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Homotopy setoids and generalized quotient completion

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Contents

Introduction

Context. The topics investigated in this thesis belong to the branch of categorical logic. The main logical language considered is the *Martin-Löf Intuitionistic Type Theory*[ML84], and the main categorical framework exploited is given by the *elementary doctrines* [MR13]. This work starts with some considerations about *setoids* and their categorical descriptions.

Setoids are a concept of constructive mathematics, introduced by Bishop in [Bis67], which provide a *constructive* notion of *set*. In the seventies, Per Martin-Löf introduced the intuitionistic type theory in order to give a foundation for constructive mathematics. This framework exploits the notion of *type* and *element* of a type which is denoted with $a : A$. The main feature of intuitionistic type theories, also called dependent type theories, is that types may depend on some parameters of other types $x : A \vdash B(x)$. Following Bishops's description of setoids as a collection of objects with an equality relation, a setoid is given by a pair (X, R), where X is a type in the empty context and $x_1, x_2 : X \vdash R(x_1, x_2)$ is a dependent type which is an equivalence relation.

There exist various type theories, depending on the rules and type constructors assumed. For instance, some of the usual type constructs assumed are the type of the *pairs* of two types, also called the $\times$-type , and the function type $\to$ of functions between two types. A principal distinction is given by *extensional* and *intensional* type theories which depends on the properties of the internal notion of the equality of terms given by the *identity type* Id_A. This notion is referred to as *propositional equality* and it differs from the *judgmental equality* which is an external syntactic relation. When two terms are judgmentally equal they are also propositionally equal. If the converse holds, the identity type is called *extensional*, otherwise it is called *intensional*. Initially, the known models could not separate the two notions but, as shown in [HS98], intensional identity types conceal an higher structure which has to be understood in terms of higher categorical structures.

Other properties of dependent type theories have been deeply studied by mathematicians, logicians and also by computer scientists. Indeed, as advocated by Martin-Löf, dependent type theories can be view as programming languages and nowadays there exist various computer formalizations such as in [CC99; HKPM02]. Reasoning about the computational properties of these systems contributed to the general understanding of the theory and setoids played an important role in the interplay between *intensional* and *extensional* constructs.

The extensional constructs are desirable features of the system that permit to reason much closer to ordinary mathematics. Among the most important ones there are the *functional extensionality* and the *quotient types*. The former is the property that two functions are equal if they have the same values. The latter is the possibility to build a quotient of a type by an equivalence relation.

One possibility to obtain this constructs is to add suitable axioms, but this turns out to break the good computational properties of the type theory. Since terms can be view as programs of some specifications, it should always possible to decide if a program meets its specifications. This property is called the *decidability of type check*. Moreover, a program which computes, for example, a numerical result, should always be reduced to a numeral. This property is the existence of a *canonical form*. Several attempts to add conservative quotient types such as in [Alt99; Mai99] and

[Hof95a] led to undecidable type theories or the introduction of non-canonical elements. Notably, the extensional identity types break the decidability of type check and the functional extensionality axiom also implies the introduction of non-canonical elements. A possible solution to recover the extensional constructs inside the intensional type theory was found by Hofmann in [Hof95b] who exploited the so called *setoid model*.

A different approach is to adopt a finer system, called the *homotopy type theory* (HoTT), which supports the homotopical view of the intensional identity types anticipated in [HS98] and later in [AW09]. Awodey and Warren provided the first "homotopical model" of intensional type theories exploiting structures of homotopical algebra. In this view, types are thought as *spaces* and the identity types are thought as *path spaces*. Independently, Voevodsky was working at a practical *univalent foundation* of mathematics based on the notion of *equivalence* rather than equality of objects. His research culminated with the introduction of the remarkable *univalent axiom* and many *higher inductive types* to the intensional type theory. Assuming a *universe type* $\mathcal{U}$ whose elements are types, the univalence axiom makes the identity type $\mathsf{Id}_{\mathcal{U}}(A, B)$ coincide with the type of *equivalences* between A and B. This notion is interpreted as the notion of *homotopy equivalence* in the homotopical models of HoTT such as [KL21].

In this foundational approach, there is a clear stratification of mathematics which is given by the syntactic notion of *homotopy level* of a type. For instance, the level of the logic is given by the types called *h-propositions* that correspond to empty or contractible types. The level of set theory is given by the types called *h-sets*, that are "discrete" types or types with trivial homotopical structure. Mathematical constructs such as quotients are given by some higher inductive types. The main reference for homotopy type theory is [Uni13].

A first categorical analysis of dependent type theory was provided by Seely in [See84] who considered the category whose objects are types in the empty context and whose functions are terms of the function type. This category is denoted here with **ML**. For the extensional type theory, the category **ML** turns out to be locally cartesian closed while, for the intensional type theory its properties become less effective. Setoids and functions preserving relations form a category here denoted as **Std**. The fact that setoids appear as a syntactic solution to take quotients of equivalence relations has a category-theoretic counterpart given by the *exact completion*. As shown by Carboni and Vitale [CV98], one can add well-behaved quotients to a category with weak finite limits in a universal way. When a category with finite limits has "well-behaved" quotients, it is called *exact* [Bar71]. A relevant example of this construction is given by the category **Std** which can be obtained as the exact completion of the syntactic category **ML**. The properties of the category of setoids has been deeply studied for instance in [Wil10] and [Hof95c] and, in [MP00], the authors proved that setoids form a ΠW*-pretopos*. This notion, as discussed in [Ber06], is a suitable candidate for a predicative analogous of the notion of *topos*.

A different categorical setting to describe logical systems is given by the theory of *fibered categories* which relies on the concept of *Grothendieck fibration* [Gro71]. There are several structures which exploit this notion, and in the recent years many of them have been used to describe dependent type theories, such as in [Car86], [Jac93] and [Dyb96]. These structures rely on suitable functors of the form $\mathsf{P} : \mathcal{E} \to \mathscr{C}$, or equivalently of the form $\mathsf{P} : \mathscr{C}^{op} \to \mathbf{Cat}$ with values in the category of small categories and functors. This approach emphasizes the level of the contexts, given by the category $\mathscr{C}$, and the levels of the formulae or types in a context given by the values of P on the objects of $\mathscr{C}$. The substitution of terms is given by the values of the functor P on the arrows of $\mathscr{C}$. An exhaustive account of fibered categories and logical systems can be found in [Jac99].

To this family belong the elementary doctrines introduced by Maietti and Rosolini in [MR13]. The elementary doctrines are a weakened notion of Lawvere's *hyperdoctrines* [Law69; Law70], given by a functor $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Pos}$ from a category with strict finite products to the category of posets and order preserving functions.

Lawvere's original remark is that the logical operations of first order logic, such as the existential and universal quantifications and the equality predicate, are expressed through suitable adjoint functors of the substitution functors. In this setting, many categorical semantics of first order logic can be described using the elementary doctrines. The "standard interpretation" introduced by Makkai and Reyes [MR77] can be described in terms of the elementary doctrine of *subobjects*. The categorical Brouwer-Heyting-Kolmogorv (BHK) interpretation discussed in [Pal04] can be described in terms of the elementary doctrine of *weak subobjects.*

Moreover, the theory of elementary doctrines gives a fruitful description of the constructions used to develop constructive mathematics in foundation based on the intensional type theory.

In this framework, it is possible to define the notion of well-behaved quotient of a P-*equivalence relation* and a universal construction which *freely* adds quotients to a suitable elementary doctrine [MR13; MR12; MR16; MR15]. The elementary quotient completion is, to some extent, a generalization of the exact completion. However, the elementary quotient completion does not necessary provide an exact category but it takes into account a wider class of categories with suitable well-behaved quotients.

Dependent type theories give rise to rich elementary doctrines and setoids are an instance of the elementary quotient completion applied to such doctrines.

Outline and main results. Taking into account the above situation, in this thesis we pursue three main objectives. The first one is to study a particular class of setoids, that we have called *homotopy setoids*, and their categorical properties using the machinery of the elementary quotient completion. The second one is to introduce a more general framework in which to develop a generalized notion of quotient completion which has the exact completion and the elementary quotient completion as particular instances. The third objective is to provide a categorical semantic of first order logic suitable for a large class of categories. Below, we outline the content and the main contribution of each chapter in more detail.

In the first part of Chapter 1, we recall the precise definition of setoids and the connection with the theory of the exact completion. In the second part, we recall the main notions and results of the theory of elementary doctrines together with the elementary quotient completion. In particular, we recall the description of setoids in this framework. In order to support the notions of the first chapter, at the end of the thesis we provide two appendices. The first one is about some results of the elementary doctrines and the second is about the syntax of the type theory we have considered.

In Chapter 2, we define the homotopy setoids, taking into account ideas from homotopy type theory. Working into an intuitionistic type theory plus the functional extensionality axiom, we study those setoids (X, R) such that X is an h-set and R is an h-proposition. However, we made no assumption of the univalence axiom and of any higher inductive types. This class of setoids is motivated by the fact that h-sets and h-propositions are the homotopy levels needed to provide the set-based mathematics.

Hence, the first goal of this thesis is to study the category of h-setoids, denoted with $\mathbf{Std}_0$, and to prove that it has good categorical properties such as the category **Std**. In particular, since setoids form a locally cartesian closed pretopos, we ask if $\mathbf{Std}_0$ appears as a weaker notion of locally cartesian closed pretopos. The main problem is that $\mathbf{Std}_0$ does not provide an exact category, but it has well-behaved quotients with respect to a suitable elementary doctrine. Notably, it appears as an instance of elementary quotient completion.

The strategy adopted is to study the properties that the quotient completion inherits from the starting structure. For the theory of the exact completion there are several results in this direction. In particular, in [CR00] and [Emm20] the authors give a characterization of those categories whose exact completion is *locally cartesian closed*. In [GV98], the authors give a characterization of those categories whose exact completion is *lextensive*. In [Men00], there is a characterization of those categories whose exact completion is a topos.

We follow these ideas and give a characterization of the elementary doctrines whose elementary quotient completion gives rise to a lextensive locally cartesian closed category (Theorem 2.3.7 and Theorem 2.5.6). A similar result about the local cartesian closure appeared recently in [MPR21]. We dedicate a section to investigate the relationship between our result and the one in *loc. cit.*. Finally, we prove that these results apply to the homotopy setoids which provide a non-trivial example of a weaker notion of pretopos (Corollary 2.6.15).

In Chapter 3, we introduce the theory of the *biased elementary doctrines* (Definition 3.1.2) which generalizes the theory of (strict) elementary doctrines. This new framework allow us to take into account crucial examples of functors of the form $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ which are not in the realm of the elementary doctrines due to the lack of strict finite products in $\mathscr{C}$, which only appear in a weak form. In order to do that, we exploit the notion of *proof-irrelevant elements*, which takes inspiration from an example arising from intuitionistic type theory.

For these structures we provide a procedure, called *strictification*, which associates a strict elementary doctrine to a weak one in a suitable way (Theorem 3.3.5). Moreover, we provide a corresponding construction of quotient completion (Theorem 3.4.15). As particular instances, we obtain the elementary quotient completion and the exact completion in its more general form. The last section is dedicated to a generalization of the result about the local cartesian closure proved in the previous chapter (Theorem 3.7.5).

Chapter 4 deals with a categorical semantic of first order logic. The semantic developed follows the standard interpretation of intuitionistic first order logic of Makkai and Reyes and the categorical BHK interpretation. Both the interpretations lie on categories with strict finite products and respectively strict and weak pullbacks. We consider the case of weak finite products and, in case they are strict, we obtain the BHK interpretation.

In order to do that, we develop the ideas of the proof-irrelevant elements of the previous chapter internally to a category with weak finite products and weak pullbacks. For this interpretation, we discuss some soundness and completeness results for various fragments of intuitionistic first order logic (Theorem 4.5.8, Proposition 4.5.9 and Theorem 4.7.3).

We can summarize the main contributions of this thesis as follows:

- In Theorem 2.3.7 and in Theorem 2.5.6, we give a characterization of the elementary doctrines whose elementary quotient completion respectively gives rise to locally cartesian closed and extensive categories. In Corollary 2.6.15, we prove that the category homotopy setoids provides an example of *relative* Π-pretopos.

- In Definition 3.1.2 we introduce the more general framework of the biased elementary doctrines. In Theorem 3.3.5, we prove the properties of the strictification. In Theorem 3.3.5, we prove the universal property of the corresponding quotient completion. In Theorem 3.7.5, we generalize Theorem 2.3.7 to this framework.

- In Theorem 4.5.8, Proposition 4.5.9 and Theorem 4.7.3 we prove the completeness and the soundness results of more general categorical BHK interpretation for fragments of first order logic in categories with weak finite products and weak pullbacks.

Chapter 1

Preliminaries

In this chapter, we will recall the main notions of type theory and category theory that will be preliminary for the developments of the next chapters. The chapter is divided in two parts.

In the first part, we will recall the concept of *setoid* introduced by Bishop in [Bis67] and its expression in dependent type theories such as the Martin-Löf intuitionistic type theory [ML84]. We will recall the categorical constructions arising from these syntactic objects and how setoids are related with the theory of the *exact completion* [CV98].

In the second part, we will deal with the *elementary doctrines* introduced by Maietti and Rosolini in [MR13]. We will recall the main results about these structures and the construction of the *elementary quotient completion*. The elementary doctrines provide a useful categorical tool to treat logical systems, in particular dependent type theories. The concepts introduced in the previous section will be view from this perspective. The theory of elementary doctrines will be the main categorical language used in this thesis.

1.1 Setoids and exact completion

Setoids are a concept of constructive mathematics, introduced by Bishop in [Bis67]. Intuitively, a setoid consists of a tangible collection of elements and a procedure to show when two elements are equal. Dependent Type theories, such as Martin-Löf intuitionistic type theory [ML84], give a fruitful logical description of these objects as types equipped with a dependent type which is an equivalence relation.

Setoids have been widely studied in logic, category theory and computer science and, in [Hof95b], Martin Hofmann used setoids to investigate relations between *extensional* and *intensional* type theories. We recall that a type theory is called *extensional* if the *reflection rule*

$$\frac{x, y : X \vdash p : \mathsf{Id}_X(x, y)}{x = y}$$

is derivable. Intuitively, the reflection rule implies that the "internal" notion of equality given by identity type Id_X coincides with the *judgmental* equality =, which is the "external" notion of equality. Elements that are judgmentally equal are also internally equal. The converse does not hold as shown in the work [HS98], which predicted the homotopical view of types later crystallized in the *homotopy type theory* [Uni13]. Intuitively, the extensional constructs are those that breaks the good computational properties of a type theory such as the *decidability of type check*. Setoids offer a system definable in intensional type theories that can recover the extensional constructs, without loosing the good computational properties of the theory.

In this work, we mainly considered an intensional Martin-Löf intuitionistic type theory with the usual type constructors and a universe, such as the one described in [NPS90] or in [Coq89]. We will denote this theory as $\mathcal{ML}$ and refer to Appendix B for the rules and the notation adopted.

Notation. Given a type X in a context Γ , we will call the type X *inhabited* if there exists a term $\Gamma \vdash r : X$. In this case will adopt the notation $\Gamma \vdash X$ true without specifying any inhabitant term. If $\mathcal{U}$ denotes the universe type, we will refer to the elements of $\mathcal{U}$ as *small* types. A *closed* type is a type in the empty context, in this case we will omit the symbol $\vdash$.

We now give a precise formulation of setoids in type theory. The definition only exploits the Π-type.

Definition 1.1.1. A *setoid* is a pair (X, R), such that X is a closed type and R is a dependent type of the form $x, y : X \vdash R(x, y)$ satisfying reflexivity, symmetry and transitivity conditions, i.e.

$$\prod_{x:X} R(x,x) \qquad \mathsf{true}, \tag{1.1}$$

$$\prod_{x,y:X} R(x,y) \to R(y,x) \qquad \mathsf{true}, \tag{1.2}$$

$$\prod_{x,y,z:X} R(x,y) \times R(y,z) \to R(x,z) \qquad \mathsf{true}. \tag{1.3}$$

Category theory provides an algebraic description of these syntactic objects and we now define the category arising from small types and the category of setoids.

Given a type theory such as $\mathcal{ML}$, we can consider the associated syntactic category **ML** whose

- objects are small closed types,
- arrows $\lfloor t \rceil : X \to A$ are equivalence classes of terms $x : X \vdash t(x) : A$ up to *functional extensionality*, i.e t and $t' : X \to A$ are in relation if

$$\prod_{x:X} \mathsf{Id}_A(t(x), t'(x)) \qquad \mathsf{true}. \tag{1.4}$$

We will denote with **ML** the syntactic category arising from any dependent type theory with enough type constructors. Every time we will consider this category we will specify the underlying type theory. The category **ML** inherits properties from the type theory considered and we will discuss them along the way. At the moment, we just provide the existence of limits in **ML**.

Lemma 1.1.2. *The category **ML** has strict finite products and weak pullbacks.*

Proof. If X and Y are two closed types, we can consider the product type $X \times Y$ and the projection arrows $\lfloor \pi_1 \rceil : X \times Y \to X$ and $\lfloor \pi_2 \rceil : X \times Y \to Y$. If $\lfloor f \rceil : Z \to X$ and $\lfloor g \rceil : Z \to Y$ are two arrows, then the introduction rule of $\times$-type provides a term

$$z : Z \vdash (f(z), g(z)) : X \times Y.$$

The induced arrow will be denoted with $\lfloor \langle f, g \rangle \rceil : Z \to X \times Y$ and it satisfies $\lfloor \pi_1 \rceil \lfloor \langle f, g \rangle \rceil = \lfloor f \rceil$ and $\lfloor \pi_2 \rceil \lfloor \langle f, g \rangle \rceil = \lfloor g \rceil$. This arrow is the unique with such property thanks to the elimination rule of $\times$-type.

The identity types are used to build pullbacks. Indeed, given to arrows $\lfloor t \rceil : X \to A$ and $\lfloor u \rceil : Y \to A$ the diagram

$$\begin{tikzcd} \sum_{x:X,y:Y} \mathsf{Id}_A(t(x),u(y)) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "u"] \\ X \arrow[r, "t"'] & A. \end{tikzcd} \tag{1.5}$$

may fail to be a strict pullback, since proofs of the same identity need not be unique. □

A category with strict finite products and weak pullbacks will be often referred to as *quasi left exact* (*qlex*). Equivalently, qlex categories can be defined as categories with strict finite products and weak equalizers, or as categories with weak finite limits. This is a consequence of the description of limits in term of pullbacks or equalizers that can be found in [Bor94, Proposition 2.8.2]. When the category has weak finite products instead of strict ones and weak pullbacks (or weak equalizers), we will call it *weakly left exact* (*wlex*). As observed by Carboni and Vitale in [CV98], wlex categories are precisely categories with weak finite limits.

Observation 1.1.3. In case of the extensional type theory, the reflection rule implies that the diagram in (1.5) is a strict pullback and the category **ML** is locally cartesian closed (*lcc*). This is discussed in detail in [See84] where the author proved a correspondence between extensional type theories and lcc categories thorough the construction of a syntactic category actually equivalent to **ML**. This correspondence, was thought to provide a semantic of extensional type theories in lcc categories but, as discovered later, there are some coherence problems about substitutions. A possible syntactic solution to this problem can be found in [Cur93]. A categorical solution is given by the theory of fibrations as explained in [Hof95c].

Another example of strict pullbacks will be given in the next chapter working inside an intensional type theory.

We now define the category of setoids **Std** whose

- objects are setoids (X, R) where X is a small closed type,
- arrows between two setoids (X, R) and (Y, S) are the equivalence classes of the terms $x : X \vdash t(x) : Y$, such that t preserves the relations, i.e.

$$\prod_{x,y:X} R(x,y) \to S(t(x),t(y)) \qquad \mathsf{true}, \tag{1.6}$$

  given by the following equivalence relation: the term t is in relation with a term $x : X \vdash t'(x) : Y$ if

$$\prod_{x,y:X} R(x,y) \to S(t(x),t'(y)) \qquad \mathsf{true}. \tag{1.7}$$

We will denote with **Std** the category of setoids arising from a any dependent type theory with enough type constructors. Every time we will consider this category we will specify the underlying type theory. As for the category of types, different type theories give rise to different category of setoids. The setoids of the type theory $\mathcal{ML}$ have been widely studied in the literature and we have the following well-known fact.

Fact 1. *The category **Std** of setoids is* ΠW*-pretopos.*

Parts of the proof of the above fact can be found for instance in [MP00],[Wil10] and [Hof95c]. For a discussion about the differences between the setoids of intensional and extensional type theories we refer to [Ber06].

We now recall the relations between setoids and the exact completion which rely on the concept of quotient. One of the main extensional type constructor is the *quotient type* which builds quotients of equivalence relations. Quotient type have been widely studied in the literature for example in [Alt99], [Mai99] and [Hof95a], but all the attempts lead to the introduction of non-canonical elements or to the undecidability of the type check. As we will clarify, setoids can recover the quotient construction in place of the extensional *quotient types*. This process has a correspondent construction in category theory which is the exact completion.

We now recall the definition of *exact* category and the construction of the exact completion. Before this, the notion of *pseudo-equivalence relation* in a category is given by a pair of arrows

$$r_1, r_2 : R \to Y$$

which satisfies suitable reflexivity, symmetry and transitivity conditions, see Definition A.0.2.

Definition 1.1.4. A category $\mathscr{C}$ is called (Barr) *exact* when

1. it is left exact,
2. every effective equivalence relation (i.e. a kernel pair) has a coequalizer,
3. pullbacks of regular epimorphisms are regular epimorphisms
4. every equivalence relation is effective.

A category which satisfies only conditions 1-3 is called *regular.*

Starting from a category $\mathscr{C}$ with weak finite limits, we can form the category $\mathscr{C}_{ex}$ (the *exact completion* of $\mathscr{C}$) as follows.

Definition 1.1.5. Let $\mathscr{C}$ be a category with (weak) finite limits. The category $\mathscr{C}_{ex}$ has objects given by pseudo-equivalence relations. An arrow between two objects

$$r_1, r_2 : R \to X \qquad s_1, s_2 : S \to Y$$

is given by an equivalence class of a pair $(f, \tilde{f})$ of arrows which makes the following diagram commute component-wise

$$\begin{tikzcd} R \arrow[r, "\tilde{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Two pairs $(f, \tilde{f})$ and $(g, \tilde{g})$ are equivalent if there exists an arrow $\Sigma : X \to S$ such that $s_1\Sigma = f$ and $s_2\Sigma = g$. The arrow Σ is called *a half-homotopy*.

There are several exact completions depending on the assumptions of the category $\mathscr{C}$. In [CV98], the authors discuss the exact completion of lex, wlex and regular categories. What we called exact completion is the more general form, i.e. the exact completion over a wlex category also denoted by $\mathscr{C}_{ex/wlex}$. Every exact completion gives an exact category with a suitable universal property, stated in terms of 2-categorical adjunctions, that can be found in detail in [CV98] and [Vit94]. We will recall them in Chapter 3 in order to give a more general formulation. Intuitively, the exact completion adds well-behaved quotients to a category $\mathscr{C}$ with weak limits. Setoids are built in a similar way and, thanks to Σ-types, there is a correspondence between the type theoretic notion of equivalence relation and the categorical notion of pseudo-equivalence relation in the category **Std**.

Remark 1.1.6. Given an equivalence relation $x, y : X \vdash R(x, y)$ we can consider the arrows

$$\pi_1, \pi_2 : \sum_{x,y:X} R(x,y) \to X \tag{1.8}$$

of the projections on the first two components and obtain a pseudo-equivalence relation. Vice versa, given a pseudo-equivalence relation $r_1, r_2 : R \to X$, we can consider the dependent type

$$x, y : X \vdash \sum_{z:R} \mathsf{Id}_X(r_1(z), x) \times \mathsf{Id}_X(r_2(z), y) \tag{1.9}$$

and obtain witnesses of the fact that is a type theoretic equivalence relation. A similar correspondence follows for the arrows that preserve the equivalence relations and half-homotopies between pseudo-equivalence relations.

Hence, we can recall the following well-known fact.

Fact 2. *The category **Std** of setoids is equivalent to the exact completion $\mathbf{ML}_{ex}$ of the syntactic category **ML**.*

The exact completion $\mathscr{C}_{ex}$ inherits properties from the category $\mathscr{C}$. It happens that assuming a weaker version of the desired property in $\mathscr{C}$ implies that the whole property holds in $\mathscr{C}_{ex}$. In this direction, we mention the following well-known results.

- In [CR00] and [Emm20] the authors characterize the categories $\mathscr{C}$ such that the exact completion $\mathscr{C}_{ex}$ is (locally) cartesian closed.
- In [GV98] the authors characterize the categories $\mathscr{C}$ such that the exact completion $\mathscr{C}_{ex}$ is extensive.
- In [Men00] the author characterizes the categories $\mathscr{C}$ such that the exact completion $\mathscr{C}_{ex}$ is a topos.

It follows that a possible approach to studying the categorical properties of setoids can be to consider the properties of the category of types **ML**. Another approach can be to work directly in the category of setoids, as done in [MP00; Wil10] and in [Mai07] for a different type theory.

In the next chapter, we will define a particular class of setoids. Firstly, we will study directly the corresponding category of setoids. Secondly, we will adopt the point of view of the elementary doctrines. Our goal is to prove that, in a suitable form, the above facts about setoids hold for the particular class we have considered.

1.2 Elementary doctrines

The elementary doctrines were introduced by Maietti and Rosolini in [MR13]. They are a weaker notion of Lawvere's *hyperdoctrines* [Law69; Law70], which are suitable categorical structures to deal with logical languages. In particular, the elementary doctrines give an abstract description of constructions based on intensional type theory such as those introduced in the previous section.

We now recall the main definitions and examples which can be found in detail in [MR13; MR12; MR15; MR16].

The underlying categorical structure, which we will refer to simply as *doctrine*, is given by functors of the form

$$\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Pos}$$

from a category $\mathscr{C}$ with strict finite products to the category Pos of partially-ordered sets (posets) and order-preserving functions. Intuitively, one should think of the base category $\mathscr{C}$ as the level of the contexts and the fibers $\mathsf{P}(X)$ as the predicates in the context $X \in \mathscr{C}$.

A first refinement of the definition of doctrine is given by what is called a *primary* doctrine. Intuitively, primary doctrines are those doctrines fruitful to set up many-sorted logic with binary conjunctions and true constant. Before proceeding with the definition, we fix the notation InfSL to indicate the category of inf-semilattices. The objects of InfSL are posets with finite meets and order preserving functions which preserve finite meets.

Definition 1.2.1. Let $\mathscr{C}$ be a category with strict finite products. A *primary doctrine* is a functor $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Pos}$ which takes value in the category InfSL of inf-semilattices, i.e.:

P1 $\mathsf{P}(A)$ has finite meets, for every object $A \in \mathscr{C}$,

P2 for every arrow $f : A \to B$ of $\mathscr{C}$, the functor $\mathsf{P}_f : \mathsf{P}(B) \to \mathsf{P}(A)$ preserves finite meets.

The main logical example comes from first order theories. We now see how to organize these languages in a primary doctrine.

Example 1.2.2. Given a first order theory $\mathcal{T}$ on a language $\mathcal{L}$, we consider the category $\mathcal{V}$ built as follows. Objects of $\mathcal{V}$ are lists of distinct variables $\bar{x} := (x_1, \dots, x_n)$ and arrows are lists of substitutions for variables $[\bar{t}/\bar{y}] : \bar{x} \to \bar{y}$. The composition of arrows is given by simultaneous substitutions. The functor $LT : \mathcal{V}^{op} \to \mathsf{InfSL}$ sends a list of variables $\bar{x}$ to the *Lindenbaum-Tarski algebra* $LT(\bar{x})$ defined as follows:

- objects are equivalence classes of well-formed formulae $[\varphi]$ of $\mathcal{L}$, with free variables $x_1, \dots, x_n$, with respect to equiprovability $\varphi \dashv\vdash_{\mathcal{T}} \varphi'$,
- arrows $[\varphi] \leq [\psi]$ are provable consequences $\varphi \vdash_{\mathcal{T}} \psi$.

If $[\bar{t}/\bar{y}] : \bar{x} \to \bar{y}$ is an arrow o f $\mathcal{V}$, the functor $LT([\bar{t}/\bar{y}]) : LT(\bar{y}) \to LT(\bar{x})$ sends the equivalence class of a formula $\lfloor \psi(\bar{y}) \rceil$ to the equivalence class $\lfloor \psi([\bar{t}/\bar{y}]) \rceil$. The functor is a primary doctrine because the posets $LT(\bar{x})$ have finite meets given by the logical conjunctions and the top element is given by the true predicate.

Among the primary doctrines there are those which can deal with the equality predicate which are called *elementary*. This can be achieved requiring the existence of an element $\delta_X \in \mathsf{P}(X \times X)$ satisfying suitable conditions. Before providing the definition of elementary doctrine, we fix some notations.

Notation. Let $\mathscr{C}$ be a category with strict finite products and let $X_1, \dots, X_n$ be objects of $\mathscr{C}$. If $j : \{1, \dots, k\} \to \{1, \dots, n\}$ is an assignment with $1 \leq k$, then

$$\langle j(1), \dots, j(k) \rangle : X_1 \times \cdots \times X_n \to X_{j(1)} \times \cdots \times X_{j(k)}$$

will denote the map induced on the product $X_{j(1)} \times \cdots \times X_{j(k)}$ by projections $p_{j(-)} : X_1 \times \cdots \times X_n \to X_{j(-)}$. If $k = 1$, the arrows $\langle j(1) \rangle$ will always be denoted with $p_{j(1)}$. If $k = 2$ and $n = 1$, the arrow $\langle 1, 1 \rangle : X \to X \times X$ will always be denoted with Δ_X.

Definition 1.2.3. Let $\mathscr{C}$ be a category with strict finite products. A primary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is called *elementary doctrine* if, for every object $X \in \mathscr{C}$, there exists an element $\delta_X \in \mathsf{P}(X \times X)$ such that:

E1 For every element $\alpha \in \mathsf{P}(X)$, the assignment

$$\exists_{\Delta_X}(\alpha) := \mathsf{P}_{p_1}(\alpha) \wedge_{X \times X} \delta_X$$

is left adjoint of the functor $\mathsf{P}_{\Delta_X} : \mathsf{P}(X \times X) \to \mathsf{P}(X)$.

E2 For every object $Y \in \mathscr{C}$ and arrow $e := \langle 1, 2, 2 \rangle : X \times Y \to X \times Y \times Y$, the assignment

$$\exists_e(\alpha) := \mathsf{P}_{\langle 1,2 \rangle}(\alpha) \wedge_{X \times Y \times Y} \mathsf{P}_{\langle 2,3 \rangle}(\delta_Y)$$

for α in $\mathsf{P}(X \times Y)$ is left adjoint to $\mathsf{P}_e : \mathsf{P}(X \times Y \times Y) \to \mathsf{P}(X \times Y)$.

If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an elementary doctrine and X is an object of $\mathscr{C}$, we will refer to $\mathsf{P}(X)$ as the *fiber* of P on X and we will refer to δ_X as the *fibered equality* on X.

The above Definition is one of the equivalent formulation of elementary doctrines which makes use of adjoint functors in the stile of Lawvere. However, in a moment we will recall a well-known equivalent definition of elementary doctrines which highlights the core properties of the element δ_X. Before proceeding, we first recall the crucial definition of *descent data*.

Definition 1.2.4. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a primary doctrine. If $\beta \in \mathsf{P}(X \times X)$, then $\mathcal{D}es_\beta$ is the sub-order of elements $\alpha \in \mathsf{P}(X)$ satisfying

$$\mathsf{P}_{p_1}(\alpha) \wedge \beta \leq \mathsf{P}_{p_2}(\alpha), \tag{1.10}$$

where p_1, p_2 are the projections $X \overset{p_1}{\leftarrow} X \times X \overset{p_2}{\rightarrow} X$.

For example, for every object $X \in \mathscr{C}$, the sub-poset $\mathcal{D}es_{\delta_X}$ is given by the elements $\alpha \in \mathsf{P}(X)$ such that

$$\mathsf{P}_{p_1}\alpha \wedge \delta_X \leq \mathsf{P}_{p_2}\alpha.$$

Using an informal internal language, the elements of $\mathcal{D}es_{\delta_X}$ correspond to those $\alpha(x)$ in context $x : X$ such that

$$\alpha(x_1) \wedge x_1 =_X x_2 \vdash \alpha(x_2)$$

in context $x_1, x_2 : X$.

We now recall an equivalent definition of elementary doctrine that is discussed in [MR12, Remark 2.3]. Since we were not able to find a proof of the equivalence between Definition 1.2.3 and the following Definition 1.2.5 we provided it in appendix Proposition A.0.3.

Definition 1.2.5. Let $\mathscr{C}$ be a category with strict finite products. A primary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is called *elementary* if, for every object $X \in \mathscr{C}$, there exists an element $\delta_X \in \mathsf{P}(X \times X)$ such that:

I $\top_X \leq \mathsf{P}_{\Delta_X}(\delta_X)$.

II $\mathsf{P}(X) = \mathcal{D}es_{\delta_X}$.

III $\delta_X \boxtimes \delta_Y \leq \delta_{X \times Y}$, where $\delta_X \boxtimes \delta_Y := \mathsf{P}_{\langle 1,3 \rangle}\delta_X \wedge \mathsf{P}_{\langle 2,4 \rangle}\delta_Y$.

The first condition expresses the reflexivity of the equality predicate. The second expresses that the descent data of the equality δ_X are the whole set of predicates in the context X. The last one expresses a relation between the equality of the pairs and their components. In an informal internal language, condition **III** becomes

$$x_1 =_X x_2 \wedge y_1 =_Y y_2 \vdash (x_1, y_1) =_{X\times Y} (x_2, y_2).$$

The validity of this reasonable condition led to some considerations that have been developed in Chapter 3.

We now mention the main examples of elementary doctrines.

Example 1.2.6. If $\mathscr{C}$ is a category with strict finite products and weak pullbacks, one can consider the functor

$$\mathsf{Psub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{InfSL}.$$

Given an object $A \in \mathscr{C}$, $\mathsf{Psub}_{\mathscr{C}}(A)$ is defined as the poset reflection of the comma category $\mathscr{C}/A$. Two arrows $f, f' \in \mathscr{C}/A$ satisfy $f \leq f'$ when there exists a map h making the following diagram commute

$$\begin{tikzcd} X \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & A, & \end{tikzcd}$$

f and f' are equivalent when $f \lesseqgtr f'$. The objects of the poset $\mathsf{PSub}_{\mathscr{C}}(X)$ are called *weak subobjects* or *variations* in [Gra00]. In [Pal04], they are also called *pre-subobjects*. If $g : B \to A$ is an arrow of $\mathscr{C}$, the functor P_g sends an equivalence class $\lfloor f \rceil$ to the equivalence class represented by the chosen weak pullback ${\pi_1}_{g,f}$

$$\begin{tikzcd} V_{g,\alpha} \arrow[r, "{\pi_2}_{g,f}"] \arrow[d, "{\pi_1}_{g,f}"'] & X \arrow[d, "f"] \\ B \arrow[r, "g"'] & A. \end{tikzcd} \tag{1.11}$$

The poset $\mathsf{Psub}_{\mathscr{C}}(A)$ is an inf-semilattices: the top element is given by the identity arrow $\lfloor 1_A \rceil$ and the meet of two arrows $\lfloor f : X \to A \rceil$ and $\lfloor g : B \to A \rceil$ is given by the equivalence class of the common value of the two composites of the diagram in (1.11). The functor $\mathsf{Psub}_{\mathscr{C}}$ is an elementary doctrine and the fibered equality is given by the diagonal arrow $\delta_A := \lfloor \Delta_A \rceil$.

Example 1.2.7. If $\mathscr{C}$ is a category with strict finite products and strict pullbacks, one can consider the elementary doctrine of *subobjects*

$$\mathsf{Sub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{InfSL}.$$

If A is an object of $\mathscr{C}$, the category $\mathsf{Sub}_{\mathscr{C}}(A)$ is the poset reflection of monomorphisms over A for every object $A \in \mathscr{C}$. The action of $\mathsf{Sub}_{\mathscr{C}}$ on arrows is similar to the action of $\mathsf{Psub}_{\mathscr{C}}$.

The above examples encode two different notions of *inner logic* of a category. The elementary doctrine of subobjects encodes the correspondence *propositions as subobjects*, introduced by Makkai [MR77], for lex categories. The elementary doctrine of weak subobjects encodes the paradigm *propositions as objects* discussed by Palmgren in [Pal04], for categories with strict finite products and weak pullbacks. We will return on these aspects in Chapter 4 where we provide a more general correspondence for categories with weak finite products and weak pullbacks.

We now discuss the example of main interest for our purposes, which comes from type theory.

Example 1.2.8. As we have seen in the previous section, the intensional Martin-Löf intuitionistic type theory gives rise to a syntactic category **ML** of closed types and terms up to functional extensionality. We now introduce the functor

$$F^{ML} : \mathbf{ML}^{op} \to \mathrm{InfSL} \tag{1.12}$$

which sends a closed type A to $F^{ML}(A)$ defined as the poset of equivalence classes of types depending on A respect to *equiprovability*: $x : A \vdash B(x)$ and $x : A \vdash B'(x)$ are in the same equivalence class if there exists a term of

$$\prod_{x:A}(B(x) \to B'(x)) \times (B'(x) \to B(x)). \tag{1.13}$$

Thanks to the introduction and elimination rules of the Π-type, the above condition is equivalent to the existence of two terms

$$x : A, p(x) : B(x) \vdash q(x, p(x)) : B'(x) \qquad x : A, q'(x) : B'(x) \vdash p'(x, q'(x)) : B'(x). \tag{1.14}$$

Two equivalence classes are in relation $\lfloor B \rceil \leq \lfloor B' \rceil$ if there exists a term $x : A, p(x) : B(x) \vdash q : B'(x)$. The action of F^{ML} on the arrows is given by substitution. We will often, abusing the notation, denote a type without brackets to indicate its equivalence class.

The functor F^{ML} is a primary doctrine: the meets are given by the product type $\times$ and the top element is given by the one-element type **1**. It is also an elementary doctrine and the fibered equality is given by the identity types Id_X. Condition **I** of Definition 1.2.5 follows from the canonical element $\mathsf{refl}_x : \mathsf{Id}_X(x, x)$, and conditions **II** and **III** of Definition 1.2.5 follow from the recursion principle of the identity type. The elementary doctrine F^{ML} enjoys more properties and we will recall them once we have introduced more expressive elementary doctrines.

Two dependent types $x : A \vdash B(x)$ and $x : A \vdash B'(x)$ that are equivalent as in (1.13) are called *logically equivalent*. This notion is different from the *homotopy equivalence* that can be found in [Uni13]. In homotopy type theory, this notion corresponds to the *propositional equality* through the Voevodsky's *univalence axiom*. However, if B and B' are *mere propositions*, then the two notions of equivalence coincide. Assuming the *propositional truncation* inductive type, it follows that B and B' are logically equivalent if and only if their propositional truncations are equivalent. We refer to [Uni13] for further details.

The elementary doctrines form a 2-category denoted by **ED**. Objects of **ED** are elementary doctrines and 1-arrows from P to P' are pairs (F, f) where $F : \mathscr{C} \to \mathscr{C}'$ is a functor which preserves finite products and $f : P \Rightarrow P'$ is a natural transformation such that, for every object $A \in \mathscr{C}$, the functor $f_A : \mathsf{P}(A) \to \mathsf{P}'(F(A))$ preserves all the structure.

$$\begin{tikzcd}
\mathscr{C}^{op} \arrow[dd, "F^{op}"'] \arrow[dr, "\mathsf{P}"{name=U}] & \\
& \mathsf{InfSL} \\
\mathscr{C}'^{op} \arrow[ur, "\mathsf{P}'"'{name=D}] &
\arrow[from=U, to=D, "f"', shorten=2mm]
\end{tikzcd}$$

In particular, f_A preserves finite meets and

$$f_{A\times A}(\delta_A) = \mathsf{P}'_{\langle F(pr_1), F(pr_2)\rangle}(\delta_{F(A)}).$$

A 2-arrow $\theta : (F, f) \to (G, g)$ is a natural transformation $\theta : F \Rightarrow G$ as in the following diagram

$$\begin{tikzcd}
\mathscr{C}^{op} \arrow[dd, bend right=50, "F^{op}"'] \arrow[dd, bend left=50, "G^{op}"] \arrow[dd, phantom, "\overset{\theta^{op}}{\Longleftarrow}"] \arrow[rrd, "\mathsf{P}"] & & \\
 & f \Downarrow \;\; \Downarrow g & \mathsf{InfSL}, \\
\mathscr{C}'^{op} \arrow[rru, "\mathsf{P}'"'] & &
\end{tikzcd}$$

such that, for every object $A \in \mathscr{C}$ and every element $\alpha \in \mathsf{P}(A)$ it holds

$$f_A(\alpha) \leq \mathsf{P}'_{\theta_A}(g_A(\alpha)).$$

Set-like doctrines. We now recall a richer class of elementary doctrines. For these doctrines the *axiom of comprehension* and an equality principle hold.

Definition 1.2.9. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an primary doctrine and let X be an object of $\mathscr{C}$. A *comprehension* of an element $\alpha \in \mathsf{P}(X)$ is an arrow $\{\!|\alpha|\!\} : C \to X$ such that $\top_C \leq \mathsf{P}_{\{\!|\alpha|\!\}}\alpha$ and which satisfies the following universal property: for every arrow $f : Y \to X$ such that $\top_Y \leq \mathsf{P}_f(\alpha)$, there exists an arrow $h : Y \to C$ such that the following diagram commutes

$$\begin{tikzcd}
C \arrow[r, dashed, "h"] \arrow[dr, "\{\!|\alpha|\!\}"'] & Y \arrow[d, "f"] \\
 & X.
\end{tikzcd}$$

The comprehension $\{\!|\alpha|\!\}$ is called *strict* if the induced arrow h is unique. When h is not unique, the comprehension $\{\!|\alpha|\!\}$ is called *weak*. A comprehension $\{\!|\alpha|\!\} : C \to X$ is called *full* if $\alpha \leq \beta$ whenever $\top_C \leq \mathsf{P}_{\{\!|\alpha|\!\}}\beta$.

We will say that P has (full) (weak) comprehensions if, for every object $X \in \mathscr{C}$, each element $\alpha \in \mathsf{P}(X)$ has a (full) (weak) comprehension.

Remark 1.2.10. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a primary doctrine with (weak) comprehensions. If $\alpha \in \mathsf{P}(A)$ and $f : B \to A$ is an arrow, then the following diagram

$$\begin{tikzcd}
X' \arrow[r, "\{\!|\mathsf{P}_f\alpha|\!\}"] \arrow[d, dashed, "h"'] & B \arrow[d, "f"] \\
X \arrow[r, "\{\!|\alpha|\!\}"'] & A
\end{tikzcd}$$

where h is the arrow induced by the comprehension $\{\!|\alpha|\!\}$, is a (weak) pullback.

Definition 1.2.11. An elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ has *comprehensive diagonals* if for every pair of arrows $f, g : A \to X$ such that $\top_A \leq \mathsf{P}_{\langle f,g \rangle}\delta_X$ implies that $f = g$.

Equivalent definitions of comprehensive diagonals are discussed in [MR16, Lemma 2.9] and reported here in Lemma A.0.5. In case of comprehensive diagonals and full comprehensions, it follows that a comprehension is strict if and only if it is a monomorphism. This fact has been observed in [MR13, Corollary 4.8] and reported here in Lemma A.0.6.

We now discuss the existence of comprehensions and comprehensive diagonals in the main examples of elementary doctrine encountered.

Example 1.2.12. Given a left exact category $\mathscr{C}$, the elementary doctrine of subobjects $\mathsf{Sub}_{\mathscr{C}}$ has strict full comprehensions and comprehensive diagonals. A comprehension of an element $\lfloor m : Y \rightarrowtail X \rceil \in \mathsf{Sub}_{\mathscr{C}}(X)$ is given by the representant m. Since m is a monomorphism an easy verification shows that it is a strict comprehension. Fullness is obvious. Diagonals are comprehensive because, if $f, g : X \to Y$ are two arrows such that $1_X \leq \mathsf{Sub}_{\mathscr{C}\langle f,g\rangle}(\lfloor \Delta_Y \rceil)$, tf means that the arrow $\langle f, g\rangle$ factors through the diagonal Δ_Y, and hence $f = g$.

If $\mathscr{C}$ is weakly left exact, the elementary doctrine of weak subobjects $\mathsf{Psub}_{\mathscr{C}}$ has full weak comprehension and comprehensive diagonals. It is proved as for the above example but considering that we are not working with monomorphisms.

The elementary doctrine F^{ML} arising from type theory has full weak comprehension and comprehensive diagonals. Indeed, if $B(X) \in F^{ML}(X)$ is a dependent type, then we can consider the projection

$$\pi : \sum_{x:X} B(x) \to X \tag{1.15}$$

which provides a full weak comprehension of $B(x)$. The diagonals are comprehensive because the arrows of **ML** are defined as equivalence classes of terms up to functional extensionality.

The 2-full 2-subcategory of **ED** whose objects are elementary doctrines with full comprehensions and comprehensive diagonals is denoted with **EqD** The 1-arrows of **EqD** are 1-arrows (F, f) of **ED** such that F preserves comprehensions.

$\exists, \forall, \Rightarrow$ **doctrines.** We now recall the elementary doctrines that can express the existential and universal quantification and the logical connective of implication. We start from the existential quantification.

Definition 1.2.13. An elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is called *existential* if, for every pair of objects $X_1, X_2 \in \mathscr{C}$ the functors $\mathsf{P}_{\mathsf{p}_i} : \mathsf{P}(X_i) \to \mathsf{P}(X_1 \times X_2)$, for $i = 1, 2$, have left adjoints $\exists_{\mathsf{p}_i} : \mathsf{P}(X_1 \times X_2) \to \mathsf{P}(X_i)$ which satisfy

- the *Beck-Chevalley* condition: for the pullback diagram

$$\begin{tikzcd} X_1 \times Y \arrow[r, "p_2"] \arrow[d, dashed, "1_{X_1} \times f"'] & Y \arrow[d, "f"] \\ X_1 \times X_2 \arrow[r, "p_2"'] & X_2 \end{tikzcd}$$

  the canonical arrow $\exists_{\mathsf{p}_2} \circ \mathsf{P}_{1_{X_1} \times f}(-) \leq \mathsf{P}_f \circ \exists_{\mathsf{p}_2}(-)$ is an isomorphism. The analogous condition holds for p_1.

- the *Frobenius reciprocity*: for any projection $p_i : X_1 \times X_2 \to X_i$, element $\alpha \in \mathsf{P}(X_i)$, and $\beta \in \mathsf{P}(X_1 \times X_2)$, the canonical arrow $\exists_{p_i}(\mathsf{P}_{p_i}\alpha \wedge \beta) \leq \alpha \wedge \exists_{p_i}\beta$ is an isomorphism.

Remark 1.2.14. If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an existential elementary doctrines then P has left adjoints to all reindexings. If $f : X \to Y$ is an arrow of $\mathscr{C}$ then the functor which sends an element $\alpha \in \mathsf{P}(X)$ to

$$\exists_f(\alpha) := \exists_{p_2}(\mathsf{P}_{p_1}\alpha \wedge \mathsf{P}_{f \times 1_Y}\delta_Y) \tag{1.16}$$

where p_1, p_2 are the projections of $X \times Y$, is left adjoint to the functor $\mathsf{P}_f : \mathsf{P}(Y) \to \mathsf{P}(X)$.

Example 1.2.15. If $\mathscr{C}$ is a left exact category which is also regular, then the elementary doctrine of subobjects $\mathsf{Sub}_{\mathscr{C}}$ is existential. This is a well-known result used to interpret *regular logic* in regular categories. If $f : Y \to X$ is an arrow of $\mathscr{C}$, then the left adjoint of $\mathsf{Sub}_{\mathscr{C}f}$ is given by the functor that send an element $\lfloor m \rceil \in \mathsf{Sub}_{\mathscr{C}}$ to

$$\exists_f \lfloor m \rceil := \lfloor \mathsf{Im}(f \circ m) \rceil$$

where $\mathsf{Im}(f \circ m)$ is the monomorphism given by the regular epi mono factorization of $f \circ m$.

If $\mathscr{C}$ has strict products and weak pullbacks, then the elementary doctrine of weak subobjects $\mathsf{Psub}_{\mathscr{C}}$ is existential. If $f : Y \to X$ is an arrow of $\mathscr{C}$, then the left adjoint of $\mathsf{Psub}_{\mathscr{C}f}$ is given by the functor that send an element $\lfloor m \rceil \in \mathsf{Psub}_{\mathscr{C}}$ to

$$\exists_f \lfloor m \rceil := \lfloor f \circ m \rceil.$$

The elementary doctrine F^{ML} arising from type theory is existential. The left adjoint to the reindexing over an arrow $\lfloor t \rceil : X \to Y$ is given by the functor which sends an element $B(y) \in F^{ML}(Y)$ to the Σ-type

$$x : X \vdash \sum_{y:Y} B(x) \times \mathsf{Id}_Y(t(x), y).$$

We now define elementary doctrines which can express universal quantifications.

Definition 1.2.16. An elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is called *universal* if for every pair of objects $X_1, X_2 \in \mathscr{C}$, the functors $\mathsf{P}_{p_i} : \mathsf{P}(X) \to \mathsf{P}(X_1 \times X_2)$, for $i = 1, 2$, have right adjoints $\forall_{p_i} : \mathsf{P}(X_1 \times X_2) \to \mathsf{P}(X_i)$ which satisfy

- the *Beck-Chevalley* condition: for the pullback diagram

$$\begin{tikzcd}
X_1 \times Y \arrow[r, "p_2"] \arrow[d, dashed, "1_{X_1} \times f"'] & Y \arrow[d, "f"] \\
X_1 \times X_2 \arrow[r, "p_2"'] & X_2
\end{tikzcd}$$

  the canonical arrow $\mathsf{P}_f \circ \forall_{p_2}(-) \leq \forall_{p_2} \circ \mathsf{P}_{1_{X_1} \times f}(-)$ is an isomorphism. The analogous condition holds for p_1.

Example 1.2.17. The elementary doctrine F^{ML} is universal. If X and Y are closed types then the right adjoint to the functor $F^{ML}_{p_1}$ is given by the functor which sends an element $B(x, y) \in F^{ML}(X \times Y)$ to the Π-type

$$x : X \vdash \prod_{y:Y} B(x, y).$$

Finally, we define elementary doctrine which can express the implication.

Definition 1.2.18. A primary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is called *implicational* if for every object $X \in \mathscr{C}$ and element $\alpha \in \mathsf{P}(X)$ the functor $\alpha \wedge - : \mathsf{P}(X) \to \mathsf{P}(X)$ has a right adjoint $\alpha \Rightarrow - : \mathsf{P}(X) \to \mathsf{P}(X)$. Moreover, for every arrow $f : Y \to X$ of $\mathscr{C}$ and elements $\alpha, \beta \in \mathsf{P}(X)$, it is required that $\mathsf{P}_f(\alpha \Rightarrow \beta) = \mathsf{P}_f\alpha \Rightarrow \mathsf{P}_f\beta$.

In the following remark we observe some relations between universal and implicational doctrines.

Remark 1.2.19. If P is an elementary doctrine implicational and universal, then for every $f : X \to Y$, the functor P_f has a right adjoint which sends $\alpha \in \mathsf{P}(X)$ to

$$\forall_f(\alpha) := \forall_{p_2}(\mathsf{P}_{f\times 1_Y}\delta_Y \implies \mathsf{P}_{p_1}\alpha) \tag{1.17}$$

where p_1, p_2 are the projections of $X \times Y$.

Moreover, as observed in [MR13, Lemma 4.9] it is possible to build implications from full weak comprehensions. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a primary doctrine with full weak comprehensions and assume that, for every object $X \in \mathscr{C}$ and $\alpha \in \mathsf{P}(X)$, the reindexing $\mathsf{P}_{\{\!|\alpha|\!\}}$ over the weak comprehension $\{\!|\alpha|\!\}$ has right adjoint $\forall_{\{\!|\alpha|\!\}}$. For every $\alpha' \in \mathsf{P}(X)$ we can define the implication

$$\alpha \implies \alpha' := \forall_{\{\!|\alpha|\!\}}\mathsf{P}_{\{\!|\alpha|\!\}}\alpha'. \tag{1.18}$$

The above definition satisfies the adjoint property of Definition 1.2.18.

Example 1.2.20. This is the case of the elementary doctrine F^{ML} which is also implicational. The implication is given by the arrow type: if X is a closed type then the implication of two dependent type $B(x), B'(x) \in F^{ML}$ is given by the type

$$x : X \vdash B(x) \to B'(x).$$

The adjoint condition is given by the *Curring* operation. Hence, Remark 1.2.19 implies that for every arrow $[t] : X \to Y$, the functor F_t^{ML} has a right adjoint which sends an element $B(x) \in F^{ML}$ to the Π-type

$$y : Y \vdash \prod_{x:X}(B(x) \to \mathsf{Id}_Y(f(x), y)). \tag{1.19}$$

We have recalled the main notions about the elementary doctrines and we have seen that the functor F^{ML} arising from type theory is a rich elementary doctrine. We can collect the properties observed for F^{ML} in the following lemma which appears as [MR13, Proposition 7.2 and 7.2].

Proposition 1.2.21. *The functor $F^{ML} : \boldsymbol{ML}^{op} \to \mathsf{InfSL}$ of Example 1.2.8 is an existential, universal and implicational elementary doctrine with full weak comprehensions and comprehensive diagonals.* $\square$

1.3 Elementary quotient completion

The language of the elementary doctrines provides a more general framework to define the notion of equivalence relation and quotient. The classical notions of *pseudo-equivalence relation* and coequalizer of category theory are obtained as a particular instance considering the elementary doctrines of subobjects and weak subobjects. Moreover, the exact completion of a category with strict finite products and weak pullbacks, is a particular instance of a more general construction, namely the *elementary quotient completion*. This construction has been introduced by Maietti and Rosolini in [MR13] and it provides a procedure to add well-behaved quotients in a suitable universal way that will be recalled in this section. This framework is particularly useful to treat the quotient construction in foundations of constructive mathematics based on intensional type theory.

We start recalling the notion of equivalence relation and quotient relative to suitable doctrine.

Definition 1.3.1. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine. A P-*equivalence relation* on an object $X \in \mathscr{C}$ is an element $\rho \in \mathsf{P}(X \times X)$ such that

ref) $\delta_X \leq \rho$,

sym) $\mathsf{P}_{\langle 2,1\rangle}\rho \leq \rho$,

trans) $\mathsf{P}_{\langle 1,2\rangle}\rho \wedge \mathsf{P}_{\langle 2,3\rangle}\rho \leq \mathsf{P}_{\langle 1,3\rangle}\rho$,

Where the arrows $\langle 2,1\rangle : X \times X \to X \times X$ and $\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle : X \times X \times X \to X \times X$ are induced by the obvious projections.

Definition 1.3.2. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine and let ρ be a P-equivalence relation on X. A *quotient* of ρ is an arrow $q : X \to C$ such that $\rho \leq \mathsf{P}_{q\times q}\delta_C$[1] and, for every arrow $g : A \to Z$ such that $\rho \leq \mathsf{P}_{g\times g}\delta_Z$, there exists a unique arrow $h : C \to Z$ such that $g = h \circ q$.

A quotient is said *stable* when, for every arrow $f : C \to C'$, there is a pullback diagram

$$\begin{tikzcd} A' \arrow[r, "q'"] \arrow[d, "f'"'] & C' \arrow[d, "f"] \\ A \arrow[r, "q"'] & C \end{tikzcd}$$

in $\mathscr{C}$ such that the arrow $q' : A' \to C'$ is a quotient of the P-equivalence relation $\mathsf{P}_{f'\times f'}\rho$.

If $f : A \to B$ is an arrow in $\mathscr{C}$, the *P-kernel* of f is the P-equivalence relation $\mathsf{P}_{f\times f}(\delta_B)$.

A quotient $q : A \to B$ of the P-equivalence relation ρ is called *effective* if its P-kernel is ρ. The quotient q is of effective descent if the functor

$$\mathsf{P}_f : \mathsf{P}(B) \to \mathcal{D}es_\rho$$

is an isomorphism.

We will denote by **QD** the 2-full 2-subcategory of **EqD** whose objects are elementary doctrines of **EqD** with stable effective quotients of P-equivalence relations and of effective descent. The 1-arrows of **QD** are 1-arrows (F, f) of **ED** such that F preserves quotients and comprehensions.

We now recall the *elementary quotient completion* construction. Given an elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$, we can consider the category $\overline{\mathscr{C}}$ whose

- objects are pairs (X, ρ), where X is an object of $\mathscr{C}$ and ρ is a P-equivalence relation on X,
- arrows between two objects (X, ρ) and (Y, σ) are equivalence classes of the arrows $f : X \to Y$ such that $\rho \leq \mathsf{P}_{f\times f}(\sigma)$. Two arrows f, f' are equivalent when $\rho \leq \mathsf{P}_{f\times f'}(\sigma)$.

We now define the functor

$$\overline{\mathsf{P}} : \overline{\mathscr{C}}^{op} \to \mathsf{InfSL}$$

which sends an object $(X, \rho) \in \overline{\mathscr{C}}$ to $\overline{\mathsf{P}}(X, \rho) := \mathcal{D}es_\rho$ and an arrow $\lfloor f \rceil$ to $\overline{\mathsf{P}}_{\lfloor f \rceil} := \mathsf{P}_f$. The functor $\overline{\mathsf{P}}$ is called the *elementary quotient completion* of P.

There is an obvious 1-arrow

$$(J, j) : \mathsf{P} \to \overline{\mathsf{P}}$$

given by the functor J, which sends an object $X \in \mathscr{C}$ to $(X, \delta_X) \in \overline{C}$ and an arrow $f : X \to Y$ to the arrow $\lfloor f \rceil : (X, \delta_X) \to (Y, \delta_Y)$. For every object $X \in \mathscr{C}$ the natural transformation j is defined as $j_X := 1_{\mathsf{P}(X)}$.

The elementary quotient completion is the construction which freely adds quotients to an elementary doctrine in the sense of the following theorem which appears as [MR13, Theorem 5.8].

[1] If $f : A \to C$ and $g : B \to D$ are arrows of $\mathscr{C}$, then $f \times g$ denotes the unique arrow $A \times B \to C \times D$ induced by $f \circ p_1 : A \times B \to C$ and $g \circ p_2 : A \times B \to D$.

Theorem 1.3.3 (Maietti and Rosolini)**.** *For every elementary doctrine* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *in **EqD**, the assignment* $\mathsf{P} \to \overline{\mathsf{P}}$ *gives a left bi-adjoint to the forgetful 2-functor* $U : \textbf{\textit{QD}} \to \textbf{\textit{EqD}}$*, i.e., pre-composition with the 1-arrow* (J, j)

$$\begin{tikzcd} \mathscr{C}^{op} \arrow[dd, "J^{op}"'] \arrow[drr, "P"] & & \\ & {} \arrow[d, "j"] & InfSL \\ \mathscr{Q}^{op} \arrow[urr, "\overline{\mathsf{P}}"'] & {} & \end{tikzcd}$$

induces an equivalence of categories

$$- \circ (J, j) : \textbf{\textit{QD}}(\overline{\mathsf{P}}, X) \cong \textbf{\textit{EqD}}(P, UX)$$

for every X *in **QD**.* $\square$

If P is an elementary doctrine, then $\overline{\mathsf{P}}$ is an elementary doctrine with quotients of all $\overline{\mathsf{P}}$-equivalence relations. Given an object $(X, \rho) \in \overline{\mathscr{C}}$, a $\overline{\mathsf{P}}$-eq relation ρ' on (X, ρ) is nothing but a P-eq.relation on X such that $\rho \leq \rho'$. A quotient of ρ' is given by the arrow

$$\lfloor 1_X \rceil : (X, \rho) \to (X, \rho'). \tag{1.20}$$

Observation 1.3.4. As shown in [MR12] it is possible to split the elementary quotient completion in different steps. Instead of starting from doctrines of **EqD**, it is possible to freely add comprehensions, comprehensive diagonals and quotients separately. Every construction comes equipped with a universal property in style of Theorem 1.3.3 between the right 2-categories. We refer to *loc.cit.* for further details about these constructions.

We now discuss the relation between the exact completion discussed in Definition 1.1.5 and the elementary quotient completion.

Example 1.3.5. When $\mathscr{C}$ is a category with finite strict products and weak pullbacks we can consider the elementary doctrine of weak subobjects $\mathsf{Psub}_{\mathscr{C}} : \mathscr{C} \to \mathsf{InfSL}$ of Example 1.2.6. As we observed, this doctrine encodes the inner logic of weak subobjects. The notion of *pseudo-equivalence relations* is nothing but the notion of equivalence relation in this logic. In particular, a pseudo-equivalence relation on A is a pair of arrows

$$r_1, r_2 : R \to X$$

satisfying certain reflexive, symmetric and transitive conditions listed in Definition A.0.2. These arrows induce an arrow

$$\langle r_1, r_2 \rangle : R \to X \times X$$

and hence an element $\lfloor \langle r_1, r_2 \rangle \rceil \in \mathsf{Psub}_{\mathscr{C}}(X \times X)$ which is a $\mathsf{Psub}_{\mathscr{C}}$-eq. relation on X. If $r_1, r_2 : R \to X$ and $s_1, s_2 : S \to X$ are pseudo-equivalence relations and $f, \tilde{f}$ is a half-homotopy between them, then an easy proof shows that $\langle r_1, r_2 \rangle \leq \mathsf{Psub}_{\mathscr{C} f \times f} \langle s_1, s_2 \rangle$. This association define a functor

$$\mathscr{C}_{ex} \to \overline{\mathscr{C}}$$

which turns out to be an equivalence of categories. We observe that it is not an isomorphism because in $\mathscr{C}_{ex}$ the objects are not considered up to the following equivalence relation. The pair $r_1, r_2 : R \to X$ is equivalent to the pair $r'_1, r'_2 : R' \to AX$ if and only if there exist two arrows $h : R \to R'$ and $h' : R' \to R$ such that the following diagrams commute for $i = 1, 2$

$$\begin{tikzcd} R \arrow[rr, dashed, "h"] \arrow[dr, "r_i"'] & & R' \arrow[dl, "r'_i"] \\ & X. & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[dr, "r_i"'] & & R' \arrow[ll, dashed, "h'"'] \arrow[dl, "r'_i"] \\ & X & \end{tikzcd}$$

for $i = 1, 2$. If two pairs are in relation as above, then $\lfloor\langle r_1, r_2\rangle\rceil = \lfloor\langle r'_1, r'_2\rangle\rceil$. A coequalizer of the pseudo-equivalence relation $r_1, r_2 : R \to A$ is a quotient of the $\mathsf{Psub}_{\mathscr{C}}$-eq. relation $\lfloor\langle r_1, r_2\rangle\rceil$.

If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an elementary doctrine, then the elementary quotient completion provides to add well-behaved quotients of P-equivalence relations. While, as we observed in the above example, the exact completion provides to add well-behaved quotients of the $\mathsf{Psub}_{\mathscr{C}}$-eq. relations (at least when $\mathscr{C}$ has strict product and weak pullbacks). These operations in general do not coincide and, moreover, the category $\overline{\mathscr{C}}$ is not necessarily exact. In the next chapter, we will work with a concrete example of this fact. However, under suitable hypothesis we obtain that the elementary quotient completion yields a regular category.

The following is an immediate corollary of [MR13, Proposition 4.15]

Proposition 1.3.6. *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an elementary doctrine with full weak comprehensions and comprehensive diagonals, then the base category $\overline{\mathscr{C}}$ of the elementary quotient completion $\overline{\mathsf{P}}$ of P is a regular category.* □

The elementary quotient completion $\overline{\mathsf{P}}$ inherits some of the properties of P. We now list some of the properties inherited by $\overline{\mathsf{P}}$ that can be found in the mentioned literature.

Proposition 1.3.7. *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an elementary doctrine, then:*

1. *if P has (full) weak comprehensions, then $\overline{\mathsf{P}}$ has (full) strict comprehensions,*
2. *if P is existential, then $\overline{\mathsf{P}}$ is existential,*
3. *if P is universal, then $\overline{\mathsf{P}}$ is universal,*
4. *if P is implicational, then $\overline{\mathsf{P}}$ is implicational.*

We now discuss the elementary quotient completion of the elementary doctrine F^{ML} arising from type theory.

