathcal{H})$, and hence $\mathcal{A}ut_0(\mathcal{H})$, has $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as a subgroup. Unluckily, we cannot proceed any further. It is our belief that there are no other matrices neither in $\mathcal{A}ut_0$ nor in $\mathcal{M}on$. All of our attempts to use the Rasmussen-Tanaka theorem here failed.

A possible way to compute properly the monodromy groups and to obtain a more powerful version of the techniques used here, is to compute the monodromy in Bar-Nathan theory and relate it to the Khovanov monodromy. In a certain sense we wish to extend the approach of Tanaka to the computation of the monodromy. A further possible approach could be to consider the Khovanov theory over $\mathbb{Q}$ instead of $\mathbb{Z}$. In this context we could use other invariant such as Jacobsson's Lefschetz polynomial – see [**Jacobss**], Section 6, page 1249 – to narrow down the possibiles morphisms in $\mathcal{M}on$.

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