Example 1.3.8. The elementary quotient completion of the elementary doctrine F^{ML} of Example 1.2.8 is given by the functor

$$\overline{F^{ML}} : \overline{\mathscr{C}}^{op} \to \mathsf{InfSL}$$

from the opposite of the category $\mathscr{C}$ whose

- objects are pair $(X, \lfloor R\rceil)$ where X is a closed type and $\lfloor R\rceil$ is the equivalence class of a dependent type

  $$x_1, x_2 : X \vdash R(x_1, x_2)$$

  which is an equivalence relation,

- arrows between two objects $(X, \lfloor R\rceil)$ and $(Y, \lfloor S\rceil)$ are equivalence classes of arrows $f : X \to Y$ such that f preserves the relations R and S as in (1.6) and (1.7).

It turns out that the category of setoids is equivalent to the base category of the elementary quotient completion of F^{ML}, i.e.

$$\mathbf{Std} \cong \overline{\mathbf{ML}}. \tag{1.21}$$

Moreover, as observed by the authors of [MR13] the elementary doctrine F^{ML} is equivalent to the elementary doctrine of weak subobjects $\mathsf{PSub}_{\mathbf{ML}}$ of the weakly left exact category of types **ML**. This result follows from the existence of comprehensions and existential functors which define the one-arrows

$$\begin{tikzcd} \mathbf{ML}^{op} \arrow[dd, equal, "Id_{\mathbf{ML}}"'] \arrow[dr, "F^{ML}"] & \\ & \mathsf{InfSL}. \arrow[u, phantom, shift left=2, "\exists_{(-)}\top_{(-)} \uparrow \downarrow \{\!|-|\!\}"] \\ \mathbf{ML}^{op} \arrow[ur, "\mathsf{Psub}_{\mathbf{ML}}"'] & \end{tikzcd} \tag{1.22}$$

The comprehensions provide a natural transformation

$$\{\!| - |\!\} : F^{ML}(X) \to \mathsf{Psub}_{ML}(X)$$

which sends the equivalence class of a dependent type $x : X \vdash B(x)$ to the equivalence class of the arrow

$$\pi : \sum_{x:X} B(x) \to X.$$

The Σ-type defines a natural transformation

$$\exists_{(-)}\top_{(-)} : \mathsf{Psub}_{ML}(X) \to F^{ML}(X)$$

which sends the equivalence class of an arrow $f : Y \to X$ to the dependent type

$$x : X \vdash \sum_{y:Y} \mathsf{Id}_Y(f(x), y)$$

up to logical equivalence. The above arrows provide an isomorphism of posets.

As an application of Example 3.4.4 we obtain a proof of the Fact 2 and in particular

$$\overline{\mathbf{ML}} \cong \mathbf{Std} \cong \mathbf{ML}_{ex}. \tag{1.23}$$

Chapter 2

Homotopy setoids

In this chapter, we study a particular class of setoids, which we will refer to as the *homotopy setoids*. Taking into account the homotopical perspective of the homotopy type theory (HoTT) , we considered only "discrete" types equipped with an equivalence relation which only contains the information of when two elements are related. In the HoTT notation [Uni13], the former types are called *h-sets* and the latter are called *h-propositions*. However, the type theory we have considered does not assume the *univalence axiom* or *higher inductive types* typical of HoTT but it is just the intensional Martin-Löf intuitionistic type theory with the *functional extensionality* axiom ($\mathcal{ML} + F.E.$).

We studied the categorical properties of the homotopy setoids which form a full subcategory $\mathbf{Std}_0$ of the category of setoids $\mathbf{Std}$ introduced in the previous chapter. Our goal is to prove that $\mathbf{Std}_0$ has good categorical properties as $\mathbf{Std}$ does. In particular, since $\mathbf{Std}$ is a ΠW-pretopos, in this chapter we ask if $\mathbf{Std}_0$

- has well-behaved quotients,
- is (locally) cartesian closed,
- is extensive.

In order to address the first one, we observe that the main difference with the classical setoids is that in the homotopy setoids not every equivalence relation has quotient, but only those related with the h-propositions. Hence, $\mathbf{Std}_0$ is not the exact completion of a suitable category as it happens for $\mathbf{Std}$. For this reason, we adopted the categorical setting of the elementary doctrines introduced by Maietti and Rosolini in [MR13], which allows us the possibility to consider quotients of equivalence relations relative to a doctrine. If $\mathbf{ML}_0$ denotes the category of the h-sets, we can consider the functor

$$F^{ML_0} : \mathbf{ML}_0^{op} \to \mathsf{InfSL}$$

which sends an h-set X to the poset of the h-propositions depending on X up to *logical equivalence*, and acts on arrows as substitution. We prove that this functor is an elementary doctrine with several properties and considering the elementary quotient completion

$$\overline{F^{ML_0}} : \overline{\mathbf{ML}_0}^{op} \to \mathsf{InfSL}$$

we obtain that $\mathbf{Std}_0 \cong \overline{\mathbf{ML}_0}$. Hence, h-setoids are an instance of a more general form of quotient completion and the quotients are well-behaved with respect to the doctrine $\overline{F^{ML_0}}$.

Instead of working directly on $\mathbf{Std}_0$, we study the properties of the elementary quotient completion following the tradition started by Maietti, Rosolini. Hence, in Section 2.3 we will generalize to the context of the elementary quotient completion the result in [CR00] and [Emm20] about the (local) cartesian closure of the exact completion. In Section 2.5, we will generalize the result in [GV98] about the extensivity of the exact completion. In Section 2.6, we will apply these results to F^{ML_0} and obtained that the homotopy setoids form a Π-pretopos *relative* to $\overline{F^{ML_0}}$.

As we will discuss, this chapter has a non-trivial intersection with [MPR21]. In *loc.cit.* the authors extend to the elementary quotient completion Menni's characterization of the exact completions which are toposes [Men00] and give a different (but equivalent) proof of the result we obtained for the (local) cartesian closure. The connection between our result and the one in [MPR21], will be discussed in Section 2.4.

2.1 Definition and properties

In this section, we recall some notion of homotopy type theory and define the particular class of setoids which will be studied in more detail along the chapter. In addition, we discuss the first categorical properties of these setoids and the differences with the usual setoids introduced in Section 1.1. For the homotopy type theoretical notions we refer to [Uni13] and [Rij18].

Before starting, we fix the type theory assumed for the rest of the chapter.

Remark 2.1.1. The type theory assumed is the intensional Martin-Löf intuitionistic type theory with the *functional extensionality* axiom ($\mathcal{ML} + F.E.$). All the notations and the type constructors used can be found in Appendix B.

We now recall the definition of the *homotopy type* and briefly discuss the geometrical intuition of types as topological spaces, typical of the homotopy type theory.

Definition 2.1.2. The *homotopy type* of a type X is defined inductively as follows:

- $\mathsf{type}_{-2}(X) := \sum_{x:X} \prod_{y:X} \mathsf{Id}_X(x,y)$
- $\mathsf{type}_{-1}(X) := \prod_{x,y:X} \mathsf{Id}_X(x,y)$
- $\mathsf{type}_{0}(X) := \prod_{x,y:X} \mathsf{type}_{-1}(\mathsf{Id}_X(x,y))$
- $\mathsf{type}_{\mathsf{n}+2}(X) := \prod_{x,y:X} \mathsf{type}_{\mathsf{n}+1}(\mathsf{Id}_X(x,y))$

If $n \geq -2$ is the first integer such that $\mathsf{type}_{\mathsf{n}}(X)$ is inhabited, then we will say that the homotopy type of X is n.

By induction, we observe that if X has homotopy type equal to n, then the $\mathsf{type}_{\mathsf{j}}(X)$ is inhabited for every $j \geq n$. If the homotopy type of X is equal to -2, then the type is called *contractible*. The homotopical intuition is that the type X has an inhabitant $x : X$ and every element of X is propositional equal to x. If the type X has homotopy type equal to -1, then it is called an *h-proposition* or *mere proposition*, using the notation in [Uni13]. The h-propositions were also called *mono types* in [Mai98]. In this case, X can be only empty or contractible, which corresponds to the truth values true and false. If X has homotopy type equal to 0, it is called an *h-set*. The homotopical intuition is that X corresponds to the set of its connected components. For $1 \leq n$ types should be thought as higher groupoids. This correspondence has been crystallized in [KL21] in which the authors provide a model of the homotopy type theory in one of the possible formalization of *infinite-groupoids* in higher category theory [Lur09].

Example 2.1.3. The empty type **0** and the one element type **1** are h-propositions. The two elements type **2** and the natural number type $\mathbb{N}$ are h-sets.

We mainly focused on h-sets equipped with an equivalence relation which is an h-proposition.

Definition 2.1.4. An *homotopy setoid* is a setoid (X, R) such that X is an h-set and R is an h-proposition.

We could define the homotopy setoids with respect to a general homotopy type *n*, saying that an *n-setoid* is a setoid (X, R) such that the type X has homotopy type equal to n and the type R has homotopy type *n-1*. In this way, the homotopy setoids correspond to the *0-setoids*. If we denote with $\mathbf{ML}_n$ the full subcategories of **ML** of closed types with homotopy type n and with $\mathbf{Std}_n$ the full subcategory of **Std** of *n-setoids*, then we have the following diagram of inclusions

$$\begin{tikzcd}
\mathbf{ML}_0 \arrow[r, tail] \arrow[d, tail, "(\bullet, \mathsf{Id}_\bullet)"'] & \mathbf{ML}_1 \arrow[r, dotted, no head] \arrow[d, tail, "(\bullet, \mathsf{Id}_\bullet)"'] & \mathbf{ML} \arrow[d, tail, "(\bullet, \mathsf{Id}_\bullet)"'] \\
\mathbf{Std}_0 \arrow[r, tail] & \mathbf{Std}_1 \arrow[r, dotted, no head] & \mathbf{Std}
\end{tikzcd}$$

where the vertical arrows are given by functors which send a type X to the *free setoids* (X, Id_X) and an arrow $t : X \to Y$ to the equivalence class $[t] : (X, \mathsf{Id}_X) \to (Y, \mathsf{Id}_Y)$. The definition of homotopy type ensures that this functor is well-defined for every degree $n \geq 0$.

Remark 2.1.5. The homotopy types preserve various type constructors. For instance, as discussed in [Uni13, Example 3.1.5, 3.1.6, 3.6.1 and 3.6.2] if X and Y are h-sets (h-propositions), then the product type $X \times Y$ is an h-set (h-proposition). If X and Y are h-sets, then the sum type $X + Y$ is an h-set. If X is any type and $x : X \vdash B(x)$ is an h-set (h-proposition), then the type

$$\prod_{x:X} B(x)$$

is an h-set (h-proposition). For the h-sets, the functional extensionality axiom is needed in order to obtain that dependent function type is an h-set. This is the main reason why we adopted this axiom.

Moreover, if X is an h-set and $x : X \vdash B(x)$ is an h-set then the type

$$\sum_{x:X} B(x)$$

in an h-set; similar properties hold for every homotopy type of level $n \geq 0$.

However, not all type constructors preserve h-propositions. For instance, if X and Y are h-propositions, the sum type $X + Y$ is not necessarily an h-proposition; the one element type **1** is an h-proposition, but the sum **1+1** is an h-set. A similar issue happens for the Σ-type. If X is a type and P is an h-proposition, the type

$$\sum_{x:X} P(x)$$

is not necessarily a h-proposition.

For these reasons we recall two useful results in order to have h-propositions preserved for Σ-types and $+$-types. The following appears as [RS15, Lemma 2.2] but before we recall that, for a type $x : X \vdash P(x)$, the type

$$\mathsf{at\text{-}most\text{-}one}(X) := \prod_{x,y:X} P(x) \to P(y) \to \mathsf{Id}_X(x, y) \tag{2.1}$$

expresses the property of $P(x)$ to have at most one element for which it holds.

Lemma 2.1.6 (Rijke-Spitters)**.** *If X is any type and $x : X \vdash P(x)$ is ah h-proposition such that the type in (2.1) is inhabited, then the type*

$$\sum_{x:X} P(x)$$

is an h-proposition. □

The following result appears as [Uni13, Exercise 3.7].

Lemma 2.1.7. *If X and Y are h-propositions and $\neg(X \times Y)$ true, then $X + Y$ is an h-proposition.* □

As we have seen in Section 1.1, the category of types **ML** has strict finite products and weak pullbacks, hence all weak finite limits. For the subcategory of h-sets we obtain a stronger result.

Lemma 2.1.8. *The category $\boldsymbol{ML}_0$ is left exact.*

Proof. Equivalently, we prove the existence of finite products and pullbacks. The existence of finite products follows from Remark 2.1.5 and elimination rule of $\times$-types. This is a consequence of the fact that for h-sets, the identity type is an h-proposition and hence, given two arrows between h-sets $\lfloor t \rceil : X \to A$ and $\lfloor u \rceil : Y \to A$ the diagram

$$\begin{tikzcd} \sum_{x:X,y:Y} \mathsf{Id}_A(t(x), u(y)) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "u"] \\ X \arrow[r, "t"'] & A. \end{tikzcd} \tag{2.2}$$

has the strict universal property of pullbacks. □

We now discuss some categorical properties of the category $\mathbf{Std}_0$. In order to do that, we will review the steps of a direct proof of the fact that **Std** is an exact category and observe that most of the results hold for the category $\mathbf{Std}_0$.

Proposition 2.1.9. *The category of $\boldsymbol{Std}_0$ is a regular category.*

Before providing a proof of the above proposition, we prove some preliminary results. We start recalling how finite products and equalizers are constructed in the category of setoids. If (X, R) and (Y, S) are setoids, the the product is given by

$$(X \times Y, R \times S) \tag{2.3}$$

where $z : X \times Y \times X \times Y \vdash R(z_1, z_3) \times S(z_2, z_4)$ and $z_i := \pi_i(z)$, for $1 \leq i \leq 4$. A trivial verification shows that the reduction rule implies the universal property of products.

Moreover, given two arrows $\lfloor f \rceil, \lfloor g \rceil : (X, R) \to (Y, S)$ then the arrow

$$\lfloor \pi \rceil : (\sum_{x:X} S(f(x), g(x)), R') \to (X, R) \tag{2.4}$$

where $R'(z_1, z_2) := R(\pi z_1, \pi z_2)$ for $z_1, z_2 : \sum_{x:X} S(f(x), g(x))$, is the equalizer of $\lfloor f \rceil$ and $\lfloor g \rceil$.

The above argument can be repeated for the homotopy setoids.

Lemma 2.1.10. *The category $\boldsymbol{Std}_0$ has finite products and equalizers.*

Proof. If (X, R) and (Y, S) are homotopy setoids, then Remark 2.1.5 implies that the setoids $(X \times Y, R \times S)$ and $(\sum_{x:X} S(f(x), g(x)), R')$ are h-setoids. $\square$

The above lemma implies that $\mathbf{Std}_0$, such as $\mathbf{Std}$, has finite limits. For instance, a pullback of two arrows $\lfloor f \rceil : (X, R) \to (Z, T)$ and $\lfloor g \rceil : (Y, S) \to (Z, T)$ is given by

$$\begin{tikzcd} (P, (R \boxtimes S)^*) \arrow[r, "\lfloor \pi_2 \rceil"] \arrow[d, "\lfloor \pi_1 \rceil"'] & (Y, S) \arrow[d, "\lfloor g \rceil"] \\ (X, R) \arrow[r, "\lfloor f \rceil"'] & (Z, T) \end{tikzcd} \tag{2.5}$$

where

$$P := \sum_{x:X} \sum_{y:Y} T(f(x), g(y))$$

and

$$(R \boxtimes S)^*(z_1, z_2) := R(\pi_1 z_1, \pi_1 z_2) \times S(\pi_2 z_1, \pi_2 x_2),$$

for $z_1, z_2 : P$ and π_1 and π_2 the canonical projections of P into X and Y respectively. If $\lfloor f \rceil = \lfloor g \rceil$ we obtain a kernel pair of $\lfloor f \rceil$. If $\lfloor f \rceil : (X, R) \to (Z, T)$ is an arrow of $\mathbf{Std}$, then the kernel pair of $\lfloor f \rceil$ is given by the following pair of arrows:

$$\lfloor \pi_1 \rceil, \lfloor \pi_2 \rceil : (P, (R \boxtimes R)^*) \to (X, R) \tag{2.6}$$

where $P := \sum_{x_1:X} \sum_{x_2:X} T(f(x_1), f(x_2))$. The quotient is obtained providing X with a suitable equivalence relation. If we define $x_1, x_2 : X \vdash \overline{R}(x_1, x_2)$ to be $\overline{R}(x_1, x_2) := T(f(x_1), f(x_2))$, then the coequalizer of $\lfloor \pi_1 \rceil, \lfloor \pi_2 \rceil$ is given by the arrow

$$\lfloor 1_X \rceil : (X, R) \to (X, \overline{R}).$$

Indeed, if an arrow $\lfloor g \rceil : (X, R) \to (Y, S)$ coequalizes $\lfloor \pi_1 \rceil, \lfloor \pi_2 \rceil$ it implies that $\lfloor g \rceil(x_1) = \lfloor g \rceil(x_2)$ when $T(f(x_1), f(x_2))$. Hence, g induces an arrow $\overline{\lfloor g \rceil} : (X, \overline{R}) \to (Y, S)$ such that $\overline{\lfloor g \rceil} \lfloor 1_X \rceil = \lfloor g \rceil$, which is clearly unique.

In the following proposition we prove that kernel pairs have coequalizers also in $\mathbf{Std}_0$.

Proposition 2.1.11. *In $\mathbf{Std}_0$ every kernel pair has coequalizer.*

Proof. If $\lfloor f \rceil$ is an arrow of $\mathbf{Std}_0$, then the kernel pair is still in $\mathbf{Std}_0$. The construction of the coequalizer relies on $\overline{R}$ which is obviously an h-propostion if R is an h-proposition. $\square$

We now prove that regular epimorphisms are pullback stable. In order to do that, we recall a characterization of epimorphisms in the category $\mathbf{Std}$ due to Wilander in [Wil10]. There, the author shows that an arrow $\lfloor f \rceil : (X, R) \to (Y, S)$ is an epimorphism if and only if it is a *surjective arrow*, i.e. the following type is inhabited:

$$\mathsf{surj}(f) := \prod_{y:Y} \sum_{x:X} S(f(x), y). \tag{2.7}$$

The left to right direction requires the assumption of a universe in the type theory; otherwise, there is a model introduced by Smith in [Smi88] in which the statement does not holds. Instead, the fact that every surjective arrow is an epimorphism holds without universes.

Among the various notions of epimorphism in a category, there is that of coequalizer. In **Std** this notion coincides with that of surjectivity.

Indeed, a term of $\mathsf{surj}(f)$ implies the existence of a section $s : Y \to X$ of f, which clearly induces an arrow

$$\lfloor s \rceil : (Y, S) \to (X, \overline{R}).$$

The arrow $\lfloor s \rceil$ is the inverse of the unique arrow induced by the universal property of the coequalizer $\lfloor 1_X \rceil : (X, R) \to (X, \overline{R})$. Hence, $\lfloor f \rceil$ is a coequalizer of its kernel pair.

Vice versa, if $\lfloor f \rceil$ is the coequalizer of two arrows $\lfloor g \rceil, \lfloor h \rceil : (Z, T) \to (X, R)$, then we can consider the setoid arrow

$$\lfloor \pi \rceil : (\sum_{y:Y} \sum_{x:X} S(f(x), y), S') \to (Y, S), \tag{2.8}$$

where $S'(z_1, z_2) := S(\pi(z_1), \pi(x_2))$, for $z_1, z_2 : \sum_{y:Y} \sum_{x:X} S(f(x), y)$. Since the above arrow coequalizes $\lfloor g \rceil$ and $\lfloor h \rceil$ it follows that there exists an arrow

$$s : Y \to \sum_{y:Y} (\sum_{x:X} S(f(x), y))$$

from which we can extract a term of $\mathsf{surj}(f)$.

The same correspondence bettween surjective arrows and coequalizers holds for the homotopy setoids.

Lemma 2.1.12. *An arrow $\lfloor f \rceil : (X, R) \to (Y, S)$ of **Std**$_0$ is surjective if and only if it is a coequalizer.*

Proof. If $\lfloor f \rceil$ is an arrow of **Std**$_0$ then the kernel pair and its coequalizer are in **Std**$_0$. The same holds for the arrow in (2.8). Hence, the argument is valid in **Std**$_0$. $\square$

The above result and the one in [Wil10] imply the following characterization of epimorphisms in **Std** and **Std**$_0$.

Corollary 2.1.13. *If $\lfloor f \rceil : (X, R) \to (Y, S)$ is an arrow of **Std** the following conditions are equivalent, if $\lfloor f \rceil$ is an arrow of **Std**$_0$, then the last two conditions are equivalent:*

1. *$\lfloor f \rceil$ is an epimorphism,*
2. *$\lfloor f \rceil$ is surjective,*
3. *$\lfloor f \rceil$ is an regular epimorphism.*

This characterization of the epimorphisms helps to understand their stability properties. In **Std** the regular epimorphisms are stable under pullback. This is better seen if we reason about surjective arrows. Indeed, if $\lfloor f \rceil : (X, R) \to (Z, T)$ is a surjection and $\lfloor g \rceil : (Y, S) \to (Z, T)$ is an arrow, the description of pullbacks in (2.5) implies that the statement is equivalent to prove that the arrow

$$\lfloor \pi_2 \rceil : (P, (R \boxtimes S)^*) \to (Y, S) \tag{2.9}$$

is a surjection, where $P := \sum_{x:X} \sum_{y:Y} T(f(x), g(y))$ and

$$(R \boxtimes S)^*(z_1, z_2) := R(\pi_1 z_1, \pi_1 z_2) \times S(\pi_2 z_1, \pi_2 x_2). \tag{2.10}$$

If $s : \mathsf{surj}(f)$ and for every $z : Z$ the term $s(z) := (s_1(z), s_2(z))$ with $s_1(z) : X$ and $s_2(z) : T(f(s_1(z), z))$ then

$$y : Y \vdash (s_1(g(z)), y, s_2(g(z)), \mathsf{refl(y)}) : \mathsf{surj}(\pi_2).$$

The same argument holds for homotopy setoids.

Proposition 2.1.14. *In* $\mathbf{Std}_0$ *regular epimorphisms are stable under pullback.*

Proof. If $\lfloor f \rceil$ and $\lfloor g \rceil$ are arrows as above between homotopy setoids and $\lfloor f \rceil$ is surjective, then the arrow in (2.9) is an arrow of homotopy setoids and it is a surjection. The statement follows from Corollary 2.1.13. □

We can now collect the above results to prove Proposition 2.1.9.

Proof of Proposition 2.1.9. In Lemma 2.1.10 we have proved that $\mathbf{Std}_0$ has product and equalizers. Proposition 2.1.11 implies that every kernel pair has coequalizer. Finally regular epimorphism are pullback stable as shown in Proposition 2.1.14. □

We now discuss the properties of monomorphisms of **Std** and $\mathbf{Std}_0$. As for surjectivity, the injectivity of an arrow $\lfloor f \rceil : (X, R) \to (Y, S)$ can be expressed through the type

$$\mathsf{inj}(f) := \prod_{x_1, x_2 : X} S(f(x_1), f(x_2)) \to R(x_1, x_2). \tag{2.11}$$

We now prove that monomorphisms and injective arrows coincide both in **Std** and $\mathbf{Std}_0$.

Proposition 2.1.15. *In* **Std** *and* $\mathbf{Std}_0$ *an arrow is a monomorphism if and only if it is injective.*

Proof. Assuming $\lfloor f \rceil : (X, R) \to (Y, S)$ injective, if $\lfloor g \rceil, \lfloor h \rceil : (Z, T) \to (X, R)$ are two arrows such that $\lfloor f \rceil \lfloor g \rceil = \lfloor f \rceil \lfloor h \rceil$, then we have a term of

$$\prod_{z_1, z_2 : Z} T(z_1, z_2) \to S(f(g(z_1)), f(h(z_2))))$$

that we can combine with a term of $\mathsf{inj}(f)$ in order to obtain a term of

$$\prod_{z_1, z_2 : Z} T(z_1, z_2) \to R(g(z_1), h(z_2))$$

which state that $\lfloor g \rceil = \lfloor h \rceil$. Vice versa, if $\lfloor f \rceil$ is a monomorphism, then we can use left cancellability of $\lfloor f \rceil$ with the arrows obtained through the composition of the arrow

$$\lfloor \pi \rceil : (\sum_{x_1, x_2 : X} S(f(x_1), f(x_2)), (R \boxtimes R)^*) \to (X \times X, R \boxtimes R)$$

with the projections $\lfloor \pi_i \rceil : (X \times X, R \boxtimes R) \to (X, R)$, for $i = 1, 2$, where $(R \boxtimes R)^*$ is defined as in (2.5). Hence, we obtain $\lfloor \pi_1 \rceil \lfloor \pi \rceil = \lfloor \pi_2 \rceil \lfloor \pi \rceil$. Since

$$\lfloor \pi \rceil : (\sum_{x_1, x_2 : X} R(x_1, x_2), (R \boxtimes R)^*) \to (X \times X, R \boxtimes R)$$

is an equalizer of $\lfloor \pi_1 \rceil$ and $\lfloor \pi_2 \rceil$, we obtain an arrow

$$h : \sum_{x_1, x_2 : X} S(f(x_1), f(x_2)) \to \sum_{x_1, x_2 : X} R(x_1, x_2).$$

From this, we can extract a term of $\mathsf{inj}(f)$. □

Remark 2.1.16. We have now completed the discussion about the epimorphisms and monomorphisms of **Std** and $\mathbf{Std}_0$ and we have proved some relations with the arrow for which there is a witness of the surjectivity (2.7) and of the injectivity (2.11). If we consider the arrows of **ML** or $\mathbf{ML}_0$ and the types

$$\mathsf{surj}(f) := \prod_{y:Y}\sum_{x:X} \mathsf{Id}_Y(f(x), y) \qquad \mathsf{inj}(f) := \prod_{x_1,x_2:X} \mathsf{Id}_Y(f(x_1), f(x_2)) \to \mathsf{Id}_X(x_1, x_2),$$

then, since we are considering arrows up to function extensionality, a trivial verification shows that if an arrow is an injection (surjection), then it is also a monomorphism (epimorphism). Vice versa, the argument of the proof of Proposition 2.1.15 can be used to show that every monomorphism is an injection in **ML** and $\mathbf{ML}_0$.

At this point we ask if all the equivalence relations in $\mathbf{Std}_0$ are effective as it happens for the pretopos **Std**. Unfortunately, the answer is negative due the homotopy restrictions on the types. We now recall the proof of this fact for **Std** that can be found for instance in [MP00], in order to underline the problems which occur in $\mathbf{Std}_0$.

Proposition 2.1.17. *In **Std** every equivalence relation is effective.*

Proof. Let $\lfloor r_1\rceil, \lfloor r_2\rceil : (Y, S) \to (X, R)$ be an equivalence relation on (X, R). Define the dependent type

$$x_1, x_2 : X \vdash \overline{R}(x_1, x_2) := \sum_{y:Y} R(r_1(y), x_1) \times R(r_2(y), x_2). \tag{2.12}$$

Since $\lfloor r_1\rceil, \lfloor r_2\rceil$ form an external equivalence relation, we can extract the terms witnessing that the above type is an equivalence relation in the type theoretic sense. Moreover, using that $\lfloor r_1\rceil, \lfloor r_2\rceil$ are jointly monomorphic, we can prove that (Y, S) is isomorphic to the domain of the kernel pair of the arrow

$$\lfloor 1_X\rceil : (X, R) \to (X, \overline{R}).$$

□

Unfortunately, the above argument does not work in $\mathbf{Std}_0$. Indeed, the relation in (2.12) is not necessarily an h-proposition, and then not every equivalence relation in $\mathbf{Std}_0$ has a coequalizer. This should not be surprising since, as we mentioned in Section 1.1, **Std** is equivalent to the exact completion of the category **ML** and, as observed in Remark 1.1.6, in **ML** the internal and the external notion of equivalence relation coincide. For $\mathbf{ML}_0$ this correspondence occurs with some restrictions.

Remark 2.1.18. If X is an h-set and $x, y : X \vdash R(x, y)$ is an equivalence relation which is an h-proposition, then we can consider the arrows between h-sets

$$\pi_1, \pi_2 : \sum_{x,y:X} R(x, y) \to X \tag{2.13}$$

given by the projections on the first two components and obtain a pseudo-equivalence relation. Since $R(x, y)$ is an h-proposition, it has at most one inhabitant and, hence, π_1, π_2 are a jointly monomorphic pair.

Vice versa, given a jointly monomorphic pseudo-equivalence relation $r_1, r_2 : R \to X$, we can consider the dependent type

$$x, y : X \vdash \sum_{z:R} \mathsf{Id}_X(r_1(z), x) \times \mathsf{Id}_X(r_2(z), y) \tag{2.14}$$

and obtain witnesses of the fact that is a type theoretic equivalence relation on X. The type

$$\mathsf{Id}_X(r_1(z), x) \times \mathsf{Id}_X(r_2(z), y)$$

has at most one element with respect to z. Indeed, for every $z, z' : R$ if the type in (2.14) is inhabited, it follows from the fact that $r_1 r_2$ are jointly monomorphic that $\mathsf{Id}_R(z, z')$ is inhabited. Hence, Lemma 2.1.6 implies that (2.14) is an h-proposition.

Intuitively, in $\mathbf{Std}_0$ only some quotients have been added, namely those of the equivalence relations which come from an h-propositions. These equivalence relation are those that arise in practice when formalizing set-based mathematics such as the usual algebraic structures. In the next section, we will make this discussion precise setting the study of homotopy setoids in the categorical setting of the elementary doctrines.

Remark 2.1.19. Another possibility could be to add the *propositional truncation* higher inductive type to the type theory we have considered so far. The propositional truncation of a type, also called *squash type* in [Men90] or *bracket type* [AB04], is an h-proposition with a suitable recursion principle. Adding this type constructor, we could consider the truncation of the type in (2.12). In this way, the category $\mathbf{Std}_0$ becomes exact; we refer to [Uni13] for further details about the propositional truncation.

2.2 Homotopy setoids as elementary quotient completion

In this section, we arrange h-sets and h-propositions in a suitable elementary doctrine and prove that the homotopy setoids are obtained applying the elementary quotient completion to it. We observe the first properties of this doctrine and recover the results of the previous section in this framework. The comparison between setoids and homotopy setoids becomes, in this section, a comparison between the new structure and the elementary doctrine F^{ML} arising from the type theory.

We recall from Example 1.2.8 that the functor

$$F^{ML} : \mathbf{ML}^{op} \to \mathsf{InfSL}$$

which sends a closed type X to the poset of dependent type $x : X \vdash B(x)$ up to logical equivalence, and acts on arrows as substitution of terms, is an elementary doctrine. The identity type $\mathsf{Id}_X \in F^{ML}(X \times X)$ plays the role of the fibered equality.

Remark 2.2.1. Since we are now working with the type theory $\mathcal{ML}$ plus the functional extensionality axiom, we will denote the corresponding elementary doctrine of 1.2.8 with F^{ML^+}. However, the elementary doctrines F^{ML^+} and F^{ML} are very similar and Proposition 1.2.21 holds also for F^{ML^+}. The main difference will be discussed in 2.6.8.

We now define suitable elementary doctrines for every homotopy type. For every subcategory $\mathbf{ML}_n \subseteq \mathbf{ML}$, for $n \geq 0$, we define the functor

$$F^{ML_n} : \mathbf{ML}_n^{op} \to \mathsf{InfSL} \tag{2.15}$$

which sends a type X of homotopy type n to the poset of the dependent types of homotopy type n-1. The action on the arrows is given by substitution on a(ny) representative. The definition of homotopy type implies that the identity type of X has homotopy type n-1 and hence $\mathsf{Id}_X \in F^{ML_n}(X \times X)$ provides the fibered equality. With the notation adopted, the elementary doctrine of homotopy setoids is F^{ML_0}. As discussed in Proposition 1.2.21 F^{ML} is a rich elementary doctrine and we want to prove similar results of F^{ML_0}.

Proposition 2.2.2. *The functor $F^{ML_0} : \mathbf{ML}_0^{op} \to \mathsf{InfSL}$ is an elementary doctrine with full strict comprehensions and comprehensive diagonals. Moreover, if $m : X \to Y$ is a monomorphism, then the functor $F_m^{ML_0}$ has a left adjoint.*

Proof. Remark 2.1.5 implies that h-propositions are preserved by the $\times$ constructor. Moreover, $\mathbf{1}$ is an h-proposition and it is obviously terminal. Hence, h-propositions up to logical equivalence form an inf-semilattice.

The equivalence class of the identity type Id_X provides the elementarity of F^{ML_0}. If $x : X \vdash B(x)$ is an h-proposition depending on the h-set X, then the equivalence class of the projection

$$\pi_1 : \sum_{x:X} B(x) \to X \tag{2.16}$$

provides a full strict comprehension of B. Indeed, given an arrow $f : Y \to X$, the equation $F_f^{ML}(B) = \top_Y$ in the language of the type theory becomes the existence of a term $y : Y \vdash t(y) : B(f(y))$. If we denote with $Z := \sum_{x:X} B(x)$, we can consider the arrow

$$f' : Y \to Z$$

which sends a term $y : Y$ to the pair $(f(y), t(f(y))) : \sum_{x:X} B(x)$. Obviously, $\pi_1 \circ f' = f$. We now prove that the comprehension π is strict because it is a monomorphism. Equivalently, as discussed in Remark 2.1.16, we can prove that it is an injection, which means that the type

$$\mathsf{inj}(\pi_1) := \prod_{z_1, z_2 : Z} \mathsf{Id}_X(\pi_1(z_1), \pi_1(z_2)) \to \mathsf{Id}_Z(z_1, z_2)$$

is inhabited. This follows from the description of the identity type of Z which can be found in [Uni13, Theorem 2.7.2] where it is proved that there is a logical equivalence of the types

$$\mathsf{Id}_Z(z_1, z_2) \equiv \sum_{p : \mathsf{Id}_X(\pi_1 z_1, \pi_1 z_2)} \mathsf{Id}_{B(\pi_2 z_2)}(p^*(\pi_2 z_1), \pi_2 z_2)$$

where p^* is the transport operator, see [Uni13, Lemma 2.3.1] for a detailed definition. However, since B is an h-proposition we can extract a witness of $\mathsf{Id}_{B(\pi_2 z_2)}(p^*(\pi_2 z_1), \pi_2 z_2)$ and hence of $\mathsf{inj}(\pi_1)$.

The comprehensions are trivially full. In order to prove that $F_m^{ML_0} : F^{ML_0}(Y) \to F^{ML_0}(X)$ has a left adjoint, we observe that given an h-proposition $x : X \vdash B(x)$ and an injection $m : X \to Y$, the type

$$B(x) \times \mathsf{Id}_Y(m(x), y)$$

is true for at most one x in the sense of (2.1) and, hence, Lemma 2.1.6 implies that the type

$$y : Y \vdash \sum_{x:X} B(x) \times \mathsf{Id}_Y(m(x), y) \tag{2.17}$$

is an h-proposition. The correspondence which sends a type $B(x) \in F^{ML_0}(X)$ to the type in (2.17) trivially defines a left adjoint of $F_m^{ML_0}$. Finally, diagonals are comprehensive because the arrows of $\mathbf{ML}$ and $\mathbf{ML}_0$ are defined up to functional extensionality. □

The equivalence observed in Example 1.3.8, between the elementary doctrine F^{ML} and the elementary doctrine of weak subobjects $\mathsf{PSub}_{\mathbf{ML}}$ of the weakly left exact category of types **ML**, continues to holds for the elementary doctrine F^{ML^+}.

$$\begin{tikzcd} \mathbf{ML}^{op} \arrow[dd, equal, "Id_{\mathbf{ML}}"'] \arrow[dr, "F^{ML^+}"] & \\ & \mathsf{InfSL}. \\ \mathbf{ML}^{op} \arrow[ur, "\mathsf{Psub}_{\mathbf{ML}}"'] & \end{tikzcd} \qquad \exists_{(-)}\top_{(-)} \leftrightarrows \{\!|-|\!\} \tag{2.18}$$

Using Proposition 2.2.2, we can prove that the elementary doctrine of h-sets F^{ML_0} is equivalent to another elementary doctrine we encountered in the first chapter, namely the elementary doctrine $\mathsf{Sub}_{\mathbf{ML}_0}$ of subobjects of the left exact category $\mathbf{ML}_0$, see Example 1.2.7.

$$\begin{tikzcd} \mathbf{ML}_0^{op} \arrow[dd, equal, "Id_{\mathbf{ML}}"'] \arrow[dr, "F^{ML_0}"] & \\ & \mathsf{InfSL} \\ \mathbf{ML}_0^{op} \arrow[ur, "\mathsf{Sub}_{\mathbf{ML}_0}"'] & \end{tikzcd} \qquad \exists_{(-)}\top_{(-)} \leftrightarrows \{\!|-|\!\} \tag{2.19}$$

Proposition 2.2.3. *The elementary doctrine F^{ML_0} is isomorphic to the elementary doctrine $\mathsf{Sub}_{\mathbf{ML}_0}$ of subobjects of $\mathbf{ML}_0$.*

Proof. The functors are well-defined thanks to Proposition 2.2.2. For an h-set $X \in \mathbf{ML}_0$, we have two functors between the posets

$$\{\!|-|\!\} : F^{ML_0}(X) \rightleftarrows \mathsf{Sub}_{\mathbf{ML}_0} : \exists_{(-)}\top_{(-)}.$$

If $x : X \vdash B(x)$ is an h-proposition, and $Z := \sum_{x:X} B(x)$ is the domain of the comprehension $\pi_1 : Z \to X$, then we want to prove that $B(x)$ is logical equivalent to

$$x : X \vdash \sum_{z:Z} \mathsf{Id}_X(\pi_1(z), x)$$

which is obvious. Vice versa, if $m : Y \rightarrowtail X$ is a monomorphism, then the left adjoint $\exists_m \top_Y$ is given by the fibers

$$\mathsf{fib}_m(x) := \sum_{y:Y} \mathsf{Id}_X(m(y), x)$$

and the strict comprehension is given by

$$\pi_1 : \sum_{x:X} \mathsf{fib}_m(x) \to X.$$

The arrow $h : Y \to \sum_{x:X} \mathsf{fib}_m(x)$, which sends a term $y : Y$ to $h(y) := (m(y), y, \mathsf{refl}(m(y)))$ and the arrow $\pi_2 : \sum_{x:X} \mathsf{fib}_m(x) \to Y$ are such that $m\pi_2 = \pi_1$ and $\pi_1 h = m$. Hence, m and π_1 are in the same equivalence class. $\square$

A F^{ML_0}-eq. relation on an h-set X is exactly the equivalence class of an equivalence relation $x_1, x_2 : X \vdash R(x_1, x_2)$, which is h-proposition, up to logical equivalence. This observation and the above proposition recover the correspondence already observed in Remark 2.1.18 between the internal and external notion of equivalence relation in $\mathbf{ML}_0$.

Applying the elementary quotient completion to F^{ML_0} we obtain the elementary doctrine

$$\overline{F^{ML_0}} : \overline{\mathbf{ML}_0}^{op} \to \mathsf{InfSL}$$

and, by construction, we obtain the equivalence

$$\mathbf{Std}_0 \cong \overline{\mathbf{ML}_0}. \tag{2.20}$$

We recall that, since elementary quotient completion of the elementary doctrine of weak subobjects of a weakly left exact category is equivalent to the exact completion, we obtain for the setoids the equivalences

$$\mathbf{Std} \cong \overline{\mathbf{ML}} \cong \mathbf{ML}_{ex}. \tag{2.21}$$

Applying Proposition 1.3.6, which state the regularity of the base category of suitable doctrines, to $\overline{F^{ML_0}}$, we recover the regularity of the category of h-setoids already proved in Proposition 2.1.9.

Remark 2.2.4. The equivalence in eq. (2.20) can be considered as a solution to the problem of understanding if $\mathbf{Std}_0$ has well-behaved quotients. Thanks to 1.3.3, we obtain that $\mathbf{ML}_0$ has stable effective quotients of $\overline{F^{ML_0}}$-equivalence relations and of effective descent.

We can now pursue the study of the categorical properties of h-setoids but, instead of working directly on the category $\mathbf{Std}_0$, we will study them as the elementary quotient completion of F^{ML_0}. In the next sections, we will provide the conditions on an elementary doctrine P which are equivalent to the (local) cartesian closure and to the extensivity of the base category of $\overline{\mathsf{P}}$. Doing so, the results obtained will be usable for all the setoids built over the type theories that can be resembled in suitable elementary doctrines.

2.3 (Locally) cartesian closed elementary quotient completion

In this section, we give the conditions on a suitable elementary doctrine P such that the base category of $\overline{\mathsf{P}}$ is (locally) cartesian closed. In order to do that, we will take advantages from the results obtained by Carboni and Rosolini [CR00] and Emmenegger [Emm20] about the (local) cartesian closure of the exact completion. A similar result about the local cartesian closure of the elementary quotient completion can be found in [MPR21], in the next section we will discuss the differences between our result and the one in *loc. cit.*.

We first recall the weak notion of cartesian closure in case of categories with strict finite products.

Definition 2.3.1. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine. The base category $\mathscr{C}$ is said *weakly cartesian closed* if for every pair of objects $X, Y \in \mathscr{C}$ there exists an object Y^X and an arrow $e : Y^X \times X \to Y$ satisfying the following weak universal property: for every arrow $f : Z \times X \to Y$ there exists an arrow $h : Z \to Y^X$ such that $e \circ (h \times 1_X) = f$. The arrow e is usually called a *weak evaluation*.

In case of the elementary doctrine of weak subobjects $\mathsf{PSub}_{\mathscr{C}}$, the above definition coincides with the notion of weak exponential given in [CR00].

We now provide the first result about the cartesian closure of the elementary quotient completion.

Theorem 2.3.2. *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be a universal elementary doctrine with full weak comprehensions. The category* $\mathscr{C}$ *is weakly cartesian closed if and only if* $\overline{\mathscr{C}}$ *is cartesian closed.*

Proof. The cartesian closure of $\overline{\mathscr{C}}$ trivially implies the weak cartesian closure of $\mathscr{C}$. Vice versa, we first prove the statement for two objects of the form (X, δ_X) and (Y, σ). Since $\mathscr{C}$ is weakly cartesian closed, there exists an object Y^X and a weak evaluation arrow

$$e_X : Y^X \times X \to Y.$$

We define the following element of $\mathsf{P}(Y^X \times Y^X)$

$$\varepsilon_\sigma^X := \forall_{\langle 1,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}\mathsf{P}_{e_X \times e_X}(\sigma) \tag{2.22}$$

where $\langle 1,3,2,3\rangle : Y^X \times Y^X \times X \to Y^X \times X \times Y^X \times X$ is the arrow induced by the obvious projections. Since σ is a P-eq. relation, it is straightforward to prove that also ε_σ^X is P-eq. relation. We now prove that the object of $\overline{\mathscr{C}}$

$$(Y^X, \varepsilon_\sigma^X)$$

is a strict exponential of (X, δ_X) and (Y, σ). In order to do that, we first prove that the weak evaluation arrow e_X induces an arrow

$$\lfloor e_X \rceil : (Y^X, \varepsilon_\sigma^X) \times (X, \delta_X) \to (Y, \sigma)$$

which means that

$$\varepsilon_\sigma^X \boxtimes \delta_X \leq \mathsf{P}_{e_X \times e_X}\sigma.$$

The above inequality is obtained through the following computation

$$\begin{aligned} \varepsilon_\sigma^X \boxtimes \delta_X &= \forall_{\langle 1,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}P_{e_Y \times e_Y}(\sigma) \boxtimes \delta_X \\ &= \mathsf{P}_{\langle 1,3\rangle}\forall_{\langle 1,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}P_{e_Y \times e_Y}(\sigma) \wedge \mathsf{P}_{\langle 2,4\rangle}\delta_X \\ &\leq \mathsf{P}_{\langle 1,3,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}P_{e_Y \times e_Y}(\sigma) \wedge \mathsf{P}_{\langle 2,4\rangle}\delta_X \end{aligned}$$

where $\langle 1,3,2\rangle : Y^X \times X \times Y^X \times X \to Y^X \times Y^X \times X$. The inequality

$$\mathsf{P}_{\langle 1,3,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}P_{e_Y \times e_Y}(\sigma) \wedge \mathsf{P}_{\langle 2,4\rangle}\delta_X \leq \mathsf{P}_{e_X \times e_X}\sigma$$

is an immediate consequence of elementarity, indeed it holds the adjunction

$$\mathsf{P}_{\langle 1,3,2\rangle}(-) \wedge \mathsf{P}_{\langle 2,4\rangle}\delta_X \dashv \mathsf{P}_{\langle 1,3,2,3\rangle}(-).$$

The arrow $\lfloor e_X \rceil$ shares the required universal property: if

$$\lfloor f \rceil : (Z, \zeta) \times (X, \delta_X) \to (Y, \sigma)$$

is an arrow in $\overline{\mathscr{C}}$, then e_X implies the existence of an arrow $h : Z \to Y^X$ such that $e_X(h \times 1_X) = f$. We now prove that h induces an arrow

$$\lfloor h \rceil : (Z, \zeta) \to (Y^X, \varepsilon_\sigma^X)$$

such that $\lfloor e_X\rceil(\lfloor h\rceil \times 1_{(X,\delta_X)}) = \lfloor f\rceil$. In order to do that, we prove the inequality $\zeta \leq \mathsf{P}_{h\times h}\varepsilon^X_\sigma$ as follows:

$$\begin{aligned}
\mathsf{P}_{h\times h}\varepsilon^X_\sigma &= \mathsf{P}_{h\times h}\forall_{\langle 1,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}\mathsf{P}_{e_Y\times e_Y}(\sigma) \\
&= \forall_{\langle 1,2\rangle}\mathsf{P}_{h\times h\times 1_X}\mathsf{P}_{\langle 1,3,2,3\rangle}\mathsf{P}_{e_Y\times e_Y}(\sigma) && \text{(B-C)} \\
&= \forall_{\langle 1,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}\mathsf{P}_{h\times 1_X\times h\times 1_X}\mathsf{P}_{e_Y\times e_Y}(\sigma) \\
&= \forall_{\langle 1,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}\mathsf{P}_{f\times f}\sigma \\
&\geq \forall_{\langle 1,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}\zeta \boxtimes \delta_X \\
&\geq \zeta. && (\mathsf{P}_{\langle 1,2\rangle} \dashv \forall_{\langle 1,2\rangle})
\end{aligned}$$

The arrow $\lfloor h\rceil$ such that $\lfloor e_X\rceil(\lfloor h\rceil \times 1_{(X,\delta_X)}) = \lfloor f\rceil$ is unique. Indeed, if $\lfloor h'\rceil$ is a different arrow such that $\lfloor e_X\rceil(\lfloor h'\rceil \times 1_{(X,\delta_X)}) = \lfloor f\rceil$, proceeding as in the above computation we obtain that

$$\zeta \leq \mathsf{P}_{h\times h'}\varepsilon^X_\sigma$$

and then $\lfloor h\rceil = \lfloor h'\rceil$.

For the general case, consider two objects (X,ρ) and (Y,σ) in $\overline{\mathscr{C}}$. The equivalence relation ε^X_σ is not enough to build a strict exponential of (X,ρ) and (Y,σ), but we can use it as follows.

Let $\{\!|\rho|\!\} : R \to X \times X$ be a weak full comprehension of $\rho \in \mathsf{P}(X\times X)$. We fix the notation $r_i := p_i \circ \{\!|\rho|\!\}$ for the post-composition with the projections p_i, for $i = 1,2$. The universal property of the weak evaluation arrow implies the existence of two arrows

$$Y^{r_1}, Y^{r_2} : Y^X \to Y^R$$

such that the the following diagram commutes for $i = 1,2$

$$\begin{tikzcd}
Y^X \times R \arrow[r, "1_{Y^X}\times r_i"] \arrow[d, "Y^{r_i}\times 1_R"'] & Y^X \times X \arrow[d, "e_X"] \\
Y^R \times R \arrow[r, "e_R"'] & Y.
\end{tikzcd} \tag{2.23}$$

Given a weak full comprehension $c : C \to Y^X$ of the element $\mathsf{P}_{\langle Y^{r_1},Y^{r_2}\rangle}\varepsilon^R_\sigma$, we prove that the object

$$(C, \mathsf{P}_{c\times c}\varepsilon^X_\sigma)$$

is a strict exponential of the objects (X,ρ) and (Y,σ).

Firstly, we observe that since ε^X_σ is a P-equivalence relation, so it is $\mathsf{P}_{c\times c}\varepsilon^X_\sigma$. Secondly, we provide an evaluation arrow of the form

$$(C, \mathsf{P}_{c\times c}\varepsilon^X_\sigma) \times (X,\rho) \to (Y,\sigma).$$

In order to do that, we consider the arrow $e_X(c\times 1_X) : C \times X \to Y$ and prove that

$$\mathsf{P}_{c\times c}\varepsilon^X_\sigma \boxtimes \rho \leq \mathsf{P}_{e_X(c\times 1_X)\times e_X(c\times 1_X)}\sigma. \tag{2.24}$$

The description of comprehensions in the elementary quotient completion, see Lemma A.0.14, implies that the arrow

$$\lfloor c\rceil : (C, \mathsf{P}_{c\times c}\varepsilon^X_\sigma) \to (Y^X, \varepsilon^X_\sigma)$$

is a full strict comprehension of $\mathsf{P}_{\langle Y^{r_1},Y^{r_2}\rangle}\varepsilon^R_\sigma \in \mathcal{D}es_{\varepsilon^X_\sigma}$, and that the arrow

$$\lfloor \{\!|\rho|\!\} \rceil : (R, \mathsf{P}_{\{\!|\rho|\!\}\times\{\!|\rho|\!\}}\delta_{X\times X} \boxtimes \delta_{X\times X}) \to (X\times X, \delta_{X\times X})$$

is a full strict comprehension of $\rho \in \mathcal{D}es_{\delta_{X\times X}}$. Hence, **??** implies that the arrow

$$\langle 1_{(C,\mathsf{P}_{c\times c}\varepsilon^X_\sigma)} \times \lfloor r_1 \rceil, 1_{(C,\mathsf{P}_{c\times c}\varepsilon^X_\sigma)} \times \lfloor r_2 \rceil\rangle$$

where $\lfloor r_i \rceil := \lfloor p_i \rceil \circ \lfloor \{\!|\rho|\!\} \rceil$ are given by the post-composition of $\lfloor \{\!|\rho|\!\} \rceil$ with the projections $\lfloor p_i \rceil : (X\times X, \delta_{X\times X}) \to (X, \delta_X)$, is a strict comprehension of $\delta_{(C,\mathsf{P}_{c\times c}\varepsilon^X_\sigma)} \boxtimes \rho$. The inequality (2.24) is equivalent to

$$\delta_{(C,\mathsf{P}_{c\times c}\varepsilon^X_\sigma)} \boxtimes \rho \le \overline{\mathsf{P}}_{(\lfloor c\rceil \times 1_{(X,\delta_X)})\times(\lfloor c\rceil \times 1_{(X,\delta_X)})}\overline{\mathsf{P}}_{\lfloor e_X\rceil\times\lfloor e_X\rceil}\delta_{(Y,\sigma)}$$

and, by fullness of comprehensions, it is equivalent to prove that

$$\top \le \overline{\mathsf{P}}_{\{\!|\delta_{(C,\mathsf{P}_{c\times c}\varepsilon^X_\sigma)}\boxtimes\rho|\!\}}\overline{\mathsf{P}}_{(\lfloor c\rceil \times 1_{(X,\delta_X)})\times(\lfloor c\rceil \times 1_{(X,\delta_X)})}\overline{\mathsf{P}}_{\lfloor e_x\rceil\times\lfloor e_x\rceil}\delta_{(Y,\sigma)}$$

Since quotients are effective we can prove the above inequality in

$$\overline{\mathsf{P}}((C,\mathsf{P}_{c\times c}\varepsilon^X_\sigma)\times(R,\delta_R)),$$

i.e. reindexing through the quotient arrow

$$1\times q : (C,\mathsf{P}_{c\times c}\varepsilon^X_\sigma)\times(R,\delta_R) \to (C,\mathsf{P}_{c\times c}\varepsilon^X_\sigma)\times(R,\mathsf{P}_{\{\!|\rho|\!\}\times\{\!|\rho|\!\}}\delta_{X\times X}\boxtimes\delta_{X\times X})$$

where $q : (R,\delta_R) \to (R,\mathsf{P}_{\{\!|\rho|\!\}\times\{\!|\rho|\!\}}\delta_{X\times X}\boxtimes\delta_{X\times X})$ is the obvious quotient arrow. The statement now follows from the commutativity of the following diagram for $i = 1, 2$

$$\begin{tikzcd}
(C,\mathsf{P}_{c\times c}\varepsilon^X_\sigma)\times(R,\delta_R) \arrow[r, "1\times\lfloor r_i\rceil q"] \arrow[d, "\lfloor c\rceil\times 1"'] & (C,\mathsf{P}_{c\times c}\varepsilon^X_\sigma)\times(X,\delta_X) \arrow[d, "\lfloor c\rceil\times 1"] \\
(Y^X,\varepsilon^X_\sigma)\times(R,\delta_R) \arrow[r, "1\times\lfloor r_i\rceil q"] \arrow[d, "\lfloor Y^{r_i}\rceil\times 1"'] & (Y^X,\varepsilon^X_\sigma)\times(X,\delta_X) \arrow[d, "\lfloor e_X\rceil"] \\
(Y^R,\varepsilon^R_\sigma)\times(R,\delta_R) \arrow[r, "\lfloor e_R\rceil"'] & (Y,\sigma).
\end{tikzcd}$$

We now prove that the evaluation arrow just found

$$\lfloor e_X(c\times 1_X)\rceil : (C,\mathsf{P}_{c\times c}\varepsilon^X_\sigma)\times(X,\rho)\to(Y,\sigma)$$

has the required strict universal property. If $\lfloor f\rceil : (Z,\zeta)\times(X,\rho)\to(Y,\sigma)$ is an arrow, the weak evaluation e_X implies the existence of an arrow $h : Z\to Y^X$ such that $e_X(h\times 1_X) = f$. Applying the Beck-Chevalley conditions we obtain that

$$\mathsf{P}_{h\times h}\mathsf{P}_{Y^{r_1}\times Y^{r_2}}\varepsilon^R_\sigma = \cdots = \forall_{\langle 1,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}\mathsf{P}_{1_Z\times r_1\times 1_Z\times r_2}\mathsf{P}_{f\times f}\sigma$$

and, since $\zeta\boxtimes\rho\le\mathsf{P}_{f\times f}\sigma$, it follows that

$$\begin{aligned}
&\mathsf{P}_{h\times h}\mathsf{P}_{Y^{r_1}\times Y^{r_2}}\varepsilon^R_\sigma\\
&= \forall_{\langle 1,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}\mathsf{P}_{1_Z\times r_1\times 1_Z\times r_2}\mathsf{P}_{f\times f}\sigma\\
&\ge \forall_{\langle 1,2\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}\mathsf{P}_{1_Z\times r_1\times 1_Z\times r_2}\zeta\boxtimes\rho\\
&\ge \forall_{\langle 1,2\rangle}\mathsf{P}_{\langle 1,2\rangle}\zeta && (\mathsf{P}_{\langle 1,2\rangle}\dashv\forall_{\langle 1,2\rangle})\\
&\ge \zeta.
\end{aligned}$$

Hence, applying P_{Δ_Z}, we obtain

$$\top \leq \mathsf{P}_h \mathsf{P}_{\langle Y^{r_1}, Y^{r_2}\rangle} \varepsilon_\sigma^R$$

and the universal property of weak comprehensions implies the existence of an arrow $g : Z \to C$ such that $c \circ g = h$. The arrow g induces an arrow

$$\lfloor g \rceil : (Z, \zeta) \to (C, \mathsf{P}_{c\times c}\varepsilon_\sigma^X)$$

such that $\lfloor f \rceil = \lfloor e_X(c \times 1_X) \rceil (\lfloor g \rceil \times 1_{(X,\rho)})$. An easy verification shows that the arrow $\lfloor g \rceil$ with such properties is unique.

□

Remark 2.3.3. A result similar to Theorem 2.3.2 is [MR13, Proposition 6.7]. The results are very similar but the authors assumes implications in place of fullness of comprehensions.

Before proceeding with the local cartesian closure of the elementary quotient completion we make some observations.

If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an elementary doctrine with weak comprehension and comprehensive diagonals, we can build weak pullbacks through weak comprehensions. Indeed, given two arrows $x : X \to A$ and $y : Y \to A$ in $\mathscr{C}$ the following diagram is a weak pullback for x and y

$$\begin{tikzcd} C \arrow[rr, dashed, "\{\!|\gamma|\!\}_2"] \arrow[dr, "\{\!|\gamma|\!\}"] \arrow[dd, dashed, "\{\!|\gamma|\!\}_1"'] & & Y \arrow[dd, "y"] \\ & X \times Y \arrow[ur, "p_2"'] \arrow[dl, "p_1"] & \\ X \arrow[rr, "x"'] & & A \end{tikzcd} \tag{2.25}$$

where $\gamma := \mathsf{P}_{x\times y}\delta_A$ and $\{\!|\gamma|\!\}_i := p_i\{\!|\gamma|\!\}$, for $i = 1, 2$. Vice versa given a weak pullback of the arrows x and y

$$\begin{tikzcd} C \arrow[rr, "\pi_2"] \arrow[dr, dashed, "\langle \pi_1,\pi_2\rangle"] \arrow[dd, "\pi_1"'] & & Y \arrow[dd, "y"] \\ & X \times Y \arrow[ur, "p_2"'] \arrow[dl, "p_1"] & \\ X \arrow[rr, "x"'] & & A \end{tikzcd} \tag{2.26}$$

the arrow $\langle \pi_1, \pi_2\rangle$ is a weak comprehension of γ. A proof of this correspondence can be found in Lemma A.0.7 and Lemma A.0.8. In light of this, we obtain that any slice category $\mathscr{C}/A$ has weak binary products given by the common value of the two composites of (2.25) and, without loss of generality, we can assume that a weak product of the objects $x : X \to A, y : Y \to A$ of $\mathscr{C}/A$ is built through the weak comprehension of $\gamma := \mathsf{P}_{x\times y}\delta_A$. Similarly, if x, y and $z : Z \to A$ are three arrows of $\mathscr{C}$ then a weak limit of x, y, z is given by a weak comprehension of

$$\gamma := \mathsf{P}_{x\times y\times z}(\mathsf{P}_{\langle 1,2\rangle}\delta_A \wedge \mathsf{P}_{\langle 2,3\rangle}\delta_A)$$

$$\begin{tikzcd} & T \arrow[dl, dashed, "\{\!|\gamma|\!\}_1"'] \arrow[d, dashed, "\{\!|\gamma|\!\}_2"'] \arrow[dr, dashed, "\{\!|\gamma|\!\}_1"] & \\ X \arrow[dr, "x"'] & Y \arrow[d, "y"'] & Z \arrow[dl, "z"] \\ & A & \end{tikzcd}$$

where $\{\!|\gamma|\!\}_i := p_i\{\!|\gamma|\!\}$, and the p_i are the projections of the product $X \times Y \times Z$ for $i = 1, 2, 3$.

The existence of weak finite products in the slices $\mathscr{C}/A$ is related to the existence of strict pullbacks in the slices $\mathscr{C}/(A, \delta_A)$ of the elementary quotient completion. Indeed, if the diagram 2.26 is a weak pullback and $\rho \in \mathsf{P}(X \times X)$ and $\sigma \in \mathsf{P}(Y \times Y)$ are P-eq. relations on X and Y such that $\rho \leq \mathsf{P}_{x\times x}\delta_A$ and $\sigma \leq \mathsf{P}_{y\times y}\delta_A$ the the following diagram is a strict pullback

$$\begin{tikzcd}
(C, \mathsf{P}_{\{\!|\gamma|\!\}\times\{\!|\gamma|\!\}}(\rho \boxtimes \sigma)) \arrow[r, "\lfloor \pi_2 \rceil"] \arrow[d, "\lfloor \pi_1 \rceil"'] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, \sigma) \arrow[d, "\lfloor y \rceil"] \\
(X, \rho) \arrow[r, "\lfloor x \rceil"'] & (A, \delta_A).
\end{tikzcd} \tag{2.27}$$

Actually, strict finite products exist in every slice (not only for the slices over objects of the form (A, δ_A)) and this is an immediate consequence of the construction of comprehensions in $\overline{\mathscr{C}}$, see Lemma A.0.14.

We now provide a preliminary definition before that of *extensional exponentials*.

Definition 2.3.4. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine with weak comprehensions and comprehensive diagonals. Given three objects $x : X \to A$, $y : Y \to A$ and $z : Z \to A \in \mathscr{C}/A$ and a weak product w of x and y as in (2.25), an arrow $h : w \to z$ *preserves projections* with respect to a P-eq. relation $\sigma \in \mathsf{P}(Z \times Z)$, such that $\sigma \leq \mathsf{P}_{z\times z}\delta_A$, if

$$\mathsf{P}_{\langle \pi_1, \pi_2 \rangle \times \langle \pi_1, \pi_2 \rangle}(\delta_X \boxtimes \delta_Y) \leq \mathsf{P}_{h\times h}(\sigma),$$

where $\gamma := \mathsf{P}_{x\times y}\delta_A$.

We now translate the notion of *extensional exponential* introduced by Emmenegger in [Emm20] in the language of the elementary doctrines.

Definition 2.3.5. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine with weak comprehensions and comprehensive diagonals. If $A \in \mathscr{C}$, the slice $\mathscr{C}/A$ has an *extensional exponential* of $x : X \to A$ and $y : Y \to A$ with respect to a P-eq. relation $\sigma \in \mathsf{P}(Y \times Y)$, if there exist an object y^x and an arrow $e : u \to y$ from a weak product $y^x \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} x$ such that

(i) The evaluation e preserves projections w.r.t. σ.

(ii) For every object $z : Z \to A$ and arrow $f : u' \to y$ from a weak product $z \overset{\pi_1}{\leftarrow} u' \overset{\pi_2}{\rightarrow} x$ that preserves projections w.r.t. σ, there exist two arrows l, m making the following diagram commute:

$$\begin{tikzcd}
z \arrow[d, dashed, "l"] & u' \arrow[l] \arrow[d, dashed, "m"] \arrow[dr, bend left] \arrow[dd, bend right, "f"'] & \\
y^x & u \arrow[l] \arrow[r] \arrow[d, "e"] & x \\
& y &
\end{tikzcd}$$

The slice $\mathscr{C}/A$ has P-extensional exponentials if for every pair of objects maps $x, y \in \mathscr{C}/A$ and P-eq. relation $\sigma \in \mathsf{P}(Y \times Y)$ there exists an extensional exponential of x and y with respect to σ.

Remark 2.3.6. In case of the elementary doctrine of weak subobjects $\mathsf{PSub}_{\mathscr{C}}$, the above definition coincides with the one given by Emmenegger in [Emm20] for categories with strict finite products and weak pullbacks.

We now state the main theorem of the section.

Theorem 2.3.7. *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be an existential and universal elementary doctrine with weak full comprehensions and comprehensive diagonals. The following are equivalent:*

(i) *Every slice* $\mathscr{C}/A$ *has* P*-extensional exponentials and* P *has right adjoints to weak pullback projections,*

(ii) $\overline{\mathscr{C}}$ *is locally cartesian closed.*

Before proving Theorem 2.3.7 we make some considerations. In order to prove that (i) implies (ii) we have to prove that the slices of $\overline{\mathscr{C}}$ are cartesian closed. Unfortunately, we are not able to apply Theorem 2.3.2 for several reasons that will be discussed in detail in Remark 2.6.14. For instance, there is no possibility to consider an elementary doctrine over the slices of $\mathscr{C}$ whose elementary quotient completion are the slices of $\overline{\mathscr{C}}$. This happens because the above assumptions only imply the existence of weak finite products in $\mathscr{C}/A$, as we observed in (2.25). Hence, in order to prove the statement, we adapt the methods used in the proof of Theorem 2.3.2 for the slices of the form $\overline{\mathscr{C}}/(A, \delta_A)$.

The first step consists in providing a strict exponential of two objects of the form $\lfloor x \rceil : (X, \delta_X) \to (A, \delta_A)$ and $\lfloor y \rceil : (Y, \sigma) \to (A, \delta_A)$. Following the proof of Theorem 2.3.2, where a strict exponential of two objects (X, δ_X) and (Y, σ) was obtained through a weak exponential Y^X and the P-eq. relation

$$\varepsilon^X_\sigma := \forall_{\langle 1,2 \rangle} P_{\langle 1,3,2,3 \rangle} P_{e_Y \times e_Y}(\sigma), \tag{2.28}$$

we adapt this construction using an extensional exponential y^x, of two object x, y of the slice $\mathscr{C}/A$, and a suitable equivalence relation in style of (2.28). The second step is to use comprehensions to build a strict exponential of two objects of the form $(X, \rho) \to (A, \delta_A)$ and $(Y, \sigma) \to (A, \delta_A)$.

In order to explicit this construction for the slice $\overline{\mathscr{C}}/(A, \delta_A)$, we first fix some notations. Let $x : X \to A$ and $y : Y \to A$ two objects of $\mathscr{C}/A$ such that $\sigma \leq \mathsf{P}_{y \times y}\delta_A$ and let $y^x : E \to A$ be an extensional exponential of x and y w.r.t. σ. We consider the following weak pullbacks

$$\begin{tikzcd} W \arrow[r, "\{\!|\chi|\!\}_2"] \arrow[d, "\{\!|\chi|\!\}_1"'] & X \arrow[d, "x"] \\ X \arrow[r, "x"'] & A \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\{\!|\iota|\!\}_2"] \arrow[d, "\{\!|\iota|\!\}_1"'] & Y \arrow[d, "y"] \\ Y \arrow[r, "y"'] & A \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\{\!|\gamma|\!\}_2"] \arrow[d, "\{\!|\gamma|\!\}_1"'] & E \arrow[d, "y^x"] \\ E \arrow[r, "y^x"'] & A \end{tikzcd}$$

obtained as in (2.25) through the comprehensions $\{\!|\chi|\!\} : W \to X \times X$, $\{\!|\iota|\!\} : V \to Y \times Y$ and $\{\!|\lambda|\!\} : G \to E \times E$ of $\chi := \mathsf{P}_{x \times x}\delta_A$, $\iota := \mathsf{P}_{y \times y}\delta_A$ and $\gamma := \mathsf{P}_{y^x \times y^x}\delta_A$. We will denote by $w : W \to A$, $v : V \to A$ and $g : G \to A$ respectively the common value of the two composites in the left, central and right above diagram. Moreover, we consider the weak pullback

$$\begin{tikzcd} U \arrow[r, "\{\!|\mu|\!\}_2"] \arrow[d, "\{\!|\mu|\!\}_1"'] & X \arrow[d, "x"] \\ E \arrow[r, "y^x"'] & A \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "\{\!|\kappa|\!\}_2"] \arrow[d, "\{\!|\kappa|\!\}_1"'] & U \arrow[d, "u"] \\ U \arrow[r, "u"'] & A \end{tikzcd}$$

obtained through the comprehensions $\{\!|\mu|\!\} : U \to E \times X$ and $\{\!|\kappa|\!\} : K \to U \times U$ of $\mu := \mathsf{P}_{y^x \times x}\delta_A$ and $\kappa := \mathsf{P}_{u \times u}\delta_A$, where $u : U \to A$ is the common value of the two composites of the left diagram and $k : K \to A$ is that of the right diagram. Now, given a weak product of y^x, y^x and x

$$\begin{tikzcd}
& T \arrow[dl, dashed, "\{\!|\tau|\!\}_1"'] \arrow[d, dashed, "\{\!|\tau|\!\}_2"] \arrow[dr, dashed, "\{\!|\tau|\!\}_3"] & \\
E \arrow[dr, "y^x"'] & E \arrow[d, "y^x"'] & X \arrow[dl, "x"] \\
& A &
\end{tikzcd}$$

obtained through the weak comprehension $\{\!|\tau|\!\} : T \to E \times E \times X$ of $\tau := \mathsf{P}_{y^x \times y^x \times x}(\mathsf{P}_{\langle 1,2\rangle}\delta_A \wedge \mathsf{P}_{\langle 2,3\rangle}\delta_A)$, we will denote by $t : T \to A$ the common value of the three composites of the above diagram. If $u{\leftarrow}z{\rightarrow}u$ is a weak product, then we will denote by

$$\langle 1,3,2,3\rangle_A : t \to k \qquad\qquad \langle 1,2\rangle_A : t \to g$$

the two arrows induced by the obvious projections and by

$$e \times_A e : k \to v$$

the arrow induced by the weak evaluation $e : u \to y$. As observed in Remark 1.2.14 and Remark 1.2.19, the functors P_f have left and right adjoints. Hence, we can define the element of $\mathsf{P}(E \times E)$

$$\varepsilon^x_\sigma := \exists_{\{\!|\gamma|\!\}}\forall_{\langle 1,2\rangle_A}\mathsf{P}_{\langle 1,3,2,3\rangle_A}\mathsf{P}_{e\times_A e}\mathsf{P}_{\{\!|\iota|\!\}}\sigma \tag{2.29}$$

which provides the corresponding of the P-eq. relation in (2.28) for the extensional exponential y^x. At this point, it only remains to prove that the object $\lfloor y^x \rceil : (Y, \varepsilon^x_\sigma) \to (A, \delta_A)$ is a strict exponential of $\lfloor x \rceil : (X, \delta_X) \to (A, \delta_A)$ and $\lfloor y \rceil : (Y, \sigma) \to (A, \delta_A)$.

However, thanks to Remark 1.2.19 we can use implications to give an handier description of ε^x_σ. Indeed, in appendix in Lemma A.0.19 we give the proof of the equality of the terms

$$\begin{aligned}
\varepsilon^x_\sigma &= \forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\delta_X \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma) \wedge \gamma \\
&= \forall_{\langle 1,3\rangle}\forall_{\{\!|\mu|\!\}^2}(\mathsf{P}_{\{\!|\mu|\!\}^2}(\mathsf{P}_{\langle 2,4\rangle}\delta_X \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \mathsf{P}_{e\times e}\sigma) \wedge \gamma.
\end{aligned} \tag{2.30}$$

Where $\langle 1,3\rangle : E \times X \times E \times X \to E \times E$ is the arrow induced by the obvious projections and $\{\!|\mu|\!\}^2 := \{\!|\mu|\!\} \times \{\!|\mu|\!\}$. The proof of this fact uses the Beck-Chevalley condition for particular diagrams. This properties will be used in the rest of the chapter and, when they occur, we have referred to various results that are proved in the appendix.

We are now ready to prove one of the implications of Theorem 2.3.7. The proof follows the ideas developed in the proof of [Emm20, Theorem 3.6].

Proof of Theorem 2.3.7. $(i) \Rightarrow (ii)$ Using the notation developed in the above discussion, we first prove that the slices of the form $\overline{\mathscr{C}}/(A, \delta_A)$ are cartesian closed.

Consider two objects of $\overline{\mathscr{C}}/(A, \delta_A)$ of the form $\lfloor x \rceil : (X, \rho) \to (A, \delta_A)$ and $\lfloor y \rceil : (Y, \sigma) \to (A, \delta_A)$ (hence we are assuming that $\rho \le \chi$ and $\sigma \le \gamma$) and an extensional exponential $y^x : E \to A$ of x and y w.r.t σ. We define the element of $\mathsf{P}(E \times E)$

$$\varepsilon^\rho_\sigma := \forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\rho \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \forall_{\{\!|\mu|\!\}\times\{\!|\mu|\!\}}\mathsf{P}_{e\times e}\sigma) \tag{2.31}$$

and consider a weak comprehension $\{\!|\mathsf{P}_{\Delta_E}\varepsilon^\rho_\sigma|\!\} : C \to E$ of $\mathsf{P}_{\Delta_E}\varepsilon^\rho_\sigma$. If we denote by c the object of $\mathscr{C}/A$ given by the composition $y^x \circ \{\!|\mathsf{P}_{\Delta_E}\varepsilon^\rho_\sigma|\!\}$, then we can prove that the object of $\overline{\mathscr{C}}/(A, \delta_A)$

$$\lfloor y \rceil^{\lfloor x \rceil} := \lfloor c \rceil : (C, \omega) \to (A, \delta_A) \tag{2.32}$$

is a strict exponential of $\lfloor x \rceil$ and $\lfloor y \rceil$, where ω denotes for short $\mathsf{P}_{\{\!|\mathsf{P}_{\Delta_E}\varepsilon^{\rho}_{\sigma}|\!\}\times\{\!|\mathsf{P}_{\Delta_E}\varepsilon^{\rho}_{\sigma}|\!\}}\varepsilon^{x}_{\sigma}$. Indeed, the weak pullbacks

$$\begin{array}{ccc} U' & \xrightarrow{\{\!|\mu'|\!\}_2} & X \\ {\scriptstyle \{\!|\mu'|\!\}_1}\downarrow & & \downarrow{\scriptstyle x} \\ C & \xrightarrow[c]{} & A \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{\{\!|\mu|\!\}_2} & X \\ {\scriptstyle \{\!|\mu|\!\}_1}\downarrow & & \downarrow{\scriptstyle x} \\ E & \xrightarrow[y^x]{} & A \end{array}$$

where $\mu' := \mathsf{P}_{c\times x}\delta_A$, induce an arrow $h : U' \to U$ which makes the obvious diagram commute. If $u' : U' \to A$ denotes the common value of the two composites of the left diagram above, then a trivial computation shows that u' induces an object

$$\lfloor u' \rceil : (U', \mathsf{P}_{\{\!|\mu'|\!\}\times\{\!|\mu'|\!\}}(\omega \boxtimes \rho)) \to (A, \delta_A)$$

of $\overline{\mathscr{C}}/(A, \delta_A)$. The evaluation arrow

$$\lfloor e' \rceil : \lfloor y \rceil^{\lfloor x \rceil} \times \lfloor x \rceil \to \lfloor y \rceil$$

is given by the composition $e' := e \circ h : U' \to Y$.

Now consider an object $\lfloor z \rceil : (Z, \zeta) \to (A, \delta_A)$ of $\overline{\mathscr{C}}/(A, \delta_A)$ and an arrow

$$\lfloor f \rceil : \lfloor z \rceil \times \lfloor x \rceil \to \lfloor y \rceil. \tag{2.33}$$

The weak evaluation arrow e induces two arrows $l : z \to y^x$ and $m : n \to u$, where n is the common value of the two composites of the following weak pullback

$$\begin{array}{ccc} N & \xrightarrow{\{\!|\nu|\!\}_2} & X \\ {\scriptstyle \{\!|\nu|\!\}_1}\downarrow & & \downarrow{\scriptstyle x} \\ Z & \xrightarrow[z]{} & A \end{array}$$

with $\nu := \mathsf{P}_{z\times x}\delta_A$, such that the proper diagram commutes. We now prove that $\top \leq \mathsf{P}_l\mathsf{P}_{\Delta_E}\varepsilon^{\rho}_{\sigma}$ in order to obtain an arrow $l' : Z \to C$ such that $l = \{\!|\mathsf{P}_{\Delta_E}\varepsilon^{\rho}_{\sigma}|\!\} \circ l'$:

$$\begin{aligned} &\mathsf{P}_l\mathsf{P}_{\Delta_E}\varepsilon^{\rho}_{\sigma} \\ &= \mathsf{P}_{\Delta_Z}\mathsf{P}_{l\times l}\forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\rho \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \forall_{\{\!|\mu|\!\}\times\{\!|\mu|\!\}}\mathsf{P}_{e\times e}\sigma) && \text{(def. of } \varepsilon^{\rho}_{\sigma}) \\ &= \mathsf{P}_{\Delta_Z}\forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\rho \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \forall_{\{\!|\nu|\!\}\times\{\!|\nu|\!\}}\mathsf{P}_{m\times m}\mathsf{P}_{e\times e}\sigma) && \text{(def. of } l, m \text{ + B-C)} \end{aligned}$$

since $f = e \circ m$ and, by assumption, the arrow in (2.33) implies $\mathsf{P}_{\{\!|\nu|\!\}\times\{\!|\nu|\!\}}(\zeta \boxtimes \rho) \leq \mathsf{P}_{f\times f}\sigma$, we obtain that

$$\begin{aligned} &\mathsf{P}_{\Delta_Z}\forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\rho \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \forall_{\{\!|\nu|\!\}\times\{\!|\nu|\!\}}\mathsf{P}_{m\times m}\mathsf{P}_{e\times e}\sigma) \\ &\geq \mathsf{P}_{\Delta_Z}\forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\rho \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \forall_{\{\!|\nu|\!\}\times\{\!|\nu|\!\}}\mathsf{P}_{\{\!|\nu|\!\}\times\{\!|\nu|\!\}}(\zeta \boxtimes \rho) \\ &\geq \mathsf{P}_{\Delta_Z}\forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\rho \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow (\zeta \boxtimes \rho)) && (\mathsf{P}_{(-)} \dashv \forall_{(-)}) \\ &\geq \mathsf{P}_{\Delta_Z}\forall_{\langle 1,3\rangle}\mathsf{P}_{\langle 1,3\rangle}\zeta && (\alpha \wedge (-) \dashv \alpha \Rightarrow (-)) \\ &\geq \mathsf{P}_{\Delta_Z}\zeta \\ &\geq \top. \end{aligned}$$

A similar computation implies that the arrow $l' : Z \to C$ induces an arrow

$$\lfloor l' \rceil : (Z, \zeta) \to (C, \omega)$$

and, hence, we obtain an arrow of $\overline{\mathscr{C}}/(A,\delta_A)$ of the form

$$\lfloor l' \rceil : \lfloor z \rceil \to \lfloor c \rceil$$

such that $\lfloor e' \rceil(\lfloor l' \rceil \times 1_{\lfloor x \rceil}) = \lfloor f \rceil$. Indeed, the arrow

$$\lfloor l' \rceil \times 1_{\lfloor x \rceil} : (N, \mathsf{P}_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}(\zeta \boxtimes \rho)) \to (U', \mathsf{P}_{\{\!\{\mu'\}\!\}\times\{\!\{\mu'\}\!\}}(\omega \boxtimes \rho)))$$

is represented by an arrow $m' : N \to U'$ such that $\{\!\{\mu'\}\!\} m' = (l' \times 1_X)\{\!\{\nu\}\!\}$ and the following computation implies that $\lfloor e' \rceil(\lfloor l' \rceil \times 1_{\lfloor x \rceil}) = \lfloor e' \circ m' \rceil = \lfloor f \rceil$:

$$\begin{aligned}
\mathsf{P}_{e'm'\times f}\sigma &= \mathsf{P}_{hm'\times m}\mathsf{P}_{e\times e}\sigma && (f = e\circ m,\, e' = e \circ h)\\
&\geq \mathsf{P}_{hm'\times m}\mathsf{P}_{\{\!\{\mu\}\!\}\times\{\!\{\mu\}\!\}}\forall_{\{\!\{\mu\}\!\}\times\{\!\{\mu\}\!\}}\mathsf{P}_{e\times e}\sigma && (\mathsf{P}_{(-)} \dashv \forall_{(-)})\\
&= \mathsf{P}_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}\mathsf{P}_{l\times l}\forall_{\{\!\{\mu\}\!\}\times\{\!\{\mu\}\!\}}\mathsf{P}_{e\times e}\sigma && (\text{def. of } l')\\
&= \mathsf{P}_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}\forall_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}\mathsf{P}_{m\times m}\mathsf{P}_{e\times e}\sigma && (\text{B-C})\\
&\geq \mathsf{P}_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}\forall_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}\mathsf{P}_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}(\zeta \boxtimes \rho) && (f = e \circ m)\\
&\geq \mathsf{P}_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}(\zeta \boxtimes \rho).
\end{aligned}$$

The arrow $\lfloor l' \rceil$ such that $\lfloor e' \rceil(\lfloor l' \rceil \times 1_{\lfloor x \rceil}) = \lfloor f \rceil$ is unique. Indeed, if $\lfloor \hat{l} \rceil : \lfloor z \rceil \to \lfloor c \rceil$ is an arrow such that $\lfloor e' \rceil(\lfloor \hat{l} \rceil \times 1_{\lfloor x \rceil}) = \lfloor f \rceil$, and the product $\lfloor \hat{l} \rceil \times 1_{\lfloor x \rceil}$ is represented by an arrow $\hat{m} : N \to U'$ such that $\{\!\{\mu'\}\!\}\hat{m} = (\hat{l} \times 1_X)\{\!\{\nu\}\!\}$, then the following computation shows that $\zeta \leq \mathsf{P}_{l\times\hat{l}}\omega$:

$$\begin{aligned}
\mathsf{P}_{l\times\hat{l}}\omega &= \mathsf{P}_{l\times\hat{l}}\mathsf{P}_{\{\!\{\mathsf{P}_{\Delta_E}\varepsilon^{\rho}_{\sigma}\}\!\}\times\{\!\{\mathsf{P}_{\Delta_E}\varepsilon^{\rho}_{\sigma}\}\!\}}\varepsilon^{x}_{\sigma}\\
&= \mathsf{P}_{l'\times\{\!\{\mathsf{P}_{\Delta_E}\varepsilon^{\rho}_{\sigma}\}\!\}\hat{l}}(\forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\delta_X \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \forall_{\{\!\{\mu\}\!\}\times\{\!\{\mu\}\!\}}\mathsf{P}_{e\times e}\sigma) \wedge \gamma)\\
&\geq \forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\delta_X \wedge \mathsf{P}_{\langle 1,2\rangle}\nu \wedge \mathsf{P}_{\langle 3,4\rangle}\nu) \Rightarrow \forall_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}\mathsf{P}_{m\times h\hat{m}}\mathsf{P}_{e\times e}\sigma) \wedge \zeta && (\text{B-C})\\
&= \forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\delta_X \wedge \mathsf{P}_{\langle 1,2\rangle}\nu \wedge \mathsf{P}_{\langle 3,4\rangle}\nu) \Rightarrow \forall_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}\mathsf{P}_{e'\hat{m}\times f}\sigma) \wedge \zeta\\
&\geq \forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\delta_X \wedge \mathsf{P}_{\langle 1,2\rangle}\nu \wedge \mathsf{P}_{\langle 3,4\rangle}\nu) \Rightarrow \forall_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}\mathsf{P}_{\{\!\{\nu\}\!\}\times\{\!\{\nu\}\!\}}(\zeta \boxtimes \rho)) \wedge \zeta\\
&\geq \zeta && (\alpha \wedge (-) \dashv \alpha \Rightarrow (-))
\end{aligned}$$

This ends the proof of the cartesian closure of $\overline{\mathscr{C}}/(A,\delta_A)$, now we consider a slice of the form $\overline{\mathscr{C}}/(A,\alpha)$. If $\lfloor x \rceil : (X\rho) \to (A,\alpha)$ and $\lfloor y \rceil : (Y,\sigma) \to (A,\alpha)$ are two objects, then we provide a strict exponential through the exponential of the reindexings $\lfloor x^* \rceil$ and $\lfloor y^* \rceil$ of the arrows $\lfloor x \rceil$ and $\lfloor y \rceil$ over the quotient arrow $q := \lfloor 1_A \rceil : (A,\delta_A) \to (A,\alpha)$

$$\begin{tikzcd}
(X^*,\rho^*) \arrow[r, "q_X"] \arrow[d, "\lfloor x^* \rceil"'] \arrow[dr, phantom, "\lrcorner", very near start] & (X,\rho) \arrow[d, "\lfloor x \rceil"] \\
(A,\delta_A) \arrow[r, "q"'] & (A,\alpha)
\end{tikzcd}
\qquad
\begin{tikzcd}
(Y^*,\sigma^*) \arrow[r, "q_Y"] \arrow[d, "\lfloor y^* \rceil"'] \arrow[dr, phantom, "\lrcorner", very near start] & (Y,\sigma) \arrow[d, "\lfloor y \rceil"] \\
(A,\delta_A) \arrow[r, "q"'] & (A,\alpha).
\end{tikzcd}$$

By Lemma A.0.14, the right diagram denotes compactly the diagram

$$\begin{tikzcd}
(Y^*, \mathsf{P}_{\{\!\{\gamma_y\}\!\}\times\{\!\{\gamma_y\}\!\}}\delta_A \boxtimes \sigma) \arrow[rr, "\lfloor \{\!\{\gamma_y\}\!\}_2 \rceil"] \arrow[dr, "\lfloor \{\!\{\gamma_y\}\!\} \rceil"] \arrow[dd, "\lfloor \{\!\{\gamma_y\}\!\}_1 \rceil"'] & & (Y,\sigma) \arrow[dd, "\lfloor y \rceil"] \\
& (A \times Y, \delta_A \boxtimes \sigma) \arrow[ur] \arrow[dl] & \\
(A,\delta_A) \arrow[rr, "q"'] & & (A,\alpha)
\end{tikzcd}$$

where $\{\!\!\{\gamma_y\}\!\!\} : Y^* \to A \times Y$ is a weak comprehension of $\gamma_y := \mathsf{P}_{1_A \times y}\alpha$. The pullback of $\lfloor x \rceil$ and q is obtained similarly . Hence, we can consider the exponential

$$\lfloor y^* \rceil^{\lfloor x^* \rceil} := \lfloor c^* \rceil : (C^*, \omega^*) \to (A, \delta_A)$$

where

$$\omega^* := \mathsf{P}_{\{\!\!\{\mathsf{P}_{\Delta_{E^*}} \varepsilon^{\rho^*}_{\sigma^*}\}\!\!\} \times \{\!\!\{\mathsf{P}_{\Delta_{E^*}} \varepsilon^{\rho}_{\sigma^*}\}\!\!\}} \varepsilon^{x^*}_{\sigma^*}$$

and

$$\varepsilon^{x^*}_{\sigma^*} := \forall_{\langle 1,3\rangle}(((\mathsf{P}_{\langle 2,4\rangle}\delta_{X^*} \wedge \mathsf{P}_{\langle 1,2\rangle}\mu^* \wedge \mathsf{P}_{\langle 3,4\rangle}\mu^*)) \Rightarrow \forall_{\{\!\!\{\mu^*\}\!\!\} \times \{\!\!\{\mu^*\}\!\!\}} \mathsf{P}_{e^* \times e^*} \mathsf{P}_{\lfloor \{\!\!\{\gamma_y\}\!\!\} \rceil \times \lfloor \{\!\!\{\gamma_y\}\!\!\} \rceil} \delta_A \boxtimes \sigma) \wedge \gamma^*.$$

The relation $\varepsilon^{x^*}_{\sigma^*}$ on E^* expresses that two functions in E^* are related when they have the same evaluation in (Y^*, σ^*). This means that the evaluations have the same A components and σ-related Y components. In order to provide the exponential of $\lfloor x \rceil$ and $\lfloor y \rceil$, we consider those functions which have A components that are α-related and Y components that are σ related:

$$\tilde{\varepsilon} := \forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\delta_{X^*} \wedge \mathsf{P}_{\langle 1,2\rangle}\mu^* \wedge \mathsf{P}_{\langle 3,4\rangle}\mu^*) \Rightarrow \forall_{\{\!\!\{\mu^*\}\!\!\} \times \{\!\!\{\mu^*\}\!\!\}} \mathsf{P}_{e^* \times e^*} \mathsf{P}_{\lfloor \{\!\!\{\gamma_y\}\!\!\} \rceil \times \lfloor \{\!\!\{\gamma_y\}\!\!\} \rceil} \alpha \boxtimes \sigma) \wedge \gamma^*. \quad (2.34)$$

If we denote with $\tilde{\omega} := \mathsf{P}_{\{\!\!\{\mathsf{P}_{\Delta_{E^*}} \varepsilon^{\rho^*}_{\sigma^*}\}\!\!\} \times \{\!\!\{\mathsf{P}_{\Delta_{E^*}} \varepsilon^{\rho}_{\sigma^*}\}\!\!\}} \tilde{\varepsilon}$ then we obtain the following commutative diagram

$$\begin{tikzcd}
(C^*, \omega^*) \arrow[r, "\lfloor 1_{C^*} \rceil"] \arrow[d, "\lfloor y^* \rceil^{\lfloor x^* \rceil}"'] & (C^*, \tilde{\omega}) \arrow[d, "\lfloor y \rceil^{\lfloor x \rceil}"] \\
(A, \delta_A) \arrow[r, "q"'] & (A, \alpha).
\end{tikzcd}$$

where $\lfloor y \rceil^{\lfloor x \rceil} := \lfloor c^* \rceil$. Thanks to the pasting law of the pullbacks, the product $\lfloor c^* \rceil \times \lfloor x^* \rceil$ in $\overline{\mathscr{C}}/(A, \delta)$ is isomorphic to the product $\lfloor y \rceil^{\lfloor x \rceil} \times \lfloor x \rceil$ in $\overline{\mathscr{C}}/(A, \alpha)$. The commutativity of the above diagram and the description of pullbacks implies that, up to isomorphism, we are in the following situation

$$\begin{tikzcd}
(N, \mathsf{P}_{\{\!\!\{\nu\}\!\!\} \times \{\!\!\{\nu\}\!\!\}} \omega^* \boxtimes \rho) \arrow[r, "\lfloor 1_N \rceil"] \arrow[d, "\lfloor \{\!\!\{\nu\}\!\!\}_1 \rceil"'] & (N, \mathsf{P}_{\{\!\!\{\nu\}\!\!\} \times \{\!\!\{\nu\}\!\!\}} \tilde{\omega} \boxtimes \rho) \arrow[r, "\lfloor \{\!\!\{\nu\}\!\!\}_2 \rceil"] \arrow[d, "\lfloor \{\!\!\{\nu\}\!\!\}_1 \rceil"'] & (X, \rho) \arrow[d, "\lfloor x \rceil"] \\
(C^*, \omega^*) \arrow[r, "\lfloor 1_{C^*} \rceil"'] & (C^*, \tilde{\omega}) \arrow[r, "\lfloor y \rceil^{\lfloor x \rceil}"'] & (A, \alpha)
\end{tikzcd}$$

where $\{\!\!\{\nu\}\!\!\} : N \to C^* \times X$ is a weak comprehension of $\nu := \mathsf{P}_{c^* \times 1_X}\alpha$. The evaluation arrow

$$\lfloor e^* \rceil : \lfloor y^* \rceil^{\lfloor x^* \rceil} \times \lfloor x^* \rceil \to \lfloor y^* \rceil$$

is represented by an arrow

$$\lfloor e^* \rceil : (N, \mathsf{P}_{\{\!\!\{\nu\}\!\!\} \times \{\!\!\{\nu\}\!\!\}} \tilde{\omega} \boxtimes \rho) \to (Y^*, \sigma^*).$$

The evaluation arrow

$$\lfloor e \rceil : \lfloor y \rceil^{\lfloor x \rceil} \times \lfloor x \rceil \to \lfloor y \rceil$$

is given by $e := \{\!\!\{\gamma_y\}\!\!\}_2 \circ e^*$.

$\square$

Before providing the proof of the other implication of Theorem 2.3.7 we give the following definition.

Definition 2.3.8. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine with weak comprehensions and comprehensive diagonals. If $A \in \mathscr{C}$ and $X \xrightarrow{x} A$ and $Y \xrightarrow{y} A$ are two objects of $\mathscr{C}/A$ and $\sigma \in \mathsf{P}(Y \times Y)$ is a P-eq. relation, then two arrows $f, g : x \to y$ are called *σ-related* if

$$\delta_X \leq \mathsf{P}_{f \times g}\sigma.$$

Proof of Theorem 2.3.7. $(ii) \Rightarrow (i)$ Let $x : X \to A$ and $y : Y \to A$ be two objects of $\mathscr{C}/A$ and let $\sigma \in \mathsf{P}(Y \times Y)$ be a P-eq. relation on Y such that $\sigma \leq \mathsf{P}_{y \times y}\delta_A$. We consider a strict exponential $\lfloor y^x \rceil : (E, \varepsilon) \to (A, \delta_A)$ of $\lfloor y \rceil$ and $\lfloor x \rceil$ and the evaluation arrow

$$\lfloor e \rceil : \lfloor y \rceil^{\lfloor x \rceil} \times \lfloor x \rceil \to \lfloor y \rceil$$

which is represented by an arrow $e : u \to y$, from a weak product $y^x \leftarrow u \rightarrow x$ given by

$$\begin{tikzcd} U \arrow[r, "\{\!|\mu|\!\}_1"] \arrow[d, "\{\!|\mu|\!\}_2"'] & X \arrow[d, "x"] \\ E \arrow[r, "y^x"'] & A \end{tikzcd}$$

where $\mu := \mathsf{P}_{y^x \times x}\delta_A$. The arrow e has the following property: for every arrow $z : Z \to A$ and arrow $f : v \to y$ from a weak product $z \xleftarrow{\pi_1} v \xrightarrow{\pi_2} x$ which preserves projections w.r.t. σ, there exist two arrows $l : z \to y^x$ and $m : v \to u$ such that $e \circ m$ and f are σ-related. Hence, if $\{\!|\sigma|\!\} : K \to Y \times Y$ is a weak comprehension of σ and $U := dom(u)$ and $V := dom(v)$, then there exists an arrow j making the following diagram commute

$$\begin{tikzcd} K \arrow[r, "\{\!|\sigma|\!\}"] & Y \times Y \\ V. \arrow[u, dashed, "j"] \arrow[ur, "\langle em, f \rangle"'] & \end{tikzcd}$$

We now consider a weak pullback of $\{\!|\sigma|\!\}_1$ and e

$$\begin{tikzcd} K' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & U \arrow[d, "e"] \\ K \arrow[r, "\{\!|\sigma|\!\}_1"'] & Y \end{tikzcd} \tag{2.35}$$

and observe that this induces a weak pullback of y^x and x:

$$\begin{tikzcd} K' \arrow[r, "\{\!|\mu|\!\}_2\pi_2"] \arrow[d, "\{\!|\mu|\!\}_1\pi_2"'] & X \arrow[d, "x"] \\ E \arrow[r, "y^x"'] & A. \end{tikzcd}$$

The arrow $e' := \{\!|\sigma|\!\}_2\pi_1 : K' \to Y$ provides the desired evaluation arrow. Indeed, the weak pullback 2.35 induces an arrow $m' : V \to K'$ such that $\pi_2 m' = j$ and $\pi_1 m' = m$. Hence, we obtain that $m'e' = f$. $\square$

Remark 2.3.9. In case of the elementary doctrine of weak subobjects $\mathsf{Psub}_{\mathscr{C}}$, we obtain [Emm20, Theorem 3.6], for categories with strict products and weak pullbacks. Actually, as we will discuss in the next section, we could use instead of extensional exponentials the notion of weak exponential (Definition 2.4.2) adapted for the slices of $\mathscr{C}$ and obtain [CR00, Theorem 3.3]. As pointed out by Emmenegger, the notions of weak and extensional exponential coincide under some circumstances, for instance, when the category $\mathscr{C}$ is left exact. In the next section, we will prove that since the base categories of the elementary doctrines have strict finite products by definition, then there is an equivalence of these exponentials and of a third one introduced in [MPR21] in order to prove a result equivalent to Theorem 2.3.7.

2.4 Relations to the work of Maietti, Pasquali and Rosolini.

As already mentioned, a result similar to Theorem 2.3.7 was already claimed by Maietti, Pasquali and Rosolini in [MPR17] and it recently appeared in [MPR21]. In this section, we discuss the differences between the two statements and prove that they are equivalent. In order to do that we discuss the different notions of weak exponential that appeared in [CR00], [Emm20] and [MPR21].

The following definition of exponentials is introduced in [MPR21] and we will refer to as a *very weak* kind of exponentials.

Definition 2.4.1. Let $\mathscr{C}$ be a category with weak pullbacks and let A be an object of $\mathscr{C}$. A *very weak exponential* of the objects $x : X \to A$ and $y : Y \to A$ of $\mathscr{C}/A$ is an object $y^x : E \to A$ with an arrow $e : u \to y$ from a weak product $y^x \leftarrow u \rightarrow x$ such that

- For every $z : Z \to A$ and arrow $f : v \to y$ from a weak product $z \leftarrow v \rightarrow x$, there exist two arrows l, m making the following diagram commute:

$$\begin{tikzcd}
z \arrow[d, dashed, "l"'] & v \arrow[l] \arrow[d, dashed, "m"] \arrow[rd, bend left] \arrow[dd, bend right=60, "f"'] & \\
y^x & u \arrow[l] \arrow[r] \arrow[d, "e"] & x \\
& y. &
\end{tikzcd}$$

$\mathscr{C}$ is said *slice-wise weakly cartesian closed* if, for every $A \in \mathscr{C}$ and $x, y \in \mathscr{C}/A$, there exists a very weak exponential of x and y.

We referred to the above exponentials as "very weak" in order to distinguish them from the *weak* exponentials introduced in [CR00]. We now recall the former notion in the case of the slices of a category with weak pullbacks.

Definition 2.4.2. Let $\mathscr{C}$ be a category with weak pullbacks and let A be an object of $\mathscr{C}$. A *weak exponential* of the objects $x : X \to A$ and $y : Y \to A$ of $\mathscr{C}/A$ is an object $y^x : E \to A$ with an arrow $e : u \to y$ from a weak product $y^x \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} x$ such that

- e equalizes any pair of arrows which π_1, π_2 jointly equalizes, i.e. for every $v_1, v_2 : v \to u$ such that $\pi_1 v_1 = \pi_2 v_2$ then $ev_1 = ev_2$.

- For every $z : Z \to A$ and arrow $f : v \to y$ from a weak product $z \overset{\pi_1}{\leftarrow} v \overset{\pi_2}{\rightarrow} x$ which equalizes all pair of arrows which π_1, π_2 jointly equalizes, there exist two arrows l, m making the following diagram commute:

$$\begin{tikzcd}
z & v \arrow[l] \arrow[d, dashed, "m"] \arrow[dr, bend left] \arrow[dd, bend right, "f"'] & \\
y^x \arrow[from=1-1, dashed, "l"'] & u \arrow[l] \arrow[r] \arrow[d, "e"] & x \\
& y. &
\end{tikzcd}$$

The above exponentials, which can be seen as formulated for the elementary doctrine $\mathsf{PSub}_{\mathscr{C}}$ of weak subobjects, can be trivially defined for every elementary doctrine P with weak comprehensions and comprehensive diagonals. Actually, as already observed by Emmenegger, the weak and very weak exponentials are particular instances of the extensional exponentials and they are obtained as follows.

Observation 2.4.3. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine with weak comprehensions and comprehensive diagonals and let A be an object of $\mathscr{C}$. If $x : X \to A$ and $y : Y \to A$ are objects of $\mathscr{C}/A$, then an arrow $y^x : E \to A$ is said

- a *weak exponential* of x and y if it is an extensional exponential of x,y w.r.t. $\sigma := \delta_Y$.
- a *very weak exponential* of x and y if it is an extensional exponential of x,y w.r.t. $\sigma := \top_{Y\times Y}$.

The following result appears as [MPR21, Theorem 7.14] and uses the very weak exponentials to give an equivalent condition to the local cartesian closure of a suitable doctrine.

Theorem 2.4.4 (Maietti, Pasquali and Rosolini)**.** *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be an existential and universal elementary doctrine with weak full comprehensions and comprehensive diagonals. The following are equivalent:*

(i) $\mathscr{C}$ *is slice-wise weakly cartesian closed*

(ii) $\overline{\mathscr{C}}$ *is locally cartesian closed.*

In general, the three notions of exponentials discussed are not equivalent, in the sense that the existence of one of them implies the existence of the others. This was pointed out by Emmenegger in [Emm20], who discovered an invalid argument in the proof of [CR00, Theorem 3.3] and fixed it by the use of extensional exponentials. However, there are cases in which the three exponentials are equivalent. For instance, if the slices $\mathscr{C}/A$ have strict products, then every arrow preserves projection with respect to any P-eq. relation and hence the three notions coincide. Below, we prove that the common hypothesis of Theorem 2.3.7 and Theorem 2.4.4 implies the equivalence of all the notions of exponential introduced. Before doing that, we provide a useful technical lemma.

Lemma 2.4.5. *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be an implicational and universal elementary doctrine with full comprehensions and comprehensive diagonals. If* $A \in \mathscr{C}$ *and* $X \xrightarrow{x} A$, $W \xrightarrow{w} A$ *and* $B \xrightarrow{b} A$ *are objects in* $\mathscr{C}/A$, *for a map* $f : u \to b$, *from a weak pullback* $w \xleftarrow{\{\!|\rho|\!\}_1} u \xrightarrow{\{\!|\rho|\!\}_2} x$ *we have the following equivalent conditions:*

(i) f *preserves projections w.r.t.* $\sigma \in \mathsf{P}(B\times B)$,

(ii) $\top_W \leq \mathsf{P}_{\Delta_W}\forall_{\{\!|\rho|\!\}_1\times\{\!|\rho|\!\}_1}(\mathsf{P}_{\{\!|\rho|\!\}\times\{\!|\rho|\!\}}\mathsf{P}_{\langle 2,4\rangle}\delta_X \Rightarrow \mathsf{P}_{f\times f}\sigma)$,

where $\rho := P_{w\times x}\delta_A$.

Proof.

$$
\begin{array}{c}
\top_W \leq \mathsf{P}_{\Delta_W}\forall_{\{\!\!\{\rho\}\!\!\}_1\times\{\!\!\{\rho\}\!\!\}_1}(\mathsf{P}_{\{\!\!\{\rho\}\!\!\}\times\{\!\!\{\rho\}\!\!\}}\mathsf{P}_{2,4}\delta_X \Rightarrow \mathsf{P}_{f\times f}\sigma) \\
\hline\hline
\delta_W \leq \forall_{\{\!\!\{\rho\}\!\!\}_1\times\{\!\!\{\rho\}\!\!\}_1}(\mathsf{P}_{\{\!\!\{\rho\}\!\!\}\times\{\!\!\{\rho\}\!\!\}}P_{\langle 2,4\rangle}\delta_X \Rightarrow \mathsf{P}_{f\times f}\sigma) \\
\hline\hline
\mathsf{P}_{\{\!\!\{\rho\}\!\!\}_1\times\{\!\!\{\rho\}\!\!\}_1}\delta_W \leq \mathsf{P}_{\{\!\!\{\rho\}\!\!\}\times\{\!\!\{\rho\}\!\!\}}\mathsf{P}_{\langle 2,4\rangle\delta_X} \Rightarrow \mathsf{P}_{f\times f}\sigma \\
\hline\hline
\mathsf{P}_{\{\!\!\{\rho\}\!\!\}\times\{\!\!\{\rho\}\!\!\}}(\mathsf{P}_{\langle 1,3\rangle}\delta_W \wedge \mathsf{P}_{\langle 2,4\rangle}\delta_X) \leq \mathsf{P}_{f\times f}\sigma \\
\hline\hline
\mathsf{P}_{\{\!\!\{\rho\}\!\!\}\times\{\!\!\{\rho\}\!\!\}}(\delta_W \boxtimes \delta_X) \leq \mathsf{P}_{f\times f}\sigma.
\end{array}
$$

□

Proposition 2.4.6. *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be an existential universal elementary doctrine with full comprehensions and comprehensive diagonals. If* P *is implicational then, for every* $A \in \mathscr{C}$*, the slice* $\mathscr{C}/A$ *has weak or very weak exponentials if and only if it has extensional exponentials.*

Proof. We only need to show that existence of very weak exponentials imply the existence of extensional exponentials. The case with weak exponentials is proved similarly.

Given two objects $x : X \to A$ and $y : Y \to A$ of $\mathscr{C}/A$. Let $y^x : E \to A$ be a very weak exponential of x and y, and let $e : u \to y$ be an evaluation map, where u is a weak product with projections $y^x \overset{\{\!\!\{\mu\}\!\!\}_1}{\leftarrow} u \overset{\{\!\!\{\mu\}\!\!\}_2}{\rightarrow} x$ and $\mu := \mathsf{P}_{y^x\times x}\delta_A$. If $\sigma \in \mathsf{P}(Y\times Y)$ is a P-eq. relation on Y, consider the object

$$\varphi := \forall_{\{\!\!\{\mu\}\!\!\}_1\times\{\!\!\{\mu\}\!\!\}_1}(\mathsf{P}_{\{\!\!\{\mu\}\!\!\}\times\{\!\!\{\mu\}\!\!\}}\mathsf{P}_{<1,3>}\delta_X \Rightarrow \mathsf{P}_{e\times e}\sigma),$$

in $\mathsf{P}(E\times E)$. If $\{\!\!\{\mathsf{P}_{\Delta_E}\varphi\}\!\!\} : C \to E$ is a weak comprehension of $\mathsf{P}_{\Delta_E}\phi$ and $c \overset{\{\!\!\{\mu'\}\!\!\}_1}{\leftarrow} u' \overset{\{\!\!\{\mu'\}\!\!\}_2}{\rightarrow} x$ is a weak pullback of $c := y^x\{\!\!\{\mathsf{P}_{\Delta_E}\varphi\}\!\!\}$ and x, with $\mu' = \mathsf{P}_{c\times x}\delta_A$, then the weak universal property of u induces an arrow h which makes the following diagram commute

$$
\begin{tikzcd}
U' \arrow[rr, "\{\!\!\{\mu'\}\!\!\}_2"] \arrow[dr, dashed, "h"] \arrow[dd, "\{\!\!\{\mu'\}\!\!\}_1"'] & & X \arrow[dd, "x"] & \\
& U \arrow[ur, "\{\!\!\{\mu\}\!\!\}_2"] \arrow[rr, "e"'] \arrow[dd, "\{\!\!\{\mu\}\!\!\}_1"', near start] & & Y \arrow[dl, bend left] \\
C \arrow[rr, "c"'] \arrow[dr, bend right, "\{\!\!\{\mathsf{P}_{\Delta_E}\varphi\}\!\!\}"'] & & A & \\
& E. \arrow[ur, "y^x"'] & &
\end{tikzcd}
$$

We now prove that the arrow $e' := e \circ h$ preserves projections w.r.t. σ, using Lemma 2.4.5:

$$
\begin{aligned}
&\mathsf{P}_{\Delta_C}\forall_{\{\!\!\{\mu'\}\!\!\}_1\times\{\!\!\{\mu'\}\!\!\}_1}(\mathsf{P}_{\{\!\!\{\mu'\}\!\!\}\times\{\!\!\{\mu'\}\!\!\}}\mathsf{P}_{\langle 2,4\rangle}\delta_X \Rightarrow \mathsf{P}_{e'\times e'}\sigma)\\
&= \mathsf{P}_{\Delta_C}\forall_{\{\!\!\{\mu'\}\!\!\}_1\times\{\!\!\{\mu'\}\!\!\}_1}(\mathsf{P}_{h\times h}\mathsf{P}_{\{\!\!\{\mu\}\!\!\}\times\{\!\!\{\mu\}\!\!\}}P_{\langle 2,4\rangle}\delta_X \Rightarrow \mathsf{P}_{h\times h}\mathsf{P}_{e\times e}\sigma)\\
&= \mathsf{P}_{\Delta_C}\forall_{\{\!\!\{\mu'\}\!\!\}_1\times\{\!\!\{\mu'\}\!\!\}_1}\mathsf{P}_{h\times h}(\mathsf{P}_{\{\!\!\{\mu\}\!\!\}\times\{\!\!\{\mu\}\!\!\}}P_{\langle 2,4\rangle}\delta_X \Rightarrow \mathsf{P}_{e\times e}\sigma)\\
&\geq \mathsf{P}_{\Delta_C}\mathsf{P}_{\{\!\!\{\mathsf{P}_{\Delta_E}\varphi\}\!\!\}\times\{\!\!\{\mathsf{P}_{\Delta_E}\varphi\}\!\!\}}\forall_{\{\!\!\{\mu\}\!\!\}_1\times\{\!\!\{\mu\}\!\!\}_1}(\mathsf{P}_{\{\!\!\{\mu\}\!\!\}\times\{\!\!\{\mu\}\!\!\}}P_{\langle 2,4\rangle}\delta_X \Rightarrow \mathsf{P}_{e\times e}\sigma)\\
&= \mathsf{P}_{\{\!\!\{\mathsf{P}_{\Delta_E}\varphi\}\!\!\}}\mathsf{P}_{\Delta_E}\varphi\\
&\geq \top_C.
\end{aligned}
$$

We now prove that e' has the weak universal property of extensional exponential from the weak universal property of the very weak exponential e. Indeed, given $z : Z \to A \in \mathscr{C}/A$ and an arrow

$f : v \to y$, from a weak product $z \leftarrow v \rightarrow x$, that preserves projections w.r.t. σ, then e implies the existence of two arrows l, m making the following diagram commute

$$
\begin{tikzcd}
z \arrow[d, dashed, "l"] & v \arrow[l, "\{\!|\nu|\!\}_1"'] \arrow[d, dashed, "m"] \arrow[dr, bend left, "\{\!|\nu|\!\}_2"] \arrow[dd, bend right, "f"'] & \\
y^x & u \arrow[l] \arrow[r] \arrow[d, "e"] & x \\
& y, &
\end{tikzcd}
$$

where $\nu := P_{z\times x}\delta_A$. We obtain the following derivation

$$
\begin{aligned}
\mathsf{P}_l\mathsf{P}_{\Delta_E}\varphi &= \mathsf{P}_l\mathsf{P}_{\Delta_E}\forall_{\{\!|\mu|\!\}_1\times\{\!|\mu|\!\}_1}(\mathsf{P}_{\{\!|\mu|\!\}\times\{\!|\mu|\!\}}\mathsf{P}_{\langle 2,4\rangle}\delta_X \Rightarrow \mathsf{P}_{e\times e}\sigma)\\
&= \mathsf{P}_{\Delta_Z}\mathsf{P}_{l\times l}\forall_{\{\!|\mu|\!\}_1\times\{\!|\mu|\!\}_1}(\mathsf{P}_{\{\!|\mu|\!\}\times\{\!|\mu|\!\}}\mathsf{P}_{\langle 2,4\rangle}\delta_X \Rightarrow \mathsf{P}_{e\times e}\sigma)\\
&= \mathsf{P}_{\Delta_Z}\forall_{\{\!|\nu|\!\}_1\times\{\!|\nu|\!\}_1}\mathsf{P}_{m\times m}(\mathsf{P}_{\{\!|\mu|\!\}\times\{\!|\mu|\!\}}\mathsf{P}_{\langle 2,4\rangle}\delta_X \Rightarrow \mathsf{P}_{e\times e}\sigma) && \text{(Lemma A.0.16)}\\
&\geq \mathsf{P}_{\Delta_Z}\forall_{\{\!|\nu|\!\}_1\times\{\!|\nu|\!\}_1}P_{m\times m}\mathsf{P}_{e\times e}\sigma\\
&= \mathsf{P}_{\Delta_Z}\forall_{\{\!|\nu|\!\}_1\times\{\!|\nu|\!\}_1}\mathsf{P}_{f\times f}\sigma\\
&\geq \mathsf{P}_{\Delta_Z}\forall_{\langle 1,3\rangle}\forall_{\{\!|\nu|\!\}\times\{\!|\nu|\!\}}\mathsf{P}_{\{\!|\nu|\!\}\times\{\!|\nu|\!\}}\delta_Z \boxtimes \delta_X\\
&\geq \top_Z.
\end{aligned}
$$

hence, the comprehension $\{\!|\mathsf{P}_{\Delta_E\varphi}|\!\}$ implies existence of a map $l' : Z \to C$ such that $\{\!|P_{\Delta_W}\varphi|\!\}l' = l$. If $m' : v \to u'$ is an arrow induced by l' and 1_x, then $e' \circ m'$ and f are σ-related. Finally, an extensional exponential of x and y with respect to σ is obtained in the same way of the proof of Theorem 2.3.7 $(ii) \Rightarrow (i)$. □

The above proposition implies that our Theorem 2.3.7 is equivalent to Theorem 2.4.4.

2.5 Extensive elementary quotient completion

In this section we give equivalent conditions to the *extensivity* of the base category $\overline{\mathscr{C}}$ of the elementary quotient completion of a suitable elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$. In particular, we generalize, in the case of categories with strict finite products and weak pullbacks, the well-known result [GV98, Proposition 2.1], which states the extensivity of the exact completion. We mention that this section has a non-trivial intersection with [MPR21, §7.2]. Indeed, in *loc.cit.* the authors provide conditions on P such that $\overline{\mathscr{C}}$ has *disjoint* and *distributive* coproducts. In addition, we establish the conditions which provides the *universality* of coproducts in $\overline{\mathscr{C}}$. Before starting we fix some notations.

Notation. In this section we will denote with $X+Y$ the coproduct of two objects X and Y and with $i_X : X \to X+Y$ and $i_Y : Y \to X+Y$ the canonical *injections*. The initial object will be denoted by 0. If $f : X \to A$ and $g : Y \to A$ are two arrows, then $[f,g] : X+Y \to A$ will denote the unique arrow such that $[f,g]i_X = f$ and $[f,g]i_Y = g$. If $f : X \to A$ and $g : Y \to B$ are two arrows, then $f+g : X+Y \to A+B$ will denote the unique arrow such that $(f+g)i_X = i_A f$ and $(f+g)i_Y = i_B g$.

We now recall an equivalent formulation of *extensivity* for left exact categories. For a detailed discussion about the equivalent definitions of extensive categories we refer to [CLW93].

Definition 2.5.1. A left exact category $\mathscr{C}$ with finite coproducts is called *lextensive* if

(i) *Sums are disjoint:*

I.a) Coprojections of sums $X \overset{i_X}{\to} X + Y \overset{i_Y}{\leftarrow} Y$ are monomorphisms,

I.b) If 0 is an initial object, then the following is a pullback

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Y \arrow[d, "i_Y"] \\ X \arrow[r, "i_X"] & X + Y. \end{tikzcd}$$

(ii) *Sums are universal:* If the first two diagrams are pullbacks, then also the third one is a pullback

$$\begin{tikzcd} P_X \arrow[r] \arrow[dd] & X \arrow[d, "i_X"] \\ & X+Y \arrow[d] \\ A \arrow[r] & B, \end{tikzcd} \quad \begin{tikzcd} P_Y \arrow[r] \arrow[dd] & Y \arrow[d, "i_Y"] \\ & X+Y \arrow[d] \\ A \arrow[r] & B, \end{tikzcd} \quad \begin{tikzcd} P_X + P_Y \arrow[r] \arrow[d] & X + Y \arrow[d] \\ A \arrow[r] & B. \end{tikzcd}$$

We now recall the notion of weak extensivity which was introduced in [GV98]. The idea is to give a weak version of some of the above conditions. In order to do that, the authors observed that condition II of the above definition is equivalent to the following conditions:

- *Distributivity* of coproducts: given three objects X, Y and Z the arrow

$$[1_X \times i_Y, 1_X \times i_Z] : (X \times Y) + (X \times Z) \to X \times (Y + Z)$$

  is an isomorphism. For the rest of the section we will denote the above arrow with e,

- if the first two diagrams are equalizers, then also the third one is an equalizer

$$E_X \longrightarrow X \rightrightarrows Z, \qquad E_Y \longrightarrow Y \rightrightarrows Z, \qquad E_X + E_Y \longrightarrow X + Y \rightrightarrows Z.$$

We recall that the distributivity of coproducts implies the following important properties:

- initial objects are *strict* , i.e. every arrow into an initial object is an isomorphism,
- for every pair of objects X, Y the injections i_X, i_Y into the coproduct $X + Y$ are monomorphisms,
- for every object X, the projection $p_2 : X \times 0 \to 0$ is an isomoprhism

a proof of these facts can be found in [CLW93, Propositions 3.2, 3.3. and 3.4].

Definition 2.5.2. A weakly left exact category with sums $\mathscr{C}$ is called *weakly lextensive* if

1. Sums are disjoint,
2. Initials are strict,

3. For each choice of weak products $X \times Y$ and $X \times Z$, the sum $(X \times Y) + (X \times Z)$, with the obvious projections, is a weak product of X and $(Y + Z)$.

4. For each choice of weak equalizers $E_X \longrightarrow X \rightrightarrows Z$, $E_Y \longrightarrow Y \rightrightarrows Z$, then $E_X + E_Y \longrightarrow X + Y \rightrightarrows Z$ is a weak equalizer,

In [GV98, Proposition 2.1] the authors proved that a weakly left exact category $\mathscr{C}$ is weakly lexextensive if and only if the exact completion $\mathscr{C}_{ex}$ is lextensive. We will follow this idea and first we will formulate the notion of weakly lextensivity in the language of doctrines.

In order to do that, we observe that assuming distributive coproducts in a category $\mathscr{C}$ with strict product and weak pullbacks implies that, for every object $X \in \mathscr{C}$, the poset $\mathsf{PSub}_{\mathscr{C}}(X)$ has finite joins and, for every arrow f of $\mathscr{C}$, the functor $\mathsf{PSub}_{\mathscr{C}}(f)$ preserves them. Hence, we will consider doctrines which have sums both at the level of the contexts and at the level of the logic.

From now on, we will say that an elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *has finite joins* meaning that $\mathsf{P}(X)$ has finite joins and that P_f preserves them, for every object $X \in \mathscr{C}$ and arrow f in $\mathscr{C}$.

Definition 2.5.3. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an existential elementary doctrine with full weak comprehensions, comprehensive diagonals and finite joins. The category $\mathscr{C}$ is called P-*weakly extensive* if

1. $\mathscr{C}$ has disjoint coproducts,

2. $\mathscr{C}$ has finite coproducts and they are distributive,

3. If $f : X \to A$, $g : Y \to A$ and $\alpha \in \mathsf{P}(A)$, then $\{\!|P_f(\alpha)|\!\} + \{\!|\mathsf{P}_g(\alpha)|\!\}$ is a comprehension of $\mathsf{P}_{[f,g]}\alpha$.

Observation 2.5.4. As already observed, distributive coproducts imply that the injections of a coproduct are always monomorphisms. Hence, in the above definition the assumption of monomorphic injections is always verified.

Condition I.b of Definition 2.5.1 stated in the internal logic of P becomes

$$\mathsf{P}_{i_X \times i_Y} \delta_{X+Y} = \bot_{X \times Y}. \tag{2.36}$$

Now, if the domain of $\{\!|\bot_{X \times Y}|\!\}$ is initial then the conditions (2.36) and I.b are equivalent. In case of strict initials, condition I.b implies that the domain of $\{\!|\bot_{X \times Y}|\!\}$ is initial.

Condition 3 of Definition 2.5.3 corresponds to the weak notion of universality of sums with respect to the internal logic of P.

We now discuss useful properties of disjoint and distributive coproducts.

Remark 2.5.5. We observe that if $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an elementary doctrine and $\mathscr{C}$ has distributive finite coproducts , then if $0 \in \mathscr{C}$ is the initial object it follows that

$$\delta_0 = \top_{0 \times 0}. \tag{2.37}$$

This is a consequence of the fact that, for every $X \in \mathscr{C}$, the projection $p_2 : X \times 0 \to 0$ is an isomorphism. Indeed, since $\delta_0 = \exists_{\Delta_0} \top_0$, applying first $\exists_{p_2}$ and then P_{p_2} we obtain

$$\mathsf{P}_{p_2} \exists_{p_2} \delta_0 = \top_{0 \times 0}$$

and the term on the left is equal to δ_0 because p_2 is an isomorphism.

Moreover, if coproducts are also disjoint and the domain of $\{\!\!\{\bot_{X\times Y}\}\!\!\}$ is the initial object, then for every $\alpha \in \mathsf{P}(X)$ and $\beta \in \mathsf{P}(Y)$ we have

$$\exists_{i_X}\alpha \wedge \exists_{i_Y}\beta = \bot_{X+Y}. \tag{2.38}$$

Intuitively, an element of $X+Y$ can not be an element of both X and Y. Formally, from the pullback

$$\begin{tikzcd}
0 \arrow[rr, "0_Y"] \arrow[dr, "\{\!\!\{\bot_{X\times Y}\}\!\!\}"] \arrow[dd, "0_X"'] & & Y \arrow[dd, "i_Y"] \\
& X\times Y \arrow[ur] \arrow[dl] & \\
X \arrow[rr, "i_X"'] & & X+Y
\end{tikzcd}$$

and the Frobenius condition we obtain that

$$\begin{aligned}
\exists_{i_X}\alpha \wedge \exists_{i_Y}\beta &= \exists_{i_X}(\alpha \wedge \mathsf{P}_{i_X}\exists_{i_Y}\beta) \\
&= \exists_{p_2}\exists_{\{\!\!\{\bot_{X\times Y}\}\!\!\}}\mathsf{P}_{\{\!\!\{\bot_{X\times Y}\}\!\!\}}\mathsf{P}_{p_1}\beta && \text{(Lemma A.0.4)}\\
&\exists_{p_2}(\mathsf{P}_{p_1}\beta \wedge \bot_{X\times Y}) \\
&= \bot_{X\times Y}.
\end{aligned}$$

A similar computation and the fact that $X\times 0 \cong 0$ imply that for every $\rho \in \mathsf{P}(X\times X)$ and $\sigma \in \mathsf{P}(Y\times Y)$ then

$$\mathsf{P}_{i_X\times i_Y}(\rho \boxplus \sigma) = \bot_{X+Y}. \tag{2.39}$$

We now formulate the main result of the section.

Theorem 2.5.6. *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an existential elementary doctrine with full weak comprehensions, comprehensive diagonals and finite joins, then $\mathscr{C}$ is P-weakly extensive if and only if $\overline{\mathscr{C}}$ is extensive.*

Before providing a proof of Theorem 2.5.6 we define suitable equivalence relations on the coproduct of two objects.

Lemma 2.5.7. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an existential elementary doctrine with full weak comprehensions, comprehensive diagonals and finite joins. Assuming that $\mathscr{C}$ has finite coproducts, if X, Y are two objects of $\mathscr{C}$, and ρ is a P-eq. relation on X and σ is a P-eq. relation on Y, then*

$$\rho \boxplus \sigma := \exists_{i_X\times i_X}\rho \vee \exists_{i_Y\times i_Y}\sigma$$

is a P-eq. relation on $X+Y$. Moreover, $\delta_X \boxplus \delta_Y = \delta_{X+Y}$.

Proof. By fullness of comprehensions, the reflexivity of $\rho \boxplus \sigma$ is equivalent to

$$\top_{X+Y} \le \mathsf{P}_{\Delta_{X+Y}}(\rho \boxplus \sigma).$$

The Beck-Chevalley condition implies that the right term is equal to $\exists_{i_X}\top_X \vee \exists_{i_Y}\top_Y$. Since

$$\top_X \le \mathsf{P}_{i_X}\exists_{i_X}\top_X \le \mathsf{P}_{i_X}(\exists_{i_X}\top_X \vee \exists_{i_Y}\top_Y)$$

and the same holds for i_Y, denoting with $k := \{\!\!\{\exists_{i_X}\top_X \vee \exists_{i_Y}\top_Y\}\!\!\} : K \to X+Y$ we have two arrows $f : X \to K$, $g : Y \to K$ such that $i_X = h \circ f$ and $i_Y = k \circ g$ and then $1_{X+Y} = k[f, g]$. The inequality

follows by fullness of comprehension and $\{\!|\top_{X+Y}|\!\} = 1_{X+Y}$.
The symmetry of $\rho \boxplus \sigma$ easily follows from the Beck-Chevalley condition. For the transitivity, we observe that

$$\mathsf{P}_{\langle 1,2\rangle}(\exists_{i_X \times i_X}\rho) \wedge \mathsf{P}_{\langle 2,3\rangle}(\exists_{i_X \times i_X}\rho)$$

where $\langle 1,2\rangle, \langle 2,3\rangle : (X+Y)\times(X+Y)\times(X+Y) \to (X+Y)\times(X+Y)$ are the obvious projections, is equal to

$$\exists_{i_X \times i_X \times i_X}(\mathsf{P}_{\langle 1,2\rangle}\rho \wedge \mathsf{P}_{\langle 2,3\rangle}\rho)$$

where $\langle 1,2\rangle\langle 2,3\rangle : X\times X\times X \to X\times X$ are the obvious projections. This is obtained applying twice the Frobenius condition and observing that, since i_X is a monomorphism, the following diagram is a pullback

$$\begin{tikzcd} X \times X \times X \arrow[r, "i_X \times 1_X \times 1_X"] \arrow[d, "1_X \times 1_X \times i_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & (X+Y) \times X \times X \arrow[d, "1_{X+Y} \times i_X \times i_X"] \\ X \times X \times (X+Y) \arrow[r, "i_X \times i_X \times 1_{X+Y}"'] & (X+Y)^3. \end{tikzcd}$$

The transitivity of $\exists_{i_X \times i_X}\rho$ follows from the transitivity of ρ and from the distributivity of meets and joins.

The last part of the statement follows observing that by adjunctions we have that

$$\exists_{i_X \times i_X}\delta_X \leq \delta_{X+Y}$$

and the same holds for $\exists_{i_Y \times i_X Y}\delta_Y$. $\square$

Proof of Theorem 2.5.6. If $\overline{\mathscr{C}}$ is extensive, then Lemma 2.5.7 implies that restricting to the objects of the form (X, δ_X) it follows that $\mathscr{C}$ has distributive coproducts. Moreover, since $\overline{\mathscr{C}}$ is extensive the initial object is strict, and it provides a strict initial object of $\mathscr{C}$. Coproducts are disjoint in $\mathscr{C}$ because they are disjoint in $\overline{\mathscr{C}}$ and initials are strict. Condition 4 of Definition 2.5.3 follows through the description of comprehensions in $\overline{\mathscr{C}}$.

Vice versa, if (X,ρ) and (Y,σ) are objects of $\overline{\mathscr{C}}$, then by Lemma 2.5.7 we can consider the object

$$(X+Y, \rho \boxplus \sigma)$$

of $\overline{\mathscr{C}}$. It is a coproduct since, given two arrows $\lfloor f\rceil : (X,\rho) \to (Z,\zeta)$ and $\lfloor g\rceil : (Y,\sigma) \to (Z,\zeta)$, the coproduct $X+Y$ implies the existence of an arrow $[f,g] : X+Y \to Z$ which makes the obvious diagram commute. This arrow induces an arrow

$$\lfloor [f,g]\rceil : (X+Y, \rho \boxplus \sigma) \to (Z,\zeta)$$

because $\exists_{i_X \times i_X}\rho \leq \mathsf{P}_{[f,g]\times[f,g]}\zeta$ if and only if $\rho \leq \mathsf{P}_{i_X \times i_X}\mathsf{P}_{[f,g]\times[f,g]}\zeta$, which is true by the definition of $\lfloor f\rceil$. The same holds for $\exists_{i_Y \times i_Y}\sigma$. This arrow is unique in the sense that if there exists an arrow $\lfloor h\rceil : (X+Y, \rho \boxplus \sigma) \to (Z,\zeta)$ such that $\lfloor h\rceil\lfloor i_X\rceil = \lfloor f\rceil$ and $\lfloor h\rceil\lfloor i_Y\rceil = \lfloor g\rceil$ then

$$\rho \leq \mathsf{P}_{hi_X \times f}\zeta \qquad \sigma \leq \mathsf{P}_{hi_Y \times g}\zeta.$$

Since $f = [f,g]i_X$ and $g = [f,g]i_Y$, by adjunctions we obtain

$$\exists_{i_X \times i_X}\rho \leq \mathsf{P}_{h\times[f,g]}\zeta \qquad \exists_{i_Y \times i_Y}\sigma\mathsf{P}_{h\times[f,g]}\zeta$$

and hence $\lfloor h\rceil = \lfloor [f,g]\rceil$. Distributivity of coproducts in $\overline{\mathscr{C}}$ means that given three objects (X,ρ), (Y,σ) and (Z,ζ) then the canonical arrow

$$\lfloor e\rceil : ((X\times Y)+(X\times Z), (\rho \boxtimes \sigma) \boxplus (\rho \boxtimes \zeta)) \to (X\times(Y+Z), \rho \boxtimes (\sigma \boxplus \zeta))$$

is an isomorphism. In order to prove that, it is enough to show the equality

$$(\rho \boxtimes \sigma) \boxplus (\rho \boxtimes \zeta) = \mathsf{P}_{e\times e}((\rho \boxtimes (\sigma \boxplus \zeta)).$$

For instance, the distibutivity of meets and joins implies that the part of the equation which concerns ρ and σ is obtained as follows:

$$\begin{aligned} \mathsf{P}_{e\times e}(\mathsf{P}_{\langle 1,3\rangle}\rho \wedge \mathsf{P}_{\langle 2,4\rangle}\exists_{i_Y\times i_Y}\sigma) &= \mathsf{P}_{e\times e}(\mathsf{P}_{\langle 1,3\rangle}\rho \wedge \exists_{1_X\times i_Y\times 1_X\times i_Y}\mathsf{P}_{\langle 2,4\rangle}\sigma && \text{(B-C)}\\ &= \mathsf{P}_{e\times e}\exists_{1_X\times i_Y\times 1_X\times i_Y}(\mathsf{P}_{1_X\times i_Y\times 1_X\times i_Y}\mathsf{P}_{\langle 1,3\rangle}\rho \wedge \mathsf{P}_{\langle 2,4\rangle}\sigma) && \text{(Frobenius)}\\ &= \exists_{i_{X\times Y}\times i_{X\times Y}}(\mathsf{P}_{\langle 1,3\rangle}\rho \wedge \mathsf{P}_{\langle 2,4\rangle}\sigma). \end{aligned}$$

In order to prove that coproducts are disjoint, we observe that the initial object of $\overline{\mathscr{C}}$ is $(0,\delta_0)$, where 0 is the initial object of $\mathscr{C}$. Hence, Remark 2.5.5 and the description of pullbacks in $\overline{\mathscr{C}}$ (see Lemma A.0.7) imply that the following diagram is a pullback

$$\begin{tikzcd} (0,\delta_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (Y,\sigma) \arrow[d, "\lfloor i_Y\rceil"] \\ (X,\rho) \arrow[r, "\lfloor i_X\rceil"'] & (X+Y, \rho\boxplus\sigma). \end{tikzcd}$$

The injections are obviously monomorphisms and it remains only to prove condition II of Definition 2.5.1. Given three arrows $\lfloor f\rceil$, $\lfloor g\rceil$ and $\lfloor h\rceil$ as below and the following pullbacks

$$\begin{tikzcd} (P_X, \mathsf{P}_{\{\!|\gamma_X|\!\}\times\{\!|\gamma_X|\!\}}\alpha\boxtimes\rho)) \arrow[r] \arrow[d] & (X,\rho) \arrow[d, "\lfloor g\rceil"] \\ (A,\alpha) \arrow[r, "\lfloor f\rceil"'] & (B,\beta) \end{tikzcd} \qquad \begin{tikzcd} (P_Y, \mathsf{P}_{\{\!|\gamma_Y|\!\}\times\{\!|\gamma_Y|\!\}}\alpha\boxtimes\sigma) \arrow[r] \arrow[d] & (Y,\sigma) \arrow[d, "\lfloor h\rceil"] \\ (A,\alpha) \arrow[r, "\lfloor f\rceil"'] & (B,\beta) \end{tikzcd}$$

where $\gamma_X := \mathsf{P}_{f\times g}\beta$ and $\gamma_Y := \mathsf{P}_{f\times h}\beta$ and $P_X := dom(\{\!|\gamma_X|\!\})$ and $P_Y := dom(\{\!|\gamma_Y|\!\})$, we want to prove that the sum of the above diagrams

$$\begin{tikzcd} (P_X+P_Y, \mathsf{P}_{\{\!|\gamma_X|\!\}\times\{\!|\gamma_X|\!\}}\alpha\boxtimes\rho)\boxplus\mathsf{P}_{\{\!|\gamma_Y|\!\}\times\{\!|\gamma_Y|\!\}}\alpha\boxtimes\sigma)) \arrow[r] \arrow[d] & (X,\rho)+(Y,\sigma) \arrow[d, "{[\lfloor g\rceil,\lfloor h\rceil]}"] \\ (A,\alpha) \arrow[r, "\lfloor f\rceil"'] & (B,\beta) \end{tikzcd} \tag{2.40}$$

is a pullback of the arrows $\lfloor f\rceil$ and $[\lfloor g\rceil,\lfloor h\rceil]$, which is given, up to isomorphism, by the diagram

$$\begin{tikzcd} (P_{X+Y}, \mathsf{P}_{\{\!|\gamma|\!\}\times\{\!|\gamma|\!\}}\alpha\boxtimes(\rho\boxplus\sigma)) \arrow[r] \arrow[d] & (X+Y,\rho\boxplus\sigma) \arrow[d, "{[\lfloor g\rceil,\lfloor h\rceil]}"] \\ (A,\alpha) \arrow[r, "\lfloor f\rceil"'] & (B,\beta) \end{tikzcd}$$

where $\gamma := \mathsf{P}_{f\times[g,h]}\beta$ and $P_{X+Y} := dom(\{\!|\gamma|\!\})$. But, since $e\circ f\times[g,h] = [f\times g, f\times h]$ if we denote with $\gamma' := \mathsf{P}_{[f\times g, f\times h]}\beta$, then we obtain that

$$\{\!|\gamma|\!\} = e\circ\{\!|\gamma'|\!\}$$

which is, by condition 3 of Definition 2.5.3, equal to $e\circ(\{\!|\gamma_X|\!\}+\{\!|\gamma_Y|\!\})$. Hence, the diagram (2.40) is a pullback of $\lfloor f\rceil$ and $[\lfloor g\rceil,\lfloor h\rceil]$. $\square$

Remark 2.5.8. In case of the elementary doctrine $\mathsf{Psub}_{\mathscr{C}}$ of a category with strict finite products an weak pullbacks, we obtain the characterization of Gran and Vitale in [GV98, Proposition 2.1]. Indeed, as already observed, in case of strict products, distributivity of coproducts implies strict of initials and that injections are monomorphisms. Hence, Definition 2.5.2 and Definition 2.5.3 coincide.

2.6 $\mathbf{Std}_0$ is a "relative" Π-pretopos

We end this chapter introducing a doctrinal version of pretopos in order to emphasize the structures which appear when taking the elementary quotient completions of the doctrines of types. The results of the previous sections imply that the elementary quotient completions of suitable elementary doctrines are instances of this definition. We prove that those results apply to the elementary doctrine F^{ML_0} of h-sets and h-prop. Hence, we obtain that the homotopy setoids provide a non-trivial example of a locally cartesian closed *relative pretopos*.

We first recall that a *pretopos* is a category which is both exact and extensive. A Π-pretopos is a locally cartesian closed pretopos. We refer to [Joh02; FS90] for further details about pretoposes. The structures we have considered are similar to pretopos but the behavior of quotients is regulated by a suitable elementary doctrine and not necessarily by the elementary doctrine of subobjects.

Definition 2.6.1. A *relative pretopos* is an elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ in $\mathbf{QD}$ such that $\mathscr{C}$ is an extensive category. In this case, $\mathscr{C}$ is said to be a pretopos relative to P.

Obviously, every pretopos $\mathscr{C}$ is a pretopos relative to $\mathsf{Sub}_{\mathscr{C}}$ and Theorem 2.3.7 and Theorem 2.5.6 imply the following corollary.

Corollary 2.6.2. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ an existential universal and implicational elementary doctrine with full weak comprehensions and comprehensive diagonals. If $\mathscr{C}$ has distributive finite coproducts, P has finite joins and every slice $\mathscr{C}/A$ has P-extensional exponentials, then $\mathscr{C}$ is a Π-pretopos relative to $\overline{\mathsf{P}}$.*

Remark 2.6.3. As already mentioned, the above result generalizes various properties of the exact completion to the case of the elementary quotient completion. However, there exists a different notion of quotient completion which has not been mentioned yet. In [BM18], the authors introduce the *path categories*, a categorical framework to deal with homotopy theory, and a construction called *homotopy exact completion* which adds quotients of *homotopy equivalence relations*. If $\mathscr{C}$ is a path category, the homotopy exact completion $\mathrm{Hex}(\mathscr{C})$ of $\mathscr{C}$ turns out to be an exact category and, among the various results, the authors prove two results about the local cartesian closure and the lextensivity of $\mathrm{Hex}(\mathscr{C})$. In order to do that, the authors define the notions of *weak homotopy exponential* and *homotopy extensivity* taking into account the homotopical structure of the path categories.

The author is not aware of the precise relation between the homotopy exact completion and the elementary quotient completion. However, it is plausible that former is an instance of the latter. In fact, a future investigation would be to study if the structure of a path category $\mathscr{C}$ gives rise to an elementary doctrine which associates to every objects $X \in \mathscr{C}$ the poset reflection of the *fibrations* over X. Achieved that, it would become an exercise to verify that the results obtained in [BM18] are particular instances of Corollary 2.6.2.

The category of setoids can be obtained as an instance of the homotopy exact completion. However, since this construction leads to an exact category, the homotopy setoids cannot be obtained in this way. One possibility could be to obtain the category of homotopy setoids as a variation of the homotopy exact completion, in which one considers only particular equivalence relations such that the underlying fibrations have somehow “contractible” fibers.

We now come back to homotopy setoids and prove that they are a relative Π-pretopos. In order to do that, we first prove that the elementary doctrine F^{ML_0} satisfies hypothesis of Theorem 2.3.7 and Theorem 2.5.6.

Extensivity of $\mathbf{Std}_0$. It is well known that the syntactic category arising from a type theory with sum types has coproducts. A discussion about that can be found for example in [EP20]. We now briefly recall the main points and conclude that Theorem 2.5.6 applies to the elementary doctrine F^{ML_0}.

The coproduct of two closed types X and Y is given by the sum type $X+Y$ and the initial object is given by the empty type **0**. In particular, if X and Y are h-sets, then the sum is an h-set and the same holds for every homotopy type, see [Uni13]. In §2.12 of *loc.cit.*, there is an explicit description of the identity types of the elements of $X+Y$ which is:

$$\begin{aligned}\mathsf{Id}_{X+Y}(\mathsf{inl}(\mathsf{x_1}),\mathsf{inl}(\mathsf{x_2})) &\cong \mathsf{Id}_X(x_1,x_2)\\ \mathsf{Id}_{X+Y}(\mathsf{inr}(\mathsf{y_1}),\mathsf{inr}(\mathsf{y_2})) &\cong \mathsf{Id}_X(y_1,y_2)\\ \mathsf{Id}_{X+Y}(\mathsf{inl}(\mathsf{x}),\mathsf{inr}(\mathsf{y})) &\cong \mathbf{0}\end{aligned} \tag{2.41}$$

The first two conditions imply that in **ML**, and hence in $\mathbf{ML}_0$, the injections

$$\begin{aligned}i_X &:= \lfloor \mathsf{inl} \rceil : X \to X+Y\\ i_Y &:= \lfloor \mathsf{inr} \rceil : Y \to X+Y\end{aligned} \tag{2.42}$$

are monomorphisms. The last condition of (2.41) and the fact that **0** is an h-set imply that coproducts are disjoint both in **ML** and $\mathbf{ML}_0$. The distributivity of coproducts corresponds to proving that the arrow represented by the term

$$\mathsf{Ind}_+(1_X\times i_Y, 1_X\times i_Z) : (X\times Y)+(X\times Z)\to X\times(Y+Z) \tag{2.43}$$

induced by the induction principle of coproducts

$$(A\to C)\to(A\to C)\to(A+B\to C)$$

is an isomorphism. Using again the recursion principle with $A := Y$, $B = Z$ and $C := X \to (X\times Y)+(X\times Z)$ we obtain

$$\mathsf{Ind}_+(i_{X\times Y}, i_{X\times Z}) : (Y+Z)\to(X\to(X\times Y)+(X\times Z)) \tag{2.44}$$

such that

$$\begin{aligned}\mathsf{Ind}_+(i_{X\times Y}, i_{X\times Z})(x, i_Y y) &:= i_{X\times Y}(x,y)\\ \mathsf{Ind}_+(i_{X\times Y}, i_{X\times Z})(x, i_Z z) &:= i_{X\times Z}(x,z).\end{aligned} \tag{2.45}$$

An easy verification show that the terms in (2.43) and in (2.44) induces inverse arrows in **ML** and $\mathbf{ML}_0$. It remains to prove condition 3 of Definition 2.5.3 which corresponds to the fact that given two arrows $\lfloor f\rceil : X\to A$ and $\lfloor g\rceil : Y\to A$ and a dependent type $a : A\vdash P(a)$, then the arrow

$$\mathsf{Ind}_+(f(\pi_X), g(\pi_Y)) : \sum_{x:X} P(f(x)) + \sum_{y:Y} P(g(y)) \to A \tag{2.46}$$

is a comprehension of the dependent type

$$z : X+Y \vdash P(\mathsf{ind}_+(f,g)(z)).$$

This follows from the equivalence between the types

$$\sum_{z:X+Y} P(\mathsf{rec}(f,g)(z)) \qquad\qquad \sum_{x:X} P(f(x)) + \sum_{y:Y} P(g(y))$$

which is a more general form of distributivity of sums and products.

Remark 2.6.4. We underline the fact that the results and definitions of this section hold also for elementary doctrines not necessarily with all finite joins. Indeed, we actually used the joins only in case of the coproduct $X+Y$ of two objects $X, Y \in \mathscr{C}$. In particular, we exploited the P-eq. relation $\rho \boxplus \sigma$ on $X+Y$ built from two P-equivalence relations $\rho \in \mathsf{P}(X\times X)$ and $\sigma \in \mathsf{P}(Y\times Y)$ and we needed the distributivity of $\boxplus$ and $\boxtimes$ and the Frobenius recirocity. In the following lemma, we prove that this holds for elementary doctrine F^{ML_0} of h-sets and h-props, which does not have all finite joins as observed in $Remark$ 2.1.5.

Lemma 2.6.5. *If X and Y are two types and $x_1, x_2 : X \vdash R$ and $y_1, y_2 : Y \vdash S$ are dependent types, then the type*

$$\begin{aligned} R \boxplus S := &\sum_{x_1,x_2:X} R(x_1,x_2) \times \mathsf{Id}_{X+Y}(z_1, i_X(x_1)) \times \mathsf{Id}_{X+Y}(z_2, i_X(x_2)) + \\ &+ \sum_{y_1,y_2:X} S(y_1,y_2) \times \mathsf{Id}_{X+Y}(z_1, i_Y(y_1)) \times \mathsf{Id}_{X+Y}(z_2, i_Y(y_2)) \end{aligned}$$

which depends on $z_1, z_2 : X+Y$, is an h-proposition.

Proof. The argument follows from Lemma 2.1.7. Indeed, due to the description of the identity type of the sum $X+Y$, the two components of the above sum can not be simultaneously inhabited. □

Hence, the categories **ML** and **ML**$_0$ are respectively F^{ML^+} and F^{ML_0}-weakly extensive. The same holds for F^{ML}. Theorem 2.5.6 implies the following result.

Corollary 2.6.6. *The categories $\overline{\textbf{ML}}$ and $\overline{\textbf{ML}_0}$ are extensive.*

A discussion about coproducts in **Std** can be found in [Wil10]. The author shows that, in order to have disjoint coproducts, it is necessary to assume a universe in the type theory. The argument relies on the Smith's model of Martin-Löf *intuitionistic type theory* without universes [Smi88]. For a discussion about extensivity of setoids arising from a different type theory, we refer to [Mai09].

Local cartesian closure of Std$_0$. We now prove that F^{ML_0} satisfies the hypothesis of Theorem 2.3.2 and Theorem 2.3.7. The following result was already observed for the elementary doctrine F^{ML} in [MR13, Proposition 7.3].

Proposition 2.6.7. *The elementary doctrine F^{ML_0} is implicational, universal and* **ML**$_0$ *is weakly cartesian closed.*

Proof. The implication is given by the arrow type: if X is an h-set, and $A(x)$ and $B(x)$ are h-propositions, then

$$x : X \vdash A(x) \to B(x)$$

is a particular form of dependent product type and by Remark 2.1.5 it is an h-proposition. The adjunction property of the implication is trivially verified by *Currying* and λ-abstraction.

We now prove that f^{ML_0} has right adjoint to all reindexings. Given an arrow $f : X \to Y$ and an h-proposition $x : X \vdash B(x)$, define the type

$$y : Y \vdash \prod_{x:X} (\mathsf{Id}_Y(f(x), y) \to B(x)) \tag{2.47}$$

which is an h-proposition because of Remark 2.1.5. Again the adjunction property follows from λ-abstraction. The Beck-Chevalley condition on pullback diagrams

$$\begin{tikzcd} V := \sum_{x:X}\sum_{y:Y} \mathsf{Id}_Z(f(x), g(y)) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is given, for every $x : X \vdash B(x)$, by the logical equivalence of the types

$$\prod_{v:V} \mathsf{Id}_Y(\pi_2(v), y) \to B(\pi_1(v)) \qquad \prod_{x:X} \mathsf{Id}_Z(f(x), g(y)) \to B(x)$$

given by the arrow

$$h(v, p) \longmapsto \tilde{h}(x, q) := h(x, y, p, \mathsf{refl}_y).$$

A weak exponential of the h-sets X and Y is given by the arrow type $X \to Y$ which is an h-set because we have assumed the functional extensionality axiom. The term

$$f : X \to Y, x : X \vdash \lambda x.f(x) : Y$$

provides an evaluation e, which is actually strict because we have assumed the functional extensionality axiom.

□

Remark 2.6.8. The main difference between the syntactic category **ML** arising from $\mathcal{ML}$ and the one arising from $\mathcal{ML}+F.E.$ is that the former has only weak exponentials while the latter has strict exponentials due to the functional extensionality axiom.

Proposition 2.6.9. *The elementary doctrine F^{ML_0} has right adjoints to weak pullback projections and the slices of $\mathbf{ML}_0$ have F^{ML_0}-extensional exponentials.*

Proof. The first part follows from Proposition 2.6.7. For the second part, consider an h-set A and two arrows $f : X \to A$ and $g : Y \to A$. If S is an equivalence relation on Y which is an h-proposition, we build the extensional exponential of f and g with respect to S by steps. First we consider the dependent type

$$a : A \vdash \sum_{x:X} \mathsf{Id}_A(f(x), a) \to \sum_{y:Y} \mathsf{Id}_A(g(y), a) \tag{2.48}$$

and denote it with $\mathsf{Fun}^{\mathsf{f}}_{\mathsf{g}}(a)$, which is an h-set. Second we consider the type

$$\sum_{a:A} \sum_{m:\mathsf{Fun}^{\mathsf{f}}_{\mathsf{g}}(a)} \prod_{x_1,x_2:X} \prod_{p_1:\mathsf{Id}_a(f(x_2),a)} \prod_{p_2:\mathsf{Id}_a(f(x_2),a)} (\mathsf{Id}_A(x_1, x_2) \to S(\pi_1(m(x_1, p_1)), \pi_1(m(x_1, p_1)))), \tag{2.49}$$

and denote it with W. Intuitively, we collected the arrows between the fibers $\mathsf{fib}_f(a)$ and $\mathsf{fib}_g(a)$ such that equal objects of $\mathsf{fib}_f(a)$ are sent to objects of $\mathsf{fib}_g(a)$ which have S-related Y-components. Remark 2.1.5 implies that the type W is an h-set and we now prove that the projection

$$\pi_1 : W \to A$$

is an extensional exponential of x and y with respect to S. Recall that a weak product of π and f is given by the common value of the common value of the two composites

$$\begin{tikzcd} V := \sum_{w:W,x:X} \mathsf{Id}_A(f(x), \pi(w)) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\ W \arrow[r, "\pi"'] & A. \end{tikzcd} \tag{2.50}$$

in fact, as we observed in 2.2 the above diagram is a strict pullback. The evaluation arrow $e : V \to Y$ is given by the term

$$(w, x, p) : V \vdash \pi_1(\pi_2(w)(x, p)) : Y$$

and obviously $g \circ e = f \circ \pi_2$. The evaluation preserves projections w.r.t. S. Indeed, if $((a, m, t), x, p)$ and $((a', m', t'), x', p')$ are terms of V such that $p_1 : \mathsf{Id}_W((a, m, t), (a', m', t'))$ and $p_2 : \mathsf{Id}_X(x, x')$, then we can transport along the A-component of $p_1^A : \mathsf{Id}_A(a, a')$ the function $m : \mathsf{Fun}^{\mathsf{f}}_{\mathsf{g}}(a)$ to obtain a function of $p_1^{A*}(m) : \mathsf{Fun}^{\mathsf{f}}_{\mathsf{g}}(a')$ which is defined as follows

$$p_1^{A*}(m)(x', p') := m(x', p' \cdot p_1^{-1})$$

where $p' \cdot p_1^{-1}$ denotes the concatenation of p' with the inverse path of p_1. Through p_1 we obtain a term of $\mathsf{Id}_{\mathsf{Fun}^{\mathsf{f}}_{\mathsf{g}}(a')}(p_1^*(m), m')$. Hence, since $m(x', p' \cdot p_1^{-1})$ and $m(x, p)$ have S-related Y-components, we obtain that $m'(x', p')$ and $m(x, p)$ have S-related Y-components. The evaluation is strict because we assumed the functional extensionality axiom. □

Remark 2.6.10. We observe that a slight modification of the extensional exponential type (2.49) of two objects $x : X \to A$ and $y : Y \to A$ with respect to an eq. relation S on Y. Indeed, as observed in Observation 2.4.3 substituting Id_Y to S in (2.49) we obtain a weak exponential of x and y. Similarly, substituting $\mathbf{1}$ to S in (2.49) we obtain a very weak exponential of x and y. A very weak exponential of x and y is given by the arrow

$$\pi : \sum_{a:A} \big(\sum_{x:X} \mathsf{Id}_A(f(x), a) \to \sum_{y:Y} \mathsf{Id}_A(g(y), a) \big) \to A. \tag{2.51}$$

Applying Theorem 2.3.2 we obtain that $\mathbf{Std}_0$ is cartesian closed. For the local cartesian closure we need to do some observations.

Remark 2.6.11. As we have proved in Proposition 2.2.2, F^{ML_0} is not existential, but it has left adjoint to the reindexings over monomorphisms. Hence, even if we proved Proposition 2.6.7 and Proposition 2.6.9, we can not apply Theorem 2.3.7 to F^{ML_0} . However, a deeper look at the proof of Theorem 2.3.7 shows that we only used left adjoints of reindexings of comprehensions and of product of comprehensions, and the Beck-Chevalley condition on the weak pullback diagrams. For F^{ML_0}, the comprehensions are monomorphisms and the Beck-Chevalley condition of the left adjoints for weak pullback diagrams follows from the Beck-Chevalley condition of the right adjoints, see Lemma A.0.17. Hence, applying Theorem 2.3.7 we obtain the following result.

Theorem 2.6.12. *The category* ***Std***$_0$ *is locally cartesian closed.* □

Observation 2.6.13. We now observe that the elementary doctrine F^{ML_0} of h-sets is in the class of the elementary doctrines for which we could give a more direct proof of Theorem 2.3.7.

Indeed, in case of elementary doctrines $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ with strict full comprehensions and left adjoint to reindexings of monomorphisms it is possible to build the *slice doctrines* of P. We now provide the main ideas of the construction without the details. The argument will be treated in depth in the next chapter.

Given an object $A \in \mathscr{C}$ we can define the functor

$$\mathsf{P}_{/A} : \mathscr{C}/A^{op} \to \mathsf{InfSL} \tag{2.52}$$

which sends an object $x : X \to A$ of $\mathscr{C}/A$ to the poset $\mathsf{P}_{/A} := \mathsf{P}(A)$, and an arrow $h : x \to y$ from x to $y : Y \to A$ to $\mathsf{P}_{/A}(h) := \mathsf{P}_h$. Since $\mathscr{C}/A$ has strict products, that can be built through the strict comprehensions, the functor $\mathsf{P}_{/A}$ inherits the elementary structure from P. If $x \times_A x$ denotes the common value of the two composites of the pullback

$$\begin{tikzcd}
X \times_A X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "x"] \\
X \arrow[r, "x"'] & A
\end{tikzcd}$$

the fibered equality of x is given by the reindexing $\delta_x := \mathsf{P}_{\langle \pi_1, \pi_2 \rangle}\delta_X \in \mathsf{P}_{/A}(x \times_A x)$. In this case, it is a trivial computation to prove that δ_x satisfies the axioms of Definition 1.2.5. A $\mathsf{P}_{/A}$-eq. relation on x corresponds to an element $\mathsf{P}(X \times_A X)$ which satisfies reflexivity, symmetry and transitivity conditions. The left adjoints to the reindexings over monomorphisms give a correspondence between $\mathsf{P}_{/A}$-eq.relations and P-eq.-relations. Indeed, it happens that r is a $\mathsf{P}_{/A}$-eq. relation ox x if and only if $\exists_{\langle \pi_1, \pi_2 \rangle} r$ is a P-eq. relation on X and

$$\delta_X \leq \exists_{\langle \pi_1, \pi_2 \rangle} r \leq \mathsf{P}_{x \times x}\delta_A. \tag{2.53}$$

This correspondence gives a practical description of the elementary quotient completion of $\mathsf{P}_{/A}$ in terms of the elementary quotient completion of P i.e.:

$$\overline{\mathsf{P}_{/A}} \cong \overline{\mathsf{P}}_{/(A,\delta_A)}. \tag{2.54}$$

Hence, it is possible to prove Theorem 2.3.7 just applying Theorem 2.3.2 and Remark 2.3.9 to the slice doctrines of P and provide explicitly the exponentials of the slices of the form $\overline{\mathsf{P}}_{/(A,\alpha)}$.

Remark 2.6.14. The above discussion was is of the main motivation that led us to the investigations of the next chapter. Indeed, we remark that the above construction relies on the fact that the base category $\mathscr{C}$ of the elementary doctrines $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ considered have strict pullbacks. Hence, the slices of $\mathscr{C}$ have strict finite product and the slice functors are still in the realm of the doctrines. However, in lots of cases, such as F^{ML}, the slices of the base category have only weak finite products.

Another important aspect is that in the proof of Theorem 2.3.7, we actually repeated the arguments of Theorem 2.3.2, assuming a different notion of exponential, for the slice categories. Lot of work has been done just because we could not consider the functor that P induces on the slices as an elementary doctrine, because they lack of strict finite products.

Moreover, the results obtained for the local cartesian closure and the extensivity of the elementary quotient completion obtain the results about the exact completion, only in the case of categories with strict finite products and weak pullbacks. However, [GV98, Proposition 2.1],[CR00, Theorem 3.3] and [Emm20, Theorem 3.6] are stated for categories in which also finite products are weak. A similar observation can be done for the exact completion which is a particular instance of the elementary quotient completion only for categories with strict products and weak pullbacks.

In the next chapter, we will provide a more general framework which takes into account these considerations.

We conclude this chapter observing that Remark 2.2.4,Theorem 2.6.12 and Corollary 2.6.6 imply the following result.

Corollary 2.6.15. *The categories $\overline{\boldsymbol{ML}}$ and $\overline{\boldsymbol{ML}_0}$ are pretopos relative to F^{ML^+} and F^{ML_0} respectively.*

Hence, we obtained that the homotopy setoids form a non-trivial example of relative Π-pretopos. Another non-trivial example of relative pretopos could be obtained for setoids arising from the *minimal type theory* (mTT) introduced by Maietti and Sambin, see [MS05] and [Mai07].

Concluding remarks and further developments. The above result about the homotopy-setoids, can be also interpreted as follows. We mentioned that in [RS15] the authors considered the category of h-sets of the homotopy type theory. Assuming the *univalence axiom* and various *higher inductive* types, it is possible to obtain, internally to that type theory, a category which resembles the category **Set** of sets and functions, which is a well-knows topos. In [RS15, Theorem 2.2] the authors prove that the h-sets form a ΠW-pretopos and, assuming also the *resizing rule*, it becomes a topos. Hence, h-sets provides the corresponding notion of "set" in that type theoretic framework. What we did actually is to detect the corresponding category of sets in a weaker type theory that does not have the sophisticated type constructors of the homotopy type theory. But, it will be part of future research to consider W-types in our context.

The formalization of setoids in proof assistants has been deeply investigated in [BCP03]. One future development will be to implement in a proof assistant, based on Agda [CC99] or Coq [HKPM02], the homotopy setoids through one of the notion of category internal to the type theory, such as the *E-categories* and *H-categories* introduced in [Pal18] or the *pre-categories* introduced in [AKS15];

Other future developments will be to investigate the connections between the elementary quotient completion and the homotopy exact completion of [BM18], and the possibility to obtain the homotopy setoids of a variation of the latter completion as discussed in Remark 2.6.3.

Chapter 3

Biased doctrines

The aim of this chapter is to generalize the notion of elementary doctrine and of elementary quotient completion, for functors where the base category may have just weak finite products. In Remark 2.6.14, we underlined the main reasons which motivates this generalization. In particular, we obtain a more general framework in which the corresponding elementary quotient completion recover the exact completion of a category with weak finite products and weak pullbacks.

In the first part, we will introduce the notion of *biased elementary doctrine* which is a suitable controvariant functor, from a category with weak finite products into the category of posets. The elementary doctrines are a particular instance of the new framework and many other examples which were not in the realm of the elementary doctrines, due to the lack of strict products in the base categories, will be discussed. In Section 3.2, we will provide the main definitions and we will discuss the following motivational examples:

- the functor $\mathsf{Psub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{InfSL}$ for a weakly left exact category $\mathscr{C}$,
- the *slice doctrine* $\mathsf{P}_{/A} : (\mathscr{C}/A)^{op} \to \mathsf{InfSL}$ of a biased elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ and object $A \in \mathscr{C}$.

In Section 3.2 we will discuss the fundamental notion of *proof-irrelevant* elements and in Section 3.3 we will provide a construction which associates a strict elementary doctrine to a biased elementary doctrine in a universal way. In Section 3.4 we provide the corresponding quotient completion, which generalize both the elementary quotient completion and the exact completion of a weakly left exact category as provided in [CV98].

The second part of the chapter will be focused on the generalizations of the theorems obtained in Sections 2.3 and 2.6 for the strict elementary doctrines. In Section 3.5, we will define biased elementary doctrines which can deal with implication and existential and universal quantifications. In Section 3.6 we will provide some results about the exact completion of the slice doctrines of a biased elementary doctrine, which will be useful in Section 3.7.

Before starting, we fix some notations about weak finite products.

Notation. If $\mathscr{C}$ is a category with weak finite products and $X_1, \dots, X_n$ are objects of $\mathscr{C}$, then we can obtain a weak product of those objects in different not isomorphic ways. Indeed, if we have

- a weak product $\mathsf{p}_i : W \to X_i$, for $i = 1, \dots m$, of the objects $X_1, \dots, X_m$.
- a weak product $\mathsf{p}_i : V \to X_i$, for $i = m+1, \dots, n$, of the objects $X_{m+1}, \dots, X_n$.

We can obtain a weak product of the objects $X_1, \dots, X_n \in \mathscr{C}$ through a binary weak product $W \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\to} V$. Moreover, given an assignment $j : \{1, \dots, k\} \to \{i, \dots, n\}$ with $1 \leq k$, and a weak product

Z of the objects $X_{j(c)}$, for $c = 1, \dots, k$, we will abuse the notation and denote with $\langle j(1), \dots, j(k) \rangle : U \to Z$ the arrow induced by the composition of the projections $\mathsf{p}_{j(t)} \circ \mathsf{p}_1$, if $j(t) \leq m$, and $\mathsf{p}_{j(t)} \circ \mathsf{p}_2$, if $m + 1 \leq j(t)$.

3.1 Biased elementary doctrines

The underlying categorical structure of this section will be controvariant functors from categories with weak finite products to the category of posets. We will refer to such functors as *biased doctrines*. In this section we provide a definition of *primarity* and *elementarity* for biased doctrines and discuss some examples which naturally fit the new context.

The modular correspondence between the categorical structures and logic is especially beneficial here: the conditions in Definition 1.2.1 are mutually independent and we can rewrite them when $\mathscr{C}$ has just weak finite products as follows.

Definition 3.1.1. Let $\mathscr{C}$ be a category with weak finite products. A *biased primary doctrine* is a functor $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Pos}$ which takes value in the category InfSL of inf-semilattices, i.e.:

P1 for every object $X \in \mathscr{C}$, $\mathsf{P}(X)$ has finite meets

P2 for every arrow $f : X \to Y$ in $\mathscr{C}$, the map $\mathsf{P}_f : \mathsf{P}(Y) \to \mathsf{P}(X)$ preserves finite meets.

Contrary to the definition of biased primary doctrine, the conditions in Definition 1.2.3, which characterize the elementary structure, are interdependent and the (strict) products in the base category $\mathscr{C}$ played a key role in it. Since two weak products of the same objects need not be isomorphic, we have to devise a way so that fibered equalities, which shall now become *biased fibered equalities*, interact appropriately. As we expect, an elementary doctrine shall satisfy also the following Definition 3.1.2, but we will discuss in detail the relationship in the section.

Definition 3.1.2. Let $\mathscr{C}$ be a category with weak finite products. A *biased elementary doctrine* is a biased primary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$, such that, for every object $X \in \mathscr{C}$ and for each choice of weak product $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\rightarrow} X$ there exists an element $\delta_X^W \in \mathsf{P}(W)$ satisfying:

wI For every arrow $Z \xrightarrow{d} W$, with $\mathsf{p}_1 \circ d = \mathsf{p}_2 \circ d$, it is $\top_Z \leq \mathsf{P}_d(\delta_X^W)$.

wII $\mathsf{P}(X) = \mathcal{D}es_{\delta_X^W} := \{\alpha \in \mathsf{P}(X) \mid \mathsf{P}_{\mathsf{p}_1}\alpha \wedge \delta_X^W \leq \mathsf{P}_{\mathsf{p}_2}\alpha\}$.

wIII If $f : Y \to X$ is an arrow of $\mathscr{C}$, then for every choice of weak product $Y \overset{\mathsf{p}_1}{\leftarrow} V \overset{\mathsf{p}_2}{\rightarrow} Y$ and for every arrow $g : V \to W$ such that $\mathsf{p}_i \circ g = f \circ \mathsf{p}_i$, we have

$$\delta_Y^V \leq \mathsf{P}_g \delta_X^W.$$

wIV For every choice of weak product $W \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\rightarrow} W$ and arrows $U \underset{\langle 2,4 \rangle}{\overset{\langle 1,3 \rangle}{\rightrightarrows}} W$,

$$\delta_X^W \in \mathcal{D}es_{\mathsf{P}_{\langle 1,3 \rangle}(\delta_X^W) \wedge \mathsf{P}_{\langle 2,4 \rangle}(\delta_X^W)}.$$

Observation 3.1.3. As expected, every elementary doctrine is actually a biased elementary doctrine. Indeed, for every object $X \in \mathscr{C}$ and weak product $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\rightarrow} X$, there exists a unique map $\langle \mathsf{p}_1, \mathsf{p}_2 \rangle : W \to X \times X$ into the strict product $X \times X$. The biased fibered equalities are given by the reindexings $\delta_X^W := \mathsf{P}_{\langle \mathsf{p}_1, \mathsf{p}_2 \rangle}(\delta_X) \in \mathsf{P}(W)$. The element δ_X^W satisfies the conditions of the above definition because δ_X satisfies the conditions in Definition 1.2.5.

We now consider functors that are not in the realm of elementary doctrines since the base categories have only weak finite products. However, we can prove that they are examples of biased elementary doctrines.

Example 3.1.4 (Slice doctrine)**.** Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine with weak comprehensions and comprehensive diagonals in the sense of Definition 1.2.5. Since weak pullbacks in $\mathscr{C}$ can be obtained through comprehensions, see A.0.7, it follows that the slices $\mathscr{C}/A$ have weak products for every object $A \in \mathscr{C}$. The functor

$$\mathsf{P}_{/A} : (\mathscr{C}/A)^{op} \to \mathsf{InfSL}$$

is defined on an object $(f : X \to A) \in \mathscr{C}/A$ as $\mathsf{P}_{/A}(f) := \mathsf{P}(X)$ and on an arrow $h : f \to g$ of $\mathscr{C}/A$ as $\mathsf{P}_{/A}(h) := \mathsf{P}_h$. We will refer to the functor $\mathsf{P}_{/A}$ as the *slice doctrine* over A. We can now prove that the slice doctrine $\mathsf{P}_{/A} : (\mathscr{C}/A)^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine for all object $A \in \mathscr{C}$. Indeed, let $x : X \to A$ be an object of $\mathscr{C}/A$ and consider a weak product of $x \overset{p_1}{\leftarrow} w \overset{p_2}{\rightarrow} x$ given by the common value of the composites of the following weak pullback diagram

$$\begin{tikzcd}
X \times_A X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "x"] \\
X \arrow[r, "x"'] & A.
\end{tikzcd}$$

The elementary structure is obtained setting $\delta^w_x := \mathsf{P}_{\langle \pi_1, \pi_2 \rangle}\delta_X \in \mathsf{P}_{/A}(w)(:= \mathsf{P}(X \times_A X))$. Conditions **wI** and **wII** of Definition 3.1.2 trivially follow from conditions **I** and **II** of Definition 1.2.5. We now prove that condition **wIII** holds. Let $y : Y \to A$ be an object of $\mathscr{C}/A$ and let $v : Y \times_A Y \to A$ a weak product of v and v. Given two arrows as in the following commutative diagrams

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "y"'] & & X \arrow[dl, "x"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \times_A Y \arrow[rr, "g"] \arrow[dr, "v"'] & & X \times_A X \arrow[dl, "w"] \\
& A &
\end{tikzcd}$$

such that $\pi_i \circ g = f \circ \pi_i$, for $i = 1, 2$, then we obtain condition **wIII** as follows

$$\begin{aligned}
(\mathsf{P}_{/A})_g \delta^w_x &:= \mathsf{P}_g \mathsf{P}_{\langle \pi_1, \pi_2 \rangle}\delta_X \\
&= \mathsf{P}_{\langle \pi_1, \pi_2 \rangle}\mathsf{P}_{f \times f}\delta_X \\
&\geq \mathsf{P}_{\langle \pi_1, \pi_2 \rangle}\delta_Y \\
&:= \delta^v_y.
\end{aligned}$$

We now prove condition **wIV**. Let $u : U \to A$ be a weak product of w and w and let $h : U \to X \times X \times X \times X$ be the unique arrow induced on the strict product $X \times X \times X \times X$. If $\langle 1, 3 \rangle_{/A}, \langle 2, 4 \rangle_{/A} : u \to w$ are arrows induced by the composition $\pi_1 \circ \pi_1, \pi_2 \circ \pi_1$ and $\pi_1 \circ \pi_2, \pi_2 \circ \pi_2$, then we obtain condition **wIV** as follows

$$\begin{aligned}
&(\mathsf{P}_{/A})_{\pi_1}\delta^w_x \wedge (\mathsf{P}_{/A})_{\langle 1,3 \rangle_{/A}}\delta^w_x \wedge (\mathsf{P}_{/A})_{\langle 2,4 \rangle_{/A}}\delta^w_x \\
&:= \mathsf{P}_h(\mathsf{P}_{\langle 1,2 \rangle}\delta_X \wedge \mathsf{P}_{\langle 1,3 \rangle \delta_X} \wedge \mathsf{P}_{\langle 2,4 \rangle \delta_X}) \\
&\leq \mathsf{P}_h \mathsf{P}_{\langle 3,4 \rangle \delta_X} \\
&:= (\mathsf{P}_{/A})_{\pi_2}\delta^w_x.
\end{aligned} \tag{III}$$

Example 3.1.5 (Weak subobjects)**.** For a given category $\mathscr{C}$ with weak finite limits, the functor of weak subobjects $\mathsf{Psub}_{\mathscr{C}}$ of Example 1.2.6 is actually a biased elementary doctrine. Indeed, if X is an object of $\mathscr{C}$ and $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\rightarrow} X$ is a weak product, the element δ_X^W is given by the equivalence class, in the poset reflection of $\mathscr{C}/W$, of the right dashed arrow of the following weak limit:

$$\begin{tikzcd}
& D \arrow[dl, dashed] \arrow[dr, dashed, "\delta_X^W"] & \\
X \arrow[d, "1_X"'] \arrow[drr, "1_X" near start] & & W \arrow[d, "\mathsf{p}_2"] \arrow[dll, "\mathsf{p}_1"' near start] \\
X & & X.
\end{tikzcd}$$

Observe that δ_x^w is an equalizer of the arrows $W \underset{\mathsf{p}_2}{\overset{\mathsf{p}_1}{\rightrightarrows}} X$.

We now compare Definition 1.2.5 and Definition 3.1.2. Condition **wI** of Definition 3.1.2 is a more general formulation of condition **I** of Definition 1.2.5. Condition **wII** of Definition 3.1.2 is the same of **II** of Definition 1.2.5, i.e., for all elements $\alpha \in \mathsf{P}(X)$ it follows that

$$\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \delta_X^W \leq \mathsf{P}_{\mathsf{p}_2}\alpha.$$

Moreover, it is straightforward to prove that conditions **wIII** and **wIV** of Definition 3.1.2 are satisfied for strict elementary doctrines. We now ask if condition **III** of Definition 1.2.5 holds for biased elementary doctrines. Actually, we know that for the strict elementary doctrines we have the equality of the elements

$$\delta_{X\times Y} = \delta_X \boxtimes \delta_Y,$$

(see Proposition A.0.3). For the biased elementary doctrines, the above identity does not hold in general, as we will discuss in Example 3.2.5. However, in the following lemma, we prove that the inequality $\delta_{X\times Y} \leq \delta_X \boxtimes \delta_Y$ holds for biased elementary doctrines.

Lemma 3.1.6. *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine, then the following conditions hold*

i) For every pair of objects $X, Y \in \mathscr{C}$, if $X \overset{\mathsf{p}_1}{\leftarrow} Z \overset{\mathsf{p}_2}{\rightarrow} Y$ is a weak product, then for each choice of weak products $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\rightarrow} X$, $Y \overset{\mathsf{p}_1}{\leftarrow} V \overset{\mathsf{p}_2}{\rightarrow} Y$, and $Z \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\rightarrow} Z$ and arrows $U \xrightarrow{\langle 1,3\rangle} W$, $U \xrightarrow{\langle 2,4\rangle} V$, it follows that

$$\delta_Z^U \leq \mathsf{P}_{\langle 1,3\rangle}\delta_X \wedge \mathsf{P}_{\langle 2,4\rangle}\delta_Y.$$

ii) If W and W' are weak products of X and X, then for any arrow $h : W' \to W$ such that $\mathsf{p}'_i = \mathsf{p}_i \circ h$, for $i = 1, 2$, the fibered equalities $\delta_X^W \in \mathsf{P}(W)$ and $\delta_X^{W'} \in \mathsf{P}(W')$ are related by the following inequality

$$\delta_X^{W'} \leq \mathsf{P}_h(\delta_X^W).$$

Proof. They follow from condition **wIII**. □

For any object $X \in \mathscr{C}$, Definition 3.1.2 requires the verification of certain conditions for every choice of weak product of X and itself. We now obtain a description of biased elementary doctrines which only depends on a choice of weak products. In order to do that, we discuss some fundamental properties of the fibered equalities δ_X. Before we define a particular class of elements in the fibers of the weak products.

Definition 3.1.7. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased primary doctrine and let $X_1, \ldots, X_n$ be objects of $\mathscr{C}$. Let W be a weak product of $X_1, \ldots, X_n$ with projections $\mathsf{p}_i : W \to X_i$, for $i = 1, \ldots, n$, and $\beta \in \mathsf{P}(W)$, we say that the reindexing $\mathsf{P}_t(\beta)$ of β along an arrow $T \xrightarrow{t} W$ is *determined by projections* if, for every arrow $T \xrightarrow{t'} W$ such that $\mathsf{p}_i \circ t = \mathsf{p}_i \circ t'$ for $i = 1, \ldots, n$,

$$\mathsf{P}_t(\beta) = \mathsf{P}_{t'}(\beta).$$

An element $\beta \in \mathsf{P}(W)$ is *reindexed by projections* if, for every object T and arrow $T \xrightarrow{t} W$, the reindexing of β along t is determined by projections.

The importance of the above definition is that, in case of weak products, two arrows $t, t' : T \to W$ that have the same projections are not necessarily equal. However, the functors P_t and $\mathsf{P}_{t'}$ have the same values on the elements of $\mathsf{P}(W)$ that are reindexed by projections. The fibered equalities δ_X^W of Definition 3.1.2 have enough properties to be reindexed by projections. Actually, less is needed, as it is shown in the following lemma.

Lemma 3.1.8. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased primary doctrine and let X be an object of $\mathscr{C}$. Assume that $X \xleftarrow{\mathsf{p}_1} W \xrightarrow{\mathsf{p}_2} X$ is a weak product and δ_X^W is an element of $\mathsf{P}(W)$ satisfying condition **wI** of Definition 3.1.1 and the condition*

wIV′ *There exist a weak product $W \xleftarrow{\mathsf{p}_1} U \xrightarrow{\mathsf{p}_2} W$ and two arrows $U \underset{\langle 2,4\rangle}{\overset{\langle 1,3\rangle}{\rightrightarrows}} W$ such that*

$$\delta_X^W \in \mathcal{D}es_{\mathsf{P}_{\langle 1,3\rangle}(\delta_X^W) \wedge \mathsf{P}_{\langle 2,4\rangle}(\delta_X^W)}.$$

Then δ_X^W is reindexed by projections.

Proof. Let $T \underset{t_2}{\overset{t_1}{\rightrightarrows}} W$ be two arrows such that $\mathsf{p}_i \circ t_1 = \mathsf{p}_i \circ t_2$, for $i = 1, 2$. The weak universal property of U implies the existence of an arrow $\langle t_1, t_2\rangle : T \to U$ such that $p_i\langle t_1, t_2\rangle = t_i$, for $i = 1, 2$. Hence, $\mathsf{P}_{t_i}(\delta_X^W) = \mathsf{P}_{\langle t_1,t_2\rangle}\mathsf{P}_{\mathsf{p}_i}(\delta_X^W)$, for $i = 1, 2$ and, by condition **wI**, it follows that $\mathsf{P}_{\langle t_1,t_2\rangle}\mathsf{P}_{\langle 1,3\rangle}(\delta_X^W) = \top_T = \mathsf{P}_{\langle t_1,t_2\rangle}\mathsf{P}_{\langle 2,4\rangle}(\delta_X^W)$. We obtain the inequality $\mathsf{P}_{t_1}\delta_X^W \leq \mathsf{P}_{t_2}\delta_X^W$ as follows:

$$\begin{aligned}
\mathsf{P}_{t_1}(\delta_X^W) &= \mathsf{P}_{\langle t_1,t_2\rangle}\mathsf{P}_{\mathsf{p}_1}(\delta_X^W) \\
&= \mathsf{P}_{\langle t_1,t_2\rangle}\mathsf{P}_{p_1}(\delta_X^W) \wedge \top_T \\
&= \mathsf{P}_{\langle t_1,t_2\rangle}(\mathsf{P}_{\mathsf{p}_1}\delta_X^W \wedge \mathsf{P}_{\langle 1,3\rangle}\delta_X^W \wedge \mathsf{P}_{\langle 2,4\rangle}\delta_X^W) && (\mathbf{wI}) \\
&\leq \mathsf{P}_{\langle t_1,t_2\rangle}\mathsf{P}_{\mathsf{p}_2}(\delta_X^W) = \mathsf{P}_{t_2}(\delta_X^W) && (\mathbf{wIV'}) \\
&= \mathsf{P}_{t_2}(\delta_X^W).
\end{aligned}$$

The opposite inequality is obtained similarly considering an arrow $\langle t_2, t_1\rangle$. □

Corollary 3.1.9. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased elementary doctrine and let X be an object of $\mathscr{C}$. If W and W' are two weak products of X and itself and $h : W \to W'$ is an arrow satisfying $\mathsf{p'}_i h = \mathsf{p}_i$ for $i = 1, 2$, then $\mathsf{P}_h\delta_X^{W'} = \delta_X^W$.*

We say that a category $\mathscr{C}$ has *a choice of weak products* if there exists a functor $\omega : \mathscr{C} \times \mathscr{C} \to \mathscr{C}^\Lambda$, where Λ is the category with three objects and the non trivial span

$$\bullet \longleftarrow \bullet \longrightarrow \bullet\ ,$$

such that:

- the value $\omega(X,Y)$ is a weak product of the objects $X, Y \in \mathscr{C}$,
- the arrow $\omega(f,g) : \omega(X,Y) \to \omega(A,B)$ satisfies $p_1 \circ \omega(f,g) = f \circ p_1$ and $p_2 \circ \omega(f,g) = g \circ p_2$, for every pair of arrows $f : X \to A$ and $g : Y \to B$ of $\mathscr{C}$.

We will use the usual notation $X \overset{p_1}{\leftarrow} X \times Y \overset{p_2}{\to} Y$ for the choice of weak product $\omega(X,Y)$ and $f \times g$ for the choice of weak product of arrows $\omega(f,g)$. In the following Theorem, we prove that a choice of weak products provides an easier description of the biased elementary doctrines.

Theorem 3.1.10. *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a functor such that $\mathscr{C}$ has a choice of weak products and such that for every object $X \in \mathscr{C}$ there exists an element $\delta_X \in \mathsf{P}(X \times X)$ satisfying:*

i) for every arrow $Z \xrightarrow{d} X \times X$ such that $p_1 \circ d = p_2 \circ d$, then $\top_Z \leq \mathsf{P}_d(\delta_X)$,

ii) $\mathsf{P}(X) = \mathcal{D}es_{\delta_X}$,

iii) If $f : Y \to X$ is an arrow of $\mathscr{C}$, then $\delta_Y \leq \mathsf{P}_{f\times f}\delta_X$,

iv) $\delta_X \in \mathcal{D}es_{\mathsf{P}_{\langle 1,3\rangle}(\delta_X)\wedge \mathsf{P}_{\langle 2,4\rangle}(\delta_X)}$,[1]

then the functor $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine

Proof. For each choice of weak product $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\to} X$, the weak universal property of weak products induces an arrow $h : W \to X \times X$ such that $p_i \circ h = \mathsf{p}_i$, for $i = 1, 2$. Even if h is not unique, Lemma 3.1.8 implies that we can uniquely reindex δ_X along such arrow and define $\delta_X^W := \mathsf{P}_h\delta_X$, which trivially satisfies conditions **wI** and **wII** of Definition 3.1.2.
We now prove **wIII** of Definition 3.1.2. Let W be a weak product of X and X, and let V be a weak product of Y and Y. So there are arrows $h : W \to X \times X$ and $k : V \to Y \times Y$ such that $p_i \circ h = \mathsf{p}_i$ and $p_i \circ k = \mathsf{p}_i$, for $i = 1, 2$. Let $f : Y \to X$ and $g : W \to V$ be two arrows of $\mathscr{C}$ such that $\mathsf{p}_i \circ g = f \circ \mathsf{p}_i$, for $i = 1, 2$. We obtain that $\delta_Y^V \leq \mathsf{P}_g\delta_X^W$ as follows

$$\begin{aligned} \delta_Y^V &:= \mathsf{P}_k\delta_Y \\ &\leq \mathsf{P}_k\mathsf{P}_{f\times f}\delta_X && \text{(iii)} \\ &= \mathsf{P}_g\mathsf{P}_h\delta_X && \text{(Lemma 3.1.8)} \\ &:= \mathsf{P}_g\delta_X^W. \end{aligned}$$

We now prove condition **wIV** of Definition 3.1.2. Let $W \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\to} W$ be a weak product and let $U \underset{\langle 2,4\rangle'}{\overset{\langle 1,3\rangle'}{\rightrightarrows}} W$, be two arrows induced by the projections $\mathsf{p}_1 \circ \mathsf{p}_1, \mathsf{p}_1 \circ \mathsf{p}_2$ and $\mathsf{p}_2 \circ \mathsf{p}_1, \mathsf{p}_2 \circ \mathsf{p}_2$. The weak universal property of weak products induces arrows $h : W \to X \times X$ and $k : U \to (X \times X) \times (X \times X)$ satisfying $p_i \circ h = \mathsf{p}_i$ and $p_i \circ k = h \circ \mathsf{p}_i$, for $i = 1, 2$. We obtain the relation

[1] The arrow $(X \times X) \times (X \times X) \overset{\langle 1,3\rangle}{\to} X \times X$ denotes the choiche $p_1 \times p_1$. Similarly, the arrow $\langle 2, 4\rangle$ denotes the choice $p_2 \times p_2$.

$\delta_X^W \in \mathcal{D}es_{\mathsf{P}_{\langle 1,3\rangle'}(\delta_X^W)\wedge \mathsf{P}_{\langle 2,4\rangle'}(\delta_X^W)}$ as follows:

$$\begin{aligned}
&\mathsf{P}_{\mathsf{p}_1}\delta_X^W \wedge \mathsf{P}_{\langle 1,3\rangle'}\delta_X^W \wedge \mathsf{P}_{\langle 2,4\rangle'}\delta_X^W && (3.1)\\
&:= \mathsf{P}_{\mathsf{p}_1}\mathsf{P}_h\delta_X \wedge \mathsf{P}_{\langle 1,3\rangle'}\mathsf{P}_h\delta_X \wedge \mathsf{P}_{\langle 2,4\rangle'}\mathsf{P}_h\delta_X\\
&= \mathsf{P}_k\mathsf{P}_{p_1}\delta_X \wedge \mathsf{P}_k\mathsf{P}_{\langle 1,3\rangle}\delta_X \wedge \mathsf{P}_k\mathsf{P}_{\langle 2,4\rangle}\delta_X && \text{(Lemma 3.1.8)}\\
&= \mathsf{P}_k(\mathsf{P}_{p_1}\delta_X \wedge \mathsf{P}_{\langle 1,3\rangle}\delta_X \wedge \mathsf{P}_{\langle 2,4\rangle}\delta_X)\\
&\leq \mathsf{P}_k\mathsf{P}_{p_2}\delta_X && \text{(iv)}\\
&= \mathsf{P}_{\mathsf{p}_2}\mathsf{P}_h\delta_X\\
&:= \mathsf{P}_{\mathsf{p}_2}\delta_X^W.
\end{aligned}$$

□

3.2 Proof-irrelevant elements

In this section, we will detect elements in the fibers of a biased elementary doctrines $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ that are suitable in the following sense. If W is a weak product of the objects $X_1, \ldots, X_n \in \mathscr{C}$ and $f_i : A \to X_i$ are arrows of $\mathscr{C}$, for $i = 1, \ldots, n$, then the weak universal property of the weak products induces a not necessarily unique arrow $\langle f_1, \ldots, f_n\rangle : A \to W$, which makes the obvious diagram commute. Hence, the reindexings of the elements of $\mathsf{P}(W)$ along such arrows are not uniquely determined. We shall define the sub-poset of $\mathsf{P}(W)$ of *proof-irrelevant* elements and we will prove that they are reindexed by projections. Proof-irrelevant elements take their name from the slices of the elementary doctrine F^{ML} of dependent types, as we will discuss in Example 3.2.5. Moreover, we will prove that proof-irrelevant elements only depends, up to isomorphism, on the objects $X_1, \ldots, X_n \in \mathscr{C}$.

Lemma 3.2.1. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased elementary doctrine and let $X_1, \ldots, X_n$ be objects of $\mathscr{C}$. If W is a weak product of $X_1, \ldots, X_n$ with projections $\mathsf{p}_i : W \to X_i$, for $i = 1, \ldots, n$, then the sub-poset*

$$\mathcal{D}es_{(\mathsf{P}_{\langle 1,n+1\rangle}(\delta_{X_1}^{W_1})\wedge\cdots\wedge \mathsf{P}_{\langle n,2n\rangle}(\delta_{X_n}^{W_n}))} \subseteq \mathsf{P}(W)$$

does not depend on the choice of weak products $X_i \overset{\mathsf{p}_1}{\leftarrow} W_i \overset{\mathsf{p}_2}{\rightarrow} X_i$, $W \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\rightarrow} W$ and arrows $\langle i, n+i\rangle : U \to W_i$, for $i = 1, \ldots, n$.

Proof. Let $\mathcal{D}es_{(\mathsf{P}_{\langle 1,n+1\rangle'}(\delta_{X_1}^{W_1'})\wedge\cdots\wedge \mathsf{P}_{\langle n,2n\rangle'}(\delta_{X_n}^{W_n'}))} \subseteq \mathsf{P}(W)$ be the sub-poset defined through a different choice of weak binary products $X_i \overset{\mathsf{p'}_1}{\leftarrow} W_i' \overset{\mathsf{p'}_2}{\rightarrow} X_i$, for $i = 1, \ldots, n$, $W \overset{\mathsf{p'}_1}{\leftarrow} U' \overset{\mathsf{p'}_2}{\rightarrow} W$ and arrows $\langle i, n+i\rangle' : U' \to W_i'$, for $i = 1, \ldots, n$. We prove the equality

$$\mathcal{D}es_{(\mathsf{P}_{\langle 1,n+1\rangle}(\delta_{X_1}^{W_1})\wedge\cdots\wedge \mathsf{P}_{\langle n,2n\rangle}(\delta_{X_n}^{W_n}))} = \mathcal{D}es_{(\mathsf{P}_{\langle 1,n+1\rangle'}(\delta_{X_1}^{W_1'})\wedge\cdots\wedge \mathsf{P}_{\langle n,2n\rangle'}(\delta_{X_n}^{W_n'}))}$$

as follows. The weak universal property of weak products induces arrows

$$k : U' \to U,$$

$$h_i : W_i' \to W_i,$$

such that $\mathsf{p}_j k = \mathsf{p'}_j$ and $\mathsf{p}_j h_i = \mathsf{p'}_j$ for $i = 1, \dots, n$ and $j = 1, 2$. We obtain the inclusion " $\subseteq$ " as follows. If $\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \mathsf{P}_{\langle 1,n+1\rangle}(\delta^{W_1}_{X_1}) \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle}(\delta^{W_n}_{X_n}) \leq \mathsf{P}_{\mathsf{p}_2}\alpha$, then

$$\begin{aligned}
&\mathsf{P}_{\mathsf{p'}_1}\alpha \wedge \mathsf{P}_{\langle 1,n+1\rangle'}\delta^{W'_1}_{X_1} \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle'}\delta^{W'_n}_{X_n} \\
&= \mathsf{P}_k\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \mathsf{P}_{\langle 1,n+1\rangle'}\mathsf{P}_{h_i}\delta^{W_1}_{X_1} \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle}\mathsf{P}_{h_n}\delta^{W_n}_{X_n} && \text{(Corollary 3.1.9)} \\
&= \mathsf{P}_k(\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \mathsf{P}_{\langle 1,n+1\rangle}\delta^{W_1}_{X_1} \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle}\delta^{W_n}_{X_n}) && \text{(Lemma 3.1.8)} \\
&\leq \mathsf{P}_k\mathsf{P}_{\mathsf{p}_2}\alpha \\
&= \mathsf{P}_{\mathsf{p'}_2}\alpha.
\end{aligned}$$

The opposite inclusion $\supseteq$ follows similarly considering arrows $k' : U \to U'$ and $h'_i : W_i \to W'_i$ such that $\mathsf{p'}_j k' = \mathsf{p}_j$, $\mathsf{p'}_j h'_i = \mathsf{p}_j$ for $i = 1, \dots, n$, $j = 1, 2$. ☐

The above lemma allows us to give the following definition.

Definition 3.2.2. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased elementary doctrine and let $X_1, \dots, X_n$ be objects of $\mathscr{C}$. For every weak product W of the objects $X_1, \dots, X_n$, we will refer to the sub-poset

$$\mathsf{PIrr}_{\mathscr{C}}(W) := \mathcal{Des}_{(\mathsf{P}_{\langle 1,n+1\rangle}(\delta^{W_1}_{X_1})\wedge\cdots\wedge\mathsf{P}_{\langle n,2n\rangle}(\delta^{W_n}_{X_n}))}$$

of $\mathsf{P}(W)$ as the sub-poset of *proof-irrelevant* elements (or *strict predicates*) of the weak product W.

Observation 3.2.3. We observe that, if P is a strict elementary doctrine and $X_1, \dots, X_n$ are objects of $\mathscr{C}$, then the proof-irrelevant elements of a strict product $X_1 \times \cdots \times X_n$ coincide with the fiber $\mathsf{P}(X_1 \times \cdots \times X_n)$.

We now prove that the proof-irrelevant elements are reindexed by projections.

Proposition 3.2.4. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased elementary doctrine and let $X_1, \dots, X_n$ be objects of $\mathscr{C}$. For every weak product W of the objects $X_1, \dots, X_n$, the proof-irrelevant elements of W are reindexed by projections.*

Proof. Let $T \underset{t2}{\overset{t_1}{\rightrightarrows}} W$ be two arrows satisfying $p_i \circ t_1 = p_i \circ t_2$, for $i = 1, \dots, n$ and let $\langle t_1, t_2\rangle : T \to U$ be an arrow induced by the weak universal property of a weak binary product $W \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\rightarrow} W$, such that $\mathsf{p}_i\langle t_1, t_2\rangle = t_i$ for $i = 1, 2$. If $\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \mathsf{P}_{\langle 1,n+1\rangle}\delta^{W_1}_{X_1} \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle}\delta^{W_n}_{X_n} \leq \mathsf{P}_{\mathsf{p}_2}\alpha$, then we obtain $\mathsf{P}_{t_1}\alpha \leq \mathsf{P}_{t_2}\alpha$ as follows:

$$\begin{aligned}
\mathsf{P}_{t_1}\alpha &= \mathsf{P}_{\langle t_1,t_2\rangle}\mathsf{P}_{\mathsf{p}_1}\alpha \\
&= \mathsf{P}_{\langle t_1,t_2\rangle}\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \top_T \\
&= \mathsf{P}_{\langle t_1,t_2\rangle}(\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \mathsf{P}_{\langle 1,n+1\rangle}\delta^{W_1}_{X_1} \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle}\delta^{W_n}_{X_n}) \\
&\leq \mathsf{P}_{\langle t_1,t_2\rangle}\mathsf{P}_{\mathsf{p}_2}\alpha \\
&= \mathsf{P}_{t_2}\alpha.
\end{aligned}$$

Similarly, the opposite inequality $\mathsf{P}_{t_2}\alpha \leq \mathsf{P}_{t_1}\alpha$ is obtained considering an arrow $\langle t_2, t_1\rangle$. ☐

Proof-irrelevant elements take their name from the following example.

Example 3.2.5. Consider the elementary doctrine $F^{ML} : \mathbf{ML}^{op} \to \mathsf{InfSL}$ of Example 1.2.8 and a closed type A. If $\lfloor f\rceil : X \to A$ is an object of $\mathbf{ML}/A$, then a choice of weak product of $\lfloor f\rceil$ and $\lfloor f\rceil$ is given by the equivalence class of the common value of the two composites of the following weak pullback diagram

$$\begin{tikzcd} W := \sum_{x_1,x_2:X} \mathsf{Id}_A(f(x_1), f(x_2)) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & A \end{tikzcd}$$

The slice doctrine $F^{ML}{}_{/A}$ sends the weak product $\lfloor w \rceil$ to $F^{ML}{}_{/A}(\lfloor w \rceil) := F^{ML}(W)$, which is given by the equivalence classes of the dependent types on W:

$$w : W \vdash B(w).$$

The fibered equality $\delta_{\lfloor f \rceil}$ is given by:

$$(x_1, x_2, p) : W \vdash \mathsf{Id}_X(x_1, x_2).$$

Similarly, if $g := f \circ \pi_1$, a weak product of $\lfloor g \rceil$ and $\lfloor g \rceil$ is given by the common value of the two composites of the following weak pullback

$$\begin{tikzcd} U := \sum_{w_1,w_2:W} \mathsf{Id}_W(g(w_1), g(w_2)) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & W \arrow[d, "g"] \\ W \arrow[r, "g"'] & A. \end{tikzcd}$$

The fibered equality $\delta_{\lfloor g \rceil}$ is an element of $F^{ML}{}_{/A}(h) := F^{ML}(U)$ given by:

$$(w_1, w_2, p) : U \vdash \mathsf{Id}_W(w_1, w_2),$$

and we will refer to $\delta_{\lfloor f \rceil}$ as *proof-relevant equality* on W. On the other hand, if $w := (x_1, x_2, q) : W$ and $w' := (x_1', x_2', q') : W$, then the element $\delta_{\lfloor f \rceil} \boxtimes \delta_{\lfloor f \rceil}$ corresponds to the dependent type

$$(w, w', p) : U \vdash \mathsf{Id}_X(x_1, x_1') \wedge \mathsf{Id}_X(x_2, x_2'),$$

which does not depend on the proof terms

$$q : \mathsf{Id}_A(f(x_1), f(x_2)), \qquad q' : \mathsf{Id}_A(fx_2, fx_2').$$

Hence, we will refer to $\delta_{\lfloor f \rceil} \boxtimes \delta_{\lfloor f \rceil}$ as the *proof-irrelevant equality* on W.

Similarly, we can describe proof-irrelevant elements of the objects $f_i : X_i \to A$, for $i = 1, \dots, n$, of $\mathbf{ML}/A$ as follows. If $\lfloor g \rceil : W \to A$ is a weak product of $f_1, \dots, f_n$, then proof-irrelevant elements over $f_1, \dots, f_n$ are types $w : W \vdash B(w)$ such that, if $w_1, w_2 : W$ and the type

$$\mathsf{Id}_X(w_{11}, w_{21}) \wedge \cdots \wedge \mathsf{Id}_X(w_{1n}, w_{2n})$$

is inhabited, then $B(w_1)$ is inhabited if and only if $B(w_2)$ is inhabited.

The following example gives an explicit description of proof-irrelevant elements of the biased elementary doctrine of weak subobjects $\mathsf{Psub}_{\mathscr{C}}$ of weakly left exact category $\mathscr{C}$. We omit the proofs, that will be provided in detail in the next chapter.

Example 3.2.6. Let $\mathscr{C}$ be a category with weak finite limits and let $X_1, \dots, X_n$ be objects of $\mathscr{C}$. If W is a weak product of $X_1, \dots, X_n$, then the proof-irrelevant elements of W are described as follows. Let $\mathscr{C}/(X_1, \dots, X_n)$ be the category of cones over $X_1, \dots, X_n$ and let $\mathscr{C}/(X_1, \dots, X_n)_{po}$ be its poset reflection. The assignment which takes the equivalence class of a cone $R \xrightarrow{r_i} X_i$, for $i = 1, \dots, n$, over $X_1, \dots, X_n$, into the equivalence class of the right dashed arrow of the following weak limit

$$
\begin{tikzcd}
& R' \arrow[dl, dashed] \arrow[dr, dashed, "\rho"] & \\
R \arrow[d, "r_1"'] \arrow[drr, "r_n" near start] & & W \arrow[d, "\mathsf{p}_n"] \arrow[dll, "\mathsf{p}_1"' near start] \\
X_1 \arrow[rr, dotted, no head] & & X_n.
\end{tikzcd}
$$

provides a bijection between $\mathscr{C}/(X_1, \ldots, X_n)_{po}$ and the proof-irrelevant elements of W. The above correspondence can be used to interpret (a fragment of) intuitionistic logic in categories with weak limits. In [Pal04], the author presents how standard interpretation of categorical logic, see [MR77], relates with the categorical BHK-interpretation in categories with strict products and weak limits. In the next chapter, we will extend the latter interpretation to wlex categories and provide the details of the above bijection.

We end this section proving that different weak products of the same objects yield isomorphic sub-posets of proof-irrelevant elements.

Proposition 3.2.7. *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be a biased elementary doctrine and let* $X_1, \ldots, X_n$ *be objects of* $\mathscr{C}$*. The sub-poset of proof-irrelevant elements of* W *is defined up to isomorphism for every choice of weak product* W *of the objects* $X_1, \ldots, X_n$*.*

Proof. Let W' be a different weak product of the objects $X_1, \ldots, X_n$ with projections $\mathsf{p'}_i : W' \to X_i$, for $i = 1, \ldots, n$, and let

$$\mathcal{D}es_{(\mathsf{P}_{\langle 1,n+1\rangle'}(\delta^{W'_1}_{X_1}) \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle'}(\delta^{W'_n}_{X_n}))} \subseteq \mathsf{P}(W')$$

be the sub-poset of proof-irrelevant elements over W', defined through different weak products $W' \overset{\mathsf{p}_1}{\leftarrow} U' \overset{\mathsf{p}_2}{\rightarrow} W'$, $X_i \overset{\mathsf{p}_1}{\leftarrow} W'_i \overset{\mathsf{p}_2}{\rightarrow} X_i$, for $i = 1, \ldots, n$, and arrows $\langle i, n+i\rangle' : U' \to W'_i$, for $i = 1, \ldots, n$. The weak universal property of weak products induces arrows

$$h : W' \to W$$

and

$$l : W \to W'$$

such that $\mathsf{p}_i h = \mathsf{p'}_i$ and $\mathsf{p'}_i l = \mathsf{p}_i$, for $i = 1, \ldots, n$. We prove that the functors P_h restricts to an isomorphism with inverse P_l between the sub-orders

$$\mathsf{P}_h : \mathcal{D}es_{(\mathsf{P}_{\langle 1,n+1\rangle}(\delta^{W_1}_{X_1}) \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle}(\delta^{W_n}_{X_n}))} \underset{\longleftarrow}{\overset{\longrightarrow}{\cong}} \mathcal{D}es_{(\mathsf{P}_{\langle 1,n+1\rangle'}(\delta^{W'_1}_{X_1}) \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle'}(\delta^{W'_n}_{X_n}))} : \mathsf{P}_l.$$

The weak universal property of weak products induces arrows

$$k : U' \to U$$

and

$$h_i : W'_i \to W_i$$

such that $\mathsf{p}_j k = h\mathsf{p'}_j$ and $\mathsf{p}_j h_i = \mathsf{p'}_j$ for $i = 1, \ldots, n$, $j = 1, 2$. We first prove that if

$$\alpha \in \mathcal{D}es_{(\mathsf{P}_{\langle 1,n+1\rangle}(\delta^{W_1}_{X_1}) \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle}(\delta^{W_n}_{X_n}))}$$

then $\mathsf{P}_h\alpha \in \mathcal{D}es_{(\mathsf{P}_{\langle 1,n+1\rangle'}(\delta^{W'_1}_{X_1})\wedge\cdots\wedge\mathsf{P}_{\langle n,2n\rangle'}(\delta^{W'_n}_{X_n}))}$ as follows:

$$
\begin{aligned}
&\mathsf{P}_{\mathsf{p'}_1}\mathsf{P}_h\alpha \wedge \mathsf{P}_{\langle 1,n+1\rangle'}\delta^{W'_1}_{X_1} \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle'}\delta^{W'_n}_{X_n} \\
&= \mathsf{P}_k\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \mathsf{P}_{\langle 1,n+1\rangle'}\mathsf{P}_{h_1}\delta^{W_1}_{X_1} \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle'}\mathsf{P}_{h_n}\delta^{W_n}_{X_n} && \text{(Corollary 3.1.9)} \\
&= \mathsf{P}_k(\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \mathsf{P}_{\langle 1,n+1\rangle}\delta^{W_1}_{X_1} \wedge \cdots \wedge \mathsf{P}_{\langle n,2n\rangle}\delta^{W_n}_{X_n}) && \text{(Lemma 3.1.8)} \\
&\leq \mathsf{P}_k\mathsf{P}_{\mathsf{p}_2}\alpha \\
&= \mathsf{P}_{\mathsf{p'}_2}\mathsf{P}_h\alpha.
\end{aligned}
$$

A similar computation shows that P_l restricts to the sub-posets of proof-irrelevant elements. The bijection follows from proposition 3.2.4 since $\mathsf{p}_i(h\circ l) = \mathsf{p}_i \circ Id$ and $\mathsf{p'}_i(l\circ h) = \mathsf{p'}_i \circ Id$ for $i = 1,\dots,n$. $\square$

From Lemma 3.2.1, Proposition 3.2.4, and Proposition 3.2.7, given n objects $X_1,\dots,X_n \in \mathscr{C}$, we can consider the the limit of the diagram of isomorphisms given by the restrictions of reindexing among the various presentations of proof irrelevant elements of the weak products of $X_1,\dots,X_n$ and denote it by $\mathsf{P}^s[X_1,\dots,X_n]$. We will refer to $\mathsf{P}^s[X_1,\dots,X_n]$ as *proof-irrelevant elements* (or *strict predicated*) of $X_1,\dots,X_n$. In the next section, we will prove that the assignment P^s is actually functorial and it is a strict elementary doctrine, which we will call the *strictification* of P.

3.3 Strictification

In this section, we will provide a construction which relates biased elementary doctrines and the strict ones. Using the properties of proof-irrelevant elements, we will associate to each biased elementary doctrine a functor which turns out to be a strict elementary doctrine. Hence, we will obtain a characterization of the biased elementary doctrines in terms of the strict elementary doctrines. In order to do that, we will need the universal construction which freely adds strict products to a category.

Notation. For every $n \in \mathbb{N}$, we will denote by $[n]$ the set $\{1,\dots,n\}$. If $j : [m] \to [n]$ is an assignment and the cardinality of the codomain is clear from the context, then we will often denote j by its values $\langle j(1),\dots,j(m)\rangle$.

The product completion $(\mathsf{Fam}_{\mathsf{fin}}(\mathscr{C}^{op}))^{op}$ of an arbitrary category $\mathscr{C}$ can be found in [BC95]. We now recall a presentation of the construction which better fits our context.

Definition 3.3.1. Let $\mathscr{C}$ be a category. The finite product completion of $\mathscr{C}$ is the category $\mathscr{C}_s$ defined as follows:

objects of $\mathscr{C}_s$ are finite lists $[X_1,\dots,X_n]$ of objects of $\mathscr{C}$.

arrows of $\mathscr{C}_s$ are pair $(f,\hat{f}) : [X_1,\dots,X_n] \to [Y_1,\dots,Y_m]$ such that $\hat{f} : [m] \to [n]$ is an assignment and $f = [f_1,\dots,f_m]$ is a list of arrows $f_i : X_{\hat{f}(i)} \to Y_i$ of $\mathscr{C}$, for $i \in [m]$. The composition of two arrows $(f,\hat{f}) : [X_1,\dots,X_n] \to [Y_1,\dots,Y_m]$ and $(g,\hat{g}) : [Y_1,\dots,Y_m] \to [Z_1,\dots,Z_k]$ is given by

$$(g,\hat{g}) \circ (f,\hat{f}) = (g * f, \hat{g} \circ \hat{f})$$

where $g * f = [g_1 \circ f_{\hat{g}(1)},\dots,g_k \circ f_{\hat{g}(k)}]$.

There is an obvious functor $S : \mathscr{C} \to \mathscr{C}_s$, which sends an object $X \in \mathscr{C}$ to the list $[X] \in \mathscr{C}_s$ and an arrow f of $\mathscr{C}$ to the arrow $(f, \langle 1 \rangle) : [X] \to [Y]$ of $\mathscr{C}_s$. We will denote with **Cat** the 2-category of small categories, and with **CART** the 2-category of categories with strict finite products and functors preserving them. It is well known that the above construction gives a left bi-adjoint to the forgetful functor $U : \mathbf{CART} \to \mathbf{Cat}$.

Proposition 3.3.2. *For every small category $\mathscr{C}$, the pre-composition with $S : \mathscr{C} \to \mathscr{C}_s$ induces an essential equivalence of categories*

$$- \circ S : \boldsymbol{CART}(\mathscr{C}_s, \mathcal{D}) \cong \boldsymbol{Cat}(\mathscr{C}, \mathcal{D})$$

for every category $\mathcal{D}$ with strict products. □

Observation 3.3.3. We observe that the above adjunction restricts between the 2-category **CART** and the full 2-subcategory **WCART** of **Cat** of categories with weak finite products and functors. We also remark that the the obvious functor $S : \mathscr{C} \to \mathscr{C}_s$ does not preserve the weak finite products of $\mathscr{C}$, neither it turns weak products into strict ones.

Notation. If $\mathscr{C}$ is a category and $f : X \to Y$ is an arrow of $\mathscr{C}$, then we will denote by $[f]$ the arrow $(f, \langle 1 \rangle) : [X] \to [Y]$ of $\mathscr{C}_s$. Similarly, if $[f] : [X] \to [Y]$ and $[g] : [X] \to [Z]$ are arrows of $\mathscr{C}_s$, we will adopt the notation $\langle [f], [g] \rangle : [X] \to [Y, Z]$ to denote the unique arrow of $\mathscr{C}_s$ induced by the arrow $[f], [g]$ on the product $[Y, Z]$ of $\mathscr{C}_s$. Finally, if $[f] : [X] \to [Y]$ and $[g] : [Z] \to [W]$ are arrows of $\mathscr{C}_s$, then we will denote by $[f] \times [g] : [X, Z] \to [Z, W]$ to denote the arrow $([f, g], \langle 1, 2 \rangle)$ of $\mathscr{C}_s$.

In the following proposition, we prove that a biased elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ induces a functor $\mathsf{P}^s : \mathscr{C}_s^{op} \to \mathsf{InfSL}$.

Proposition 3.3.4. *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine, then we can define a functor $\mathsf{P}^s : \mathscr{C}_s^{op} \to \mathsf{InfSL}$.*

Proof. The functor P^s is defined on a list $[X_1, \dots, X_n]$ as the isomorphism class of the sub-orders of proof-irrelevant elements of weak products of $X_1, \dots, X_n$. In particular, $P^s[X]$ denotes nothing but the poset $\mathsf{P}(X)$. We now prove that the assignment is functorial.
Let $(f, \hat{f}) : [X_1, \dots, X_n] \to [Y_1, \dots, Y_m]$ be an arrow of $\mathscr{C}_s$ and let W be a weak product of $X_1, \dots, X_n$ and V a weak product of $Y_1, \dots, Y_m$. Hence, we obtain an arrow $g : W \to V$ such that $\mathsf{p}_i \circ g = f_i \circ \mathsf{p}_{\hat{f}(i)}$, for $i \in [m]$. We now prove that if α is a proof-irrelevant element of V, then $\mathsf{P}_g(\alpha)$ is a proof-irrelevant element of W. Indeed, given some weak products $W \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\rightarrow} W$ and $V \overset{\mathsf{p}_1}{\leftarrow} Z \overset{\mathsf{p}_2}{\rightarrow} V$, the arrow g induces an arrow $h : U \to Z$ such that $\mathsf{p}_i \circ h = g \circ \mathsf{p}_i$, for $i = 1, 2$. Given weak products $X_i \overset{\mathsf{p}_1}{\leftarrow} W_i \overset{\mathsf{p}_2}{\rightarrow} X_i$ and $Y_i \overset{\mathsf{p}_1}{\leftarrow} V_i \overset{\mathsf{p}_2}{\rightarrow} Y_i$, and arrows $\langle i, i+n \rangle : U \to W_i$ and $\langle j, j+m \rangle : Z \to V_j$, for $i \in [n]$ and $j \in [m]$, then assuming $\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \mathsf{P}_{\langle 1, n+1 \rangle}\delta_{Y_1}^{V_1} \wedge \cdots \wedge \mathsf{P}_{\langle n, 2n \rangle}\delta_{Y_n}^{V_n} \leq \mathsf{P}_{\mathsf{p}_2}\alpha$ we obtain

$$\begin{aligned}
&\mathsf{P}_{\mathsf{p}_1}\mathsf{P}_g\alpha \wedge \mathsf{P}_{\langle 1, n+1 \rangle}\delta_{X_1}^{W_1} \wedge \cdots \wedge \mathsf{P}_{\langle n, 2n \rangle}\delta_{X_n}^{W_n} && \text{Definition 3.1.2-iii)}\\
&\leq \mathsf{P}_h\mathsf{P}_{p_1}\alpha \wedge \mathsf{P}_{\langle 1, m+1 \rangle}\mathsf{P}_{\hat{f}(1)}\delta_{Y_1}^{V_1} \wedge \cdots \wedge \mathsf{P}_{\langle m, 2m \rangle}\mathsf{P}_{\hat{f}(m)}\delta_{Y_m}^{V_m} && \text{Lemma 3.1.8}\\
&= \mathsf{P}_h(\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \mathsf{P}_{\langle 1, n+1 \rangle}\delta_{Y_1}^{V_1} \wedge \cdots \wedge \mathsf{P}_{\langle n, 2n \rangle}\delta_{Y_n}^{V_n})\\
&\leq \mathsf{P}_h\mathsf{P}_{\mathsf{p}_2}\alpha\\
&= \mathsf{P}_{\mathsf{p}_2}\mathsf{P}_g\alpha.
\end{aligned}$$

Hence, $\mathsf{P}^s(f, \hat{f})$ sends the isomorphism class of a proof-irrelevant element α of $X_1, \dots, X_n$ to the isomorphism class of the proof-irrelevant element $\mathsf{P}_g(\alpha)$ of $Y_1, \dots, Y_m$. The assignment is well defined thanks to Proposition 3.2.4. □

We now prove that the functor P^s is actually a strict elementary doctrine.

Theorem 3.3.5. *Let $\mathscr{C}$ be a category with weak products. If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine, then P^s is a strict elementary doctrine. Vice versa, for every strict elementary doctrine $R : \mathscr{C}_s^{op} \to \mathsf{InfSL}$, the pre-composition $R \circ S : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine.*

Proof. Assuming that $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine, we prove that $\mathsf{P}^s : \mathscr{C}_s^{op} \to \mathsf{InfSL}$ is a strict elementary doctrine as follows. For every object $[X_1, \dots, X_n] \in \mathscr{C}_s$ we define the fibered equality $\delta_{[X_1,\dots,X_n]}$ as the isomorphism class of the proof-irrelevant equality $\mathsf{P}_{\langle 1,n+1\rangle}\delta_{X_1}^{W_1} \wedge \dots \wedge \mathsf{P}_{\langle n,2n\rangle}\delta_{X_n}^{W_n}$, which is in $\mathsf{P}^s[X_1, \dots, X_n, X_1, \dots, X_n]$. In particular, $\delta_{[X]}$ is the isomorphism class of the fibered equalities δ_X^W. Now we prove that $\delta_{[X_1,\dots,X_n]}$ satisfies conditions **I**,**II** and **III** of Definition 1.2.5. Indeed, the diagonal $\Delta_{[X_1,\dots,X_n]}$ is given by the arrow $(1_{\bar{X}}, \langle 1, \dots, n, 1, \dots, n\rangle)$, where $1_{\bar{X}} := [1_{X_1}, \dots, 1_{X_n}, 1_{X_1}, \dots, 1_{X_n}]$ and $\mathsf{P}^s_{\Delta_{[X_1,\dots,X_n]}}\delta_{[X_1,\dots,X_n]}$ is equal to the isomorphism class of $\mathsf{P}_{\langle 1,\dots,n,1,\dots,n\rangle}(\mathsf{P}_{\langle 1,n+1\rangle}\delta_{X_1}^{W_1} \wedge \dots \wedge \mathsf{P}_{\langle n,2n\rangle}\delta_{X_n}^{W_n})$. By Lemma 3.1.6 and condition **wI** we obtain $\top_{[X_1,\dots,X_n]} \le \mathsf{P}^s_{\Delta_{[X_1,\dots,X_n]}}\delta_{[X_1,\dots,X_n]}$. Condition **II**, follows by definition of P^s. Finally, condition **III** is obtained as follows. The element $\delta_{[X_1,\dots,X_n]} \boxtimes \delta_{[Y_1,\dots,Y_m]}$ is given by the isomorphism class of the element

$$\mathsf{P}_{\langle 1,3\rangle}(\mathsf{P}_{\langle 1,n+1\rangle}\delta_{X_1}^{W_1} \wedge \dots \wedge \mathsf{P}_{\langle n,2n\rangle}\delta_{X_n}^{W_n}) \wedge \mathsf{P}_{\langle 2,4\rangle}(\mathsf{P}_{\langle 1,n+1\rangle}\delta_{Y_1}^{V_1} \wedge \dots \wedge \mathsf{P}_{\langle n,2n\rangle}\delta_{Y_n}^{V_n}).$$

By Lemma 3.1.8, the above element is equal to

$$\mathsf{P}_{\langle 1,n+m+1\rangle}\delta_{X_1}^{W_1} \wedge \mathsf{P}_{\langle n,2n+m\rangle}\delta_{X_n}^{W_n} \wedge \mathsf{P}_{\langle 1+n,2n+m+1\rangle}\delta_{Y_1}^{V_1} \wedge \mathsf{P}_{\langle m+n,2n+2m\rangle}\delta_{Y_n}^{V_n}.$$

Hence, we obtain the equalities

$$\delta_{[X_1,\dots,X_n]} \boxtimes \delta_{[Y_1,\dots,Y_m]} = \delta_{[X_1,\dots,X_n,Y_1,\dots,Y_m]} = \delta_{[X_1]} \boxtimes \dots \boxtimes \delta_{[X_n]} \boxtimes \delta_{[Y_1]} \boxtimes \dots \boxtimes \delta_{[Y_m]}.$$

Now consider a strict elementary doctrine $R : \mathscr{C}_s^{op} \to \mathsf{InfSL}$ and the composition $\mathsf{P} := R \circ S$. The fact that P is a biased elementary doctrine follows setting $\delta_X^W := R_{\langle [\mathsf{p}_1],[\mathsf{p}_2]\rangle}\delta_{[X]}$. ☐

Definition 3.3.6. If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine, then the strict elementary doctrine $\mathsf{P}^s : \mathscr{C}_s^{op} \to \mathsf{InfSL}$ is called the *strictification* of P.

Observation 3.3.7. It is not obvious how to collect biased elementary doctrines in a 2-category. If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ to $\mathsf{P}' : \mathscr{C}'^{op} \to \mathsf{InfSL}$ are biased elementary doctrines and (F, f) is a pair where $F : \mathscr{C} \to \mathscr{C}'$ is a functor and, for every object $X \in \mathscr{C}$, the functors $f_X : \mathsf{P}(X) \to \mathsf{P}'(F(X))$ preserve the structure on the fibers, then we obtain the biased elementary doctrine $\mathsf{P}' \circ F : \mathscr{C}^{op} \to \mathsf{InfSL}$. If W is a weak product of the objects $X_1, \dots, X_n \in \mathscr{C}$, then the proof-irrelevant elements of $\mathsf{P}' \circ F$ over W are not necessarily in relation with the proof-irrelevant elements of P' over a weak product V of the objects $F(X_1), \dots, F(X_n) \in \mathscr{C}'$. The assumption that F preserves weak products would fix this issue, but it is too restrictive. For instance the functor $S : \mathscr{C} \to \mathscr{C}_s$ does not preserve weak products. The definition of a 2-category of biased elementary doctrines is still under investigation.

We end this section observing that the notion of *full, weak comprehension* for biased elementary doctrines is the same of Definition 1.2.9. Comprehensive diagonals are defined as follows.

Definition 3.3.8. A biased elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ has *comprehensive diagonals* if, for every pair of arrows $f, g : A \to X$ of $\mathscr{C}$ such that $\top_{[A]} \le \mathsf{P}^s_{\langle [f],[g]\rangle}\delta_{[X]}$, then $f = g$.

3.4 Quotient completion

In this section, we will define P-equivalence relations and the relative notion of quotient for biased elementary doctrines. We will provide the corresponding quotient completion and we will obtain the exact completion of a weakly left exact category as an instance. Finally, we will discuss the universal property of this construction which is slightly different from the universal property of strict elementary doctrines stated in Theorem 1.3.3.

Definitions and construction. In order to define P-equivalence relations of a biased elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ we will use proof-irrelevant elements. Indeed, since proof-irrelevant elements are reindexed by projections, they are suitable to define P-equivalence relations in the style of Definition 1.3.1. Given an object $X \in \mathscr{C}$ and a weak product $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\rightarrow} X$, a proof-irrelevant element $\rho \in \mathsf{P}(W)$ of W is a P-equivalence relation if it satisfies

ref) $\delta_X^W \leq \rho$,

sym) $\mathsf{P}_{\langle 2,1\rangle}(\rho) \leq \rho$,

trans) $\mathsf{P}_{\langle 1,2\rangle}(\rho) \wedge \mathsf{P}_{\langle 2,3\rangle}(\rho) \leq \mathsf{P}_{\langle 1,3\rangle}(\rho)$,

for some (and thus all) weak product $\mathsf{p}_i : K \to X$ for $i = 1, 2, 3$, and arrows $\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle : K \to W$, $\langle 2,1\rangle : W \to W$.

We can synthesize the above conditions working with the strictification P^s of P as follows.

Definition 3.4.1. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased elementary doctrine. A P-equivalence relation on an object $X \in \mathscr{C}$ is an element $\rho \in \mathsf{P}^s[X, X]$ which is a P^s-equivalence relation on $[X] \in \mathscr{C}_s$.

The following example makes explicit the use of the strictification to define P-equivalence relations for the biased elementary doctrine of weak subobjects.

Example 3.4.2. Let $\mathscr{C}$ be a category with weak limits and let X be an object of $\mathscr{C}$. In Example 3.2.6 we discussed that, for every weak product $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\rightarrow} X$, there is a bijection which sends an equivalence class $[r_1, r_2 : R \to X]_{po}$ of a pair $r_1, r_2 : R \to X$ to the the equivalence class, in the poset reflection of $\mathscr{C}/W$, of the right dashed arrow of the following weak limit:

$$\begin{tikzcd}
 & R' \arrow[dl, dashed] \arrow[dr, dashed, "\rho"] & \\
R \arrow[d, "r_1"'] \arrow[drr, "r_2" near start] & & W \arrow[d, "\mathsf{p}_2"] \arrow[dll, "\mathsf{p}_1"' near start] \\
X & & X.
\end{tikzcd}$$

In the next chapter, we will prove that the element $\lfloor \rho \rceil$ is a proof-irrelevant element of W and that there is a bijection

$$(\mathscr{C}/(X, X))_{po} \cong \mathsf{Psub}^s_{\mathscr{C}}[X, X].$$

It is straightforward to prove that, if $r_1, r_2 : R \to X$ is a pseudo-equivalence relation on X, then $\lfloor \rho \rceil$ is a $\mathsf{Psub}_{\mathscr{C}}$-equivalence relation on X.

The definition of quotient of a P-equivalence relation can be stated as follows.

Definition 3.4.3. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased elementary doctrine and let ρ be a P-equivalence relation on A. A *quotient* of ρ is an arrow $q : A \to C$ in $\mathscr{C}$ such that $\rho \leq \mathsf{P}^s_{[q]\times[q]}(\delta_{[C]})$ and, for every arrow $g : A \to Z$ such that $\rho \leq \mathsf{P}^s_{[g]\times[g]}(\delta_{[Z]})$, there exists a unique arrow $h : C \to Z$ such that $g = h \circ q$.

Example 3.4.4. Let $\mathscr{C}$ be a category with finite weak limits and let X be an object of $\mathscr{C}$. The bijection of Example 3.4.2 and a straightforward computation show that an arrow $q : A \to C$ is the coequalizer of a pseudo-equivalence relation $r_1, r_2 : R \to X$ if and only if it is the quotient of the correspondent $\mathsf{Psub}_{\mathscr{C}}$-equivalence relation $\lfloor \rho \rceil \in \mathsf{Psub}^s_{\mathscr{C}}[X, X]$.

The *biased elementary quotient completion* is obtained similarly to the strict case.
If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine, we consider the category $\overline{\mathscr{C}}$ whose

objects are pairs (X, ρ) where A is an object of $\mathscr{C}$ and ρ is a P-equivalence relation on X,

arrows $\lfloor f \rceil : (X, \rho) \to (Y, \sigma)$ are equivalence classes of arrows $f : X \to Y$ such that $\rho \leq \mathsf{P}^s_{[f]\times[f]}(\sigma)$. Two arrows f, f' are equivalent when $\rho \leq \mathsf{P}^s_{[f]\times[f']}(\sigma)$.

The assignment $\overline{\mathsf{P}}$ which sends an object $(X, \rho) \in \overline{\mathscr{C}}$ to $\overline{\mathsf{P}}(X, \rho) := \mathcal{Des}_\rho$ and an arrow $\lfloor f \rceil : (X, \rho) \to (Y, \sigma)$ to $\overline{\mathsf{P}}_{\lfloor f \rceil} := \mathsf{P}_f$ is a well defined functor as it is shown in the following lemma.

Lemma 3.4.5. *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine, then:*

(i) *If (X, ρ) and (Y, σ) are two objects of $\overline{\mathscr{C}}$ and $f : X \to Y$ is an arrow of $\mathscr{C}$ such that $\rho \leq \mathsf{P}^s_{[f]\times[f]}\sigma$, then P_f restricts to a map*

$$\mathsf{P}_f : \mathcal{Des}_\sigma \to \mathcal{Des}_\rho.$$

(ii) *If $f, g : X \to Y$ are arrows of $\mathscr{C}$ such that $\rho \leq \mathsf{P}^s_{[f]\times[g]}\sigma$ and $\beta \in \mathcal{Des}_\sigma$, then*

$$\mathsf{P}_f(\beta) = \mathsf{P}_g(\beta).$$

Proof. In order to prove (i), let $X \xleftarrow{\mathsf{p}_1} W \xrightarrow{\mathsf{p}_2} X$ and $Y \xleftarrow{\mathsf{p}_1} V \xrightarrow{\mathsf{p}_2} Y$ be weak products such that $\rho \in \mathsf{P}(W)$, $\beta \in \mathsf{P}(V)$ and let $g : W \to V$ be an arrow such that $\mathsf{p}_i \circ g = f \circ \mathsf{p}_i$ for $i = 1, 2$, and $\rho \leq \mathsf{P}_g\sigma$. If $\beta \in \mathcal{Des}_\sigma$, then we obtain $\mathsf{P}_f\beta \in \mathcal{Des}_\rho$ as follows:

$$\begin{aligned}
\mathsf{P}_{\mathsf{p}_1}\mathsf{P}_f\beta \wedge \rho &\leq \mathsf{P}_{\mathsf{p}_1}\mathsf{P}_f\beta \wedge \mathsf{P}_g\sigma \\
&= \mathsf{P}_g(\mathsf{P}_{\mathsf{p}_1}\beta \wedge \sigma) \\
&\leq \mathsf{P}_g\mathsf{P}_{\mathsf{p}_2}\beta \\
&= \mathsf{P}_{\mathsf{p}_2}\mathsf{P}_f\beta.
\end{aligned}$$

To obtain condition (ii), we recall that condition **wI** of Definition 3.1.2 implies that $\top_X = \mathsf{P}_{\Delta_X}\rho$ for every diagonal $\Delta_X : X \to W$. Now we consider an arrow $h : W \to V$ such that $\mathsf{p}_1 \circ h = f \circ \mathsf{p}_1$ and $\mathsf{p}_2 \circ h = g \circ \mathsf{p}_2$. Hence, we first obtain that $\mathsf{P}_f\beta \leq \mathsf{P}_g\beta$, as follows:

$$\begin{aligned}
\mathsf{P}_f\beta &= \mathsf{P}_{\Delta_X}\mathsf{P}_h\mathsf{P}_{\mathsf{p}_1}\beta \wedge \top_X \\
&= \mathsf{P}_{\Delta_X}(\mathsf{P}_h\mathsf{P}_{\mathsf{p}_1}\beta \wedge \rho) && (\top_X = \mathsf{P}_{\Delta_X}\rho) \\
&\leq \mathsf{P}_{\Delta_X}\mathsf{P}_h(\mathsf{P}_{\mathsf{p}_1}\beta \wedge \sigma) \\
&\leq \mathsf{P}_{\Delta_X}\mathsf{P}_h\mathsf{P}_{\mathsf{p}_2}\beta \\
&= \mathsf{P}_g\beta.
\end{aligned}$$

The opposite inequality follows from the symmetry condition of P-equivalence relations. $\square$

Observation 3.4.6. We observe that if $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine and $\mathscr{C}$ has a choice of weak products, then we can describe P-equivalence relations, quotients and the quotient completion without using P^s. The objects of $\overline{\mathscr{C}}$ are pair (X, ρ) where $X \in \mathscr{C}$ and $\rho \in \mathsf{P}(X \times X)$ is a proof-irrelevant element satisfying condition of reflexivity, symmetry and transitivity as in Definition 1.3.1. An arrow $\lfloor f \rceil : (X, \rho) \to (Y, \sigma)$ of $\mathscr{C}_s$ is the equivalence classes of an arrow $f : X \to Y$ of $\mathscr{C}$ such that $\rho \leq \mathsf{P}_{f \times f}(\sigma)$. Two arrows f, f' are equivalent when $\rho \leq \mathsf{P}_{f \times f'}(\sigma)$.

As expected, the quotient completion of a biased elementary doctrine yields a strict elementary doctrine with quotients. The following theorem has been proved in Maietti and Rosolini for the strict elementary doctrines in [MR13, Lemma 5.3, 5.4 and 5.5].

Theorem 3.4.7. *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be a biased elementary doctrine with weak full comprehensions and comprehensive diagonals, then* $\overline{\mathsf{P}} : \overline{\mathscr{C}}^{op} \to \mathsf{InfSL}$ *is a strict elementary doctrine in* ***QD***.

Proof. Given two objects $(X, \rho), (Y, \sigma) \in \overline{\mathscr{C}}$, the strict products are given up to isomorphism by

$$(W, \rho \boxtimes \sigma),$$

where W is a weak product of the objects $X, Y \in \mathscr{C}$.
Conditions **I**, **II** and **III** of Definition 1.2.5 are proved as in Theorem 3.3.5. An $\overline{\mathsf{P}}$-equivalence relation μ on (X, ρ) is a P-equivalence relation on X such that $\rho \leq \mu$. Hence, the quotient is given by

$$\lfloor 1_X \rceil : (X, \rho) \to (X, \mu),$$

and it is an effective quotient of effective descent. If $\alpha \in \overline{\mathsf{P}}(X, \rho)$, and $\{\!|\alpha|\!\} : C \to X$ is a weak comprehension of $\alpha \in \mathsf{P}(X)$, then the strict comprehension of α is given by

$$\lfloor \{\!|\alpha|\!\} \rceil : (C, \rho') \to (X, \rho)$$

where $\rho' := \mathsf{P}^s_{\lfloor \{\!|\alpha|\!\} \rceil \times \lfloor \{\!|\alpha|\!\} \rceil} \rho$. The diagonals are comprehensive by construction. We now prove that quotients are stable. In order to do that, we use the description of pullbacks through comprehensions of elementary doctrines with weak comprehensions and comprehensive diagonals, see Lemma A.0.7. Let λ be a $\overline{\mathsf{P}}$-eq. relation on the object (Y, σ) and consider its quotient $\lfloor 1_Y \rceil : (Y, \sigma) \to (Y, \lambda)$. If $\lfloor f \rceil : (X, \rho) \to (Y, \lambda)$ is an arrow, then the diagram

$$\begin{tikzcd}
(C, v) \arrow[rr, "\lfloor \pi_2 \rceil"] \arrow[dr, "\lfloor c \rceil"] \arrow[dd, "\lfloor \pi_1 \rceil"'] & & (X, \rho) \arrow[dd, "\lfloor f \rceil"] \\
& (W, \sigma \boxtimes \rho) \arrow[ur, "\lfloor \mathsf{p}_2 \rceil"'] \arrow[dl, "\lfloor \mathsf{p}_1 \rceil"] & \\
(Y, \sigma) \arrow[rr, "\lfloor 1_Y \rceil"'] & & (Y, \lambda)
\end{tikzcd}$$

where $Y \xleftarrow{\mathsf{p}_1} W \xrightarrow{\mathsf{p}_2} X$ is a weak product, $c := \{\!|\mathsf{P}^s_{\langle \mathsf{p}_1, \mathsf{p}_2 \rangle} \mathsf{P}^s_{[1_Y] \times [f]} \lambda|\!\}$ and $v := \mathsf{P}^s_{[c] \times [c]} \mathsf{P}^s_{\langle \mathsf{p}_1, \mathsf{p}_2 \rangle} \sigma \boxtimes \rho$ is a pullback diagram. We now prove that the element (X, ρ) is isomorphic to the element (C, w) where $w := \mathsf{P}^s_{[c] \times [c]} \mathsf{P}^s_{\langle \mathsf{p}_1, \mathsf{p}_2 \rangle} \lambda \boxtimes \rho$. If h denotes an arrow of the form $X \to W$ induced by the arrows f and 1_X, such that $\mathsf{p}_1 h = f$ and $\mathsf{p}_2 h = 1_X$, then since

$$\top_X \leq \mathsf{P}^s_{\langle [f], [f] \rangle} \lambda = \mathsf{P}_h \mathsf{P}^s_{\langle \mathsf{p}_1, \mathsf{p}_2 \rangle} \mathsf{P}^s_{[1_Y] \times [f]} \lambda$$

there exists an arrow $g : X \to C$ such that $c \circ g = h$. Since

$$\sigma \leq \mathsf{P}^s_{[h]\times[h]}\mathsf{P}^s_{\langle \mathsf{p}_1,\mathsf{p}_2\rangle}\lambda \boxtimes \rho = \mathsf{P}^s_{[g]\times[g]}\mathsf{P}^s_{[c]\times[c]}\mathsf{P}^s_{\langle \mathsf{p}_1,\mathsf{p}_2\rangle}\lambda \boxtimes \rho = \mathsf{P}^s_{[g]\times[g]}w$$

the arrow g induces an arrow

$$\lfloor g \rceil : (X,\rho) \to (C,w).$$

This arrow is the inverse of $\lfloor \pi_2 \rceil : (C,w) \to (X,\rho)$. □

Observation 3.4.8. If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an elementary doctrine, the elementary quotient completion and the biased one yield isomorphic elementary doctrines.

Observation 3.4.9. Example 3.4.2 and Example 3.4.4 imply that the exact completion of a weak lex category is a particular case of biased elementary quotient completion, in the sense that there is an equivalence of the two categories $\overline{\mathscr{C}} \cong \mathscr{C}_{ex/wlex}$.

Universal property. We now discuss the universal property of the biased elementary quotient completion which is different from the universal property stated in Theorem 1.3.3. This should not be surprising since a similar issue occurs in the universal property of the exact completion of weakly left exact categories. Indeed, as observed by Carboni and Vitale in [CV98], the exact completion construction

$$\Gamma : \mathscr{C} \to \mathscr{C}_{ex}$$

of weakly lex categories does not provide the unit of a biadjunction between the 2-category of exact categories and exact functors, denoted by **EX**, and any definable 2-category of weakly lex categories, denoted by **WLEX**. However, the authors consider a special class of functors called *left coverings* and provide a universal property of the exact completion in the sense of the following Theorem. We refer to [Vit94] for further details.

Definition 3.4.10. Let $F : \mathscr{C} \to \mathcal{A}$ be a functor from a weakly left exact category $\mathscr{C}$ to an exact category $\mathcal{A}$. The functor F is called *left covering* if, for all functors $\mathcal{L} : \mathcal{D} \to \mathscr{C}$ defined on a finite category $\mathcal{D}$ and for all weak limits

$$\mathrm{wlim}\mathcal{L} = (\pi_D : L \to \mathcal{L}D)_{D\in\mathcal{D}},$$

the canonical factorization $p : FL \to \tilde{L}$ is a regular epimorphism, where

$$\mathrm{lim}F\mathcal{L} = (\tilde{\pi}_D : \tilde{L} \to F(\mathcal{L}D))_{D\in\mathcal{D}}.$$

The next result appears as [CV98, Theorem 29].

Theorem 3.4.11. (*Carboni and Vitale*). *Let $\mathscr{C}$ be a weakly left exact category and let $\mathcal{A}$ be an exact category. The exact completion Γ induces an equivalence between the category of left covering functors from $\mathscr{C}$ to $\mathcal{A}$, and the category of exact functors from $\mathscr{C}_{ex}$ to $\mathcal{A}$. The same holds for the regular completion, with respect to any regular category $\mathcal{A}$.* □

Taking advantages from the above result, we now define in the context of the biased elementary doctrines the analogous of left covering functors.From condition **wIII** of Definition 3.1.2 we obtain a canonical 1-arrow

$$(J,j) : \mathsf{P} \to \overline{\mathsf{P}}$$

where the functor J is defined on objects $X \in \mathscr{C}$ as $J(X) := (X, \delta_{[X]})$ and on an arrow $f : X \to Y$ as $J(f) := \lfloor f \rceil : (X,\delta_{[X]}) \to (Y,\delta_{[Y]})$. The functors j_X are just the identities of $\mathsf{P}(X)$. When P has weak comprehensions we can observe the following facts:

- if W is a weak product of the objects $X, Y, \in \mathscr{C}$, then the uniqe arrow into the strict product

$$(W, \delta_{[W]}) \longrightarrow (W, \delta_{[X]} \boxtimes \delta_{[Y]})$$

is a quotient of the $\overline{\mathsf{P}}$-equivalence relation $\delta_{[X]} \boxtimes \delta_{[Y]}$ over $(W, \delta_{[W]})$.

- If $\{\!|\alpha|\!\} : X \to A$ is a weak comprehension of an element $\alpha \in \mathsf{P}(A)$, then $J(\{\!|\alpha|\!\})$ factors through the comprehension of $j(\alpha)$ via a $\overline{\mathsf{P}}$-quotient

$$\begin{tikzcd} (X, \mathsf{P}^s_h \delta_{[A]}) \arrow[r, "\{\!|\alpha|\!\}"] & (A, \delta_{[A]}) \\ (X, \delta_{[X]}) \arrow[u, dashed] \arrow[ur, "J(\{\!|\alpha|\!\})"'] & \end{tikzcd}$$

where h is the product of $\{\!|\alpha|\!\}$ and $\{\!|\alpha|\!\}$, i.e. $h = (\{\{\!|\alpha, \alpha|\!\}\}, \langle 1, 2\rangle)$.

The above observations and the relation between weak limits and weak comprehensions, (see Lemma A.0.7), lead to the following definition of left covering functors for biased elementary doctrine as follows.

Definition 3.4.12. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased elementary doctrine with weak full comprehensions and comprehensive diagonals and let $R : \mathcal{D}^{op} \to \mathsf{InfSL}$ be an object of **QD**. A pair $(F, f) : \mathsf{P} \to R$ is called *left covering* when

1. The functor F sends a weak product W of the objects $X, Y \in \mathscr{C}$ to the object $F(W) \in \mathcal{D}$ such that the unique arrow
$$\langle F(\mathsf{p}_1), F(\mathsf{p}_2)\rangle : F(W) \to F(X) \times F(Y)$$
is a quotient of an R-equivalence relation.

2. For every object $X \in \mathscr{C}$, the functors $f_X : \mathsf{P}(X) \to RF(X)$ preserve all the structure. In particular, the functor f_X preserves finite meets and for a weak product $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\rightarrow} X$ we have
$$f_W(\delta^W_X) = R_{\langle F(\mathsf{p}_1), F(\mathsf{p}_2)\rangle}(\delta_{F(X)}).$$
Moreover, we require that the restriction of the functor f_W
$$f_W : \mathsf{PIrr}(W) \to RF(W)$$
takes value in $\mathcal{D}es_{k_X}$ where k_X is the R-kernel of $\langle F(\mathsf{p}_1), F(\mathsf{p}_2)\rangle$, i.e.
$$R_{\langle F(p_1), F(p_2)\rangle \times \langle F(p_1), F(p_2)\rangle} \delta_{F(X) \times F(X)}.$$

3. if $\{\!|\alpha|\!\} : X \to A$ is a weak comprehension of the element $\alpha \in \mathsf{P}(A)$, then the arrow $F(\{\!|\alpha|\!\}) : F(X) \to F(A)$ factors through $\{\!|f(\alpha)|\!\}$ via a quotient of an R-equivalence relation.

Observation 3.4.13. Since R has stable quotients of effective descent, the arrow $\langle F(\mathsf{p}_1), F(\mathsf{p}_2)\rangle$ of condition 1 is the quotient of its R-kernel

$$R_{\langle F(\mathsf{p}_1), F(\mathsf{p}_2)\rangle \times \langle F(\mathsf{p}_1), F(\mathsf{p}_2)\rangle} \delta_{F(X) \times F(Y)}.$$

If $\rho \in \mathsf{P}^s[X, X]$ is a P-eq. relation, then $f_W \rho \in \mathcal{D}es_{k_X}$. Hence, thanks to condition 1 and effectiveness of quotients, we will abuse the notation and write $f_W \rho \in R(F(X) \times F(X))$.

Observation 3.4.14. From Proposition A.0.9 we know that (weak) equalizers can be built through (weak) comprehensions. Hence, Proposition 27 of [CV98] implies that the functor F of a left covering 1-arrow (F, f) is a left covering functor. Moreover, Lemma 21 of *loc. cit.* implies that F preserves monomorphic families of arrows. However, the composition of left covering functors is not necessary left covering, as shown by a counterexample in §3.2 of *loc. cit.*. For this reason, biased elementary doctrines and left covering 1-arrows do not form a 2-category.

We will denote by $\mathbf{Lco}(\mathsf{P}, R)$ the category of left covering 1-arrows from P to R. In the following theorem we prove the universal property of the biased elementary quotient completion in style of Theorem 3.4.11.

Theorem 3.4.15. *Let P be a biased elementary doctrine with full weak comprehensions and comprehensive diagonals, and let R be an elementary doctrine in $\boldsymbol{QD}$. The pre-composition with the 1-arrow (J, j) induces an equivalence between the following categories*

$$\boldsymbol{QD}(\overline{\mathsf{P}}, R) \xrightarrow{\quad (-)\circ(J,j) \quad} \boldsymbol{Lco}(\mathsf{P}, R).$$

Proof. We first prove that the functor $(-) \circ (J, j)$ is essentially surjective. Given a 1-arrow (F, f) of $\mathbf{Lco}(\mathsf{P}, R)$ we can define a 1-arrow $(\bar{F}, \bar{f}) : \overline{\mathsf{P}} \to R$ as follows. The functor $\bar{F}$ sends a projective object $(X, \delta_{[X]})$ to the image of F, i.e. $\bar{F}(X, \delta_{[X]}) := F(X)$. On the objects of the form (X, ρ), the image $\bar{F}(X, \rho)$ is defined to be the codomain of the quotient of the R-equivalence relation $f_W \rho$, for a weak product W of X and X. Similarly, if $\lfloor g \rceil : (A, \delta_A) \to (B, \delta_B)$ is an arrow between projectives, then we define $\bar{F}(\lfloor g \rceil) := F(g)$. If $\lfloor g \rceil : (A, \rho) \to (B, \sigma)$ then we define $\bar{F}(\lfloor g \rceil)$ as the unique arrow induced by the quotients, which makes the following diagram commute

$$\begin{tikzcd}
\bar{F}(A, \delta_A) \arrow[r, dashed] \arrow[d] & \bar{F}(B, \delta_B) \arrow[d] \\
\bar{F}(A, \rho) \arrow[r, "F(g)"] & \bar{F}(B, \sigma).
\end{tikzcd}$$

The functors $\bar{f}_{(-)} : \mathsf{P}(-) \to R(F(-))$ are defined as the functors f on the projectives $(X, \delta_{[X]})$. On the elements (X, ρ), it is a trivial verification to prove that the functor f_X restricts to a functor

$$\bar{f}_{(X,\rho)} := f_X : \mathcal{D}es\,\rho \to \mathcal{D}es\, f_W \rho.$$

We now prove that $(\bar{F}, \bar{f})$ sends strict comprehensions to strict comprehensions. Indeed, as in the proof of Theorem 3.4.7, a strict comprehension of $\alpha \in \bar{P}(X, \rho)$ is given by

$$\lfloor \{\!|\alpha|\!\} \rceil : (C, \rho') \to (X, \rho)$$

where $\rho' := \mathsf{P}^s_h \rho$ and h is the product of $\{\!|\alpha|\!\}$ and $\{\!|\alpha|\!\}$, i.e. $h = (\{\{\!|\alpha, \alpha|\!\}\}, \langle 1, 2 \rangle)$. Since R has strict comprehensions, it follows that a comprehension $\{\!|\bar{f}_{(X,\rho)}(\alpha)|\!\} : D \to \bar{F}(X, \rho)$ of $\bar{f}_{(X,\rho)}(\alpha)$ is monic. Hence, by Lemma A.0.6, it follows that D and $\bar{F}(C, \rho')$ are quotients of the same R-equivalence relation

$$R_{F\{\!|\alpha|\!\} \times F\{\!|\alpha|\!\}} f_W \rho = f_V \rho',$$

where V is a weak product of C and C. Hence, we have proved that $(\bar{F}, \bar{f}) \in \mathbf{QD}(\overline{\mathsf{P}}, R)$. We now prove that the functor $(-) \circ (J, j)$ is fully faithful. Indeed if (F, f) and (G, g) are 1-arrows of $\mathbf{Lco}(P, R)$ and $\theta : (F, f) \Rightarrow (G, g)$ is a 2-arrow, then it can be extended to a 2-arrow $\bar{\theta} : (\bar{F}, \bar{f}) \Rightarrow (\bar{G}, \bar{g})$. The arrows $\bar{\theta}_{(A, \delta_A)}$ on the projectives are defined as θ_A. On the objects of the form $\bar{\theta}_{(A,\rho)}$ the arrow is defined as the unique arrow induced by quotients, which makes the following diagram commute

$$\begin{tikzcd}
\bar{F}(A,\rho) \arrow[r, dashed] & \bar{G}(A,\rho) \\
\bar{F}(A,\delta_A) \arrow[u] \arrow[r, "\theta_A"] & \bar{G}(A,\delta_A). \arrow[u]
\end{tikzcd}$$

□

Remark 3.4.16. We could define the quotient completion of a biased elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ in a different way. Indeed, we can add quotients applying first the strictification and then the elementary quotient completion obtaining the elementary doctrine $\overline{\mathsf{P}^s}$. However, this construction does not have the universal property discussed. For instance, completing with quotient we would obtain more projectives than the objects of $\mathscr{C}$.

3.5 Doctrines with $\Rightarrow, \exists, \forall$

We will now start the second part of the chapter, which is devoted to provide a general formulation of the theorems of Section 2.3 in the context of biased elementary doctrines. In this section, we will define doctrines that can express the connective of implication and the existential and universal quantification. The definitions will be very similar to the strict case but, as usual, we will pay particular attention to what happens when we restrict to proof-irrelevant elements.

We start with the definition of biased elementary doctrine with implications.

Definition 3.5.1. A biased primary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is called *implicational* if for every object $X \in \mathscr{C}$ and element $\alpha \in \mathsf{P}(X)$ the functor $\alpha \wedge - : \mathsf{P}(X) \to \mathsf{P}(X)$ has a right adjoint $\alpha \Rightarrow - : \mathsf{P}(X) \to \mathsf{P}(X)$. Moreover, for every arrow $f : Y \to X$ of $\mathscr{C}$ and elements $\alpha, \beta \in \mathsf{P}(X)$, it holds the equality $\mathsf{P}_f(\alpha \Rightarrow \beta) = \mathsf{P}_f\alpha \Rightarrow \mathsf{P}_f\beta$.

Observation 3.5.2. If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an implicational biased elementary doctrine, then a trivial computation shows that the implication of proof-irrelevant element is proof-irrelevant. Hence, P is implicational if and only if the strictification P^s is implicational.

We now consider biased elementary doctrines that can express existential and universal quantifications. The definition is given, as usual, requiring left and right adjoints to reindexing functors but, because of weak products, we shall consider a weak version of the Beck-Chevalley and Frobenius conditions.

Definition 3.5.3. A biased elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is called *existential* if for every pair of objects $X_1, X_2 \in \mathscr{C}$ and weak product $X_1 \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\to} X_2$ the functors $\mathsf{P}_{\mathsf{p}_i} : \mathsf{P}(X) \to \mathsf{P}(W)$, for $i = 1, 2$, have left adjoints $\exists_{\mathsf{p}_i} : \mathsf{P}(W) \to \mathsf{P}(X_i)$ which satisfy

- the *weak Beck-Chevalley* condition: for any commutative diagram of this form

$$\begin{tikzcd}
V \arrow[r, "\mathsf{p}_2"] \arrow[d, dashed, "f'"] \arrow[dd, bend right=60, "\mathsf{p}_1"'] & Y \arrow[d, "f"] \\
W \arrow[r, "\mathsf{p}_2"'] \arrow[d, "\mathsf{p}_1"] & X_2 \\
X_1, &
\end{tikzcd}$$

where $X_1 \overset{\mathsf{p}_1}{\leftarrow} V \overset{\mathsf{p}_2}{\rightarrow} Y$ is a weak product, the canonical arrow $\exists_{\mathsf{p}_2} \circ \mathsf{P}_{f'}\alpha \leq \mathsf{P}_f \circ \exists_{\mathsf{p}_2}\alpha$ is an isomorphism, for all proof-irrelevant $\alpha \in \mathsf{P}(W)$.

- the *weak Frobenius reciprocity*: for any projection $\mathsf{p}_i : W \to X_i$, element $\alpha \in \mathsf{P}(X_i)$, and $\beta \in \mathsf{P}(W)$ proof-irrelevant, the canonical arrow $\exists_{\mathsf{p}_i}(\mathsf{P}_{\mathsf{p}_i}\alpha \wedge \beta) \leq \alpha \wedge \exists_{\mathsf{p}_i}\beta$ is an isomorphism.

Definition 3.5.4. A biased elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is called *universal* if for every pair of objects $X_1, X_2 \in \mathscr{C}$ and weak product $X_1 \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\rightarrow} X_2$ the functors $\mathsf{P}_{\mathsf{p}_i} : \mathsf{P}(X) \to \mathsf{P}(W)$, for $i = 1, 2$, have right adjoints $\forall_{\mathsf{p}_i} : \mathsf{P}(W) \to \mathsf{P}(X_i)$ which satisfy

- the *"weak"Beck-Chevalley* condition: for any commutative diagram of this form

$$\begin{tikzcd} V \arrow[r, "\mathsf{p}_2"] \arrow[d, dashed, "f'"] \arrow[dd, bend right=60, "\mathsf{p}_1"'] & Y \arrow[d, "f"] \\ W \arrow[r, "\mathsf{p}_2"'] \arrow[d, "\mathsf{p}_1"] & X_2 \\ X_1, & \end{tikzcd}$$

where $X_1 \overset{\mathsf{p}_1}{\leftarrow} V \overset{\mathsf{p}_2}{\rightarrow} Y$ is a weak product, the canonical arrow $\mathsf{P}_f \circ \forall_{\mathsf{p}_2}\alpha \leq \forall_{\mathsf{p}_2} \circ \mathsf{P}'_f\alpha$ is an isomorphism, for all proof-irrelevant $\alpha \in \mathsf{P}(W)$.

When $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an existential (universal) biased elementary doctrine, the adjoint functors $\exists_{\mathsf{p}_i}$ ($\forall_{\mathsf{p}_i}$) behave particularly well on proof-irrelevant elements. At first, we observe that if X_1, X_2 are two objects of $\mathscr{C}$ then we can define left (right) adjoint functors $\exists_{p_i} : \mathsf{P}^s[X_1, X_2] \to \mathsf{P}^s[X_i]$ to the functors $\mathsf{P}^s_{p_i} : \mathsf{P}^s[X_i] \to \mathsf{P}^s[X_1, X_2]$. Indeed, let $X \overset{\mathsf{p}_1}{\leftarrow} K \overset{\mathsf{p}_2}{\rightarrow} Y$ and $X \overset{\mathsf{p}'_1}{\leftarrow} K' \overset{\mathsf{p}'_2}{\rightarrow} Y$ be to weak products and $\alpha \in \mathsf{P}^s[X, Y]$. If $h : K' \to K$ and $h' : K \to K'$ are arrows such that $\mathsf{p}_i \circ h' = \mathsf{p}'_i$ and $\mathsf{p}'_i \circ h = \mathsf{p}_i$, for $i = 1, 2$, then Proposition 3.2.7 implies that P_h is an isomorphism on proof-irrelevant elements with inverse $\mathsf{P}_{h'}$ and then $\mathsf{P}_h = \exists_{h'}$ and $\mathsf{P}_{h'} = \exists_h$. Hence, if $\alpha \in \mathsf{P}(K)$ and $\alpha' := \mathsf{P}_h\alpha \in \mathsf{P}(K')$ we obtain

$$\exists_{p_1}\alpha = \exists_{\mathsf{p}'_1}\alpha'.$$

Similarly, we can consider three (or more) objects $X, Y, Z \in \mathscr{C}$ and a weak product given by a weak product $K \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\rightarrow} Z$ of a weak product $X \overset{\mathsf{p}_1}{\leftarrow} K \overset{\mathsf{p}_2}{\rightarrow} Y$ and Z. Hence, we have a left (right) adjoint to the functor $\mathsf{P}_{\mathsf{p}_2} : \mathsf{P}(K) \to \mathsf{P}(U)$. In the following lemma we prove that the adjunctions restrict to proof-irrelevant elements providing a left (right) adjoint to the functor $\mathsf{P}^s_{p_2}$ as in the following diagram

$$\begin{tikzcd} \mathsf{P}^s[Z, X, Y] \arrow[r, dashed, shift left, "\exists_{p_2}"] \arrow[r, phantom, "\perp"] \arrow[d, hook] & \mathsf{P}^s[X, Y] \arrow[l, shift left, "\mathsf{P}^s_{p_2}"] \arrow[d, hook] \\ \mathsf{P}(U) \arrow[r, shift left, "\exists_{\mathsf{p}_2}"] \arrow[r, phantom, "\perp"] & \mathsf{P}(K). \arrow[l, shift left, "\mathsf{P}_{\mathsf{p}_2}"] \end{tikzcd}$$

Lemma 3.5.5. *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an existential (universal) biased elementary doctrine, and $X_1, \dots, X_n$, $Y_1, \dots, Y_m$ are objects of $\mathscr{C}$, then we obtain the following adjunctions on proof-irrelevant elements*

$$\begin{tikzcd} \mathsf{P}^s_{p_1} : \mathsf{P}^s[X_1, \dots, X_n] \arrow[r, shift left] \arrow[r, phantom, "\top"] & \mathsf{P}^s[X_1, \dots, X_n, Y_1, \dots, Y_m] : \exists_{p_1} \arrow[l, shift left] \end{tikzcd}$$

$$\left(\mathsf{P}^s_{p_1} : \mathsf{P}^s[X_1,\dots,X_n] \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{P}^s[X_1,\dots,X_n,Y_1,\dots,Y_m] : \forall_{p_1} \right).$$

Proof. Let W be a weak product of the objects $X_l \in \mathscr{C}$, for $1 \leq l \leq n$, let V be a weak product of the objects $Y_j \in \mathscr{C}$ for $1 \leq j \leq m$, and let $W \overset{\mathsf{p}_1}{\leftarrow} K \overset{\mathsf{p}_2}{\rightarrow} V$ be a weak product of W and V. If $\alpha \in \mathsf{P}(K)$ is proof-irrelevant, then we prove that $\exists_{\mathsf{p}_1}\alpha$ is proof-irrelevant. Indeed, if $X_l \overset{\mathsf{p}_1}{\leftarrow} W_l \overset{\mathsf{p}_2}{\rightarrow} X_l$ and $Y_j \overset{\mathsf{p}_1}{\leftarrow} V_j \overset{\mathsf{p}_2}{\rightarrow} Y_j$ are weak products and $K \overset{\mathsf{p}_1}{\leftarrow} Q \overset{\mathsf{p}_2}{\rightarrow} K$ and

$$\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \bigwedge_{l=1}^{n} \mathsf{P}_{\langle l,n+l\rangle}\delta^{W_l}_{X_l} \wedge \bigwedge_{j=1}^{m} \mathsf{P}_{\langle j,m+j\rangle}\delta^{V_j}_{X_j} \leq \mathsf{P}_{\mathsf{p}_2}\alpha$$

in $\mathsf{P}(Q)$ for some arrows $\langle l,n+l\rangle : Q \to W_l$ and $\langle j,m+j\rangle : Q \to V_j$ induced by the weak universal property of the weak products. If $W \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\rightarrow} W$ is a weak product we consider a commutative diagram of the form

$$\begin{tikzcd}
T \arrow[r, "p_1"] \arrow[d, dashed, "h_1"] \arrow[dd, bend right, "\mathsf{p}_2"'] & U \arrow[d, "\mathsf{p}_1"] \\
K \arrow[r, "\mathsf{p}_1"'] \arrow[d, "\mathsf{p}_2"] & W \\
V, &
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "p_1"] \arrow[d, dashed, "h_2"] \arrow[dd, bend right, "\mathsf{p}_2"'] & U \arrow[d, "\mathsf{p}_2"] \\
K \arrow[r, "\mathsf{p}_1"'] \arrow[d, "\mathsf{p}_2"] & W \\
V, &
\end{tikzcd}$$

and we obtain that $\mathsf{P}_{\mathsf{p}_1}\exists_{\mathsf{p}_1}\alpha \wedge \bigwedge_{l=1}^{n} \mathsf{P}_{\langle l,n+l\rangle}\delta^{W_l}_{X_l} \leq \mathsf{P}_{\mathsf{p}_2}\exists_{\mathsf{p}_1}\alpha$, for some arrows $\langle l,n+l\rangle : U \to W_l$ induced by the weak universal property of the weak products, as follows:

$$\begin{aligned}
\mathsf{P}_{\mathsf{p}_1}\exists_{\mathsf{p}_1}\alpha \wedge \bigwedge_{l=1}^{n} \mathsf{P}_{\langle l,n+l\rangle}\delta^{W_l}_{X_l} &= \exists_{\mathsf{p}_1}\mathsf{P}_{h_1}\alpha \wedge \bigwedge_{l=1}^{n} \mathsf{P}_{\langle l,n+l\rangle}\delta^{W_l}_{X_l} && \text{(B-C)}\\
&= \exists_{\mathsf{p}_1}(\mathsf{P}_{h_1}\alpha \wedge \mathsf{P}_{\mathsf{p}_1}\bigwedge_{l=1}^{n} \mathsf{P}_{\langle l,n+l\rangle}\delta^{W_l}_{X_l}) && \text{(Frob.)}
\end{aligned}$$

since T is a weak product of W, W and V, we can consider and arrow $\langle 1,3,2,3\rangle : T \to Q$ induced by the weak universal property of weak products and we obtain

$$\begin{aligned}
&\exists_{\mathsf{p}_1}(\mathsf{P}_{h_1}\alpha \wedge \mathsf{P}_{\mathsf{p}_1}\bigwedge_{l=1}^{n} \mathsf{P}_{\langle l,n+l\rangle}\delta^{W_l}_{X_l})\\
&= \exists_{\mathsf{p}_1}\mathsf{P}_{\langle 1,3,2,3\rangle}(\mathsf{P}_{\mathsf{p}_1}\alpha \wedge \bigwedge_{l=1}^{n} \mathsf{P}_{\langle l,n+l\rangle}\delta^{W_l}_{X_l} \wedge \bigwedge_{j=1}^{m} \mathsf{P}_{\langle j,m+j\rangle}\delta^{V_j}_{X_j})\\
&\leq \exists_{\mathsf{p}_1}\mathsf{P}_{\langle 1,3,2,3\rangle}\mathsf{P}_{\mathsf{p}_2}\alpha = \exists_{\mathsf{p}_1}\mathsf{P}_{h_2}\alpha = \mathsf{P}_{\mathsf{p}_2}\exists_{\mathsf{p}_1}\alpha. && \text{(B-C)}
\end{aligned}$$

We now consider another tern $W' \overset{\mathsf{p'}_1}{\leftarrow} K' \overset{\mathsf{p'}_2}{\rightarrow} V'$ of weak products and a (not necessarily commutative) diagram of arrows

$$\begin{tikzcd}
K' \arrow[r, "\mathsf{p'}_1"] \arrow[d, "h"'] & W' \arrow[d, "g"] \\
K \arrow[r, "\mathsf{p}_1"'] & W
\end{tikzcd}$$

such that $\mathsf{p}_l \circ \mathsf{p}_1 \circ h = \mathsf{p'}_l \circ \mathsf{p'}_1$ and $\mathsf{p}_j \circ \mathsf{p}_2 \circ h = \mathsf{p'}_j \circ \mathsf{p'}_2$ and $\mathsf{p}_l \circ g = \mathsf{p'}_l$ for $1 \le l \le n$ and $1 \le j \le m$, then we want to prove that

$$\mathsf{P}_g \exists_{\mathsf{p}_1} \alpha = \exists_{\mathsf{p}'_1} \mathsf{P}_h \alpha.$$

In order to do that, we consider two arrows $h' : K \to K'$ and $g' : W \to W'$ such that $\mathsf{p'}_l \circ \mathsf{p'}_1 \circ h' = \mathsf{p}_l \circ \mathsf{p}_1$ and $\mathsf{p'}_j \circ \mathsf{p'}_2 \circ h' = \mathsf{p}_j \circ \mathsf{p}_2$ and $\mathsf{p'}_l \circ g' = \mathsf{p}_l$ for $1 \le l \le n$ and $1 \le j \le m$. By Proposition 3.2.7, the functors P_h and $\mathsf{P}_{h'}$ are inverse functors on proof-irrelevant elements and the same holds for the functors P_g and $\mathsf{P}_{g'}$. Hence, the above equality follows from the following relations on proof-irrelevant elements

$$\mathsf{P}_g \exists_{\mathsf{p}_1} = \exists_{g'} \exists_{p_1} \dashv \mathsf{P}_{\mathsf{p}_1} \mathsf{P}_{g'} = \mathsf{P}_{h'} \mathsf{P}_{\mathsf{p'}_1} \vdash \exists_{\mathsf{p'}_1} \exists_{h'} = \exists_{\mathsf{p}'_1} \mathsf{P}_h.$$

Hence, we obtain a left adjoint $\exists_{p_1}$ to the reindexing $\mathsf{P}^s_{p_1}$. A similar argument proves the statement for the right adjoint $\forall_{p_1}$ of the reindexing $\mathsf{P}^s_{p_1}$. □

As a corollary of the above lemma, we obtain the following characterization of the existential (universal) biased elementary doctrines in terms of their strictifications.

Corollary 3.5.6. *A biased elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is existential (universal) if and only if the strictification P^s is an existential (universal) elementary doctrine.* □

The above results allow us to work directly with the strictifications of the existential (universal) biased elementary doctrines, which makes handier the description of proof-irrelevant elements.

When $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine, and $g : Y \to W$ is an arrow into a weak product of the objects $X_1, \dots, X_n \in \mathscr{C}$, then we will denote with g_s the composition in $\mathscr{C}_s$ of the arrows

$$[Y] \xrightarrow{[g]} [W] \xrightarrow{\langle [\mathsf{p}_1], \dots, [\mathsf{p}_n] \rangle} [X_1, \dots, X_n]$$

where $\langle [\mathsf{p}_1], \dots, [\mathsf{p}_n] \rangle$ is the unique arrow induced by the arrows $[\mathsf{p}_i] : [W] \to [X_i]$, for $1 \le i \le n$.

Remark 3.5.7. If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an existential biased elementary doctrine and $f : Y \to X$ is an arrow of $\mathscr{C}$, then the functor $\mathsf{P}_f : \mathsf{P}(X) \to \mathsf{P}(Y)$ has a left adjoint $\exists_f$ which coincides with the functor $\exists_{[f]} : \mathsf{P}^s[Y] \to \mathsf{P}^s[X]$. An easy verification shows that the functor $\exists_f$ sends an element $\alpha \in \mathsf{P}(Y)$ to

$$\exists_f(\alpha) := \exists_{\mathsf{p}_2}(\mathsf{P}_{f'} \delta^W_X \wedge \mathsf{P}_{\mathsf{p}_1} \alpha), \tag{3.2}$$

where $f' : K \to W$ is an arrow from weak products $Y \overset{\mathsf{p}_1}{\leftarrow} K \overset{\mathsf{p}_2}{\rightarrow} X$ and $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\rightarrow} X$, such that $\mathsf{p}_1 \circ f' = f \circ \mathsf{p}_1$ and $\mathsf{p}_2 \circ f' = \mathsf{p}_2$.

In particular, if $g : Y \to W$ is an arrow into a weak product W of the objects $X_i \in \mathscr{C}$, for $1 \le i \le n$, then we can consider the functor $\exists_{g_s} : \mathsf{P}^s[Y] \to \mathsf{P}^s[X_1, \dots, X_n]$, which sends an element $\alpha \in \mathsf{P}(Y)$ into the equivalence class of the element

$$\exists_g(\alpha) := \exists_{\mathsf{p}_2}(\mathsf{P}_{g'}(\delta^{W_1}_{X_1} \boxtimes \cdots \boxtimes \delta^{W_n}_{X_n}) \wedge \mathsf{P}_{\mathsf{p}_1} \alpha) \tag{3.3}$$

of $\mathsf{P}(W)$, where $X_i \overset{\mathsf{p}_1}{\leftarrow} W_i \overset{\mathsf{p}_2}{\rightarrow} X_i$, $Y \overset{\mathsf{p}_1}{\leftarrow} K \overset{\mathsf{p}_2}{\rightarrow} W$ and $W \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\rightarrow} W$ are weak products and $g' : K \to U$ is an arrow such that $\mathsf{p}_1 \circ g' = g \circ \mathsf{p}_1$ and $\mathsf{p}_2 \circ g' = \mathsf{p}_2$.

A similar remark holds also for the universal quantification.

Remark 3.5.8. If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an implicational and universal biased elementary doctrine and $f : Y \to X$ is an arrow of $\mathscr{C}$, then the functor $\mathsf{P}_f : \mathsf{P}(X) \to \mathsf{P}(Y)$ has a left adjoint $\exists_f$ which coincides with the functor $\forall_f : \mathsf{P}^s(Y) \to \mathsf{P}^s(X)$. An easy verification shows that the functor $\forall_f$ sends an element $\alpha \in \mathsf{P}(Y)$ to

$$\forall_f(\alpha) := \forall_{\mathsf{p}_2}(\mathsf{P}_{f'}\delta_X^W \Rightarrow \mathsf{P}_{\mathsf{p}_1}(\alpha)) \tag{3.4}$$

where $f' : K \to W$ is an arrow between weak products $Y \overset{\mathsf{p}_1}{\leftarrow} K \overset{\mathsf{p}_2}{\to} X$ and $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\to} X$, such that $\mathsf{p}_1 \circ f' = f \circ \mathsf{p}_1$ and $\mathsf{p}_2 \circ f' = \mathsf{p}_2$.

Similarly, if $g : Y \to W$ is an arrow into a weak product W of the objects $X_i \in \mathscr{C}$, for $1 \leq i \leq n$, then we can consider the functor $\forall_{g_s} : \mathsf{P}^s(Y) \to \mathsf{P}^s[X_1, \dots, X_n]$, which sends an element $\alpha \in \mathsf{P}(Y)$ into the equivalence class of the element

$$\forall_g(\alpha) := \forall_{\mathsf{p}_2}(\mathsf{P}_{g'}(\delta_{X_1}^{W_1} \boxtimes \cdots \boxtimes \delta_{X_n}^{W_n}) \wedge \mathsf{P}_{\mathsf{p}_1}\alpha) \tag{3.5}$$

of $\mathsf{P}(W)$, where $X_i \overset{\mathsf{p}_1}{\leftarrow} W_i \overset{\mathsf{p}_2}{\to} X_i$, $Y \overset{\mathsf{p}_1}{\leftarrow} K \overset{\mathsf{p}_2}{\to} W$ and $W \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\to} W$ are weak products and $g' : K \to U$ is an arrow such that $\mathsf{p}_1 \circ g' = g \circ \mathsf{p}_1$ and $\mathsf{p}_2 \circ g' = \mathsf{p}_2$.

3.6 Slice doctrines and quotient completion

In this section, we will provide some results which relate suitable existential (universal) biased elementary doctrine and their slices. These results will be useful in the next section in order to prove the local cartesian closure of the quotient completion.

The first lemma we prove is a version of Lemma A.0.7 for biased elementary doctrines.

Lemma 3.6.1. *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine with weak comprehensions and comprehensive diagonals, then the category $\mathscr{C}$ has weak finite limits.*

Proof. We will prove that $\mathscr{C}$ has weak equalizers and observe that weak finite products and weak equalizers imply the existence of weak finite limits, see [CV98, Proposition 1]. Let $f, g : X \to Y$ be two arrows of $\mathscr{C}$ and let $h : X \to V$ be an arrow into a weak product $Y \overset{\mathsf{p}_1}{\leftarrow} V \overset{\mathsf{p}_2}{\to} Y$, such that $\mathsf{p}_1 \circ h = f$ and $\mathsf{p}_2 \circ h = g$. If $\rho := \mathsf{P}_h\delta_Y^V$ then we have a comprehension $\{\!|\rho|\!\} : C \to X$, which is trivially a weak equalizer of f and g. ☐

The above lemma implies that if $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a biased elementary doctrine with weak comprehensions and comprehensive diagonals, then the category $\mathscr{C}$ has weak pullbacks. Hence, the slice categories of $\mathscr{C}$ have weak finite products that can be described as follows. If $f : X_1 \to Y$ and $g : X_2 \to Y$ are arrows of $\mathscr{C}$ and $\rho := \mathsf{P}^s_{[f]\times[g]}\delta_{[Y]}$, then a weak pullback of f and g is obtained as in the following commutative diagram

$$\begin{tikzcd}
C \arrow[rr, "\pi_2"] \arrow[dr, "\{\!|\rho|\!\}"] \arrow[dd, "\pi_1"'] & & X_2 \arrow[dd, "g"] \\
& W \arrow[ur, "\mathsf{p}_2"'] \arrow[dl, "\mathsf{p}_1"] & \\
X_1 \arrow[rr, "f"'] & & Y,
\end{tikzcd}$$

where $\{\!|\rho|\!\}$ is a weak comprehension of the element $\mathsf{P}^s_{\langle[\mathsf{p}_1],[\mathsf{p}_2]\rangle}\rho \in \mathsf{P}(W)$. Hence, a trivial computation shows that the slice doctrine of Example 3.1.4 can be defined in the same way for biased elementary doctrines.

We now prove some technical lemmas for the existential and universal biased elementary doctrines, which will be useful to derive properties about the quantifiers of their slices. Our goal is to prove that if P is a suitable existential (universal) biased elementary, then the slice doctrines $\mathsf{P}_{/A}$ are existential (universal).

Lemma 3.6.2. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an existential biased elementary doctrine with full comprehensions. If X is an object of $\mathscr{C}$ and $\alpha \in \mathsf{P}(X)$, then*

$$\exists_{\{\!|\alpha|\!\}} \mathsf{P}_{\{\!|\alpha|\!\}} \gamma = \alpha \wedge \gamma,$$

for every $\gamma \in \mathsf{P}(X)$ and weak comprehension $\{\!|\alpha|\!\} : C \to X$.

Corollary 3.6.3. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an existential biased elementary doctrine with full comprehensions. For every object $X \in \mathscr{C}$ and $\rho \in \mathsf{P}^s[X, X]$, if $\{\!|\rho|\!\} : C \to W$ is a weak comprehension of $\mathsf{P}^s_{\langle [\mathsf{p}_1], [\mathsf{p}_2] \rangle} \rho \in \mathsf{P}(W)$, then*

$$\exists_{\{\!|\rho|\!\}_s} \mathsf{P}^s_{\{\!|\rho|\!\}_s} \gamma = \rho \wedge \gamma$$

for every $\gamma \in \mathsf{P}^s[X, X]$.

Proof. Since $\{\!|\rho|\!\}_s := \langle \mathsf{p}_1, \mathsf{p}_2 \rangle \circ \{\!|\rho|\!\}$, the statement follows from Lemma 3.6.2 observing that $\exists_{\langle \mathsf{p}_1, \mathsf{p}_2 \rangle}$ is left adjoint to the fully-faithful functor $\mathsf{P}^s_{\langle \mathsf{p}_1, \mathsf{p}_2 \rangle} : \mathsf{P}^s[X, X] \to \mathsf{P}(W)$. ☐

Lemma 3.6.4. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an existential biased elementary doctrine. For every arrow $(f, \hat{f}) : [X]_{i \in [n]} \to [Y]_{j \in [m]}$ of $\mathscr{C}_s$, then*

(*i*) $\mathsf{P}^s_{(f,\hat{f})} \exists_{(f,\hat{f})} \beta = \beta$, *for every $\beta \in \mathcal{D}es_{\mathsf{P}^s_{(f,\hat{f}) \times (f,\hat{f})} \delta_{[Y]_{j \in [m]}}}$.*

Moreover, if P is also universal and implicational

(*ii*) $\mathsf{P}^s_{(f,\hat{f})} \forall_{(f,\hat{f})} \beta = \beta$, *for every $\beta \in \mathcal{D}es_{\mathsf{P}^s_{(f,\hat{f}) \times (f,\hat{f})} \delta_{[Y]_{j \in [m]}}}$.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} [X]_{i\in[n]} \times [X]_{i\in[n]} & \xrightarrow{\mathsf{p}_2} & [X]_{i\in[n]} \\ \Big\downarrow{\scriptstyle 1_{[X_i]_{i\in[n]}} \times (f,\hat{f})} & & \Big\downarrow{\scriptstyle (f,\hat{f})} \\ [X]_{i\in[n]} \times [Y]_{j\in[m]} & \xrightarrow[\mathsf{p}_2]{} & [Y]_{j\in[m]} \end{array}$$

if $g : W \to V$ is an arrow into a weak product $Y \overset{\mathsf{p}_1}{\leftarrow} V \overset{\mathsf{p}_2}{\to} Y$ such that $\mathsf{p}_i \circ g = f \circ \mathsf{p}_i$, for $i = 1, 2$, and $h : K \to V$ is an arrow such that $\mathsf{p}_1 \circ h = f \circ \mathsf{p}_1$ and $\mathsf{p}_2 \circ h = 1_Y \circ \mathsf{p}_2$

(i) The left adjoint implies that $\exists_{(f,\hat{f})} \mathsf{P}^s_{(f,\hat{f})} \beta \leq \beta$. The opposite inequality is obtained as follows:

$$\begin{aligned} \mathsf{P}^s_{(f,\hat{f})} \exists_{(f,\hat{f})} \beta &= \mathsf{P}^s_{(f,\hat{f})} \exists_{p_2} (\mathsf{P}^s_{(f,\hat{f}) \times 1_{[X_i]_{i\in[n]}}} \delta_{[Y_j]_{j\in[m]}} \wedge \mathsf{P}_{p_1} \beta) && \text{(Remark 3.5.7)} \\ &= \exists_{p_2} \mathsf{P}^s_{1_{[X_i]_{i\in[n]}} \times (f,\hat{f})} (\mathsf{P}^s_{(f,\hat{f}) \times 1_{[X_i]_{i\in[n]}}} \delta_{[Y_j]_{j\in[m]}} \wedge \mathsf{P}_{p_1} \beta) && \text{(B-C)} \\ &= \exists_{p_2} (\mathsf{P}^s_{(f,\hat{f}) \times (f,\hat{f})} \delta_{[Y_j]_{j\in[m]}} \wedge \mathsf{P}_{p_1} \beta) \\ &\leq \exists_{p_2} \mathsf{P}^s_{p_2} \beta \leq \beta. \end{aligned}$$

(ii) The right adjoint implies that $\mathsf{P}^s_{(f,\hat{f})}\forall_{(f,\hat{f})}\beta \leq \beta$. The opposite inequality is obtained as follows:

$$\begin{aligned}
\mathsf{P}^s_{(f,\hat{f})}\forall_{(f,\hat{f})}\beta &= \mathsf{P}^s_{(f,\hat{f})}\forall_{p_2}(\mathsf{P}^s_{(f,\hat{f})\times 1_{[X_i]_{i\in[n]}}}\delta_{[Y_j]_{j\in[m]}} \Rightarrow \mathsf{P}_{p_1}\beta) && \text{(Remark 3.5.8)}\\
&= \forall_{p_2}\mathsf{P}^s_{1_{[X_i]_{i\in[n]}}\times(f,\hat{f})}(\mathsf{P}^s_{(f,\hat{f})\times 1_{[X_i]_{i\in[n]}}}\delta_{[Y_j]_{j\in[m]}} \Rightarrow \mathsf{P}_{p_1}\beta) && \text{(B-C)}\\
&= \forall_{p_2}(\mathsf{P}^s_{(f,\hat{f})\times(f,\hat{f})}\delta_{[Y_j]_{j\in[m]}} \Rightarrow \mathsf{P}^s_{1_{[X_i]_{i\in[n]}}\times(f,\hat{f})}\mathsf{P}^s_{p_1}\beta)\\
&= \forall_{p_2}(\mathsf{P}^s_{(f,\hat{f})\times(f,\hat{f})}\delta_{[Y_j]_{j\in[m]}} \Rightarrow \mathsf{P}^s_{p_1}\beta)\\
&\geq \beta
\end{aligned}$$

where the last inequality follows because $\beta \leq \forall_{p_2}(\mathsf{P}^s_{(f,\hat{f})\times(f,\hat{f})}\delta_{[Y_j]_{j\in[m]}} \Rightarrow \mathsf{P}^s_{p_1}\beta)$ if and only if $\mathsf{P}^s_{p_2}\beta \leq (\mathsf{P}^s_{(f,\hat{f})\times(f,\hat{f})}\delta_{[Y_j]_{j\in[m]}} \Rightarrow \mathsf{P}^s_{p_1}\beta)$, which holds by the definition of implication and by the assumptions on β. □

The following corollary relates P-equivalence relations and $\mathsf{P}_{/A}$-equivalence relations through the existential quantifier.

Corollary 3.6.5. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an existential biased elementary doctrine with full weak comprehensions and comprehensive diagonals and let $X \xrightarrow{x} A$ be an object of $\mathscr{C}/A$ and $x \xleftarrow{\pi_1} w \xrightarrow{\pi_2} x$ be a weak product of x and x. Considering the element $\rho := \mathsf{P}^s_{[x]\times[x]}\delta_{[A]}$, we obtain the following conditions:*

i) if σ is a P-equivalence relation on X such that $\sigma \leq \rho$, then $\mathsf{P}^s_{\langle[\pi_1],[\pi_2]\rangle}\sigma$ is a $\mathsf{P}_{/A}$-equivalence relation on x and $\mathcal{D}es_\sigma = (\mathcal{D}es_{/A})_{\mathsf{P}^s_{\langle[\pi_1],[\pi_2]\rangle}\sigma}$,

ii) if r is a $\mathsf{P}_{/A}$-equivalence relation on x, then $\exists_{\langle[\pi_1],[\pi_2]\rangle}r$ is a P-equivalence relation on X and $(\mathcal{D}es_{/A})_r = \mathcal{D}es_{\exists_{\langle[\pi_1],[\pi_2]\rangle}r}$.

Proof. We first provide the statements considering a weak product $x \xleftarrow{\pi_1} w \xrightarrow{\pi_2} x$ obtained through a weak comprehension of ρ as shown in the left diagram below. On the right, we observe the corresponding situation in $\mathscr{C}_s$

$$\begin{tikzcd}
C \arrow[rr, "\pi_2"] \arrow[dr, "\{\!|\rho|\!\}"] \arrow[dd, "\pi_1"'] & & X \arrow[dd, "x"] \\
& W \arrow[ur, "\mathsf{p}_2"'] \arrow[dl, "\mathsf{p}_1"] & \\
X \arrow[rr, "x"'] & & A.
\end{tikzcd}
\qquad
\begin{tikzcd}
{[C]} \arrow[rr, shift left, "{[\pi_1]}"] \arrow[rr, shift right, "{[\pi_2]}"'] \arrow[dd, "{[\{\!|\rho|\!\}]}"'] \arrow[ddrr, dashed, "\{\!|\rho|\!\}_s"] & & {[X]} \\
& & \\
{[W]} \arrow[rr, "{\langle[\mathsf{p}_1],[\mathsf{p}_2]\rangle}"'] & & {[X,X]} \arrow[uu, shift left, "p_1"] \arrow[uu, shift right, "p_2"']
\end{tikzcd}$$

i) The first part of the statement is a trivial computation and, just applying the functor $\mathsf{P}^s_{\{\!|\rho|\!\}_s}$, we obtain the inclusion $\mathcal{D}es_\sigma \subseteq (\mathcal{D}es_{/A})_{\mathsf{P}^s_{\{\!|\rho|\!\}_s}\sigma}$. Vice versa, if $\alpha \in (\mathcal{D}es_{/A})_{\mathsf{P}^s_{\{\!|\rho|\!\}_s}\sigma}$, it holds that

$$\mathsf{P}^s_{[\pi_1]}\alpha \wedge \mathsf{P}^s_{\{\!|\rho|\!\}_s}\sigma \leq \mathsf{P}^s_{[\pi_1]}\alpha.$$

Hence, applying $\exists_{\{\!|\rho|\!\}_s}$, we obtain

$$\exists_{\{\!|\rho|\!\}_s}\mathsf{P}^s_{\{\!|\rho|\!\}_s}(\mathsf{P}^s_{p_1}\alpha \wedge \sigma) \leq \exists_{\{\!|\rho|\!\}_s}\mathsf{P}^s_{\{\!|\rho|\!\}_s}\mathsf{P}^s_{p_2}\alpha \leq \mathsf{P}^s_{p_2}\alpha.$$

The statement follows from Corollary 3.6.3, observing that $\sigma \leq \rho$.

$ii)$ We now prove reflexivity and symmetry of $\exists_{\{\!|\rho|\!\}_s} r$. Reflexivity follows from reflexivity of r, i.e. $\delta_{[x]} \leq r$ and observing that $\delta_{[x]} := \mathsf{P}^s_{\{\!|\rho|\!\}_s}\delta_{[X]}$. Hence, applying $\exists_{\{\!|\rho|\!\}_s}$ we obtain

$$\exists_{\{\!|\rho|\!\}_s}\mathsf{P}^s_{\{\!|\rho|\!\}_s}\delta_{[X]} \leq \exists_{\{\!|\rho|\!\}_s} r.$$

The statement follows from Corollary 3.6.3, observing that $\delta_{[X]} \leq \rho$.

In order to obtain the symmetry of $\exists_{\{\!|\rho|\!\}_s} r$ we first observe that, by $i)$, then $r \in \mathcal{D}es_{\mathsf{P}^s_{\{\!|\rho|\!\}_s \times \{\!|\rho|\!\}_s}\delta_{[X,X]}}$. Hence, considering the following commutative diagram

$$\begin{tikzcd}
{[C]} \arrow[r, "\{\!|\rho|\!\}_s"] \arrow[d, "\langle 2,1\rangle_{/A}"'] & {[X,X]} \arrow[d, "\langle 2,1\rangle"] \\
{[C]} \arrow[r, "\{\!|\rho|\!\}_s"'] & {[X,X]}
\end{tikzcd}$$

the symmetry of r implies that $\mathsf{P}^s_{[\langle 2,1\rangle_{/A}]} r \leq r$. We can apply Lemma 3.6.4 to obtain

$$\mathsf{P}^s_{[\langle 2,1\rangle_{/A}]}\mathsf{P}^s_{\{\!|\rho|\!\}_s}\exists_{\{\!|\rho|\!\}_s} r \leq \mathsf{P}^s_{\{\!|\rho|\!\}_s}\exists_{\{\!|\rho|\!\}_s} r$$

and since $\mathsf{P}^s_{[\langle 2,1\rangle_{/A}]}\mathsf{P}^s_{\{\!|\rho|\!\}_s} = \mathsf{P}^s_{\{\!|\rho|\!\}_s}\mathsf{P}^s_{\langle 2,1\rangle}$ the above inequality holds if and only if

$$\exists_{\{\!|\rho|\!\}_s}\mathsf{P}^s_{\{\!|\rho|\!\}_s}\mathsf{P}^s_{\langle 2,1\rangle}\exists_{\{\!|\rho|\!\}_s} r \leq \exists_{\{\!|\rho|\!\}_s} r.$$

The statement follows from Corollary 3.6.3 observing that $\exists_{\{\!|\rho|\!\}_s} r \leq \rho$. Transitivity is proved similarly.

We now observe that the statement is true for every other weak limit w' of x and x, with $w' : C' \to A$. Indeed, there exist a commutative diagram

$$\begin{tikzcd}
{[C']} \arrow[d, "{[h]}"'] \arrow[dr, "{\langle[\pi'_1],[\pi'_2]\rangle}"] & \\
{[C]} \arrow[r, "\{\!|\rho|\!\}_s"'] & {[X,X]}
\end{tikzcd}$$

such that $\pi_i \circ h = \pi_i'$, for $i = 1, 2$. Since the functor P_h is an isomorphism between the proof-irrelevant elements of w and w', if $r' := \mathsf{P}_h r$ then we obtain

$$\mathsf{P}_{[h]}\exists_{\{\!|\rho|\!\}_s} r = \exists_{\langle[\pi'_1],[\pi'_2]\rangle} r'.$$

$\square$

We can now prove that the slices of suitable existential (universal) biased elementary doctrines are existential (universal).

Proposition 3.6.6. *Let P be an existential biased elementary doctrine with full comprehensions and comprehensive diagonals. For every object $A \in \mathscr{C}$, the slice doctrine $\mathsf{P}_{/A}$ is existential. Moreover, if P is universal and, for every arrow $f : Y \to X$, the functors P_f have right adjoints $\forall_f$, then $\mathsf{P}_{/A}$ is universal.*

Proof. In order to prove the weak Beck-Chevalley condition it is enough to consider weak products in the slice $\mathscr{C}_s/A$ as in the following commutative diagram

$$\begin{tikzcd}
D \arrow[r, "\pi_2"] \arrow[d, "\tilde{f}"] \arrow[dd, bend right, "\pi_1"'] & Y \arrow[d, "f"'] \arrow[dd, bend left, "y"] \\
C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X_2 \arrow[d, "x_2"'] \\
X_1 \arrow[r, "x_1"'] & A.
\end{tikzcd}$$

If $x_i : X_i \to A$, for $i = 1, 2$ and $y : Y \to A$ are objects of $\mathscr{C}_s/A$, it is enough to prove the statement for weak products $w : C \to A$ of x_1 and x_2 and $v : D \to A$, of x_1 and y, which are obtained through weak comprehensions. Hence, if $\rho := \mathsf{P}^s_{[x_1]\times[x_2]}\delta_{[A]}$ and $\sigma := \mathsf{P}^s_{[x_1]\times[y]}\delta_{[A]}$, then we are in the following situation

$$\begin{tikzcd}
D \arrow[r, "\{\!|\sigma|\!\}"] \arrow[d, "\tilde{f}"'] & V \arrow[r, "\mathsf{p}_2"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\
C \arrow[r, "\{\!|\rho|\!\}"'] & W \arrow[r, "\mathsf{p}_2"'] \arrow[d, "\mathsf{p}_1"] & X_2 \\
& X_1 &
\end{tikzcd}$$

where $\{\!|\rho|\!\}$ is a weak comprehension of $\mathsf{P}^s_{\langle[\mathsf{p}_1],[\mathsf{p}_2]\rangle}\rho$ and $\{\!|\sigma|\!\}$ is a weak comprehension of $\mathsf{P}^s\langle[\mathsf{p}_1],[\mathsf{p}_2]\rangle\sigma$. If $\alpha \in \mathsf{P}_{/A}(w)(:= \mathsf{P}(C))$ is a proof-irrelevant element, then we can prove that $\mathsf{P}_f\exists_{\pi_2}\alpha \leq \exists_{\pi_2}\mathsf{P}_{\tilde{f}}\alpha$ as follows:

$$\begin{aligned}
\mathsf{P}_f\exists_{\pi_2}\alpha &= \mathsf{P}_f\exists_{\mathsf{p}_2}\exists_{\{\!|\rho|\!\}}\alpha \\
&= \exists_{\mathsf{p}_2}\mathsf{P}_{f'}\exists_{\{\!|\rho|\!\}}\alpha && \text{(B-C for P)} \\
&= \exists_{\mathsf{p}_2}\exists_{\{\!|\sigma|\!\}}\mathsf{P}_{\{\!|\sigma|\!\}}\mathsf{P}_{f'}\exists_{\{\!|\rho|\!\}}\alpha && \text{(Lemma 3.6.2)} \\
&= \exists_{\pi_2}\mathsf{P}_{\tilde{f}}\mathsf{P}_{\{\!|\rho|\!\}}\exists_{\{\!|\rho|\!\}}\alpha \\
&= \exists_{\pi_2}\mathsf{P}_{\tilde{f}}\alpha. && \text{(Lemma 3.6.4)}
\end{aligned}$$

The statement for the right adjoint is proved similarly. ☐

Example 3.6.7. The main example of existential and universal biased elementary doctrine is given by the slice doctrines F^{ML}_A. As observed in Example 1.2.15 and Example 1.2.17, the elementary doctrine F^{ML} has left and right adjoint to all reindexings. Hence, the Beck-Chevalley conditions follow from Proposition 3.6.6. However, they could be obtained also through the description of the descent data of the slices given in Corollary 3.6.5 and from Lemma A.0.16.

For categories with weak finite limits, it is straightforward to prove that the exact completion of a slice is equivalent to the slice of the exact completion. We end this section providing the corresponding result for suitable biased elementary doctrines.

Proposition 3.6.8. *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a existential biased elementary doctrine with weak full comprehensions and comprehensive diagonals, then for every object $A \in \mathscr{C}$*

$$\overline{\mathsf{P}_{/A}} \cong \overline{\mathsf{P}}_{/(A,\delta_{[A]})}.$$

Proof. Thanks to Lemma 3.6.4, we can define a 1-arrow $(M, m) : \overline{\mathsf{P}_{/A}} \to \overline{\mathsf{P}}_{/(A,\delta_{[A]})}$ of **QD** as follows. The functor M sends an object (x, r) of $\overline{\mathscr{C}/A}$, with $x : X \to A$, to the object

$$\lfloor x \rceil : (X, \exists_{\{\!|\rho|\!\}_s} r) \to (A, \delta_{[A]})$$

of $\overline{\mathscr{C}}/(A, \delta_{[A]})$, where $\rho := \mathsf{P}^s_{[x]\times[x]}\delta_{[A]}$. An arrow $f : (x, r) \to (y, s)$, where $y : Y \to A$, is sent to $\lfloor f \rceil : (X, \exists_{\{\!|\rho|\!\}_s} r) \to (Y, \exists_{\{\!|\sigma|\!\}_s} s)$, with $\sigma := \mathsf{P}^s_{[y]\times[y]}\delta_{[A]}$. The natural transformation m is given by the identity.

Using again Lemma 3.6.4, we can define an inverse (N, n) to (M, m) as follows. The functor N sends an object $\lfloor x \rceil : (X, \lambda) \to (A, \delta_{[A]})$ of $\overline{\mathscr{C}}/(A, \delta_{[A]})$ to the object

$$(x, \mathsf{P}_{\{\!|\rho|\!\}_s} r)$$

where $\rho := \mathsf{P}^s_{[x]\times[x]}\delta_{[A]}$. An arrow $\lfloor f \rceil : \lfloor x \rceil \to \lfloor y \rceil$, with $\lfloor y \rceil : (Y, \sigma) \to (A, \delta_{[A]})$ is sent to the arrow f. The natural transformation n is given by the identity.

Lemma 3.6.2 and Lemma 3.6.4 imply that the 1-arrows (M, m) and (N, n) are inverse. $\square$

The above proposition provides a precise formulation of the method mentioned in Observation 2.6.13.

3.7 Local cartesian closure

In this section, we retrace the steps of Section 2.3, in which we proved the local cartesian closure of the elementary quotient completion, in the more general context of the biased elementary doctrines. Thanks to the results obtained in the previous section, we can provide the local cartesian closure working on the cartesian closure of the slice doctrines. As an instance, we obtain the result of Carboni-Rosolini [CR00] and Emmenegger [Emm20] about the local cartesian closure of the exact completion of a weakly left exact category in their general form. This is a slight improvement with respect to the elementary quotient completion, which could only obtain the exact completion of a category with strict products and weak pullbacks as an instance.

As already done in Section 2.3 for the strict elementary doctrines, we now reformulate the ideas developed by Emmenegger in [Emm20] in the language of the biased elementary doctrines.

Definition 3.7.1. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased elementary doctrine and let X, Y and Z be objects of $\mathscr{C}$. If $X \overset{\mathsf{p}_1}{\leftarrow} K \overset{\mathsf{p}_2}{\rightarrow} Y$ is a weak product, then an arrow $f : K \to Z$ *preserves projections* with respect to a P-equivalence relation $\sigma \in \mathsf{P}^s[Z, Z]$ if

$$\delta_{[X]} \boxtimes \delta_{[Y]} \leq \mathsf{P}^s_{[h]\times[h]}(\sigma).$$

We observe that, by Lemma 3.4.5, the above definition can be equivalently reformulated requiring that in $\mathscr{C}_s$ the arrow $[h] : [K] \to [Z]$ can be lifted as in the following diagram

$$\begin{tikzcd}
{[K]} \arrow[d, "\langle \mathsf{p}_1, \mathsf{p}_2 \rangle"'] \arrow[dr, "h"] & \\
{[X, Y]} \arrow[r, dashed, "h"'] & {[Z]}.
\end{tikzcd}$$

Definition 3.7.2. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased elementary doctrine. If X, Y are objects of $\mathscr{C}$ and $\sigma \in \mathsf{P}^s[Y, Y]$ is a P-eq. relation on Y, an *extensional exponential* of X and Y with respect to σ is an object E with an arrow $e : U \to Y$, from a weak product $E \overset{\mathsf{p}_1}{\leftarrow} U \overset{\mathsf{p}_2}{\rightarrow} X$, such that:

- the map e preserves projections w.r.t. σ,
- for every objects $Z \in \mathscr{C}$ and arrow $f : U' \to Y$, from a weak product $Z \overset{\mathsf{p}_1}{\leftarrow} U' \overset{\mathsf{p}_2}{\to} X$, there exist arrows l, m making the following diagram commute:

$$\begin{tikzcd}
Z \arrow[d, dashed, "l"] & U' \arrow[l, "\mathsf{p}_1"'] \arrow[d, dashed, "m"] \arrow[dr, bend left, "\mathsf{p}_2"] \arrow[dd, bend right, "e'"'] & \\
E & U \arrow[l, "p_1"] \arrow[r, "\mathsf{p}_2"] \arrow[d, "e"] & X \\
 & Y. &
\end{tikzcd}$$

We will say that $\mathscr{C}$ has P-*extensional exponentials* if, for every pair of objects $X, Y \in \mathscr{C}$ and P-equivalence relation $\sigma \in \mathsf{P}^s[Y, Y]$, then there exists an extensional exponential of X and σ.

Definition 3.7.3. Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a biased elementary doctrine. If X and Y are objects of $\mathscr{C}$ and $\sigma \in \mathsf{P}^s[Y, Y]$ is a P-equivalence relation on Y, then two arrows $f, g : X \to Y$ are called *σ-related* if

$$\delta_{[X]} \leq \mathsf{P}^s_{[f]\times[g]}\sigma.$$

We now prove the corresponding Theorem 2.3.2 for the biased elementary doctrines. In light of the observations done by Emmenegger [Emm20] and since we are now working with weak finite products, we require the existence of P-extensional exponentials instead of weak exponentials.

Theorem 3.7.4. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be a universal biased elementary doctrine with full weak comprehensions. $\overline{\mathscr{C}}$ is cartesian closed if and only if $\mathscr{C}$ has P-extensional exponentials.*

Proof. For the "only if" part, given two objects $X, Y \in \mathscr{C}$ and a P-eq. relation σ on Y, then we can consider the exponential of the objects $(X, \delta_{[X]})$ and (Y, σ). This exponential induces an object and an arrow which provide almost a P-extensional exponentials, but the universal property holds up to σ-relation. In order to obtain a P-extensional exponentials of x and y w.r.t. σ we repeat the proof of Theorem 2.3.7 $(ii) \Rightarrow (i)$.

The proof of the "if" part is the same of the proof of Theorem 2.3.2, considering weak finite product instead of strict ones and extensional exponentials instead of the weak exponentials. We only recall the main steps.

Given two objects of the form $(X, \delta_{[X]})$ and (Y, σ), we can consider an extensional exponential Y^X of X, Y with respect to σ and a weak evaluation arrow $e : U \to Y$ from a weak product U of X and Y. We now consider the P-equivalence relation on Y^X given by

$$\varepsilon^X_\sigma := \forall_{\langle 1,2\rangle}\mathsf{P}^s_{\langle 1,3,2,3\rangle}\mathsf{P}^s_{[e]\times[e]}\sigma \tag{3.6}$$

Then we obtain that the object $(Y^X, \varepsilon^X_\sigma)$ and the arrow

$$\lfloor e \rceil : (Y^X, \varepsilon^X_\sigma) \times (X, \delta_{[X]}) \to (Y, \sigma)$$

provide a strict exponential of $(X, \delta_{[X]})$ and (Y, σ). For the general case, given two objects $(X, \delta_{[X]})$ and (Y, σ) we consider a weak product of $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\to} X$ and consider the representant of ρ in $\mathsf{P}(W)$, i.e. $\rho_W := \mathsf{P}^s_{\langle \mathsf{p}_1, \mathsf{p}_2\rangle}(\rho)$, where $\langle \mathsf{p}_1, \mathsf{p}_2\rangle$ is the unique arrow $W \to [X, X]$. Hence, we can consider a comprehension $\{\!|\rho_W|\!\} : R \to W$ and consider the arrows $r_i := \mathsf{p_i} \circ \{\!|\rho_W|\!\}$, for $i = 1, 2$. If Y^X denotes the extensional exponential of X and Y w.r.t. σ and Y^Rdenotes the extensional exponential of R and Y w.r.t. σ, the evaluation arrows e_X and e_R induces two arrows

$$Y^{r_1}, Y^{r_2} : Y^X \to Y^R$$

such that the obvious diagrams commute. If $c : C \to Y^X$ is a weak comprehension of the element $\mathsf{P}_{\langle [Y^{r_1}, Y^{r_2}] \rangle} \varepsilon^R_\sigma$, then the strict exponential is given by the object

$$(C, \mathsf{P}^s_{[c]\times[c]} \varepsilon^X_\sigma)$$

with the evaluation arrow

$$\lfloor e_x(c \times 1_X) \rceil : (C, \mathsf{P}^s_{[c]\times[c]} \varepsilon^X_\sigma) \times (X, \rho) \to (Y, \sigma).$$

□

In case we restrict to the biased elementary doctrine $\mathsf{Psub}_{\mathscr{C}}$ of weak subobjects of a wlex category $\mathscr{C}$, then we obtain [*Theorem 2.14*,[Emm20]].

We can now use the above theorem and the results obtained in the previous section for the slices of existential and universal biased elementary doctrines to prove the main theorem about the local cartesian closure of the quotient completion.

Theorem 3.7.5. *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be a universal and existential biased elementary doctrine with full weak comprehensions and comprehensive diagonals. If* P *has right adjoints to all reindexings, then the following are equivalent:*

(i) *For every object $A \in \mathscr{C}$, the slice $\mathscr{C}/A$ has $\mathsf{P}_{/A}$-extensional exponentials,*

(ii) *$\overline{\mathscr{C}}$ is locally cartesian closed.*

Proof. The implication $(ii) \Rightarrow (i)$ follows as in the proof of Theorem 2.3.7. For the implication $(i) \Rightarrow (ii)$ we first observe that the cartesian closure of the slices of the form $\overline{\mathscr{C}}/(A, \delta_{[A]})$ follows applying first Theorem 3.7.4 to the slice doctrine P_A and then applying Proposition 3.6.8 to $\overline{\mathsf{P}_A}$. The proof of the general case follows the construction provided in Theorem 2.3.7 for constructing an exponential of two objects in slice categories of the form $\overline{\mathscr{C}}/(A, \alpha)$. □

In case we restrict to the biased elementary doctrine $\mathsf{Psub}_{\mathscr{C}}$ of weak subobjects of a wlex category $\mathscr{C}$, then we obtain [Emm20, Theorem 3.6].

Concluding remarks and further developments. In this chapter, we have provided a more general framework which generalizes both the elementary quotient completion of Maietti and Rosolini and the exact completion of a wlex category provided by Carboni and Vitale in [CV98]. At the end we have generalized the theorems about the (local) cartesian closure of the exact completion provided in [CR00] and [Emm20] also for categories with weak finite products and weak pullbacks. One future development will be to generalize Theorem 2.5.6 for the biased elementary doctrines and obtain [GV98, Proposition 2.1] also for categories with weak finite products. Moreover, as observed by Maietti and Rosolini in [MR13], the doctrines correspond to particular Grothendieck fibrations, namely the faithful fibrations. Similarly, the primary and the elementary doctrines corresponds to suitable faithful fibrations, see [EPR22]. A future development would be to understand which notion of fibration corresponds to the biased elementary doctrines.

Chapter 4

A weaker categorical BHK interpretation

The aim of this chapter is to take a little step forward in categorical semantic of mathematical logic. The interpretation of a (fragment of) many-sorted first-order logical language $\mathcal{L}(S)$, in a category $\mathscr{C}$ with strict products, requires an assignment of categorical entities to logical symbols. A sort S in interpreted as an object $\mathcal{M}(S) \in \mathscr{C}$ and a multi-variable term $t(x_1, \dots, x_n) : Z$, with $x_i : S_i$, for $1 \leq i \leq n$, is inductively interpreted as an arrow $\mathcal{M}(t) : \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n) \to \mathcal{M}(Z)$ of $\mathscr{C}$. Relation symbols have a double interpretation.

The *standard interpretation*, introduced by Makkai and Reyes in [MR77], assigns to each relation symbol $R \subseteq S_1 \times \cdots \times S_n$ the equivalence class of a monomorphism $\mathcal{M}(R)$ with codomain $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$. The interpretation is extended inductively to a each formula φ with $FV(\varphi) \subseteq \bar{x} = (x_1, \dots, x_n)$ which is interpreted as a monomorphism with codomain $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$. This process is also called *propositions as subobjects* interpretation. In order to interpret regular logic, $\mathscr{C}$ is required to be at least regular.

Alternatively, a relation symbol $R \subseteq S_1 \times \cdots \times S_n$ can be interpreted as the equivalence class of an arrow $\mathcal{M}(R)$ with codomain $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$, which is not necessarily a monomorphism, see [Pal04]. This interpretation is called *propositions-as-objects* or the *categorical Brouwer-Heyting-Kolmogorov* interpretation and expresses categorically the Curry-Howard paradigm "proposition as types". This interpretation is suitable for a larger class of categories, i.e. categories with strict finite products and weak pullbacks (qlex). Actually, the standard interpretation exploits the functor $\mathsf{Sub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{Pos}$ of subobjects, while the BHK interpretation uses the functor $\mathsf{Psub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{Pos}$ of weak subobjects that we encountered in the previous chapters.

From a categorical perspective a more natural definition is that of a weakly left exact category (wlex) in which also products are weak. Hence, we ask if it possible to interpret intuitionistic logic in wlex categories. In this chapter, we will give a positive answer to this question and we will provide a sound and complete construction for various fragments of first order logic. Our interpretation is suitable for a larger class of categories such as the slice categories of qlex categories and, in case the finite products are strict, it coincides with the BHK interpretation. For these reasons, sometimes we will refer to this interpretation as the *weak BHK interpretation*.

Our interpretation relies on the idea of proof-irrelevant elements developed in Chapter 3. In particular, we will implicitly study the biased elementary doctrine of weak subobjects $\mathsf{PSub}_{\mathscr{C}}$ of a wlex category. However, we will intentionally leave the language of biased elementary doctrines as far as possible, but it will be clear where the results obtained could be restated using the tools developed in the previous chapters. Most of the results have already been obtained in a more general form, but we think that a direct proof in the case of the biased elementary doctrine of weak subobjects $\mathsf{PSub}_{\mathscr{C}}$ is of big interest. Our main example of weak BHK interpretation is provided in the slices of the syntactic category **ML** of an intensional type theory, which are wlex.

4.1 Syntax

In this section we recall the syntax and the categorical semantic of logical theories. In particular, we recall the *standard interpretation* of a fragment of intuitionistic logic in left exact categories and the *BHK interpretation* in *quasi left exact* categories as developed in [MR77; But98; Pal04].

We first recall the syntactic notions of a logical language. A (typed) *signature S* consists of a set of sort symbols $\{S_1, S_2, \dots\}$, a set of sorted constants symbols $\{c_1, c_2, \dots\}$, a set of finitary sorted function symbols $\{f, g, h, \dots\}$ and a set of relation symbols $\{R, P, \dots\}$.We will adopt the usual notations for these symbols:

$$c : S, \qquad f : S_1 \times \cdots \times S_n \to S, \qquad R \subseteq S_1 \times \cdots \times S_n.$$

In order to consider logical languages with equality predicate, we assume a special binary relation symbol $\approx_S$ for each sort S.

A (fragment of) first order language $\mathcal{L}(S)$, over a signature S, consists of a list of countable variables $x, y, \dots$ for each sort X (denoted by $x : X$), and a set of terms and formulae defined inductively. The set of terms is defined through the following clauses:

(T1) each constant of sort S is a term of sort S,

(T2) each variable of sort S is a term of sort S,

(T3) if $t_1 : S_1, \dots, t_n : S_n$ are terms and $f : S_1 \times \cdots \times S_n \to Y$ is a function symbol, then $f(t_1, \dots, t_n)$ is a term of type Y.

Te set of formulae is obtained inductively through a subset of the following clauses depending on the fragment of the logic under consideration:

(F1) if $R \subseteq S_1 \times \cdots \times S_n$ is a relation symbol and $t_1 : S_1, \dots, t_n : S_n$ are terms, then $R(t_1, \dots, t_n)$ is a formula,

(F2) if t_1 and t_2 are terms of the same sort S, then $t_1 \approx_S t_2$ is a formula,

(F3) the "truth" predicate $\top$ is a formula,

(F4) if φ and ψ are folrmulae then $\varphi \wedge \psi$ is a formula,

(F5) if φ is a formula and $x : S$ a variable, then $\exists x \varphi$ is a formula,

(F6) the "false" predicate $\perp$ is a formula,

(F7) if φ and ψ are formulae then $\varphi \vee \psi$ is a formula,

(F8) if φ and ψ are formulae then $\varphi \implies \psi$ is a formula,

(F9) if φ is a formula and $x : S$ a variable, then $\forall x \varphi$ is a formula,

(F10) if φ is a formula then $\neg \varphi$ is a formula.

The formulae obtained through clauses (F1-F4) are called *Horn formulae*. The formulae obtained through clauses (F1-F5) are called *Regular formulae* and those obtained through clauses (F1-F7) are called *Coherent formulae*. The set of *first order formulae* is obtained using all the clauses (F1-F10).

Given the formulae $\varphi_1, \ldots, \varphi_n$, the set $FV(\varphi_1, \ldots, \varphi_n)$ denotes the set of *free variables* of $\varphi_1, \ldots, \varphi_n$. Similarly, $FV(t_1, \ldots, t_n)$ denotes the set of *free variables* of the terms $t_1, \ldots, t_n$. A theory $\mathbb{T}$ in the language $\mathcal{L}(S)$ is a set of sequents of the form

$$\varphi_1, \ldots, \varphi_n \vdash_{\bar{x}} \psi,$$

where $FV(\varphi_1, \ldots, \varphi_n, \varphi) \subseteq \bar{x} = (x_1, \ldots, x_m)$. Sequences of formulae are denoted by capital Greek letters Γ, Δ, etc. Special axioms concern the equality predicate:

(E1) If $x : S$ then:

$$\vdash_x x \approx_S x.$$

(E2) If $x, y : S$ then:

$$x \approx_S y \vdash_{x,y} y \approx_S x.$$

(E3) If $x, y, z : S$ then:

$$x \approx_S y,\ y \approx_S z \vdash_{x,y,z} x \approx_S z.$$

(E4) If $f : S_1 \times \cdots \times S_n \to Y$ is a function symbol then:

$$x_1 \approx_{S_1} y_1, \ldots, x_m \approx_{S_m} y_m \vdash_{\bar{x},\bar{y}} f(x_1, \ldots, x_m) \approx_Y f(y_1, \ldots, y_m).$$

(E5) If $R \subseteq S_1 \times \cdots \times S_n$ is a relation symbol then:

$$x_1 \approx_{S_1} y_1, \ldots, x_m \approx_{S_m} y_m, R(x_1, \ldots, x_m) \vdash_{\bar{x},\bar{y}} R(y_1, \ldots, y_m).$$

A *deduction system* is a theory equipped with rules and axioms of inference. The following rules are called *structural* and are often assumed for most of deduction systems.

(S1) (Assumption) For $1 \leq i \leq n$:

$$\varphi_1, \ldots, \varphi_n \vdash_{\bar{x}} \varphi_i.$$

(S2) (Weakening) For any ψ with $FV(\psi) \subseteq \bar{x}$:

$$\frac{\Gamma \vdash_{\bar{x}} \varphi}{\Gamma, \psi \vdash_{\bar{x}} \varphi}$$

(S3) (Cut)

$$\frac{\Gamma, \varphi, \Delta \vdash_{\bar{x}} \psi \qquad \Gamma, \Delta \vdash_{\bar{x}} \varphi}{\Gamma, \Delta \vdash_{\bar{x}} \psi}$$

(S4) (Substitution) If $\bar{t} = (t_1, \ldots, t_m)$ is a list of terms of the same sorts of $\bar{x}$ and $FV(\bar{t} \subseteq \bar{w})$

$$\frac{\Gamma \vdash_{\bar{x}} \varphi}{\Gamma(\bar{t}/\bar{x}) \vdash_{\bar{w}} \varphi(\bar{t}/\bar{x})}$$

The last set of rules is given by the *logical rules*, which concern the logical connectives and the existential and universal quantification.

(L1) (Conjunction)

$$(\wedge\text{-I})\ \frac{\Gamma \vdash_{\bar{x}} \varphi_1 \qquad \Gamma \vdash_{\bar{x}} \varphi_2}{\Gamma \vdash_{\bar{x}} \varphi_1 \wedge \varphi_2} \qquad\qquad (\wedge\text{-E})\ \frac{\Gamma \vdash_{\bar{x}} \varphi_1 \wedge \varphi_2}{\Gamma \vdash_{\bar{x}} \varphi_i}\ (i = 1, 2)$$

(L2) (Existential quantification)

$$(\exists\text{-I})\ \frac{\varphi(b/y) \vdash_{\bar{x}} \psi}{\exists y \varphi \vdash_{\bar{x}} \psi}\ (y \notin FV(\psi)) \qquad\qquad (\exists\text{-E})\ \frac{\exists y \varphi \vdash_{\bar{x}} \psi}{\varphi \vdash_{\bar{x},y} \psi}\ (y \notin FV(\psi))$$

(L3) (Disjunction)

$$(\vee\text{-I})\ \frac{\Gamma \vdash_{\bar{x}} \varphi_1 \qquad \Gamma \vdash_{\bar{x}} \varphi_2}{\Gamma \vdash_{\bar{x}} \varphi_1 \vee \varphi_2} \qquad\qquad (\vee\text{-E})\ \frac{\Gamma \vdash_{\bar{x}} \varphi_1 \vee \varphi_2 \qquad \Gamma, \varphi_1 \vdash_{\bar{x}} \psi \qquad \Gamma, \varphi_2 \vdash_{\bar{x}} \psi}{\Gamma \vdash_{\bar{x}} \psi}$$

(L4) (Absurdity)

$$\frac{\Gamma \vdash_{\bar{x}} \perp}{\Gamma \vdash_{\bar{x}} \psi}$$

(L5) (Implication)

$$(\implies\text{-I})\ \frac{\Gamma, \varphi \vdash_{\bar{x}} \psi}{\Gamma \vdash_{\bar{x}} \varphi \implies \psi} \qquad\qquad (\implies\text{-E})\ \frac{\Gamma \vdash_{\bar{x}} \varphi \implies \psi \qquad \Gamma \vdash_{\bar{x}} \varphi}{\Gamma \vdash_{\bar{x}} \psi}$$

(L6) (Universal quantification)

$$(\forall\text{-I})\ \frac{\Gamma \vdash_{\bar{x},y} \varphi}{\Gamma \vdash_{\bar{x}} \forall y \varphi}\ (y \notin FV(\Gamma)) \qquad\qquad (\forall\text{-E})\ \frac{\Gamma \vdash_{\bar{x}} \forall y \varphi}{\Gamma \vdash_{\bar{x}} \varphi(b/y)}\ (FV(b) \subseteq \bar{x})$$

Regular logic with equality is given by the clauses (T1-T3), (F1-F5), (E1-E5), (S1-S4) and (L1-L2). *Coherent logic* is given by (T1-T3), (F1-F7), (E1-E5), (S1-S4) and (L1-L4). *First order logic* is given using all the above clauses and rules.

4.2 Standard interpretation

In this section we recall the standard interpretation of categorical logic in regular categories, as developed in [MR77]. We shall sometimes adopt the notation of the more recent notes [But98]. In order to interpret a language $\mathcal{L}(S)$ in a left exact category $\mathscr{C}$ we fix some categorical notation.

Remark 4.2.1. In the rest of these notes, we will always assume that a left exact category comes equipped with a choice of limits such as products, pullbacks, equalizers and terminal objects. Similarly, when $\mathscr{C}$ is a category with weak pullbacks and strict products, we will assume a choice of strict products and of weak pullbacks. Finally, a weakly lex category will be equipped with a choice of weak limits.

The idea behind the standard interpretation is similar to the set valued interpretation of first order logic. The paradigm *propositions as subsets* becomes, in the language of category theory, *propositions as subobjects*. If $\mathscr{C}$ is a category and X is an object of $\mathscr{C}$, then a *subobjects* is the equivalence class of a monomorphism $m : M \rightarrowtail X$ up to the following equivalence relation: the monomorphisms m and $m' : M' \rightarrowtail X$ are equivalent if there exist two arrows $h : M \to M'$ and $k : M' \to M$ such that $m' \circ h = m$ and $m \circ h = m'$. The substitution of terms into formulae is interpreted through the pullbacks. This interpretation can be fruitfully described using the functor in Example 1.2.7 which we recall here. If $\mathscr{C}$ is a left exact category, then the functor of subobjects

$$\mathsf{Sub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{Pos}$$

sends

- an object $X \in \mathscr{C}$ to the poset $\mathsf{Sub}_{\mathscr{C}}(X)$ of subobjects over X with the following partial equivalence relation: $\lfloor m : M \rightarrowtail X \rceil \leq \lfloor n : N \rightarrowtail X \rceil$ if there exists an arrow h which makes the following diagram commute

$$\begin{tikzcd} M \arrow[rr, dashed, "h"] \arrow[dr, tail, "m"'] & & N \arrow[dl, tail, "n"] \\ & X & \end{tikzcd}$$

- an arrow $f : Y \to X$ of $\mathscr{C}$ to the functor $\mathsf{Sub}_{\mathscr{C}}(f) : \mathsf{Sub}_{\mathscr{C}}(X) \to \mathsf{Sub}_{\mathscr{C}}(Y)$ which sends a monomorphism $m : M \rightarrowtail X$ to the left vertical arrow of the following pullback:

$$\begin{tikzcd} P \arrow[r] \arrow[d, tail, dashed, "f^*(m)"'] & M \arrow[d, tail, "m"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

The functor $\mathsf{Sub}_{\mathscr{C}}(f)$ will be denoted as f^*, and its action on a monomorphism m will be denoted by $f^*(m)$.

Observation 4.2.2. As already mentioned inExample 1.2.7, when $\mathscr{C}$ is a left exact category, and X is an object of $\mathscr{C}$, then the poset $\mathsf{Sub}_{\mathscr{C}}(X)$ has finite meets. Given two subobjects $m : A \rightarrowtail X$ and $n : B \rightarrowtail X$, the meet $m \wedge n$ is given by the equivalence class of the common value of the composite of following pullback

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] & B \arrow[d, tail, "n"] \\ A \arrow[r, tail, "m"'] & X. \end{tikzcd}$$

Every arrow $h : Y \to X$ of $\mathscr{C}$ induces a meet-preserving functor $h^* : \mathsf{Sub}_{\mathscr{C}}(X) \to \mathsf{Sub}_{\mathscr{C}}(Y)$.

In order to interpret connectives and quantifiers, we may require additional structure on the category in which we would interpret the logical language. In order to interpret regular formulae, we recall one of the equivalent definition of regular category and refer to [Bar71] or to the first chapter of [But98].

Definition 4.2.3. A left exact category $\mathscr{C}$ is said *regular* if

- any arrow of $\mathscr{C}$ factorizes as a regular epimorphism followed by a monomorphism,
- these factorizations are pullback-stable.

Regular categories form a category **REG** whose arrows are regular functors, i.e. fuctors preserving finite limits and coequalizers of kernel pairs. The following result is well-known, and a proof can be found in [Joh02].

Lemma 4.2.4. *Let $\mathscr{C}$ be a regular category. If $f : Y \to X$ is an arrow of $\mathscr{C}$, then the functor f^* has a left adjoint*

$$\begin{tikzcd} \mathsf{Sub}_{\mathscr{C}}(Y) \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \mathsf{Sub}_{\mathscr{C}}(X), \arrow[l, shift left, "f^*"] \end{tikzcd}$$

given by $f_!(m) = \mathsf{Im}(f \circ m)$. □

If $\mathscr{C}$ is a regular category, the standard interpretation $\mathcal{M}$ in $\mathscr{C}$ of the signature of a language $\mathcal{L}(S)$ is defined as follows:

- each sort S is interpreted as an object $\mathcal{M}(S) \in \mathscr{C}$,
- each constant symbol $c : S$ is interpreted as an arrow $\mathcal{M}(c) : 1 \to \mathcal{M}(S)$,
- each function symbol $f : S_1 \times \cdots \times S_n \to Y$ is interpreted as an arrow $\mathcal{M}(f) : \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n) \to \mathcal{M}(Y)$,
- each relation symbol $R \subseteq S_1 \times \cdots \times S_n$ is interpreted as an object $\mathcal{M}(R) \in \mathsf{Sub}_{\mathscr{C}}(\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))$. The equality symbol $\approx_S$ is interpreted as the diagonal $\mathcal{M}(S) \overset{\Delta_S}{\to} \mathcal{M}(S) \times \mathcal{M}(S)$.

For a list of variables $\bar{x} = (x_1, \ldots, x_n)$, of sorts $x_1 : S_1, \ldots, x_n : S_n$, we will denote by $\mathcal{M}(\bar{S})$ the product $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$.

The interpretation extends inductively on terms and formulae taking in account the set of free variables occurring. A term or a formula can be considered with extra 'dummy' variables. For example, a variable x_j of sort S_j, is a term with free variables occurring in the set $\bar{x} = (x_1, \ldots, x_n)$, where $x_i : S_i$ for $1 \leq i \leq n$. The interpretation of a term t of sort S, with $FV(t) \subseteq \bar{x} = (x_1, \ldots, x_n)$, where $x_i : S_i$ for $1 \leq i \leq n$, is an arrow $\mathcal{M}_{\bar{x}}(t) : \mathcal{M}(\bar{S}) \to \mathcal{M}(S)$ defined inductively as follows:

(T1) a constant symbol $c : S$ is interpreted as the map $\mathcal{M}_{\bar{x}}(c) : \mathcal{M}(\bar{S}) \to \mathcal{M}(S)$ factoring through $\mathcal{M}(c) : 1 \to \mathcal{M}(S)$,

(T2) a variable $x_i : S_i$ is interpreted as the projection $\mathcal{M}_{\bar{x}} := p_i : \mathcal{M}(\bar{S}) \to \mathcal{M}(S_i)$,

(T3) if $\mathcal{M}_{\bar{x}}(t_i)$ are the interpretations of the terms $t_i : S_i$, for $1 \leq i \leq n$, and $\mathcal{M}(f) : \mathcal{M}(\bar{x}) \to \mathcal{M}(Y)$ is the interpretation of the function symbol $f : S_1 \times \cdots \times S_n \to Y$, then the term $f(t_1, \ldots, t_n)$ is interpreted as the composition $\mathcal{M}_{\bar{x}}(f(t_1, \ldots, t_n)) := \mathcal{M}(f) \circ \langle \mathcal{M}(t_1), \ldots, \mathcal{M}(t_n) \rangle$.

A formula φ, in which occur the terms $t_1 : Z_1, \ldots, t_n : Z_n$, such that $FV(t_1, \ldots, t_n) \subseteq FV(\varphi) \subseteq \bar{x} = (x_1, \ldots, x_m)$, where $x_1 : S_1, \ldots, x_m : S_m$, is interpreted inductively as follows:

(F1) If $R \subseteq Z_1 \times \cdots \times Z_n$ is a relation symbol and $t_1 : Z_1, \ldots, t_n : Z_n$ are terms, then $\mathcal{M}_{\bar{x}}(R(t_1, \ldots, t_n)) := \mathcal{M}_{\bar{x}}(\bar{t})^* \mathcal{M}(R)$, where $\mathcal{M}_{\bar{x}}(\bar{t}) := \langle \mathcal{M}_{\bar{x}}(t_1), \ldots, \mathcal{M}_{\bar{x}}(t_n) \rangle$,

(F2) if t_1, t_2 are terms of the same sort Z, then $t_1 \approx_S t_2$ is interpreted as the equalizer of the arrows

$$\begin{tikzcd} \mathcal{M}(\bar{S}) \arrow[r, shift left, "\mathcal{M}_{\bar{x}}(t_1)"] \arrow[r, shift right, "\mathcal{M}_{\bar{x}}(t_2)"'] & \mathcal{M}(Z) \end{tikzcd} ,$$

(F3) $\mathcal{M}_{\bar{x}}(\top) = 1_{\mathcal{M}(\bar{S})}$,

(F4) if φ and ψ are formulae with $FV(\varphi, \psi) \subseteq \bar{x}$, then $\varphi \wedge \psi$ is interpreted as the common value of the composite of following pullback

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] & \mathcal{M}(\varphi) \arrow[d, tail, "\mathcal{M}_{\bar{x}}\varphi"] \\ \mathcal{M}(\psi) \arrow[r, tail, "\mathcal{M}_{\bar{x}}\psi"'] & \mathcal{M}(\bar{S}), \end{tikzcd}$$

(F5) if φ is a formula with $FV(\varphi) \subseteq (y, x_1, \ldots, x_n)$ with $y : Z$, then $\mathcal{M}_{\bar{x}}\exists y\varphi := \pi_!(\mathcal{M}_{y,\bar{x}}\phi)$, where π is the projection $\mathcal{M}(Z) \times \mathcal{M}(\bar{Z}) \to \mathcal{M}(\bar{Z})$.

We now briefly recall that the above interpretation is sound and complete. The following result appears as [But98, Lemma 4.1].

Theorem 4.2.5 (Soundness of the standard interpretation)**.** *Let $\mathbb{T}$ be a regular theory and $\mathcal{M}$ an interpretation in a regular category $\mathscr{C}$. If $\bar{x} = (x_1, \ldots, x_n)$ are variable of sort $x_i : S_i$, for $1 \leq i \leq n$, and a sequent $\varphi \vdash_{\bar{x}} \psi$ is derivable, then $\mathcal{M}_{\bar{x}}(\varphi) \leq \mathcal{M}_{\bar{x}}(\psi)$ in $\mathsf{Sub}_{\mathscr{C}}(\mathcal{M}(\bar{S}))$.* □

The standard interpretation is also complete. The proof follows from the construction of the syntactic category $\mathscr{C}(\mathbb{T})$ defined as follows. The category $\mathscr{C}(\mathbb{T})$ has

objects are equivalence classes of provably equivalent formulae in context, i.e. pairs $\{\bar{x} : \bar{X}, \varphi\}$, where $\bar{x} : \bar{X}$ is a context $x_1 : X_1, \ldots, x_n : X_n$ and φ is a formula $\bar{x} \subseteq FV(\varphi)$.

arrows from $(\bar{x} : \bar{X}, \varphi)$ to $(\bar{y} : \bar{Y}, \psi)$ are equivalence classes of formulae γ in the context : $\bar{x} : \bar{X}, \bar{y} : \bar{Y}$ which are functional in the following sense:

- $\gamma(\bar{x}, \bar{y}) \vdash_{\bar{x},\bar{y}} \varphi(\bar{x}) \wedge \psi(\bar{y})$,
- $\gamma(\bar{x}) \vdash_{\bar{x},\bar{y}} \exists\bar{y}\gamma(\bar{x}, \bar{y})$,
- $\gamma(\bar{x}, \bar{y_1}), \gamma(\bar{x}, \bar{y_2}) \vdash_{\bar{x},\bar{y_1},\bar{y_2}} \bar{y_1} \approx_{\bar{Y}} \bar{y_2}$.

Given two arrows $\{\gamma\} : \{\bar{x} : \bar{X}, \varphi\} \to \{\bar{y} : \bar{Y}, \psi\}$ and $\{\chi\} : \{\bar{y} : \bar{Y}, \varphi\} \to \{\bar{z} : \bar{Z}, \rho\}$, the composition is given by the equivalence class of the formula

$$\exists\bar{y}(\gamma(\bar{x}, \bar{y}) \wedge \chi(\bar{y}, \bar{z})).$$

If $\mathbb{T}$ is a regular theory, then it can be proved that $\mathcal{C}(\mathbb{T})$ is a regular category. Moreover, one can define a canonical interpretation $\mathcal{U}$ into $\mathcal{C}(\mathbb{T})$ as follows

- $\mathcal{U}(X) := \{x : X, x = x\}$, for a sort X
- $\mathcal{U}(c) := \{x : X, x = c\}$, for a constant symbol $c : X$
- $\mathcal{U}(f) := \{\bar{x}, y : \bar{X}, Y, f(\bar{x}) = y\}$, for a function symbol $f : \bar{X} \to Y$
- $\mathcal{U}(R) := \{\bar{x} : \bar{X}, R(\bar{x})\}$, seen as a subobject of $\mathcal{U}(\bar{X})$, for a relation symbol $R \subseteq X_1 \times \cdots \times X_n$.

Since provability and satisfiability coincide in the canonical interpretation, the standard interpretation is also complete. We refer to §6 of [But98] or to §8 of [MR77] for further details. In particular, it follows that the models of $\mathbb{T}$ in a regular category $\mathscr{C}$ are equivalent to regular functors from $\mathcal{C}(\mathbb{T})$ to $\mathscr{C}$

$$Mod(\mathbb{T}, \mathscr{C}) \cong \mathbf{REG}(\mathcal{C}(\mathbb{T}), \mathscr{C}).$$

A proof of the following result appears in [But98, Proposition 6.4].

Proposition 4.2.6 (Completeness of the standard interpretation)**.** *The canonical interpretation $\mathcal{U}$ in the regular category $\mathcal{C}(\mathbb{T})$ is a complete model of $\mathbb{T}$. In particular, the syntactic calculus given above is complete with respect to interpretations in (small) regular categories.* □

4.3 BHK interpretation

In this section we recall the *categorical Brouwer-Heyting-Kolmogorov* interpretation following [Pal04]. This interpretation is suitable for a larger class of categories, as it is shown below. Before starting, we fix some notations.

We now recall the notations of the functor introduced in Example 1.2.6.

Notation. If $\mathscr{C}$ is a category and X is an object of $\mathscr{C}$, we will denote by $(\mathscr{C}/X)_{po}$ the poset reflection of the slice category $\mathscr{C}/X$, whose objects are equivalence classes of arrows with respect to the following equivalence relation: two arrows $f : A \to X$ and $g : B \to X$ of $\mathscr{C}$ are equivalent when there exist two arrows h, k making the following diagrams commute

$$\begin{tikzcd} A \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[dr, "f"'] & & B \arrow[ll, dashed, "k"'] \arrow[dl, "g"] \\ & X. & \end{tikzcd}$$

The equivalence class of an arrow f will be denoted by $\lfloor f \rceil$. We will write $\lfloor f \rceil \leq \lfloor g \rceil$ if and only if there exists an arrow h such that $f = g \circ h$.

If $\mathscr{C}$ is a category with strict products and weak pullbacks (qlex) the subobjects can be collected in a controvariant functor

$$\mathsf{Psub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{Pos}$$

which sends

- an object $X \in \mathscr{C}$ to the poset $\mathsf{Psub}_{\mathscr{C}}(X) := (\mathscr{C}/X)_{po}$,
- an arrow $f : Y \to X$ of $\mathscr{C}$ to the functor $\mathsf{Psub}_{\mathscr{C}}(f) : \mathsf{Psub}_{\mathscr{C}}(X) \to \mathsf{Psub}_{\mathscr{C}}(Y)$ which sends an equivalence class $\lfloor g : B \to X \rceil$ to the equivalence class of the left vertical arrow of a weak pullback:

$$\begin{tikzcd} P \arrow[r] \arrow[d, dashed] & B \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

The functor $\mathsf{Psub}_{\mathscr{C}}(f)$ will be denoted as f^*, and its action on $\lfloor g \rceil$ will be denoted by $f^*\lfloor g \rceil$. In order to interpret regular formulae, recall that the definition of weak pullback along $f : Y \to X$ ensures that the functor f^* has a left adjoint

$$\begin{tikzcd} \mathsf{Psub}_{\mathscr{C}}(Y) \arrow[r, shift left=1.5ex, "f_!"] \arrow[r, phantom, "\perp"] & \mathsf{Psub}_{\mathscr{C}}(X), \arrow[l, shift left=1.5ex, "f^*"] \end{tikzcd}$$

given by $f_!\lfloor - \rceil = \lfloor f \circ - \rceil$.

Observation 4.3.1. As already observed, the functor $\mathsf{Psub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{Pos}$ takes value in infimum semilattices. Indeed, given $\lfloor f \rceil, \lfloor g \rceil \in \mathsf{Psub}_{\mathscr{C}}(X)$, the meet $\lfloor f \rceil \wedge \lfloor g \rceil$ is given by the equivalence class of the common value of the composite of the following weak pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & X. \end{tikzcd}$$

Every arrow $h : Y \to X$ of $\mathscr{C}$ induces a meet-preserving functor $h^* : \mathsf{Psub}_{\mathscr{C}}(X) \to \mathsf{Psub}_{\mathscr{C}}(Y)$.

The interpretation of a language $\mathcal{L}(S)$ in a qlex category $\mathscr{C}$ works as the interpretation in lex categories, but replacing the functor $\mathsf{Sub}_{\mathscr{C}}$ by $\mathsf{Psub}_{\mathscr{C}}$. In particular, terms are interpreted as in the previous section and a relation symbol $R \subseteq S_1 \times \cdots \times S_n$ is interpreted as an object

$$\mathcal{M}(R) \in \mathsf{Psub}_{\mathscr{C}}(\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)).$$

Formulae are interpreted as the equivalence classes of the arrows involved in the standard interpretation of (F1-F5).
The categorical BHK interpretation is sound as it is shown in [Pal04, Theorem 3.3].

Theorem 4.3.2 (Soundness of the BHK interpretation)**.** *Let $\mathbb{T}$ be a regular theory and $\mathcal{M}$ an interpretation in a quasi left exact category $\mathscr{C}$. If $\bar{x} = (x_1, \ldots, x_n)$ are variables of sort $x_i : S_i$, for $1 \leq i \leq n$, and a sequent $\varphi \vdash_{\bar{x}} \psi$ is derivable in $\mathbb{T}$, then $\mathcal{M}_{\bar{x}}(\varphi) \leq \mathcal{M}_{\bar{x}}(\psi)$ in $\mathsf{Psub}_{\mathscr{C}}(\mathcal{M}(\bar{S}))$.* □

We end this section mentioning the connection between the two interpretations. Actually, the standard interpretation of regular logic in a regular category is obtained applying the *image* functor to the BHK interpretation. Recall that for a regular category $\mathscr{C}$ the regular-epi/mono factorization induces a left adjoint to the inclusion functor $U_X : \mathsf{Sub}_{\mathscr{C}}(X) \to \mathsf{Psub}_{\mathscr{C}}(X)$

$$\begin{tikzcd} \mathsf{Sub}_{\mathscr{C}}(X) \arrow[r, shift left=1.5, "U_X"] \arrow[r, phantom, "\top"] & \mathsf{Psub}_{\mathscr{C}}(X), \arrow[l, shift left=1.5, "\mathsf{Im}"] \end{tikzcd}$$

for every object $X \in \mathscr{C}$. The following result appears as [Pal04, Theorem 4.3].

Theorem 4.3.3. *Let $\mathscr{C}$ be a regular category. Suppose that $\mathcal{M}$ is an interpretation where all relation symbols are interpreted as subobjects. Denote by $\tilde{\mathcal{M}}$ the BHK interpretation function and let $\bar{\mathcal{M}}$ be the standard interpretation function. Then for regular formulae φ with $FV(\varphi) \subseteq \bar{x}$:*

$$\bar{\mathcal{M}}_{\bar{x}}(\varphi) = \mathsf{Im}(\tilde{\mathcal{M}}_{\bar{x}}(\varphi)).$$

□

The above theorem implies that the BHK interpretation is actually complete. Indeed, there is a canonical BHK interpretation $\mathcal{U}'$ in the syntactic category $\mathcal{C}(\mathbb{T})$, which coincides with the canonical interpretation $\mathcal{U}$ in $\mathcal{C}(\mathbb{T})$, on function and sort symbols and is defined as the compositions $U_X \circ \mathcal{U}$ on the relation symbols. Hence, by the completeness theorem of the standard interpretation we obtain the following result.

Proposition 4.3.4 (Completeness of the BHK interpretation)**.** *The canonical BHK interpretation $\mathcal{U}'$ in the regular category $\mathcal{C}(\mathbb{T})$ is a complete model of $\mathbb{T}$. In particular, the syntactic calculus given above is complete with respect to the BHK interpretations in (small) quasi left exact categories.* □

Proof. It is enough to observe that if $\mathcal{U}'_{\bar{x}}(\varphi) \leq \mathcal{U}'_{\bar{x}}(\psi)$ in $\mathsf{Psub}_{\mathcal{C}(\mathbb{T})}(\mathcal{U}'(\bar{S}))$, then $Im \circ \mathcal{U}'_{\bar{x}}(\varphi) \leq Im \circ \mathcal{U}'_{\bar{x}}(\psi)$ in $\mathsf{Sub}_{\mathcal{C}(\mathbb{T})}(\mathcal{U}(\bar{S}))$ and apply Theorem 4.3.3. □

4.4 Interpretation in wlex categories

In this section, we provide an interpretation of a language $\mathcal{L}(S)$ in a weakly left exact category $\mathscr{C}$. As usual, the interpretation of a sort symbol S is given by an object $\mathcal{M}(S) \in \mathscr{C}$, but the interpretation of terms and formulae requires more accuracy. This is due to weak products and to the weak universal property of weak products. As we have seen in the previous section, strict products seem necessary to treat multi-variable terms and formulae. But now, for a function symbol

$$f : S_1 \times \cdots \times S_n \to S$$

the weak products $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$ are not unique up to isomorphism and the domain of a possible interpretation $\mathcal{M}(f)$ would depend on a choice of the weak product. Similarly, a naive interpretation of a predicate symbol

$$R \subseteq S_1 \times \cdots \times S_n,$$

given by an equivalence class $\mathcal{M}(R)$ of some arrows over a weak product $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$, would depend on the choice of the weak product and not only on the interpretations $\mathcal{M}(S_i)$ of the sorts S_i, for $1 \leq i \leq n$. The same happens for substitution of terms. Assuming that we have interpreted the terms $t_1 : S_1, \ldots, t_n : S_n$ as arrows $\mathcal{M}_{\bar{x}}(t_i)$ with codomain $\mathcal{M}(S_i)$, for $1 \leq i \leq n$, and a predicate symbol $R \subseteq S_1 \times \cdots \times S_n$ as an arrow with codomain a weak product $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$, then there exists a not-unique arrow $\langle \mathcal{M}_{\bar{x}}(t_1), \ldots, \mathcal{M}_{\bar{x}}(t_n)\rangle$ induced by the weak universal property of the weak products suitable to interpret $R(t_1, \ldots, t_n)$ as

$$\langle \mathcal{M}_{\bar{x}}(t_1), \ldots, \mathcal{M}_{\bar{x}}(t_n)\rangle^* \mathcal{M}(R).$$

A naive interpretation could be to assume a choice of weak finite products and arrows induced by their weak universal property and proceed as for the BHK interpretation. Unfortunately, this process leads to problems interpreting the axioms of equality and substitution. In order to give an interpretation which is somehow well-behaved with respect to the possible choices of weak products and their universal property and validates all the axioms of regular logic, we will introduce in the next sections particular classes of arrows and subobjects.

Before starting we recall that a category is weakly left exact category if and only if it has weak finite products and weak equalizers or weak pullbacks. Obviously, lex and qlex categories are wlex and the slices of qlex categories are wlex, as it happens for the slices of the syntactic category **ML** arising from intensional type theory.

Notation. For the rest of the section, $\mathscr{C}$ will denote a wlex category. If $X_1, \ldots, X_n$ are objects of $\mathscr{C}$, we will adopt the usual notation $X_1 \times \cdots \times X_n$ to denote a choice of weak products of the objects and we will denote with $p_i : X_1 \times \cdots \times X_n \to X_i$ the projections for $1 \leq i \leq n$. Similarly, if $f : X \to Y_1$ and $g : X \to Y_2$ are two arrows of $\mathscr{C}$, then we will denote with $\langle f, g\rangle : X \to Y_1 \times Y_2$ a choice of an arrow induced by the weak universal property of the weak products, such that $p_1 \circ \langle f, g\rangle = f$ and $p_2 \circ \langle f, g\rangle = g$.

Terms. As usual, the interpretation of terms of a language $\mathcal{L}(S)$ will be given by arrows of $\mathscr{C}$. In order to treat multi-variable terms, we would interpret a term $t(x_1, \ldots, x_n) : S$ with free variables $x_1 : S_1, \ldots, x_n : S_n$, as an arrow

$$\mathcal{M}(t) : \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n) \to \mathcal{M}(S),$$

from a weak product $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$ of the objects $\mathcal{M}(S_i) \in \mathscr{C}$, for $1 \leq i \leq n$, which is well-behaved with respect to the choice of a weak product in the following sense: if

$$(\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))'$$

is a different weak product with projections $p'_i : (\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))' \to \mathcal{M}(S_i)$, for $1 \le i \le n$, then the interpretation $\mathcal{M}(t)$ must induce a unique arrow $\mathcal{M}(t)' : (\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))' \to \mathcal{M}(S)$ such that for every arrow $h : (\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))' \to \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$ with $p'_i \circ h = p_i$, for $1 \le i \le n$ the following diagram commutes

$$\begin{tikzcd}
(\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))' \arrow[d, "h"'] \arrow[dr, "\mathcal{M}(t)'"] & \\
\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n) \arrow[r, "\mathcal{M}(t)"'] & \mathcal{M}(S).
\end{tikzcd}$$

In order to do that, we now introduce a special class of arrows, which have been previously defined in [Emm20].

Definition 4.4.1. Let $X_1 \dots X_n, Y$ be objects of $\mathscr{C}$. An arrow $f : X_1 \times \cdots \times X_n \to Y$ is called *determined by projections* (*dbp*) if, for every $x, x' : T \to X_1 \times \cdots \times X_n$, such that $p_i \circ x = p_i \circ x'$, for $1 \le i \le n$, then $f \circ x = f \circ x'$.

Example 4.4.2. If $\mathscr{C}$ is a weak lex category the following are examples of arrows determined by projections:

- if $g : X_1 \times \cdots \times X_n \to Y$ is dbp, then the post composition $f \circ g$ with any arrow $f : Y \to Z$ is dbp,
- every projection $p_i : X_1 \times \cdots \times X_n \to X_i$ is dbp.

At present, we don't know conditions on the wlex category $\mathscr{C}$ which ensure the existence of arrows determined by projections.

Remark 4.4.3. When $n = 1$, a natural choice of (weak) 1-product of an object $X \in \mathscr{C}$ is given by X and the identity 1_X. In this case, every arrow out of X is determined by projections. If $n = 0$, the 0-product is a weak terminal object 1 with no projections, and an arrow $f : 1 \to A$ is determined by projections if and only if, for every pair of arrows $g, h : X \to 1$, $f \circ g = f \circ h$.

We now provide a useful property of arrows determined by projections in order to obtain an interpretation of constant and function symbols.

Lemma 4.4.4. *Let $X_1, \dots, X_n, Y$ be objects of $\mathscr{C}$ and let $f : X_1 \times \cdots \times X_n \to Y$ be an arrow determined by projections. If $(X_1 \times \cdots \times X_n)'$ is a different weak product, then the arrow f induces a unique arrow $f' : (X_1 \times \cdots \times X_n)' \to Y$ which is determined by projections.*

Proof. By the weak universal property of the weak products, there exists a not necessarily unique arrow

$$h : (X_1 \times \cdots \times X_n)' \to X_1 \times \cdots \times X_n$$

such that $p_i \circ h = p'_i$, for $1 \le i \le n$. If $h' : (X_1 \times \cdots \times X_n)' \to X_1 \times \cdots \times X_n$ is a different arrow such that $p_i \circ h' = p'_i$, for $1 \le i \le n$, since f is determined by projection, we obtain

$$f \circ h = f \circ h'.$$

Hence, we can define a unique arrow $f' := f \circ h$ from the weak product $(X_1 \times \cdots \times X_n)'$. A trivial verification shows that also f' is determined by projections. □

We will now define an interpretation in which constant and function symbols are interpreted through arrows determined by projections.

- A constant symbol $c : S$ is interpreted as an arrow $\mathcal{M}(c) : 1 \to \mathcal{M}(S)$ determined by projections, where 1 is a weak terminal object.

- A function symbol $f : S_1 \times \cdots \times S_n \to S$ is interpreted as an arrow $\mathcal{M}(f) : \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n) \to \mathcal{M}(Y)$ determined by projection.

Remark 4.4.5. The interpretation of constant and function symbols as arrows determined by projections is well-behaved with respect to the choice of weak product in the following sense. Once we have fixed the interpretations through a choice of weak products, then it spreads for different weak products, as shown below.

- If a constant symbol $c : S$ is interpreted through an arrow $\mathcal{M}(c) : 1 \to \mathcal{M}(S)$ determined by projections, from a weak terminal object 1, then we can consider a different weak terminal object $1'$ and, through Remark 4.4.3 and Lemma 4.4.4, obtain a unique arrow $\mathcal{M}(c)' : 1' \to \mathcal{M}(S)$.

- If a function symbol $f : S_1 \times \cdots \times S_n \to S$ is interpreted through an arrow $\mathcal{M}(f) : \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n) \to \mathcal{M}(Y)$ determined by projection, then we can consider a different weak product $(\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))'$ and, applying Lemma 4.4.4, obtain a unique arrow

$$\mathcal{M}(f)' : (\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))' \to \mathcal{M}(Y).$$

We can now interpret terms in a context $\bar{x} = (x_1, \dots, x_n)$, with $x_i : S_i$ for $1 \leq i \leq n$, as follows. As in the previous section, we will denote a weak product $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$ by $\mathcal{M}(\bar{S})$.

(T1) The interpretation $\mathcal{M}_{\bar{x}}(c) : \mathcal{M}(\bar{S}) \to \mathcal{M}(S)$ is given by the composition $\mathcal{M}(c) \circ u$, where $u : \mathcal{M}(\bar{S}) \to 1$ is an arrow into the weak terminal object 1 induced by the weak universal property of 1.

(T2) A variable $x_i : S_i$ is interpreted as the projection $\mathcal{M}_{\bar{x}}(x_i) := p_i : \mathcal{M}(\bar{S}) \to \mathcal{M}(S_i)$, which is determined by projections.

(T3) If $\mathcal{M}_{\bar{x}}(t_i)$ are the interpretations of the terms $t_i : Z_i$, for $i = 1, \dots, m$, and $\mathcal{M}(f) : \mathcal{M}(\bar{Z}) \to \mathcal{M}(S)$ is the interpretation of the function symbol $f : Z_1 \times \cdots \times Z_m \to S$, then the term $f(t_1, \dots, t_m)$ is interpreted as the composition

$$\mathcal{M}_{\bar{x}}(f(t_1, \dots, t_m)) := \mathcal{M}(f) \circ \langle \mathcal{M}(t_1), \dots, \mathcal{M}(t_m) \rangle,$$

where $\langle \mathcal{M}(t_1), \dots, \mathcal{M}(t_m) \rangle$ is an arrow induced by the weak universal property of the weak product $\mathcal{M}(\bar{S})$. The composition is well defined because another arrow $\langle \mathcal{M}(t_1), \dots, \mathcal{M}(t_m) \rangle'$ induced by the weak universal property of the weak product verifies

$$p_i \circ \langle \mathcal{M}(t_1), \dots, \mathcal{M}(t_m) \rangle' = p_i \circ \langle \mathcal{M}(t_1), \dots, \mathcal{M}(t_m) \rangle$$

for every projection $p_i : \mathcal{M}(\bar{S}) \to M(Z_i)$, for $i = 1, \dots, m$.

Remark 4.4.6. We underline the fact that the interpretations of (T1) and (T3) is well behaved with respect to all the possible arrows induced by the weak universal property of weak products. Indeed:

- if $\mathcal{M}(c) : 1 \to \mathcal{M}(S)$ is the interpretation of a constant symbol $c : S$ and $u, u' : \mathcal{M}(S) \to 1$ are two arrows induced by the weak universal property of the weak terminal objects, then Remark 4.4.3 and the fact that $\mathcal{M}(c)$ is determined by projections imply that

$$\mathcal{M}(c) \circ u = \mathcal{M}(c) \circ u'.$$

- If $\mathcal{M}_{\bar{x}}(t_i)$ are the interpretations of the terms $t_i : Z_i$, for $1 \leq i \leq m$, and $\langle \mathcal{M}(t_1), \ldots, \mathcal{M}(t_m) \rangle'$ is a different arrow induced by the weak universal property of the weak product then

$$p_i \circ \langle \mathcal{M}(t_1), \ldots, \mathcal{M}(t_m) \rangle' = p_i \circ \langle \mathcal{M}(t_1), \ldots, \mathcal{M}(t_m) \rangle$$

  for every projection $p_i : \mathcal{M}(\bar{S}) \to M(Z_i)$, for $1 \leq i \leq m$. Since the interpretation $\mathcal{M}(f) : \mathcal{M}(\bar{Z}) \to \mathcal{M}(S)$ of the function symbol $f : Z_1 \times \cdots \times Z_m \to S$ is determined by projections, it holds that

$$\mathcal{M}(f) \circ \langle \mathcal{M}(t_1), \ldots, \mathcal{M}(t_m) \rangle = \mathcal{M}(f) \circ \langle \mathcal{M}(t_1), \ldots, \mathcal{M}(t_m) \rangle'.$$

Relations. As for the BHK interpretation, we will interpret a relation symbol $R \subseteq S_1 \times \cdots \times S_n$ as an equivalence class of arrows with codomain a weak product $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$, i.e. as an element

$$\mathcal{M}(R) \in \mathsf{Psub}_{\mathscr{C}}(\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)).$$

In this section we will provide the explicit construction, which is motivated by the analogies with the interpretations in categories with strict products. In the next section, we will justify the construction introducing the notion of *proof-irrelevant* elements.

We first recall some categorical notations that we have mentioned in Example 3.2.6.

Notation. Given a category $\mathscr{C}$ and objects $X_1, \ldots, X_n \in \mathscr{C}$, we will denote by $\mathscr{C}/(X_1, \ldots, X_n)$ the category of cones over $X_1, \ldots, X_n$ whose

objects are lists $(f_1, \ldots, f_n)$ of arrows $f_i : A \to X_i$ of $\mathscr{C}$, for $1 \leq i \leq n$, with a common domain A,

arrows $h : (f_1, \ldots, f_n) \to (g_1, \ldots, g_n)$ between two cones with domains A and B are arrows $h : A \to B$ of $\mathscr{C}$ such that $g_i \circ h = f_i$, for $1 \leq i \leq n$.

The poset $(\mathscr{C}/(X_1, \ldots, X_n))_{po}$ is the category whose objects are equivalence classes of cones given by the following equivalence relation: $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ are equivalent when there exist two arrows $h : A \to B$ and $k : B \to A$ making commutative the following diagrams componentwise

$$\begin{tikzcd} A \arrow[rr, dashed, "h"] \arrow[d, "f_1"'] \arrow[drr, "f_n" near start] & & B \arrow[d, "g_n"] \arrow[dll, "g_1"' near start] \\ X_1 & \cdots & X_n \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "f_1"'] \arrow[drr, "f_n" near start] & & B \arrow[ll, dashed, "k"'] \arrow[d, "g_n"] \arrow[dll, "g_1"' near start] \\ X_1 & \cdots & X_n \end{tikzcd}$$

We will denote the equivalence class of a cone $(f_1, \ldots, f_n)$ as $\lfloor (f_1, \ldots, f_n) \rceil$, and we will write $\lfloor (f_1, \ldots, f_n) \rceil \leq \lfloor (g_1, \ldots, g_n) \rceil$ if and only if there exists an arrow $h : A \to B$ such that $g_i \circ h = f_i$, for $1 \leq i \leq n$. Moreover, we will denote with $(\mathscr{C}/(X_1, \ldots, X_n))_{j.m.}$ the poset whose objects are equivalence classes of jointly monic cones over $X_1, \ldots, X_n$, up to isomorphism.

Observation 4.4.7. Now we want to highlight the advantages of working with categories with strict products. If $\mathscr{C}$ is a category with strict products and $X_1, \ldots, X_n$ are objects of $\mathscr{C}$, the strict universal property of strict products induces a functor which sends a cone to a unique arrow over the strict product $X_1 \times \cdots \times X_n$. The inverse functor is given by the post-composition with the projections $p_i : X_1 \times \cdots \times X_n \to X_i$, for $1 \leq i \leq n$. Hence, we obtain the following isomorphisms:

$$\begin{aligned} \mathscr{C}/(X_1, \ldots, X_n) &\cong \mathscr{C}/(X_1 \times \cdots \times X_n), \\ (\mathscr{C}/(X_1, \ldots, X_n))_{j.m.} &\cong \mathsf{Sub}_{\mathscr{C}}(X_1 \times \cdots \times X_n), \\ (\mathscr{C}/(X_1, \ldots, X_n))_{po} &\cong \mathsf{Psub}_{\mathscr{C}}(X_1 \times \cdots \times X_n). \end{aligned}$$

These bijections explicit the fact that the standard and the BHK interpretation of a relation symbol $R \subseteq S_1 \times \cdots \times S_n$ is completely determined respectively by a cone or an equivalence class of cones over $\mathcal{M}(S_i)$, for $1 \leq i \leq n$. However, when $\mathscr{C}$ is a category with weak products and weak pullbacks and $X_1 \times \cdots \times X_n$ is a weak product of the objects $X_1, \ldots, X_n \in \mathscr{C}$, then the above bijections do not necessarily hold. In particular, the weak universal property of the weak products does not provide an obvious functor which sends a cone over $X_1, \ldots, X_n$ to an arrow over a weak product $X_1 \times \cdots \times X_n$. In the next section, in Theorem 4.5.3, we will extend the above bijections for wlex categories restricting to suitable sub-poset of $\mathsf{Psub}_{\mathscr{C}}(X_1 \times \cdots \times X_n)$.

From the above observation, we suggest to consider the slogan *propositions as cones*. Taking this insight of relations, we define an interpretation of the relation symbols in wlex categories following three steps.

We first consider a relation symbol $R \subseteq S_1 \times \cdots \times S_n$ and the interpretations $\mathcal{M}(S_i) \in \mathscr{C}$ of the sorts, for $1 \leq i \leq n$. Secondly, we choose a cone $(r_1, \ldots, r_n)$ with $r_i : R \to \mathcal{M}(S_i)$, for $1 \leq i \leq n$. Finally, we choose a weak product $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$ and we associate an element of $\mathsf{PSub}_{\mathscr{C}}(\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))$.

Equality symbols. Given a sort S, the interpretation of the the equality relation symbol $\approx_S$ is given first considering the cone

$$
\begin{tikzcd}
 & \mathcal{M}(S) \arrow[dl, "1_{\mathcal{M}(S)}"'] \arrow[dr, "1_{\mathcal{M}(S)}"] & \\
\mathcal{M}(S) & & \mathcal{M}(S).
\end{tikzcd}
$$

Now we consider the following weak limit and define $\mathcal{M}(\approx_S)$ as the equivalence class, in the poset reflection of $\mathscr{C}/(\mathcal{M}(S) \times \mathcal{M}(S))$, of the right dashed arrow

$$
\begin{tikzcd}
 & D \arrow[dl, dashed] \arrow[dr, dashed, "\delta_S"] & \\
\mathcal{M}(S) \arrow[d, "1_{\mathcal{M}(S)}"'] \arrow[drr, "1_{\mathcal{M}(S)}" near start] & & \mathcal{M}(S) \times \mathcal{M}(S) \arrow[d, "p_2"] \arrow[dll, "p_1"' near start] \\
\mathcal{M}(S) & & \mathcal{M}(S).
\end{tikzcd}
\tag{4.1}
$$

Observe that δ_S is an equalizer of the arrows

$$
\mathcal{M}(S) \times \mathcal{M}(S) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{M}(S),
$$

and, when $\mathcal{M}(S) \times \mathcal{M}(S)$ is a strict product, we obtain the unique diagonal Δ_S.

The weak universal property of weak limits implies that the above interpretation of the equality relation symbol satisfies the following property for different choices of weak products.

Lemma 4.4.8. *Given a sort symbol S and a different weak product $\mathcal{M}(S) \xleftarrow{p'_1} (\mathcal{M}(S) \times \mathcal{M}(S))' \xrightarrow{p'_2} \mathcal{M}(S)$, if $\lfloor \delta'_S \rceil$ is the interpretation of the equality relation symbol $\approx_S$ over $(\mathcal{M}(S) \times \mathcal{M}(S))'$ obtained as in (4.1), then for every arrow $h : (\mathcal{M}(S) \times \mathcal{M}(S))' \to \mathcal{M}(S) \times \mathcal{M}(S)$ such that $p_i \circ h = p'_i$, for $i = 1, 2$ it follows that*

$$
h^* \lfloor \delta_S \rceil = \lfloor \delta'_S \rceil.
$$

$\square$

Observation 4.4.9. Since we have interpreted the equality relation symbols $\approx_S$ as the equivalence classes of certain weak equalizers, it is not difficult to show that axioms (E1)-(E4) are satisfied.

Relation symbols. A relation symbol $R \subseteq S_1 \times \cdots \times S_n$ is first interpreted as a cone $(r_1, \ldots, r_n)$ over the objects $\mathcal{M}(S_1), \ldots, \mathcal{M}(S_n)$:

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_n"] & \\ \mathcal{M}(S_1) & \cdots & \mathcal{M}(S_n). \end{tikzcd}$$

The interpretation $\mathcal{M}(R)$ is given by the equivalence class of the right dashed arrow of the following weak limit:

$$\begin{tikzcd} & R' \arrow[dl, dashed, "r"'] \arrow[dr, dashed, "\rho"] & \\ R \arrow[d, "r_1"'] \arrow[drr, "r_n"] & & \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n) \arrow[d, "p_n"] \arrow[dll, "p_1"'] \\ \mathcal{M}(S_1) & \cdots & \mathcal{M}(S_n). \end{tikzcd} \tag{4.2}$$

We observe that when $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$ is a strict product, then the arrow ρ is equivalent to the unique arrow induced by $r_1, \ldots, r_n$ on $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$. The weak universal property of weak limits trivially implies that the above interpretation of a predicate symbol satisfies the following property for different choices of weak products.

Lemma 4.4.10. *Using the above notation, if $(\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))'$ is another weak product with projections $p'_i : (\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))' \to \mathcal{M}(S_i)$, for $1 \leq i \leq n$, and $\mathcal{M}(R)'$ is the interpretation of R over $(\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))'$ built as in 4.2, then for every arrow $h : (\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))' \to \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$, such that $p'_i \circ h = p_i$, it follows that*

$$h^*\mathcal{M}(R) = \mathcal{M}(R)'.$$

□

4.5 Proof-irrelevant elements

In this section we will give a correspondence between cones and arrows in style of Observation 4.4.7 for weakly lex categories. Given a weak product $X_1 \times \cdots \times X_n$ of the objects $X_1, \ldots, X_n \in \mathscr{C}$, we can send a cone over $X_1, \ldots, X_n$ to the equivalence class of the weak limit in (4.2). This construction provides a functor

$$\mathcal{M} : \mathscr{C}/(X_1, \ldots, X_n)_{po} \to \mathsf{Psub}_{\mathscr{C}}(X_1 \times \cdots \times X_n).$$

An easy verification shows that the post-composition with the projections $p_i : X_1 \times \cdots \times X_n \to X_i$, for $1 \leq i \leq n$, does not necessarily provide an inverse to the above functor but only a left inverse. We will define the sub-poset of *proof-irrelevant* elements of $\mathsf{Psub}_{\mathscr{C}}(X_1 \times \cdots \times X_n)$ such that the corestriction of the above functor is an isomorphism. This correspondence implies that the interpretation of relation symbols as proof-irrelevant elements validates axiom (E5) of the equality predicate.

Before starting we fix some notations.

Notation. If $\mathscr{C}$ is a category with weak products, then we can obtain a weak product of the objects $X_1, \dots, X_n \in \mathscr{C}$ as

$$X_1 \times \cdots \times X_m \overset{p_1}{\leftarrow} (X_1 \times \cdots \times X_m) \times (X_{m+1} \times \cdots \times X_n) \overset{p_2}{\rightarrow} X_{m+1} \times \cdots \times X_n.$$

If $i \le m$ and $m < j \le n$, then, abusing the notation, we will denote by

$$\langle i, j\rangle : (X_1 \times \cdots \times X_m) \times (X_{m+1} \times \cdots \times X_n) \to X_i \times X_j$$

a choice of an arrow induced by the composition of the projections $p_i \circ p_1$ and $p_j \circ p_2$, where $p_i : X_1 \times \cdots \times X_m \to X_i$ and $p_j : X_{m+1} \times \cdots \times X_n \to X_j$.

When $\mathscr{C}$ is wlex and $X_1, \dots, X_n$ are objects of $\mathscr{C}$, we can consider the equivalence classes $\lfloor \delta_{X_i} \rceil$ of the arrows obtained as in (4.1), considering weak products $X_i \times X_i$, for $1 \le i \le n$. If $\bar{X} := X_1 \times \cdots \times X_n$, then we will refer to the element

$$\langle 1, n+1\rangle^* \lfloor \delta_{X_1} \rceil \wedge \cdots \wedge \langle n, 2n\rangle^* \lfloor \delta_{X_n} \rceil \in \mathsf{Psub}_{\mathscr{C}}(\bar{X} \times \bar{X}). \tag{4.3}$$

as the *proof-irrelevant* equality or the *componentwise* equality of $X_1 \times \cdots \times X_n$. Instead, the equivalence class $\lfloor \delta_{\bar{X}} \rceil \in \mathsf{Psub}_{\mathscr{C}}(\bar{X} \times \bar{X})$ obtained as in (4.1), for a weak product $\bar{X} \times \bar{X}$, will be called *proof-relevant* equality of $X_1 \times \cdots \times X_n$. This terminology has already been motivated in Example 3.2.5, but it will be discussed again in Section 4.6.

We recall the definition of *descent data* given in Definition 1.2.4. If $X \in \mathscr{C}$ and $\beta \in \mathsf{Psub}_{\mathscr{C}}(X \times X)$, the sub-order of the descent data of β is given by

$$\mathcal{D}es_{\beta} := \{\alpha \in \mathsf{Psub}_{\mathscr{C}}(X) \mid p_1^*(\alpha) \wedge \beta \le {p_2}^*(\alpha)\}. \tag{4.4}$$

By definition, it follows that the sub-orders of descent data are closed by finite meets and if $\beta' \le \beta$ then

$$\mathcal{D}es_{\beta} \subseteq \mathcal{D}es_{\beta'}.$$

We now recall that the *proof-irrelevant* elements of a weak product $X_1 \times \cdots \times X_n$ are the sub-order of $\mathsf{PSub}(X_1 \times \cdots \times X_n)$ given by the descent data of the proof-irrelevant equality of $X_1 \times \cdots \times X_n$. The sub-poset of proof-irrelevant elements of $X_1 \times \cdots \times X_n$ is denoted as

$$\mathsf{PIrr}_{\mathscr{C}}(X_1 \times \cdots \times X_n).$$

Remark 4.5.1. Observe that the assignment $\mathsf{PIrr}_{\mathscr{C}}$ is not functorial. Indeed, given three objects $X, Y, Z \in \mathscr{C}$, if $W := X \times Y$ is a weak product of X and Y, then a weak product $W \times Z$ of W and Z is also a weak product of X, Y and Z. In this situation, by definition the proof-irrelevant elements of the binary product $(W \times Z)$ are also proof-irrelevant elements of $(W \times Z)$ seen as a ternary product of X, Y and Z, but the converse does not necessarily holds. As we have seen in Chapter 3, in order to set proof-irrelevant elements functorially we need the framework of the biased elementary doctrines.

In the following proposition we show that the functor $\mathcal{M}$ takes value in the sub-poset of proof-irrelevant elements.

Proposition 4.5.2. *For every weak product $\bar{X} := X_1 \times \cdots \times X_n$ of the objects $X_1, \dots, X_n \in \mathscr{C}$, the functor*

$$\mathcal{M} : (\mathscr{C}/X_1, \dots, X_n)_{po} \to \mathsf{Psub}_{\mathscr{C}}(X_1 \times \cdots \times X_n),$$

takes value in $\mathsf{PIrr}_{\mathscr{C}}(X_1 \times \cdots \times X_n)$.

Proof. Consider the following weak pullback diagrams for $1 \leq i \leq n$:

$$\begin{tikzcd}
P_{i,n+i} \arrow[r, "<i{,}n+i>^*\delta_{X_i}"] \arrow[d, "d_i"'] & \bar{X} \times \bar{X} \arrow[d, "<i{,}n+i>"] \\
D_i \arrow[r, "\delta_{X_i}"'] & X_i \times X_i
\end{tikzcd}$$

and then consider the weak limit of the following diagram

$$\begin{tikzcd}
 & H \arrow[dl, dashed, "h_1"'] \arrow[dr, dashed, "h_n"] & \\
P_{\langle 1,n+1\rangle} \arrow[dr, "\langle 1{,}n+1\rangle^*\delta_{X_1}"'] & \cdots & P_{\langle n,2n\rangle} \arrow[dl, "\langle n{,}2n\rangle^*\delta_{X_n}"] \\
 & \bar{X} \times \bar{X}. &
\end{tikzcd}$$

We denote by h the common value of the composite of the weak limit above which corresponds to the proof-irrelevant equality $\bigwedge_{i=1}^{n} \langle i, n+i\rangle^*\delta_{X_i}$. If $\lfloor\rho\rceil := \mathcal{M}\lfloor(r_1, \ldots, r_n)\rceil$ for a cone $(r_1, \ldots, r_n)$ and the reindexings $p_i^*\rho$, for $i = 1, 2$, are given by the following weak pullbacks

$$\begin{tikzcd}
R_i \arrow[r, "p_i^*\rho"] \arrow[d, "p_i'"'] & \bar{X} \times \bar{X} \arrow[d, "p_i"] \\
R' \arrow[r, "\rho"'] & \bar{X}
\end{tikzcd}$$

then the conjunction $c := p_1^*\rho \wedge h$ is obtained as the common value of the composite of the following weak pullback

$$\begin{tikzcd}
C \arrow[r, "t"] \arrow[d, "h'"'] & H \arrow[d, "h"] \\
R_1 \arrow[r, "p_1^*\rho"'] & \bar{X} \times \bar{X}.
\end{tikzcd}$$

The statement is equivalent to prove that $\lfloor c\rceil \leq \lfloor p_2^*\rho\rceil$ in $\mathsf{Psub}_{\mathscr{C}}(\bar{X} \times \bar{X})$. But, since

$$p_i \circ c \circ p_2 = r_i \circ r \circ p_1' \circ h'$$

for $1 \leq i \leq n$, then there exists an arrow $k : C \to R'$ such that $\rho \circ k = p_2 \circ c$. Hence, there exists an arrow $s : C \to R_2$ such that $p_2^*\rho \circ s = c$ and $s \circ p_2' = k$. $\square$

In the following theorem we extend Observation 4.4.7 proving that, when $\mathscr{C}$ is a wlex category, cones are in bijections with proof-irrelevant elements.

Theorem 4.5.3. *For every weak product $\bar{X} := X_1 \times \cdots \times X_n$ of the objects $X_1, \ldots, X_n \in \mathscr{C}$, the functor $\mathcal{M}$ and the post-composition with the projections provide a bijection*

$$(\mathscr{C}/X_1, \ldots, X_n)_{po} \cong \mathsf{PIrr}(\bar{X}).$$

Proof. If $(r_1, \ldots, r_n)$ is a cone over $X_1, \ldots, X_n$ with domain an object $R \in \mathscr{C}$, then $\mathcal{M}(r_1, \ldots, r_n) : R' \to \bar{X}$ is a proof-irrelevant element. Using the notation of the diagram in (4.2), we have $\mathcal{M}(r_1, \ldots, r_n) := \lfloor \rho \rceil$ and an arrow $r' : R \to R'$ such that $r \circ r' = 1_R$ and $\rho \circ r' = \langle r_1, \ldots, r_n \rangle$ for an arrow $\langle r_1, \ldots, r_n \rangle$ induced by the weak universal property of the weak products. Hence, the arrows r, r' show that $(r_1, \ldots, r_n) \sim (p_1 \circ \rho, \ldots, p_n \circ \rho)$.
Vice versa, if $\sigma : S \to X_1 \times \cdots \times X_n$ is a proof-irrelevant element then we obtain the cone $(\sigma_1, \ldots, \sigma_n)$, where $\sigma_i := p_i \circ \sigma$, for $1 \leq i \leq n$. If $\lfloor \bar{\sigma} \rceil := \mathcal{M}\lfloor (\sigma_1, \ldots, \sigma_n) \rceil$ then obviously $(\sigma_1, \ldots, \sigma_n) \leq \bar{\sigma}$.

In order to obtain $\bar{\sigma} \leq \sigma$, we observe that $\sigma_i \circ s = p_i \circ \bar{\sigma}_i$, for $1 \leq i \leq n$, implies the existence of arrows $c_i : \bar{S} \to D_i$ such that $\langle \sigma \circ s, \bar{\sigma} \rangle \circ \langle i, n+i \rangle = \delta_i \circ c_i$, for some arrows $\langle \sigma \circ s, \bar{\sigma} \rangle, \langle i, n+i \rangle$ induced by the weak universal property of the weak products. Hence, from the weak pullbacks

$$\begin{tikzcd}
P_{i,n+i} \arrow[r, "<i,n+i>^*\delta_{X_i}"] \arrow[d, "d_i"'] & \bar{X} \times \bar{X} \arrow[d, "<i,n+i>"] \\
D_i \arrow[r, "\delta_{X_i}"'] & X_i \times X_i
\end{tikzcd}$$

we obtain arrows $u_i : \bar{S} \to P_{i,n+i}$ such that $d_i \circ u_i = c_i$ and $\langle i, n+i \rangle^* \delta_{X_i} \circ u_i = \langle \sigma \circ s, \bar{\sigma} \rangle$, for $1 \leq i \leq n$. Hence, using the notation of proof of Proposition 4.5.2, we obtain an arrow $l : \bar{S} \to H$, such that $h_i \circ l = u_i$, for $1 \leq i \leq n$. Now from the week pullbacks

$$\begin{tikzcd}
S_i \arrow[r, "p_i^*\sigma"] \arrow[d, "p_i'"'] & \bar{X} \times \bar{X} \arrow[d, "p_i"] \\
S \arrow[r, "\sigma"'] & \bar{X}
\end{tikzcd}$$

for $i = 1, 2$, we obtain an arrow $n : \bar{S} \to S_1$ such that $p_1' \circ n = s$ and $p_1^* \sigma \circ n = \langle \sigma \circ s, \bar{\sigma} \rangle$. This arrow implies the existence on an arrow j such that $h' \circ j = n$ and $t \circ j = l$. Hence, we are in the situation of the following diagram

$$\begin{tikzcd}
\bar{S} \arrow[r, dashed, "j"] \arrow[dr, dashed, "n"'] & C \arrow[r, "t"] \arrow[d, "h'"'] & H \arrow[d, "h"] & \\
& S_1 \arrow[r, "p_1^*\sigma"'] & \bar{X} \times \bar{X} \arrow[d, "p_2"'] & S_2 \arrow[l, "p_2^*\sigma"'] \arrow[d, "p_2'"] \\
& & \bar{X} & S \arrow[l, "\sigma"]
\end{tikzcd}$$

since σ is proof-irrelevant, there exists an arrow $v : C \to S_2$ such that $p_2^* \sigma \circ v = h \circ t$. Hence, using the arrow $v \circ j$ we obtain that $\bar{\sigma} \leq \sigma$. □

The proof of the above theorem and the proof Proposition 4.5.2 do not depend on the choice of the weak products $\bar{X} \times \bar{X}$ and $X_i \times X_i$ and arrows $\langle i, n+i \rangle : \bar{X} \times \bar{X} \to X_i \times X_i$, for $1 \leq i \leq n$. This justifies the notation $\mathsf{PIrr}(\bar{X})$ adopted for proof-irrelevant elements, which makes explicit only the weak product $\bar{X}$ considered. Moreover, the correspondence with cones implies that proof-irrelevant elements of different weak products of the same objects are actually isomorphic. We collect this observations in the following corollary.

Corollary 4.5.4. *Using the notation of **??**, if $\bar{X} \overset{p'_1}{\leftarrow} (\bar{X} \times \bar{X})' \overset{p'_2}{\rightarrow} \bar{X}$ is another weak product and $\langle i, i+n\rangle' : (\bar{X} \times \bar{X})' \rightarrow (X_i \times X_i)'$ are arrows induced by the weak universal property of different weak products $(X_i \times X_i)'$, for $1 \leq i \leq n$, then*

(i) *The following sub-orders of $\mathsf{Psub}_{\mathscr{C}}(\bar{X})$ are equal*

$$\mathcal{Des}_{\langle 1,n+1\rangle^* \lfloor \delta_{X_1}\rceil \wedge \cdots \wedge \langle n,2n\rangle^* \lfloor \delta_{X_n}\rceil} = \mathcal{Des}_{\langle 1,n+1\rangle'^* \lfloor \delta'_{X_1}\rceil \wedge \cdots \wedge \langle n,2n\rangle'^* \lfloor \delta'_{X_n}\rceil}.$$

(ii) *If $(X_1 \times \cdots \times X_n)'$ is a different weak product of the objects $X_1, \ldots, X_n \in \mathscr{C}$, then*

$$\mathsf{PIrr}(\bar{X}) \cong \mathsf{PIrr}(X_1 \times \cdots \times X_n)'.$$

□

We observe that the notion of proof-irrelevant elements is trivial for strict products. Indeed, if $\bar{X} := X_1 \times \cdots \times X_n$ is a strict product then we have the following relation between proof-relevant and irrelevant equalities

$$\lfloor \delta_{\bar{X}}\rceil = \langle 1, n+1\rangle^* \lfloor \delta_{X_1}\rceil \wedge \cdots \wedge \langle n, 2n\rangle^* \lfloor \delta_{X_n}\rceil = \lfloor \Delta_{\bar{X}}\rceil \tag{4.5}$$

where $\Delta_{\bar{X}} : \bar{X} \rightarrow \bar{X} \times \bar{X}$ is the unique diagonal arrow of $\bar{X}$. The above relation implies that

$$\mathsf{PIrr}(\bar{X}) = \mathsf{Psub}_{\mathscr{C}}(\bar{X}).$$

On the contrary, in case of weak products, we only have $\mathcal{Des}_{\lfloor \delta_{\bar{X}}\rceil} = \mathsf{Psub}_{\mathscr{C}}(\bar{X})$ and

$$\lfloor \delta_{\bar{X}}\rceil \leq \langle 1, n+1\rangle^* \lfloor \delta_{X_1}\rceil \wedge \cdots \wedge \langle n, 2n\rangle^* \lfloor \delta_{X_n}\rceil. \tag{4.6}$$

The intuition behind the above relation is that, in case of weak products, the equality of two objects of a weak product implies but is not the same of the equality of the components of the objects. In Section 4.6 we will provide an explicit example of this difference.

Remark 4.5.5. Hence, we have interpreted relation symbols as proof-irrelevant elements. This construction validate axiom (E5) of equality predicate. Indeed, if $\alpha \in \mathsf{PIrr}_{\mathscr{C}}(\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n))$ is the interpretation of a relation symbol $R \subseteq S_1 \times \cdots \times S_n$ then the inequality

$$p_1^*(\alpha) \wedge \langle 1, n+1\rangle^* \lfloor \delta_{S_1}\rceil \wedge \cdots \wedge \langle n, 2n\rangle^* \lfloor \delta_{S_n}\rceil \leq {p_2}^*(\alpha)$$

is just the interpretation of the axiom (E5) for the relation R

$$x_1 \approx_{S_1} y_1, \ldots, x_n \approx_{S_n} y_n, R(x_1, \ldots, x_n) \vdash_{\bar{x}} R(y_1, \ldots, y_n).$$

Folmulae Before interpreting formulae, we provide a useful description of arrows determined by projections using the internal logic of the functor of weak subobjects.

Proposition 4.5.6. *If $\bar{X} := X_1 \times \cdots \times X_n$ is a weak product of the objects $X_1, \ldots, X_n \in \mathscr{C}$, then an arrow $f : \bar{X} \rightarrow Y$ is determined by projections if and only if*

$$\langle 1, n+1\rangle^* \delta_{X_1} \wedge \cdots \wedge \langle n, 2n\rangle^* \delta_{X_n} \leq (f \times f)^* \delta_Y \tag{4.7}$$

in $\mathsf{Psub}_{\mathscr{C}}(\bar{X} \times \bar{X})$, for every weak product $\bar{X} \times \bar{X}$.

Proof. Using the notation of the proof of Proposition 4.5.2, we observe that the arrows $p_1 \circ h, p_2 \circ h : H \to \bar{X}$ are such that $p_i \circ p_1 \circ h = p_i \circ p_2 \circ h$, for $1 \le i \le n$. Hence, if f is dbp, then we obtain $f \circ p_1 \circ h = f \circ p_2 \circ h$ and the arrow $(f \times f) \circ h$ induces an arrow $s : H \to D_Y$ such that $\delta_Y \circ s = (f \times f) \circ h$. The inequality in 4.7 follows from the weak pullback diagram

$$\begin{tikzcd}
H \arrow[r, dashed] \arrow[rr, bend left, "s"] \arrow[dr, bend right, "h"'] & P \arrow[r, "u"] \arrow[d, "(f\times f)^*\delta_Y"'] & D_Y \arrow[d, "\delta_Y"] \\
& \bar{X} \times \bar{X} \arrow[r, "f\times f"'] & Y \times Y.
\end{tikzcd}$$

Vice versa, let $g, k : Z \to \bar{X}$ be two arrows such that $p_i \circ g = p_i \circ k$, for $i = 1, \dots, n$ and assume that there exists an arrow $s' : H \to P$ making $h \le (f \times f)^* \delta_Y$. By a trivial computation, we obtain an arrow $z : Z \to H$ such that $\langle g, k \rangle = h \circ z$. Hence, $g \circ f = k \circ f$ follows from $(f \times f) \circ \langle h, k \rangle = \delta_Y \circ u \circ s \circ z$. $\square$

We now provide some important properties of proof-irrelevant elements, which will be useful to interpret formulae.

Proposition 4.5.7. *If $\bar{X} := X_1 \times \dots \times X_n$ is a weak product of the objects $X_1, \dots, X_n \in \mathscr{C}$ and $\alpha \in \mathsf{PIrr}(X_1 \times \dots \times X_n)$, then*

(i) *If $Y_1 \times \dots \times Y_m$ is a weak product of the objects $Y_1, \dots, Y_m \in \mathscr{C}$ and $h : Y_1 \times \dots \times Y_m \to X_1 \times \dots \times X_n$ is determined by projections, then*

$$h^* \alpha \in \mathsf{PIrr}(Y_1 \times \dots \times Y_m).$$

(ii) *If $Y_1 \times \dots \times Y_m$ is a weak product of the objects $Y_1, \dots, Y_m \in \mathscr{C}$ and $f_i : Y_1 \times \dots \times Y_m \to X_i$, for $1 \le i \le n$, are arrows determined by projections, then for every arrow $\langle f_1, \dots, f_n \rangle : Y_1 \times \dots \times Y_m \to X_1 \times \dots \times X_n$*

$$\langle f_1, \dots, f_n \rangle^* \alpha \in \mathsf{PIrr}(Y_1 \times \dots \times Y_m).$$

(iii) *If $f_1, f_2 : Z \to X_1 \times \dots \times X_n$ are arrows such that $p_i \circ f_1 = p_i \circ f_2$, for $1 \le i \le n$, then*

$$f_1^* \alpha = f_2^* \alpha.$$

(iv) *If $\bar{X} \times Y$ is a weak product of $\bar{X}$ with an object $Y \in \mathscr{C}$, then the functor $p_{2!}$ resctricts to proof-irrelevant elements*

$$p_{2!} : \mathsf{PIrr}(\bar{X} \times Y) \to \mathsf{PIrr}(\bar{X}).$$

Moreover, $p_{2!}$ satisfies the following Beck-Chevalley condition: for any weak product $X \times Z$ and commutative diagram of the form

$$\begin{tikzcd}
X \times Y \arrow[r, "p_2"] \arrow[d, dashed, "\hat{f}"'] & Y \arrow[d, "f"] \\
X \times Z \arrow[r, "p_2"'] & Z
\end{tikzcd}$$

where $p_1 \hat{f} = p_1$, then for all $\alpha \in \mathsf{PIrr}(X \times Z)$ the canonical inequality

$$p_{2!} \hat{f}^* \alpha \le f^* p_{2!} \alpha$$

is an equality. The same holds for $p_{1!}$.

Proof. (i) If α is such that $p_1^*\alpha \wedge \langle 1, n+1\rangle^*\delta_{X_1} \wedge \cdots \wedge \langle n, 2n\rangle^*\delta_{X_n} \leq p_2^*\alpha$ then, applying $(h\times h)^*$ and Equation (4.6) and Proposition 4.5.6, we obtain $p_1^*h^*\alpha \wedge \langle 1, m+1\rangle^*\delta_{Y_1} \wedge \cdots \wedge \langle m, 2m\rangle^*\delta_{Y_m} \leq p_2^*h^*\alpha$.
(ii) follows from a similar computation.
(iii) Let $\langle f_1, f_2\rangle : Z \to \bar{X}\times\bar{X}$ be an arrow induced by the weak universal property of the weak products. An easy computation implies that $\langle f_1, f_2\rangle(\bigwedge_{i=1}^{n}\langle i, n+i\rangle^*\delta_{X_i}) = \top_Z$ and, hence, we have the equality $f_1^*\alpha = \langle f_1, f_2\rangle p_1^*\alpha \wedge \langle f_1, f_2\rangle(\bigwedge_{i=1}^{n}\langle i, n+i\rangle^*\delta_{X_i})$. Since α is proof-irrelevant, we obtain $f_1^*\alpha \leq f_2^*\alpha$. The opposite inequality is obtained similarly considering an arrow $\langle f_2, f_1\rangle$.
(iv) The first part of the statement is obtained as follows. Let $\alpha \in \mathsf{PIrr}(\bar{X}\times Y)$, Theorem 4.5.3 implies that $\alpha = \alpha^*$ where α^* is the equivalence class of the right dashed arrow of the following weak limit

$$\begin{tikzcd}
& A^\star \arrow[dl, dashed] \arrow[dr, dashed, "\alpha^*"] & \\
A \arrow[d, "\alpha_1"'] \arrow[drr, "\alpha_{n+1}" near start] & & \bar{X}\times Y \arrow[d, "p_{n+1}"] \arrow[dll, "p_1"' near start] \\
X_1 & \cdots & Y.
\end{tikzcd} \tag{4.8}$$

We now consider $p_{1!}\alpha : A \to \bar{X}$ and prove the statement showing that $p_{1!}\alpha = \beta$, where β is the equivalence class of the right dashed arrow of the following weak limit

$$\begin{tikzcd}
& A' \arrow[dl, dashed, "h"'] \arrow[dr, dashed, "\alpha^*"] & \\
A \arrow[d, "\alpha_1"'] \arrow[drr, "\alpha_{n}" near start] & & \bar{X} \arrow[d, "p_{n}"] \arrow[dll, "p_1"' near start] \\
X_1 & \cdots & X_n.
\end{tikzcd}$$

The inequality $p_{1!}\alpha \leq \beta$ is trivial. In order to show the converse, we consider the arrows $\beta_{n+1} : \alpha_{n+1}\circ h$ and $\langle \beta, \beta_{n+1}\rangle : A' \to \bar{X}\times Y$. The arrows h, $\langle \beta, \beta_{n+1}$ and the universal property of the weak limit in 4.8 implies that $\langle \beta, \beta_{n+1} \leq \alpha$. Hence, we obtain $\beta = p_1 \circ \langle \beta, \beta_{n+1} \leq \alpha^* = \alpha$. We now prove the second part of the statement.
The element $f^*p_{2!}\alpha$ can be obtained through the composition of two weak pullbacks as follows

$$\begin{tikzcd}
\tilde{A} \arrow[r, "x"] \arrow[d, "s"'] \arrow[rrr, bend left, "f^*p_{2!}\alpha"] \arrow[dr, phantom, "\lrcorner" very near start] & P \arrow[rr, "p'_2"] \arrow[dd, "f'"'] \arrow[dr, "\langle f'p_1, p'_2\rangle"] \arrow[dr, phantom, "\lrcorner" very near start] & & Y \arrow[dd, "f"] \\
& & X\times Y \arrow[ur, "p_2"'] \arrow[dl, "\hat{f}"'] & \\
A \arrow[r, "\alpha"'] & X\times Z \arrow[rr, "p_2"'] & & Z.
\end{tikzcd}$$

Instead, the element $\bar{\alpha} := \hat{f}^*\alpha$ is given by the weak pullback

$$\begin{tikzcd}
\bar{A} \arrow[r, "\bar{\alpha}"] \arrow[d] & X\times Y \arrow[d, "\langle 1, f\rangle"] \\
A \arrow[r, "\alpha"'] & X\times Z.
\end{tikzcd}$$

By Theorem 4.5.3, α is the same equivalence class of the arrow α^* obtained through the following weak limit

$$\begin{tikzcd}
& A^\star \arrow[dl, dashed] \arrow[dr, dashed, "\alpha^*"] & \\
A \arrow[d, "p_1\circ\alpha"'] \arrow[drr, "p_2\circ\alpha" near start] & & X\times Z \arrow[d, "p_2"] \arrow[dll, "p_1" near start] \\
X & & Z.
\end{tikzcd}$$

Hence, we can obtain an arrow $h : \tilde{A} \to A^*$ such that $\alpha^* \circ h = \hat{f}\langle f'p_1, p_2'\rangle \circ x$. The arrow h can be used to obtain an arrow $k : \tilde{A} \to \bar{A}$ such that $\bar{\alpha} \circ k = \langle p_1 f', p'_2\rangle \circ x$. Similarly, $\bar{\alpha}$ is the same equivalence class of the arrow $\bar{\alpha}^*$ obtained through the following weak limit

$$\begin{tikzcd}
& \bar{A}^\star \arrow[dl, dashed] \arrow[dr, dashed, "\bar{\alpha}^*"] & \\
\bar{A} \arrow[d, "p_1\circ\bar{\alpha}"'] \arrow[drr, "p_2\circ\bar{\alpha}" near start] & & X\times Y \arrow[d, "p_2"] \arrow[dll, "p_1" near start] \\
X & & Y.
\end{tikzcd}$$

Hence, we obtain the inequality $\langle p_1 f', p'_2\rangle \circ x \leq \bar{\alpha}$. The post-composition with the projection p_2 implies the statement $f^* p_{2!}\alpha \leq p_{2!}\hat{f}^*\alpha$. $\square$

We now provide the interpretation of conditions (F1)-(F5). We will denote a weak product $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$ by $\mathcal{M}(\bar{S})$. Given a context $\bar{x} = (x_1, \ldots, x_m)$, with $x_i : S_i$ for $i = 1, \ldots, m$, a formula φ, in which occur the terms $t_1 : Z_1, \ldots, t_n : Z_n$, such that $FV(t_1, \ldots, t_n) \subseteq FV(\varphi) \subseteq \bar{x} = (x_1, \ldots, x_m)$, is interpreted inductively as follows:

(F1) If $R \subseteq Z_1 \times \cdots \times Z_n$ is a relation symbol, then the formula $R(t_1, \ldots, t_n)$ is interpreted as

$$\mathcal{M}_{\bar{x}}(R(t_1, \ldots, t_n)) := \mathcal{M}_{\bar{x}}(\bar{t})^* \mathcal{M}(R),$$

where $\mathcal{M}_{\bar{x}}(\bar{t}) := \langle \mathcal{M}_{\bar{x}}(t_1), \ldots, \mathcal{M}_{\bar{x}}(t_n)\rangle$ is an arrow induced by the weak universal property of weak products.

(F2) If t_1, t_2 are terms of the same sort Z, then $t_1 \approx_S t_2$ is interpreted as the equivalence class of a weak equalizer of the arrows

$$\begin{tikzcd}
\mathcal{M}(\bar{S}) \arrow[r, shift left, "\mathcal{M}_{\bar{x}}(t_1)"] \arrow[r, shift right, "\mathcal{M}_{\bar{x}}(t_2)"'] & \mathcal{M}(Z).
\end{tikzcd}$$

(F3) $\mathcal{M}_{\bar{x}}(\top) = \lfloor 1_{\mathcal{M}(\bar{S})} \rceil$.

(F4) If φ and ψ are formulae with $FV(\varphi, \psi) \subseteq \bar{x}$, then $\varphi \wedge \psi$ is interpreted as the equivalence class of the common value of the composite of the following weak pullback

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & \mathcal{M}(\varphi) \arrow[d, "\mathcal{M}_{\bar{x}}\varphi"] \\
\mathcal{M}(\psi) \arrow[r, "\mathcal{M}_{\bar{x}}\psi"'] & \mathcal{M}(\bar{S}).
\end{tikzcd}$$

(F5) If φ is a formula with $FV(\varphi) \subseteq (y, x_1, \dots, x_n)$ with $y : S$, then $\mathcal{M}_{\bar{x}}\exists y\varphi := \pi_!(\mathcal{M}_{y,\bar{x}}\phi)$, where π is the projection $\mathcal{M}(S) \times \mathcal{M}(\bar{S}) \to \mathcal{M}(\bar{S})$.

We can now collect all the above results in the following theorem.

Theorem 4.5.8 (Soundness of the weak BHK interpretation)**.** *Let $\mathbb{T}$ be a regular theory and $\mathcal{M}$ an interpretation in a weakly left exact category $\mathscr{C}$. If $\bar{x} = (x_1, \dots, x_n)$ are variable of sort $x_i : S_i$, for $1 \leq i \leq n$, and a sequent $\varphi \vdash_{\bar{x}} \psi$ is derivable, then $\mathcal{M}_{\bar{x}}(\varphi) \leq \mathcal{M}_{\bar{x}}(\psi)$ in $\mathsf{PIrr}(\mathcal{M}(\bar{S}))$ for every weak product $\mathcal{M}(\bar{S})$ of the objects $\mathcal{M}(S_i)$, for $1 \leq i \leq n$.*

Proof. The proof is the same of Theorem 4.2.5 and Theorem 4.3.2. We just mention that, in order to soundly interpret the substitution law (S4) for existential formulae, we have proved a Beck-Chevalley condition in item (iv) of Proposition 4.5.7. Similarly, from items (i-iii) of Proposition 4.5.7 it follows that proof-irrelevant elements remain proof-irrelevant after suitable reindexings. Hence, the interpretation validates axiom (E5) of equality predicates. $\square$

We end this section observing that the weak BHK interpretation is also complete. Indeed, the syntactic category $\mathcal{C}(\mathbb{T})$ is wlex and the weak BHK interpretation coincides with the BHK interpretation in qlex categories. This happens because, in case of strict finite products, every arrow $f : X_1 \times \dots \times X_n \to Y$ is determined by projections. Moreover, as observed in eq. (4.5), proof-relevant and proof-irrelevant equalities coincide and the elements of $\mathsf{Psub}_{\mathscr{C}}(X_1 \times \dots \times X_n)$ are all proof-irrelevant. Hence, we obtain the following completeness result.

Proposition 4.5.9 (Completeness of the BHK interpretation)**.** *There is a canonical weak BHK interpretation in the regular category $\mathcal{C}(\mathbb{T})$ which coincides with the canonical BHK interpretation $\mathcal{U}'$. This is a complete model of $\mathbb{T}$. In particular, the syntactic calculus given above is complete with respect to interpretations in (small) weakly left exact categories.* $\square$

4.6 An example from type theory

In this section we give an explicit example of the weak BHK-interpretation in a weakly lex category. In order to do that, we first consider the main example of BHK-interpretation in categories with strict products and weak pullbacks. Our main example of weak BHK interpretation is derived from it.

Consider the syntactic category **ML** arising from intensional Martin-Löf intuitionistic type theory, introduced in Chapter 1. The objects of **ML** are closed types and the arrows are equivalence classes $\lfloor t \rceil : A \to B$ of terms $x : A \vdash t(x) : B$ up to functional extensionality. In Lemma 1.1.2 we proved that **ML** is quasi left exact and a weak pullback of two arrows $\lfloor t \rceil : X \to A$ and $\lfloor u \rceil : Y \to A$ is given by the following commutative diagram

$$\begin{tikzcd} \sum\limits_{x:X,y:Y} \mathsf{Id}_A(t(x), u(y)) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "u"] \\ X \arrow[r, "t"'] & A. \end{tikzcd}$$

The BHK interpretation in the category **ML** recovers the Curry-Howard correspondence: *proposition as types*. Indeed, assuming the notation of Example 1.2.8, if A is a closed type then define

$F^{ML}(A)$ as the poset of equivalence classes of types depending on A respect to *equiprovability*: $x : A \vdash B(x)$ and $x : A \vdash B'(x)$ are in the same equivalence class if there exists a term of

$$\prod_{x:A}(B(x) \to B'(x)) \wedge (B'(x) \to B(x)), \tag{4.9}$$

and $\lfloor B \rceil \leq \lfloor B' \rceil$ if there exists a term $x : A, p : B \vdash q : B'$. The Σ-type provides the following correspondence between weak subobjects and dependent types.

Lemma 4.6.1. *If A is a closed type then $\mathsf{Psub}_{\mathbf{ML}}(A) \cong F^{ML}(A)$.*

Proof. The correspondence sends a weak subobject $\lfloor f : X \to A \rceil$ to the equivalence class of the dependent type $a : A \vdash \sum_{x:X} \mathsf{Id}_A(f(x), a)$. Vice versa, every dependent type $a : A \vdash B(a)$ is sent to the equivalence class of the projection $\pi : \sum_{a:A} B(a) \to A$. A trivial computation shows that these correspondences are inverse. $\square$

The above correspondence was already observed in Example 1.3.8 where we recalled the equivalence the elementary doctrine F^{ML} and $\mathsf{Psub}_{\mathbf{ML}}$. Hence, a relation symbol $R \subseteq S_1 \times \cdots \times S_n$, and more in general a formula φ such that $FV(\varphi) \subseteq \bar{x} = (x_1, \ldots, x_m)$, is interpreted as the equivalence class of a dependent type expression

$$x_1 : \mathcal{M}(S_1), \ldots, x_n : \mathcal{M}(S_n) \vdash \mathcal{M}(R)$$

and the logical connectives and quantifiers are interpreted as in the Curry-Howard correspondence, which is summarized at the end of Appendix B.

Our motivational example to interpret intuitionistic logic in wlex categories is given by the slices $\mathbf{ML}/A$ of the category $\mathbf{ML}$ over a type $A \in \mathbf{ML}$. As we mentioned, since $\mathbf{ML}$ is quasi left exact, it follows that the slices $\mathbf{ML}/A$ are weakly left exact and the weak products are given by the weak pullbacks of $\mathbf{ML}$. If $f : X \to A$ is an object of $\mathbf{ML}/A$, then

$$\mathsf{Psub}_{\mathbf{ML}/A}(f) = \mathsf{Psub}_{\mathbf{ML}}(X), \tag{4.10}$$

which is in turn equal to $F^{ML}(X)$. Hence, in $\mathbf{ML}/A$ a sort S is interpreted as an arrow $s : X \to A$ and the terms and the formulae are interpreted as follows.

Equality symbols. If $s : X \to A$ is the interpretation of a sort S, we consider a weak pullback of s with itself. A canonical choice is given by the following diagram

$$\begin{tikzcd} D := \sum_{x_1,x_2:X} \mathsf{Id}_A(sx_1, sx_2) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "s"] \\ X \arrow[r, "s"'] & A. \end{tikzcd} \tag{4.11}$$

Since the equalizers in the slices of a category are computed as the equalizers in that category, the equality relation symbol $\approx_S$ is interpreted as the equivalence class $\lfloor \pi \rceil$ of the weak equalizer

$$\sum_{d:D} \mathsf{Id}_X(\pi_1 d, \pi_2 d) \xrightarrow{\pi} D \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X.$$

Then Lemma 4.6.1 implies that the equality relation symbol $\approx_S$ corresponds to the dependent type

$$(x_1, x_2, p) : D \vdash \mathsf{Id}_X(x_1, x_2). \tag{4.12}$$

Similarly, if $s_i : X_i \to A$ are the interpretations of the sorts S_i, for $1 \leq i \leq n$, a canonical choice of weak product is the arrow $\bar{s} := s_i \circ \pi_i : W \to A$, for $1 \leq i \leq n$, whose domain is given by

$$W := \sum_{x_1:X_1,\ldots,x_n:X_n} \mathsf{Id}_A(s_1(x_1), s_2(x_2)) \times \cdots \times \mathsf{Id}_A(s_{n-1}(x_{n-1}), s_n(x_n)). \tag{4.13}$$

As above, a canonical choice of weak product of $\bar{s}$ with itself is given by the following diagram

$$\begin{tikzcd}
D := \sum_{x,y:W} \mathsf{Id}_A(\bar{s}x, \bar{s}y) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & W \arrow[d, "\bar{s}"] \\
W \arrow[r, "\bar{s}"'] & A.
\end{tikzcd}$$

We can now show the main difference between the proof-relevant and the proof-irrelevant (or component-wise) equalities. The former is given by the equalizer of the projections π_1 and π_2. Hence, it corresponds to the dependent type

$$(x, y, p) : D \vdash \mathsf{Id}_W(x, y). \tag{4.14}$$

Instead, the component-wise equality is given by the conjunction

$$(x, y, p) : D \vdash \mathsf{Id}_{X_1}(x_1, y_1) \times \cdots \times \mathsf{Id}_{X_n}(x_n, y_n), \tag{4.15}$$

where we denoted for short $x_i := \pi_i x : X_i$ and $y_i := \pi_i y : X_i$, for $1 \leq i \leq n$. We called the above dependent type the proof-irrelevant or component-wise equality of W since two elements $x, y : W$ are equal if their components $x_i, y_i : X_i$, for $1 \leq i \leq n$ are equal; independently on the proof terms

$$\begin{aligned} x_{n+1} &: \mathsf{Id}_A(s_1(x_1), s_2(x_2)) \times \cdots \times \mathsf{Id}_A(s_{n-1}(x_{n-1}), s_n(x_n)) \\ y_{n+1} &: \mathsf{Id}_A(s_1(y_1), s_2(y_2)) \times \cdots \times \mathsf{Id}_A(s_{n-1}(y_{n-1}), s_n(y_n)). \end{aligned} \tag{4.16}$$

Obviously, if the type in (4.14) is inhabited, then also the type in (4.15) is inhabited. The converse does not necessarily hold.

Formulae. The interpretation of a relation symbol $R \subseteq S_1 \times \cdots \times S_n$, and more in general of a formula φ such that $FV(\varphi) \subseteq \bar{x} = (x_1, \ldots, x_n)$ with $x_i : S_i$, is provided by elements which we called proof-irrelevant for the current example.

Indeed, if $s_i : X_i \to A$ are the interpretations of the sorts S_i, for $1 \leq i \leq n$, and W is the domain of the canonical choice of the weak product of $s_1, \ldots, s_n$ as in (4.13), then the interpretation of R is given by a dependent type expression

$$x : W \vdash \mathcal{M}(R)(x),$$

which is proof-irrelevant in the following sense. By definition, $\mathcal{M}(R)$ is proof-irrelevant if

$$p_1^*(\alpha) \wedge \langle 1, n+1\rangle^* \lfloor \delta_{S_1} \rceil \wedge \cdots \wedge \langle n, 2n\rangle^* \lfloor \delta_{S_n} \rceil \leq p_2{}^*(\alpha)$$

and since the proof-irrelevant equality of two elements $x, y : W$ is interpreted as the conjunction $\mathsf{Id}_{X_1}(x_1, y_1) \times \cdots \times \mathsf{Id}_{X_n}(x_n, y_n)$ as in (4.15), we obtain that $\mathcal{M}(R)$ is proof-irrelevant if when $\mathcal{M}(R)(x)$ is inhabited it follows that $\mathcal{M}(R)(y)$ is inhabited for every element $y : W$ with the same components (i.e. such that the types $\mathsf{Id}_{X_i}(x_i, y_i)$, for $1 \leq i \leq n$, are inhabited) independently on the proof terms x_{n+1}, y_{n+1} of (4.16).

In case of relation symbol $R \subseteq S \times S$ it is worthwhile to see how it works the correspondence in (4.2). Given an interpretation $s : X \to A$ of S and a weak product of s with itself as in (4.11), we first interpret R as cone $r_1, r_2 : R \to X$ and then we take the following weak limit

$$\begin{tikzcd}
 & \sum_{d:D}\sum_{z:R} \mathsf{Id}_X(r_1 z, \pi_1 d) \times \mathsf{Id}_X(r_2 z, \pi_2 d) \arrow[dl] \arrow[dr, "\pi"] & \\
R \arrow[d, "r_1"'] \arrow[drr, "r_2" near start] & & D \arrow[dll, "\pi_1"' near start] \arrow[d, "\pi_2"] \\
X & & X.
\end{tikzcd}$$

From the correspondence of Lemma 4.6.1, it follows that the predicate R is interpreted as the dependent type

$$(x, y, p) : D \vdash \sum_{z:R} \mathsf{Id}_X(r_1 z, x) \times \mathsf{Id}_X(r_2 z, y)$$

which clearly does not depend on the proof-term p.

Terms. We conclude this section describing the interpretation of a term $t : S_1 \times \cdots S_n \to S_{n+1}$ as an arrow determined by projections. As above, let $s_i : X_i \to A$ be the interpretations of the sorts S_i, for $1 \leq i \leq n+1$, and let W be the domain of the canonical choice of the weak product of $s_1, \ldots, s_n$ as in (4.13). The term t is interpreted as an arrow $\mathcal{M}(t) : W \to X_{n+1}$ such that $s_i \circ \pi_i = \mathcal{M}(t) \circ s_{n+1}$, for $1 \leq i \leq n+1$, which is determined by projections in the following sense. Proposition 4.5.6 implies $\mathcal{M}(t)$ is dbp if and only if

$$\langle 1, n+1 \rangle^* \delta_{S_1} \wedge \cdots \wedge \langle n, 2n \rangle^* \delta_{S_n} \leq (\mathcal{M}(t) \times \mathcal{M}(t))^* \delta_{S_{n+1}}$$

hence, by (4.12) and (4.15), the above inequality means that if two elements $x, y : W$ have the same components $\mathsf{Id}_{X_i}(x_i, y_i)$, for $1 \leq i \leq n$, then they have the same image through $\mathcal{M}(t)$, i.e. the type

$$(x, y, p) : D \vdash \mathsf{Id}_{X_{n+1}}(\mathcal{M}(t)(x), \mathcal{M}(t)(y))$$

is inhabited. Hence, $\mathcal{M}(t)$ is determined by projections in the sense that the value $\mathcal{M}(t)(x)$, of an element $x : W$, is determined only by the projections $x_i = \pi_i(x)$, for $1 \leq i \leq n$, independently on the proof term $x_{n+1} : \mathsf{Id}_A(s_1(x_1), s_2(x_2)) \times \cdots \times \mathsf{Id}_A(s_{n-1}(x_{n-1}), s_n(x_n))$.

As observed by Palmgren in [Pal04], the category **ML** is suitable to BHK interpret not only regular logic but also coherent and first order logic. This is due to the fact that in **ML** we can interpret the disjunction with the sum types, and the universal quantification with the Π-type. As we will see in the next section, any slice $\mathbf{ML}/A$ is suitable for a weak BHK interpretation of first order logic.

4.7 Richer logics

In this section we recall the BHK intepretation of coherent logic in general qlex categories as developed in [Pal04] and extend it to wlex categories.

Coherent logic. For a qlex category $\mathscr{C}$, in order to interpret the disjunction of formulae, it is required $\mathscr{C}$ to have finite coproducts, i.e. binary coproducts and an initial object 0. The unique arrow from 0 to an object $X \in \mathscr{C}$ will be denoted by 0_x. If $\lfloor f : X \to Z \rceil$ and $\lfloor g : Y \to Z \rceil$ are elements of $\mathsf{PSub}_{\mathscr{C}}(Z)$ then we define

$$\lfloor f \rceil \vee \lfloor g \rceil := \lfloor [f, g] \rceil,$$

where $[f, g]$ is the unique map induced by the universal property of $X + Y$. Moreover, coproducts must satisfy the *weak stability* condition: if $h : V \to Z$ is an arrow of $\mathscr{C}$ and $\lfloor f \rceil, \lfloor g \rceil \in \mathsf{PSub}_{\mathscr{C}}(Z)$, then

(i) $h^*(\lfloor f \rceil \vee \lfloor g \rceil) = h^* \lfloor f \rceil \vee h^*(\lfloor g \rceil)$,

(ii) $h^* \lfloor 0_z \rceil = \lfloor 0_V \rceil$.

From (i) it easily follows that disjunction and conjunction are distributive over each other.

Coherent formulae are interpreted as in Section 4.3 and through the assignments: given the variables $\bar{x} := (x_1, \dots, x_n)$ of sort $x_i : S_i$, for $1 \le i \le n$

(F6) The false predicate $\perp$ in the context $\bar{x}$ is interpreted as

$$\mathcal{M}_{\bar{x}}(\perp) := 0_{\mathcal{M}(\bar{S})},$$

(F7) if φ and ψ are formulae such that $FV(\varphi, \psi) \subseteq \bar{x}$, are interpreted as $\mathcal{M}_{\bar{x}}(\varphi), \mathcal{M}_{\bar{x}}(\psi) \in \mathsf{Psub}_{\mathscr{C}}(\mathcal{M}(\bar{S}))$, then the formula $\varphi \vee \psi$ is interpreted as

$$\mathcal{M}_{\bar{x}}(\varphi \vee \psi) := \mathcal{M}_{\bar{x}}(\varphi) \vee \mathcal{M}_{\bar{x}}(\psi).$$

The above conditions i, ii makes the interpretation satisfy rules (L3) and (L4). Hence, Theorem 4.3.2, about soundness of the BHK interpretation, can be restated for coherent logic in qlex categories with weakly stables coproducts. Since the syntactic category $\mathcal{C}(\mathbb{T})$ of a coherent theory $\mathbb{T}$ is a coherent category (see [Joh02, §D]) and Theorem 4.3.3 holds also for coherent formulae (see [Pal04, Lemma 5.3]), it follows that Proposition 4.3.4 about completeness extends to coherent logic.

First order logic. In order to interpret first order logic in a qlex category $\mathscr{C}$, we must add to the above assumptions the following condition:

- For every object $X \in \mathscr{C}$ and element $\alpha \in \mathsf{PSub}_{\mathscr{C}}(X)$ there exists a functor $\alpha \Rightarrow (-)$: $\mathsf{PSub}_{\mathscr{C}}(X) \to \mathsf{PSub}_{\mathscr{C}}(X)$ which is right adjoint to the functor $\alpha \wedge (-)$, i.e.

$$\alpha \wedge \beta \le \gamma \longleftrightarrow \beta \le (\alpha \Rightarrow \gamma) \tag{4.17}$$

- for every pair of objects $X, Y \in \mathscr{C}$ there exists an order preserving functor $p_{1*} : \mathsf{PSub}_{\mathscr{C}}(X \times Y) \to \mathsf{PSub}_{\mathscr{C}}(X)$ which is right adjoint to the reindexing $p_1{}^*$, i.e.

$$p_1{}^*(\alpha) \le \beta \longleftrightarrow \alpha \le p_{1*}(\beta) \tag{4.18}$$

  for all $\alpha \in \mathsf{Psub}_{\mathscr{C}}(X)$ and $\beta \in \mathsf{Psub}_{\mathscr{C}}(X \times Y)$. The same is assumed for the projection $p_2 : X \times Y \to Y$.

A category $\mathscr{C}$ which satisfies the above conditions is said to have implications and universals.

The interpretation of the first order logic formulae is the same of Section 4.3 plus the following assignments: given the variables $\bar{x} := (x_1, \dots, x_n)$ of sort $x_i : S_i$, for $1 \le i \le n$

(F8) if φ and ψ are formulae such that $FV(\varphi,\psi) \subseteq \bar{x}$, are interpreted as $\mathcal{M}_{\bar{x}}(\varphi), \mathcal{M}_{\bar{x}}(\psi) \in \mathsf{Psub}_{\mathscr{C}}(\mathcal{M}(\bar{S}))$, then the formula $\varphi \implies \psi$ is interpreted as

$$\mathcal{M}_{\bar{x}}(\varphi) \Rightarrow \mathcal{M}_{\bar{x}}(\psi) \in \mathsf{Psub}_{\mathscr{C}}(\mathcal{M}(\bar{S})),$$

(F9) if $y : S$ and φ is a formula such that $FV(\varphi) \subseteq \bar{x}, y$ which is interpreted as an element $\mathcal{M}_{\bar{x},y}(\varphi) \in \mathsf{PSub}_{\mathscr{C}}(\mathcal{M}(\bar{S}) \times \mathcal{M}(S))$, then $\forall y \varphi$ is interpreted as

$$p_* \mathcal{M}_{\bar{x},y}(\varphi) \in \mathsf{PSub}_{\mathscr{C}}(\mathcal{M}(\bar{S})),$$

where $p : (\mathcal{M}(\bar{S}) \times \mathcal{M}(S)) \to \mathcal{M}(\bar{S})$ is the obvious projection,

(F10) if φ is a formula, the negation $\neg\varphi$ is interpreted through the identification $\neg\varphi \equiv (\varphi \implies \bot)$.

The existence of left adjoints $f_! \dashv f^*$ which satisfy the Beck-Chevalley condition easily implies that the above adjunctions satisfy the following properties.

Proposition 4.7.1. *If $\mathscr{C}$ is a quasi left exact category with implications and universal, then:*

(i) *For every arrow $f : Y \to X$ and elements $\alpha, \beta \in \mathsf{PSub}_{\mathscr{C}}(X)$*

$$f^*(\alpha \Rightarrow \beta) = f^*(\alpha) \Rightarrow f^*(\beta) \tag{4.19}$$

(ii) *the right adjoints $\forall_{p_2}$ satisfy the Beck-Chevalley condition: given a weak pullback diagram of the form*

$$\begin{tikzcd} X \times Y \arrow[r, "p_2"] \arrow[d, dashed, "\hat{f}"'] & Y \arrow[d, "f"] \\ X \times Z \arrow[r, "p_2"'] & Z \end{tikzcd}$$

such that $p_1 f = p_1$, then for every $\alpha \in \mathsf{PSub}_{\mathscr{C}}(X)$, the canonical inequality

$$f^* p_{2*}(\alpha) \leq p_{2*} \hat{f}^*(\alpha) \tag{4.20}$$

is an equality. The same holds for $\forall_{p_1}$.

□

The above properties are crucial to satisfy the substitution rule in case of formulae with $\forall$ and $\implies$. The logical rules (L5) and (L6) follow from conditions (4.17) and(4.18). Hence, the soundness result Theorem 4.3.2 can be restated for first order logic in qlex categories with weakly stable coproducts and right adjoint to all reindexings f^*.

In case of wlex categories, the above constructions can be trivially restated. Indeed, the definitions of weakly stable coproducts do not depend on the weakness of products as well as the assumption of having right adjoint to all the reindexings and conjunctions. Proposition 4.7.1 holds also for wlex categories and the only properties that must be verified are the preservation of proof-irrelevant elements through the disjunction, implication and universal quantification.

Proposition 4.7.2. *Let $\mathscr{C}$ be a weakly left exact category, if $\bar{X} := X_1 \times \cdots \times X_n$ is a weak product of the objects $X_1, \ldots, X_n \in \mathscr{C}$ then:*

(i) *If $\alpha, \beta \in \mathsf{PIrr}_{\mathscr{C}}(\bar{X})$, then $\alpha \vee \beta \in \mathsf{PIrr}_{\mathscr{C}}(\bar{X})$*

(ii) *if* $\alpha, \beta \in \mathsf{PIrr}_{\mathscr{C}}(\bar{X})$, *then* $\alpha \Rightarrow \beta \in \mathsf{PIrr}_{\mathscr{C}}(\bar{X})$

(iii) *If* $\bar{X} \times Y$ *is a weak product of* $\bar{X}$ *with an object* $Y \in \mathscr{C}$ *and* $\alpha \in \mathsf{PIrr}_{\mathscr{C}}(\bar{X} \times Y)$, *then* $p_{1*}\alpha \in \mathsf{PIrr}_{\mathscr{C}}(\bar{X})$, *where* $p_1 : \bar{X} \times Y \to X$ *is the obvious projection.*

Proof. (i) follows from the weak stability of coproducts and from the distributivity of disjunction over conjunction. (ii) follows from Proposition 4.7.1-i, (4.17) and from the symmetry of δ_{X_i}, for $1 \leq i \leq n$. (iii) follows from Proposition 4.7.1-ii and (4.18). □

The above proposition implies that the weak BHK interpretation can be extended to the formulae with $\vee, \Rightarrow$ and $\forall$. Proposition 4.7.1 for wlex categories implies that the interpretation satisfies the logical rules (L5) and (L6). Hence, we can collect all the above results and obtain the following generalization of Theorem 4.5.8.

Theorem 4.7.3 (Soundness of the weak BHK interpretation for FOL)**.** *Let* $\mathbb{T}$ *be a first order theory and* $\mathcal{M}$ *an interpretation in a weakly left exact category* $\mathscr{C}$ *with weakly stable coproducts and implications and universal. If* $\bar{x} = (x_1, \dots, x_n)$ *are variable of sort* $x_i : S_i$, *for* $1 \leq i \leq n$, *and a sequent* $\varphi \vdash_{\bar{x}} \psi$ *is derivable, then* $\mathcal{M}_{\bar{x}}(\varphi) \leq \mathcal{M}_{\bar{x}}(\psi)$ *in* $\mathsf{PIrr}(\mathcal{M}(\bar{S}))$ *for every weak product* $\mathcal{M}(\bar{S})$ *of the objects* $\mathcal{M}(S_i)$, *for* $1 \leq i \leq n$. □

Unfortunately, we cannot generalize easily Proposition 4.5.9. Indeed, it is well known that the syntactic category $\mathcal{C}(\mathbb{T})$, of a first order theory $\mathbb{T}$, is a Heyting category, but this does not implies the existence of the adjunctions in (4.17) and (4.18). A formulation of a completeness result in style of Proposition 4.5.9 is still under investigation both for the BHK interpretation and for the weak BHK interpretation.

Example 4.7.4. As already observed in [Pal04], the category **ML** is suitable for a BHK interpretation of first order logic. We actually have proved it in different parts of this thesis. In Section 2.6 we have seen that **ML** has weakly stable coproducts given by the sum types and in Example 1.2.20 we have seen that **ML** has all right adjoints to reindexings. Hence, in **ML** we can interpret disjunctions, implications and universal quantifications. Similarly, it follows that any slice category **ML**$/A$ is suitable for a weak BHK interpretation of first order logic. Indeed, **ML**$/A$ has weakly stable coproducts because **ML** has them, and, as follows from Example 3.6.7, **ML**$/A$ is suitable to interpret implication and universal quantifications.

Concluding remarks and further developments. We presented a generalization of the BHK interpretation in categories with weak pullbacks and weak products. In order to interpret disjunction, we assumed weakly stable coproducts, which are strict. It is still open the question if it is possible to consider weak coproducts with a notion of weak stability and it will be part of future investigations.

Appendix A

Categorical results

We begin this chapter recalling some definitions from [CV98].

Definition A.0.1. A category $\mathscr{C}$ is called

- *Left exact* (*lex*), if it has (strict) finite products and pullbacks. Equivalently, if it has al finite limits.

- *Quasi left exact* (*qlex*), if it has (strict) finite products and weak pullbacks.

- *Weakly left exact* (*wlex*), if it has weak finite products and weak pullbacks. Equivalently, if it has weak finite limits, see [CV98, Proposition 1].

Definition A.0.2. Let $\mathscr{C}$ be a category, a *pseudo-equivalence relation* on an object $Y \in \mathscr{C}$ is a pair of parallel arrows $r_1, r_2 : X \to Y$ that is

- reflexive, if there exists an arrow $r_X : Y \to X$ such that

$$r_1 r_X = 1_Y = r_2 r_X,$$

- symmetric, if there exists an arrow $s_X : X \to X$ such that

$$r_1 s_x = r_2 \qquad r_2 s_x = r_1,$$

- transitive, if there exists a weak pullback

$$\begin{array}{ccc} P & \xrightarrow{l_1} & X \\ {\scriptstyle l_2}\downarrow & & \downarrow{\scriptstyle r_2} \\ X & \xrightarrow[r_1]{} & Y \end{array}$$

and an arrow $t_X : P \to X$ such that

$$r_1 l_1 = r_1 t_X \qquad r_2 l_2 = r_2 t_X.$$

Elementary doctrines. We now recall various results about the elementary doctrines. Some of them are reported with the corresponding reference, others are proved. We start with a proof of the equivalence of the definitions of elementary doctrine mentioned in [MR12, Remark 2.3] that we have discussed in Section 1.2 .

Proposition A.0.3. *Definition 1.2.3 and Definition 1.2.5 are equivalent.*

Before providing a proof, we recall that in Definition 1.2.3, a functor $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an elementary doctrine if for every object $X \in \mathscr{C}$, there exists an element $\delta_X \in \mathsf{P}(X \times X)$ such that:

E1 For every element $\alpha \in \mathsf{P}(X)$, the assignment

$$\exists_{\Delta_X}(\alpha) := \mathsf{P}_{p_1}(\alpha) \wedge_{X \times X} \delta_X$$

is left adjoint of the functor $\mathsf{P}_{\Delta_X} : \mathsf{P}(X \times X) \to \mathsf{P}(X)$.

E2 For every object $Y \in \mathscr{C}$ and arrow $e := \langle 1, 2, 2 \rangle : X \times Y \to X \times Y \times Y$, the assignment

$$\exists_e(\alpha) := \mathsf{P}_{\langle 1,2 \rangle}(\alpha) \wedge_{X \times Y \times Y} \mathsf{P}_{\langle 2,3 \rangle}(\delta_Y)$$

for α in $\mathsf{P}(X \times Y)$ is left adjoint to $\mathsf{P}_e : \mathsf{P}(X \times Y \times Y) \to \mathsf{P}(X \times Y)$.

In Definition 1.2.5, a functor $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an elementary doctrine if for every object $X \in \mathscr{C}$, there exists an element $\delta_X \in \mathsf{P}(X \times X)$ such that:

I $\top_X \leq \mathsf{P}_{\Delta_X}(\delta_X)$.

II $\mathsf{P}(X) = \mathcal{D}es_{\delta_X}$.

III $\delta_X \boxtimes \delta_Y \leq \delta_{X \times Y}$, where $\delta_X \boxtimes \delta_Y := \mathsf{P}_{\langle 1,3 \rangle}\delta_X \wedge \mathsf{P}_{\langle 2,4 \rangle}\delta_Y$.

Proof. We first prove that Definition 1.2.3 implies Definition 1.2.5.

Conditions **I** and **II** are obvious. Applying the isomorphism $\mathsf{P}_{\langle 1,3,2,4 \rangle}$, we obtain that **III** is equivalent to

$$\mathsf{P}_{\langle 1,2 \rangle}\delta_X \wedge \mathsf{P}_{\langle 3,4 \rangle}\delta_Y \leq \mathsf{P}_{\langle 1,3,2,4 \rangle}\delta_{X \times Y}$$

in $\mathsf{P}(X \times X \times Y \times Y)$. This inequality is equal to

$$\mathsf{P}_{\langle 1,2,3 \rangle}\mathsf{P}_{\langle 1,2 \rangle}\delta_X \wedge \mathsf{P}_{\langle 3,4 \rangle}\delta_Y \leq \mathsf{P}_{\langle 1,3,2,4 \rangle}\delta_{X \times Y}.$$

The left term is equal to $\exists_{\langle 1,2,3,3 \rangle}\mathsf{P}_{\langle 1,2 \rangle}\delta_X$ and, by **E2**, the statement is equivalent to

$$\mathsf{P}_{\langle 1,2 \rangle}\delta_X \leq \mathsf{P}_{\langle 1,2,3,3 \rangle}\mathsf{P}_{\langle 1,3,2,4 \rangle}\delta_{X \times Y} = \mathsf{P}_{\langle 1,3,2,3 \rangle}\delta_{X \times Y}$$

in $\mathsf{P}(X \times X \times Y)$. Applying the isomorphism $\mathsf{P}_{3,2,1}$, the statement is equivalent to

$$\mathsf{P}_{\langle 2,3 \rangle}\delta_X \leq \mathsf{P}_{\langle 2,1,3,1 \rangle}\delta_{X \times Y}.$$

The left term is equal to $\exists_{\langle 1,2,2 \rangle}\top_{Y \times X}$ and, by **E2**, the statement is equivalent to

$$\top_{Y \times X} \leq \mathsf{P}_{\langle 2,1,2,1 \rangle}\delta_{X \times Y}$$

which is true by condition **I**.

We now prove that Definition 1.2.5 implies Definition 1.2.3. We first obtain condition **E1** as follows. The part of the adjunction

$$\exists_{\Delta_X}\alpha \leq \beta \implies \alpha \leq \mathsf{P}_{\Delta_X}\beta$$

for every $\alpha \in \mathsf{P}(X)$ and $\beta \in \mathsf{P}(X \times X)$, trivially follows applying P_{Δ_X} to the left inequality. The inverse implication follows observing that by **II** we have

$$\mathsf{P}_{\langle 1,2\rangle}\beta \wedge \delta_{X\times X} \leq \mathsf{P}_{\langle 3,4\rangle}\beta$$

in $\mathsf{P}(X \times X \times X \times X)$. Applying $\mathsf{P}_{\langle 1,2,1,3\rangle}$ to the above inequality and **III** and **I** we obtain

$$\mathsf{P}_{\langle 1,2\rangle}\beta \wedge \mathsf{P}_{\langle 2,3\rangle}\delta_X \leq \mathsf{P}_{\langle 1,3\rangle}\beta$$

in $\mathsf{P}(X \times X \times X)$. Applying $\mathsf{P}_{\langle 1,1,2\rangle}$ we obtain that

$$\mathsf{P}_{\langle 1,1\rangle}\beta \wedge \delta_X \leq \beta$$

in $\mathsf{P}(X \times X)$. Hence, the statement follows since $\alpha \leq \mathsf{P}_{\Delta_X}\beta$.

We now prove condition **E2** in a similar way. The part of the adjunction

$$\exists_{\langle 1,2,2\rangle}\alpha \leq \beta \implies \alpha \leq \mathsf{P}_{\langle 1,2,2\rangle}\beta$$

for every $\alpha \in \mathsf{P}(X \times Y)$ and $\beta \in \mathsf{P}(X \times Y \times Y)$, trivially follows applying $\mathsf{P}_{\langle 1,2,2\rangle}$. The inverse implication follows observing that by **II** we have

$$\mathsf{P}_{\langle 1,2,3\rangle}\beta \wedge \delta_{X\times Y\times Y} \leq \mathsf{P}_{4,5,6}\beta$$

in $\mathsf{P}(X \times Y \times Y \times X \times Y \times Y)$. Applying $\mathsf{P}_{\langle 1,2,3,1,2,4\rangle}$ and and **III** and **I** we obtain

$$\mathsf{P}_{\langle 1,2,3\rangle}\beta \wedge \mathsf{P}_{\langle 3,4\rangle}\delta_Y \leq \mathsf{P}_{\langle 1,2,4\rangle}\beta$$

in $\mathsf{P}(X \times Y \times Y \times Y)$ and applying $\mathsf{P}_{\langle 1,2,2,3\rangle}$ we obtain

$$\mathsf{P}_{\langle 1,2,2\rangle}\beta \wedge \mathsf{P}_{\langle 2,3\rangle}\delta_Y \leq \beta$$

in $\mathsf{P}(X \times Y \times Y)$. Hence, the statement follows since $\alpha \leq \mathsf{P}_{1,2,2}\beta$. □

Lemma A.0.4. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine with full weak comprehension. Assuming that for every $X \in \mathscr{C}$ and $\alpha \in \mathsf{P}(X)$, the reindexings $\mathsf{P}_{\{\!|\alpha|\!\}}$ over the comprehensions $\{\!|\alpha|\!\} : C \to X$ have left adjoints, then for every $\beta \in \mathsf{P}(X)$*

$$\exists_{\{\!|\alpha|\!\}}\mathsf{P}_{\{\!|\alpha|\!\}}\beta = \beta \wedge \alpha. \tag{A.1}$$

Proof. ($\leq$) The adjuction property implies that $\exists_{\{\!|\alpha|\!\}}\mathsf{P}_{\{\!|\alpha|\!\}}\beta \leq \beta$ and that $\exists_{\{\!|\alpha|\!\}}\mathsf{P}_{\{\!|\alpha|\!\}}\beta \leq \alpha$ is equivalent to $\mathsf{P}_{\{\!|\alpha|\!\}}\beta \leq \mathsf{P}_{\{\!|\alpha|\!\}}\alpha$, which is obvious since $\mathsf{P}_{\{\!|\alpha|\!\}}\alpha = \top_C$.

($\geq$) Let $\{\!|\alpha \wedge \beta|\!\} : D \to X$ be the comprehension of $\alpha \wedge \beta$. Since $\alpha \wedge \beta \leq \alpha$ we have that there exists an arrow $h : D \to C$ such that $\{\!|\alpha|\!\} \circ h = \{\!|\alpha \wedge \beta|\!\}$. By fullness of comprehensions, we can equivalently prove that

$$\top_D \leq \mathsf{P}_{\{\!|\alpha\wedge\beta|\!\}}\exists_{\{\!|\alpha|\!\}}\mathsf{P}_{\{\!|\alpha|\!\}}\beta$$

which follows from the following computation:

$$\begin{aligned}
\mathsf{P}_{\{\!|\alpha\wedge\beta|\!\}}\exists_{\{\!|\alpha|\!\}}\mathsf{P}_{\{\!|\alpha|\!\}}\beta &= \mathsf{P}_h\mathsf{P}_{\{\!|\alpha|\!\}}\exists_{\{\!|\alpha|\!\}}\mathsf{P}_{\{\!|\alpha|\!\}}\beta && (\exists_{\{\!|\alpha|\!\}} \dashv \mathsf{P}_{\{\!|\alpha|\!\}})\\
&\geq \mathsf{P}_h\mathsf{P}_{\{\!|\alpha|\!\}}\beta\\
&= \mathsf{P}_{\{\!|\alpha\wedge\beta|\!\}}\beta\\
&= \top_D.
\end{aligned}$$

□

The following appear as [MR16, Lemma 2.9] and relates different notion of comprehensive diagonals.

Lemma A.0.5. *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be an elementary doctrine, the following are equivalent:*

- *every diagonal arrow* $\Delta_X : X \to X \times X$ *is a comprehension.*
- *For every* A *in* $\mathscr{C}$, Δ_A *is a comprehension of* δ_A.
- *For every pair of arrows* $f, g : X \to A$ *in* $\mathscr{C}$, *then* $f = g$ *if and only if*

$$\top_X \leq \mathsf{P}_{\langle f,g\rangle}(\delta_A).$$

□

The following appear as [MR13, Lemma 4.8]

Lemma A.0.6. (*4.8 of* [*MR13*]). *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be an elementary doctrine with full comprehensions and comprehensive diagonals. An arrow* $f : A \to B$ *of* $\mathscr{C}$ *is a monomorphism if and only if*

$$\delta_A = \mathsf{P}_{f\times f}\delta_B.$$

□

We now prove that the base category of suitable elementary doctrines have pullbacks.

Lemma A.0.7. *If* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *is an elementary doctrine with weak* (*strict*) *comprehensions and comprehensive diagonals then* $\mathscr{C}$ *has weak* (*strict*) *pullbacks.*

Proof. Given two arrows $f : X \to A$ and $g : Y \to A$, we can consider a comprehension of the element $\gamma := \mathsf{P}_{f\times g}\delta_A$ and the following diagram

$$\begin{tikzcd}
C \arrow[rr, dashed, "\{\!|\gamma|\!\}_2"] \arrow[dr, "\{\!|\gamma|\!\}"] \arrow[dd, dashed, "\{\!|\gamma|\!\}_1"'] & & Y \arrow[dd, "g"] \\
& X \times Y \arrow[ur, "p_2"] \arrow[dl, "p_1"] & \\
X \arrow[rr, "f"'] & & A
\end{tikzcd}$$

where $\{\!|\gamma|\!\}_i := p_i \circ \{\!|\gamma|\!\}$ for $i = 1, 2$. The diagram commutes thanks to comprehensive diagonals. Indeed,

$$\begin{aligned}
&\mathsf{P}_{\langle (f\{\!|\gamma|\!\}_1),(g\{\!|\gamma|\!\}_2)\rangle}\delta_A \\
&= \mathsf{P}_{\{\!|\gamma|\!\}}\mathsf{P}_{f\times g}\delta_A \\
&= \mathsf{P}_{\{\!|\gamma|\!\}}\gamma \\
&= \top_C
\end{aligned}$$

and hence $f\{\!|\gamma|\!\}_1 = g\{\!|\gamma|\!\}_2$. If $u_1 : U \to X$ and $u_2 : U \to Y$ are two arrow such that $fu_1 = gu_2$ then $\top_U \leq \mathsf{P}_{\langle u_1,u_2\rangle}\gamma$ and there exists an arrow $h : U \to C$ such that $\{\!|\gamma|\!\}h = \langle u_1, u_2\rangle$. If the comprehensions are strict there exists a unique h with such property. □

Hence, the slices $\mathscr{C}/A$ have weak finite products and it is always possible to assume that the domain of a weak product is the domain of a comprehension. In particular, if P has strict comprehensions, then $\mathscr{C}/A$ has strict finite products for all $A \in \mathscr{C}$.

Lemma A.0.8. *Let P be an elementary doctrine with full weak (strict) comprehensions and comprehensive diagonals. Then for every weak pullback diagram*

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & A \end{tikzcd} \tag{A.2}$$

the arrow $\langle p_1, p_2\rangle : P \to X \times Y$ is a full weak (strict) comprehension of $\gamma := \mathsf{P}_{f\times g}\delta_A$.

The following appear as [MR13, Proposition 4.6]

Proposition A.0.9. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine with weak (strict) comprehensions and comprehensive diagonals. For every pair $f, g : X \to A$ of arrows of $\mathscr{C}$, the weak (strict) comprehension of $\mathsf{P}_{\langle f,g\rangle}\delta_A$ is a weak (strict) equalizer of f and g.*

Proof. The argument is the same of Lemma A.0.7 for the comprehension of $\mathsf{P}_{\langle f,g\rangle}\delta_A$. □

Corollary A.0.10. *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an elementary doctrine with full weak comprehensions and comprehensive diagonals, then for every weak pullback diagram as in (A.2) it follows that*

$$\exists_{\langle \pi_1,\pi_2\rangle}\mathsf{P}_{\langle \pi_1,\pi_2\rangle}\beta = \beta \wedge \mathsf{P}_{f\times g}\delta_A$$

for every $\beta \in \mathsf{P}(X \times Y)$.

Proof. It follows from Lemma A.0.8 and Lemma A.0.4. □

Corollary A.0.11. *Let P be an elementary doctrine with full weak comprehensions and comprehensive diagonals. If P has right adjoints to all reindexings, then for every weak pullback diagram as in (A.2) it follows that the functor $\mathsf{P}_{f\times g}\delta_A \times (-) : \mathsf{P}(X \times Y) \to \mathsf{P}(X \times Y)$ has a right adjoint given by*

$$\mathsf{P}_{f\times g}\delta_A \implies (-) := \forall_{\langle \pi_1,\pi_2\rangle}\mathsf{P}_{\langle \pi_1,\pi_2\rangle}(-).$$

Proof. It follows from Remark 1.2.19 and Lemma A.0.8. □

Lemma A.0.12. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an existential elementary doctrine with full weak comprehensions and comprehensive diagonals. Then for every weak pullback diagram*

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & A \end{tikzcd}$$

the left adjoints satisfy the Beck-Chevalley condition.

Proof. If $\alpha \in \mathsf{P}(X)$, then the statement is obtained as follows:

$$\begin{aligned}
\mathsf{P}_g\exists_f\alpha &= \mathsf{P}_g\exists_{p_2}(\mathsf{P}_{p_1}\alpha \wedge \mathsf{P}_{f\times 1_A}\delta_A) && \text{(Remark 1.2.14)}\\
&= \exists_{p_2}\mathsf{P}_{1_A\times g}(\mathsf{P}_{p_1}\alpha \wedge \mathsf{P}_{f\times 1_A}\delta_A) && \text{(B-C)}\\
&= \exists_{p_2}(\mathsf{P}_{p_1}\alpha \wedge \mathsf{P}_{f\times g}\delta_A)\\
&= \exists_{p_2}\exists_{\langle\pi_1,\pi_2\rangle}\mathsf{P}_{\langle\pi_1,\pi_2\rangle}\mathsf{P}_{p_1}\alpha && \text{(Corollary A.0.10)}\\
&= \exists_{\pi_2}\mathsf{P}_{\pi_1}\alpha.
\end{aligned}$$

□

Lemma A.0.13. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an universal and implicational elementary doctrine with full weak comprehensions and comprehensive diagonals. Then for every weak pullback diagram*

$$\begin{tikzcd}
P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & A
\end{tikzcd}$$

the right adjoints satisfy the Beck-Chevalley condition.

Proof. If $\alpha \in \mathsf{P}(X)$, then the statement is obtained as follows:

$$\begin{aligned}
\mathsf{P}_g\forall_f\alpha &= \mathsf{P}_g\forall_{p_2}(\mathsf{P}_{f\times 1_A}\delta_A \implies \mathsf{P}_{p_1}\alpha) && \text{(Remark 1.2.19)}\\
&= \forall_{p_2}\mathsf{P}_{1_A\times g}(\mathsf{P}_{f\times 1_A}\delta_A \implies \mathsf{P}_{p_1}\alpha) && \text{(B-C)}\\
&= \forall_{p_2}(\mathsf{P}_{f\times g}\delta_A \implies \mathsf{P}_{p_1}\alpha)\\
&= \forall_{p_2}\forall_{\langle\pi_1,\pi_2\rangle}\mathsf{P}_{\langle\pi_1,\pi_2\rangle}\mathsf{P}_{p_1}\alpha && \text{(Corollary A.0.11)}\\
&= \forall_{\pi_2}\mathsf{P}_{\pi_1}\alpha.
\end{aligned}$$

□

We now want prove that the Beck-Chevalley condition holds on diagrams which are not weak pullbacks restricting on suitable elements. Before that, we recall the description of comprehensions and pullbacks of the elementary quotient completion of a suitable elementary doctrine.

Lemma A.0.14. (*5.3 of* [*MR13*]) *If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is an elementary doctrine with* (*full*) *weak comprehensions, then the elementary quotient completion $\overline{\mathsf{P}}$ has* (*full*) *strict comprehensions and comprehensive diagonals. In particular, if ρ is a P-eq. relation on the object $X \in \mathscr{C}$ and $c : C \to X$ is a weak comprehension of $\alpha \in \mathscr{D}es_\rho$, then*

$$\lfloor c\rceil : (C, \mathsf{P}_{c\times c}\rho) \to (X,\rho)$$

is a strict comprehension of $\alpha \in \overline{\mathsf{P}}(X,\rho)$.

Corollary A.0.15. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine with* (*full*) *weak comprehensions, and let $\overline{\mathsf{P}}$ be the its elementary quotient completion. If $\lfloor f\rceil : (X,\rho) \to (Y,\sigma)$ and $\lfloor g\rceil : (Z,\zeta) \to (Y,\sigma)$ are two arrows of $\overline{\mathscr{C}}$, then the following diagram*

$$\begin{tikzcd}
(C, \mathsf{P}_{c\times c}\rho \boxtimes \mu) \arrow[rr, dashed, "\lfloor \pi_2\rceil"] \arrow[dr, "\lfloor c\rceil"] \arrow[dd, dashed, "\lfloor \pi_1\rceil"'] & & (Z,\mu) \arrow[dd, "\lfloor g\rceil"] \\
& (X\times Z, \rho\boxtimes\mu) \arrow[ur, "\lfloor p_2\rceil"'] \arrow[dl, "\lfloor p_1\rceil"] & \\
(X,\rho) \arrow[rr, "\lfloor f\rceil"'] & & (Y,\sigma)
\end{tikzcd}$$

where $c := \{\!|\mathsf{P}_{f\times g}\sigma|\!\}$, *is a strict pullback.*

Proof. It follows from Lemma A.0.7 and Lemma A.0.14. □

Lemma A.0.16. *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be an elementary doctrine with full weak comprehensions and comprehensive diagonals. Assume that* P *is existential and that in the following commutative diagram the right and the outer diagrams are weak pullbacks*

$$\begin{tikzcd} Z' \arrow[r, dashed, "h_2"'] \arrow[rr, bend left, "\pi_2'"] \arrow[d, "\pi_1'"'] & Z \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & Y \arrow[d, "g"] \\ X' \arrow[r, dashed, "h_1"] \arrow[rr, bend right, "f'"'] & X \arrow[r, "f"] & A \end{tikzcd}$$

then, for every $\alpha \in \mathsf{P}(X')$

$$\mathsf{P}_{\pi_1}\exists_{h_1}\alpha = \exists_{h_2}\mathsf{P}_{\pi_1'}\alpha$$

and for every $\beta \in \mathcal{D}es_{\mathsf{P}_{\langle\pi_1,\pi_2\rangle\times\langle\pi_1,\pi_2\rangle}\delta_{X\times Y}}$

$$\mathsf{P}_{h_1}\exists_{\pi_1}\beta = \exists_{h_2}\mathsf{P}_{\pi_1'}\beta.$$

Similarly, if P *is universal and implicational, then for every* $\alpha \in \mathsf{P}(X')$

$$\mathsf{P}_{\pi_1}\forall_{h_1}\alpha = \forall_{h_2}\mathsf{P}_{\pi_1'}\alpha$$

and for every $\beta \in \mathcal{D}es_{\mathsf{P}_{\langle\pi_1,\pi_2\rangle\times\langle\pi_1,\pi_2\rangle}\delta_{X\times Y}}$

$$\mathsf{P}_{h_1}\forall_{\pi_1}\beta = \forall_{h_2}\mathsf{P}_{\pi_1'}\beta.$$

Proof. Since P has comprehensive diagonals, in $\overline{\mathscr{C}}$ the right and the outer diagrams below are pullbacks.

$$\begin{tikzcd} (Z', \mathsf{P}_{\langle\pi_1',\pi_2'\rangle\times\langle\pi_1',\pi_2'\rangle}\delta_{X'\times Y}) \arrow[r, dashed, "\lfloor h_2\rceil"'] \arrow[rr, bend left, "\lfloor \pi_2'\rceil"] \arrow[d, "\lfloor \pi_1'\rceil"'] & (Z, \mathsf{P}_{\langle\pi_1,\pi_2\rangle\times\langle\pi_1,\pi_2\rangle}\delta_{X\times Y}) \arrow[r, "\lfloor \pi_2\rceil"'] \arrow[d, "\lfloor \pi_1\rceil"] & (Y, \delta_Y) \arrow[d, "\lfloor g\rceil"] \\ (X', \delta_{X'}) \arrow[r, dashed, "\lfloor h_1\rceil"] \arrow[rr, bend right, "\lfloor f'\rceil"'] & (X, \delta_X) \arrow[r, "\lfloor f\rceil"] & (A, \delta_A) \end{tikzcd}$$

Since $\mathscr{C}$ is regular (Proposition 1.3.6) the pasting law of pullbacks implies that also the left diagram is a pullback. Hence, the statements follow from Lemmas A.0.12 and A.0.13. □

Lemma A.0.17. *Let* $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *be primary doctrine. Assume that* P *has left and right adjoints to all reindexings. For every commutative diagram of the form*

$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & A \end{tikzcd}$$

the left adjoints satisfies the Beck-Chevalley condition if and only if the right adjoints do.

Proof. Suppose that the Beck-Chevalley condition holds for the left adjoints. We want to prove that for every $\alpha \in \mathsf{P}(X)$

$$\mathsf{P}_g \forall_f \alpha = \forall_{f'} \mathsf{P}_{g'} \alpha.$$

The inequality $\leq$ holds for every commutative diagram thanks to the adjunction $\mathsf{P}_{(-)} \dashv \forall_{(-)}$. In order to prove the opposite inequality, we observe the following equivalences which follow from the adjoint conditions

$$\begin{gathered}\forall_{f'} \mathsf{P}_{g'} \alpha \leq \mathsf{P}_g \forall_f \alpha \\ \exists_g \forall_{f'} \mathsf{P}_{g'} \alpha \leq \forall_f \alpha \\ \mathsf{P}_f \exists_g \forall_{f'} \mathsf{P}_{g'} \alpha \leq \alpha.\end{gathered}$$

By assumptions, the last inequality is equal to $\exists_{g'} \mathsf{P}_{f'} \forall_{f'} \mathsf{P}_{g'} \alpha$. The statement follows from the adjunction conditions. The inverse statement is proved similarly. □

Lemma A.0.18. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an elementary doctrine with full weak comprehensions and comprehensive diagonals and let $f : X \to Y$ be an arrow of $\mathscr{C}$. If P is existential then*

$$\mathsf{P}_f \exists_f \alpha = \alpha, \tag{A.3}$$

for every $\alpha \in \mathcal{D}es_{\mathsf{P}_{f\times f}\delta_Y}$. If P is existential and universal then

$$\mathsf{P}_f \forall_f \alpha = \alpha \tag{A.4}$$

for every $\alpha \in \mathcal{D}es_{\mathsf{P}_{f\times f}\delta_Y}$.

Proof. A proof of this fact can be obtained applying the elementary quotient completion to P and observing that the diagram

$$\begin{tikzcd} (X, \mathsf{P}_{f\times f}\delta_Y) \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & (X, \mathsf{P}_{f\times f}\delta_Y) \arrow[d, "\lfloor f \rceil"] \\ (X, \mathsf{P}_{f\times f}\delta_Y) \arrow[r, "\lfloor f \rceil"'] & (Y, \delta_Y) \end{tikzcd}$$

is a pullback diagram in $\overline{\mathscr{C}}$ since $\lfloor f \rceil$ is a monomorphism. Hence the statement follows applying Lemma A.0.12. We now give a direct proof of the statements.

By adjunction, $\alpha \leq \mathsf{P}_f \exists_f \alpha$. The opposite inequality is obtained as follows:

$$\begin{aligned}
\mathsf{P}_f \exists_f(\alpha) &= \mathsf{P}_f \exists_{p_2}(\mathsf{P}_{p_1}\alpha \wedge \mathsf{P}_{f\times 1_Y}\delta_Y) && \text{(Remark 1.2.14)}\\
&= \exists_{p_2} \mathsf{P}_{1_X \times f}(\mathsf{P}_{p_1}\alpha \wedge \mathsf{P}_{f\times 1_Y}\delta_Y) && \text{(B-C)}\\
&= \exists_{p_2}(\mathsf{P}_{p_1}\alpha \wedge \mathsf{P}_{f\times f}\delta_Y)\\
&\leq \exists_{p_2}\mathsf{P}_{p_2}\alpha && (\alpha \in \mathcal{D}es_{\mathsf{P}_{f\times f}\delta_Y})\\
&\leq \alpha.
\end{aligned}$$

We now prove the second part of the statement. By adjunction, $\mathsf{P}_f \forall_f \alpha \leq \alpha$. The oppoite inequality is obtained as follows:

$$\begin{aligned}
\mathsf{P}_f \forall_f \alpha &= \mathsf{P}_f \forall_{p_2}(\mathsf{P}_{f\times 1_X}\delta_Y \implies \mathsf{P}_{p_1}\alpha) && \text{(Remark 1.2.19)}\\
&= \forall_{p_2} \mathsf{P}_{1_Y \times f}(\mathsf{P}_{f\times 1_X}\delta_Y \implies \mathsf{P}_{p_1}\alpha) && \text{(B-C)}\\
&= \forall_{p_2}(\mathsf{P}_{f\times f}\delta_Y \implies \mathsf{P}_{p_1}\alpha)\\
&\geq \forall_{p_2}\mathsf{P}_{p_2}\alpha && (\alpha \in \mathcal{D}es_{\mathsf{P}_{f\times f}\delta_Y})\\
&\geq \alpha.
\end{aligned}$$

□

We now prove the equivalence of the elements in (2.30).

Lemma A.0.19. *Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ be an existential, universal and implicational elementary doctrine with full weak comprehensions and comprehensive diagonals. Let $x : X \to A$ and $y : Y \to A$ two arrows such that $\sigma \leq \mathsf{P}_{y\times y}\delta_A$ and let $y^x : E \to A$ be an extensional exponential of x and y w.r.t. σ. We consider the following weak pullbacks*

$$
\begin{tikzcd}
W \arrow[r, "\{\!|\chi|\!\}_2"] \arrow[d, "\{\!|\chi|\!\}_1"'] & X \arrow[d, "x"] \\
X \arrow[r, "x"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\{\!|\iota|\!\}_2"] \arrow[d, "\{\!|\iota|\!\}_1"'] & Y \arrow[d, "y"] \\
Y \arrow[r, "y"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\{\!|\gamma|\!\}_2"] \arrow[d, "\{\!|\gamma|\!\}_1"'] & E \arrow[d, "y^x"] \\
E \arrow[r, "y^x"'] & A
\end{tikzcd}
$$

with comprehensions $\{\!|\chi|\!\} : W \to X \times X$, $\{\!|\iota|\!\} : V \to Y \times Y$ and $\{\!|\lambda|\!\} : G \to E \times E$ of $\chi := \mathsf{P}_{x\times x}\delta_A$, $\iota := \mathsf{P}_{y\times y}\delta_A$ and $\gamma := \mathsf{P}_{y^x\times y^x}\delta_A$. We will denote by $w : W \to A$, $v : V \to A$ and $g : G \to A$ respectively the common value of the two composites in the left, central and right above diagram. Moreover, we consider the weak pullback

$$
\begin{tikzcd}
U \arrow[r, "\{\!|\mu|\!\}_2"] \arrow[d, "\{\!|\mu|\!\}_1"'] & X \arrow[d, "x"] \\
E \arrow[r, "y^x"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "\{\!|\kappa|\!\}_2"] \arrow[d, "\{\!|\kappa|\!\}_1"'] & U \arrow[d, "u"] \\
U \arrow[r, "u"'] & A
\end{tikzcd}
$$

obtained through the comprehensions $\{\!|\mu|\!\} : U \to E \times X$ and $\{\!|\kappa|\!\} : K \to U \times U$ of $\mu := \mathsf{P}_{y^x\times x}\delta_A$ and $\kappa := \mathsf{P}_{u\times u}\delta_A$, where $u : U \to A$ is the common value of the two composites of the left diagram and $k : K \to A$ is that of the right diagram. Now, given a weak product of y^x, y^x and x

$$
\begin{tikzcd}
 & T \arrow[dl, dashed, "\{\!|\tau|\!\}_1"'] \arrow[d, dashed, "\{\!|\tau|\!\}_2"'] \arrow[dr, dashed, "\{\!|\tau|\!\}_3"] & \\
E \arrow[dr, "y^x"'] & E \arrow[d, "y^x"'] & X \arrow[dl, "x"] \\
 & A &
\end{tikzcd}
$$

obtained through the weak comprehension $\{\!|\tau|\!\} : T \to E \times E \times X$ of $\tau := \mathsf{P}_{y^x\times y^x\times x}(\mathsf{P}_{\langle 1,2\rangle}\delta_A \wedge \mathsf{P}_{\langle 2,3\rangle}\delta_A)$, we will denote by $t : T \to A$ the common value of the three composites of the above diagram. If $u \leftarrow z \rightarrow u$ is a weak product, then we will denote by

$$\langle 1,3,2,3\rangle_A : t \to k \qquad\qquad \langle 1,2\rangle_A : t \to g$$

the two arrows induced by the obvious projections and by

$$e \times_A e : k \to v$$

the arrow induced by the weak evaluation $e : u \to y$. The following elements are equal:

$$
\begin{aligned}
\varepsilon^x_\sigma &:= \exists_{\{\!|\gamma|\!\}}\forall_{\langle 1,2\rangle_A}\mathsf{P}_{\langle 1,3,2,3\rangle_A}\mathsf{P}_{e\times_A e}\mathsf{P}_{\{\!|\iota|\!\}}\sigma \\
&= \forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\delta_X \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma) \wedge \gamma \\
&= \forall_{\langle 1,3\rangle}\forall_{\{\!|\mu|\!\}^2}(\mathsf{P}_{\{\!|\mu|\!\}^2}(\mathsf{P}_{\langle 2,4\rangle}\delta_X \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \mathsf{P}_{e\times e}\sigma) \wedge \gamma.
\end{aligned}
\tag{A.5}
$$

Proof. Since e is a weak evaluation, it preserves projection w.r.t. σ. This implies that

$$\mathsf{P}_{e\times e}\sigma \in \mathcal{D}es_{\mathsf{P}_{(\{\!|\mu|\!\}^2\times\{\!|\mu|\!\}^2)}\delta_{E\times X\times E\times X}}.$$

Hence, Lemma A.0.18 implies that

$$\begin{aligned}
\varepsilon^x_\sigma &= \exists_{\{\!|\gamma|\!\}}\forall_{\langle 1,2\rangle_A}\mathsf{P}_{\langle 1,3,2,3\rangle_A}\mathsf{P}_{\{\!|k|\!\}}\mathsf{P}_{\{\!|\mu|\!\}^2}\forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma \\
&= \exists_{\{\!|\gamma|\!\}}\forall_{\langle 1,2\rangle_A}\mathsf{P}_{\{\!|\tau|\!\}}\mathsf{P}_{\langle 1,3,2,3\rangle}\forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma \\
&= \exists_{\{\!|\gamma|\!\}}\mathsf{P}_{\{\!|\gamma|\!\}}\forall_{\{\!|\gamma|\!\}}\forall_{\langle 1,2\rangle_A}\mathsf{P}_{\{\!|\tau|\!\}}\mathsf{P}_{\langle 1,3,2,3\rangle}\forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma && \text{(Lemma A.0.18)} \\
&= \forall_{\{\!|\gamma|\!\}}\forall_{\langle 1,2\rangle_A}\mathsf{P}_{\{\!|\tau|\!\}}\mathsf{P}_{\langle 1,3,2,3\rangle}\forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma \wedge \gamma && \text{(Lemma A.0.4)} \\
&= \forall_{\langle 1,3\rangle}\forall_{\langle 1,3,2,3\rangle}\forall_{\{\!|\tau|\!\}}\mathsf{P}_{\{\!|\tau|\!\}}\mathsf{P}_{\langle 1,3,2,3\rangle}\forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma \wedge \gamma \\
&= \forall_{\langle 1,3\rangle}\forall_{\langle 1,3,2,3\rangle}(\tau \Rightarrow \mathsf{P}_{\langle 1,3,2,3\rangle}\forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma) \wedge \gamma && \text{(Remark 1.2.19)} \\
&= \forall_{\langle 1,3\rangle}\forall_{\langle 1,3,2,3\rangle}\mathsf{P}_{\langle 1,3,2,3\rangle}(\mathsf{P}_{\langle 1,3,4\rangle}\tau \Rightarrow \forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma) \wedge \gamma && (\langle 1,3,2,3\rangle\langle 1,3,4\rangle = 1_{E\times E\times X}) \\
&= \forall_{\langle 1,3\rangle}(\mathsf{P}_{\langle 2,4\rangle}\delta_X \Rightarrow (\mathsf{P}_{\langle 1,3,4\rangle}\tau \Rightarrow \forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma)) \wedge \gamma && \text{(Remark 1.2.19)} \\
&= \forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\delta_X \wedge \mathsf{P}_{\langle 1,3,4\rangle}\tau) \Rightarrow \forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma) \wedge \gamma.
\end{aligned}$$

The last formula is trivially equal to

$$\begin{gathered}
\forall_{\langle 1,3\rangle}((\mathsf{P}_{\langle 2,4\rangle}\delta_X \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \forall_{\{\!|\mu|\!\}^2}\mathsf{P}_{e\times e}\sigma) \wedge \gamma = \\
\forall_{\langle 1,3\rangle}\forall_{\{\!|\mu|\!\}^2}(\mathsf{P}_{\{\!|\mu|\!\}^2}(\mathsf{P}_{\langle 2,4\rangle}\delta_X \wedge \mathsf{P}_{\langle 1,2\rangle}\mu \wedge \mathsf{P}_{\langle 3,4\rangle}\mu) \Rightarrow \mathsf{P}_{e\times e}\sigma) \wedge \gamma.
\end{gathered}$$

□

Appendix B

Type theory

In this chapter, we recall the basic constructs of the Martin-Löf intuitionistic type theory. We fix the notation used in the previous chapters and define precisely the type theory assumed in this work.

Per Martin-Löf introduced the intuitionistic type theory (also known as dependent type theory) as a logical framework to do constructive mathematics. Mathematical objects are always of a specified nature, which is expressed in type theory with the notion of *element* of certain *type*. We have four type of *judgment* which are

$$x : X \qquad | \qquad X \quad \mathsf{type} \qquad | \qquad x = y : X \qquad | \qquad X = Y \quad \mathsf{type}.$$

The first expression is read as "x is a term of type X" and resembles the usual inclusion relation $\in$ of set theory. The declaration of a type is given by the notation X type. However, we will often omit the word type. The above equality symbols are often referred to as the *judgmental equality*. This notion can be considered as "external" to the type theory, in opposition with the "internal" notion of equality given by the *identity types* discussed below.

The main feature of intuitionistic type theory is the possibility to have type depending on other types as in the expression

$$x : X \vdash B(x) \quad \mathsf{type}$$

which means that B is a type depending on the elements of the type X. Dependent types allow us the possibility to work with usual mathematical objects which are somehow indexed on elements of another type. For instance, in $\mathbb{N}$ is the set of the natural numbers, we can consider for every $n \in \mathbb{N}$ the set of the divisors of n

$$n : \mathbb{N} \vdash \mathsf{Div}(n).$$

Dependent types make a clear distinction between the Martin-Löf intuitionistic type theory and the *simple type theory* introduced in [WR97] and later in [Chu40] where dependencies where not allowed. For every type, we assume to have a suitable list of variables of that type. A general *context* is an expression of the form

$$\Gamma := x_1 : X_1, x_2 : X_2(x_1) \ldots, x_n : X_n(x_1, \ldots, x_{n-1})$$

where x_i are variables of type X_i and every type X_j depends on the types $X_1, \ldots, X_{j-1}$. Every judgment can appear in a suitable context which plays the role of the assumptions made in order to have that judgment; in this case we write

$$\Gamma \vdash \mathcal{T}.$$

If the context in empty we would omit the symbol $\vdash$.

We now introduce the rules of the type theory which consist of expressions of the form

$$\frac{H_1 \qquad \dots \qquad H_n}{C}$$

where $H_1, \dots, H_n$ are judgments called *hypothesis* and C is a judgment called *conclusion*. A *derivation* is a tree of rules. The rules can be of two kind. We now list those rules which are called *structural rules* and concern the calculus and not the logic of the system.

Rules for the judgmental equality. These rules ensure that judgmental equality of terms or types is an equivalence relation.

$$\frac{\Gamma \vdash x : X}{\Gamma \vdash x = x : X} \qquad \frac{\Gamma \vdash x = x' : X}{\Gamma \vdash x' = x : X} \qquad \frac{\Gamma \vdash x = x' : X \qquad \Gamma \vdash x' = x'' : X}{\Gamma \vdash x = x'' : X}$$

$$\frac{\Gamma \vdash X \quad \mathsf{type}}{\Gamma \vdash X = X \quad \mathsf{type}} \qquad \frac{\Gamma \vdash X = X' \quad \mathsf{type}}{\Gamma \vdash X' = X \quad \mathsf{type}} \qquad \frac{\Gamma \vdash X = X' \quad \mathsf{type} \qquad \Gamma \vdash X' = X'' \quad \mathsf{type}}{\Gamma \vdash X = X'' \quad \mathsf{type}}$$

Rules for substitution. These rules govern the substitution of terms into other terms that may appear also in a dependent type. Given a context

$$\Gamma, x : X, \Delta := x_1 : X_1, \dots, x_n : X_n, x : X, x_{n+1} : X_{n+1}, \dots, x_{n+m} : X_{n+m}$$

and a judgment $\Gamma, x : X, \Delta \vdash \mathcal{G}$, then if we have a term $\Gamma \vdash t : X$ we can substitute t in place of the occurrences of x and obtain a judgment $\mathcal{G}[t/x]$ in context $\Gamma, \Delta[t/x]$

$$\frac{\Gamma \vdash t : X \qquad \Gamma, x : X, \Delta \vdash \mathcal{G}}{\Gamma, \Delta[t/x] \vdash \mathcal{G}[t/x]}.$$

For instance, in case of the judgment which declares a type

$$\Gamma, x : X, \Delta \vdash B(x_1, \dots, x_n, x, x_{n+1}, \dots, x_{n+m}) \quad \mathsf{type}$$

we obtain the rule

$$\frac{\Gamma \vdash t : X \qquad \Gamma, x : X, \Delta \vdash B \quad \mathsf{type}}{\Gamma, \Delta[t/x] \vdash B[t/x] \quad \mathsf{type}}$$

where

$$\begin{aligned}\Delta[t/x] :=& x_{n+1} : X_{n+1}(x_1, \dots, x_n, x_n, t(x_1, \dots, x_n)), \dots \\ & \dots, x_{n+m} : X_{n+m}(x_1, \dots, x_n, , t(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m-1})\end{aligned}$$

and

$$B[t/x] := B(x_1, \dots, x_n, t(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m}).$$

Moreover, we require that the substitution of judgmentally equal terms in terms or types gives judgmentally equal terms or types

$$\frac{\Gamma \vdash t = t' : X \qquad \Gamma, x : X, \Delta \vdash b : B}{\Gamma, \Delta[t/x] \vdash b(t) = b(t') : B[t/x]}$$

$$\frac{\Gamma \vdash t = t' : X \qquad \Gamma, x : X, \Delta \vdash B \quad \mathsf{type}}{\Gamma, \Delta[t/x] \vdash B[t/x] = B[t'/x] \quad \mathsf{type}}.$$

Weakening rule. This rule implies that we can expand the context of a judgment $\mathcal{G}$ with new variables

$$\frac{\Gamma \vdash x : X \qquad \Gamma, \Delta \vdash \mathcal{G}}{\Gamma, x : X, \Delta \vdash \mathcal{G}}.$$

For instance, if A is a closed type, then A is also a type in any context $\Gamma \vdash A$.

Variable rule. This rule asserts that for any type X in the context Γ, the variable term $x : X$ is a term in the context $\Gamma, x : X$

$$\frac{\Gamma \vdash X \quad \textsf{type}}{\Gamma, x : X \vdash x : X}.$$

As a consequence, this rule implies that in the syntactic categories considered in Chapters 1 and 3, there is an identity arrow for each closed type.

These four rules are the structural rules of the type theory. We now proceed with the logical rules which concern the formation of particular types. For every type, we specify

- the *formation rule:* which tells how to form the type,
- the *introduction rule:* which tells how to introduce new terms of the type,
- the *elimination rule:* which tells how to use terms of the type,
- the *computation rules:* which tells how the introduction and elimination rules interact.

Π**-type.** Given a type X and a dependent type $x : X \vdash B(x)$ we can build the type of *functions* $\prod\limits_{x:X} B(x)$, from X to B. Intuitively, the elements of this new type are choice functions f which for an element $x : X$ pick an element $f(x) : B(x)$. In set theory this type corresponds to the indexed set families.

$$(\Pi\text{-formation}) \; \frac{\Gamma \vdash X \quad \textsf{type} \qquad \Gamma, x : X \vdash B(x) \quad \textsf{type}}{\Gamma \vdash \prod\limits_{x:X} B(x) \quad \textsf{type}}$$

$$(\Pi\text{-introduction}) \; \frac{\Gamma, x : X \vdash b(x) : B(x)}{\Gamma \vdash \lambda x.b(x) : \prod\limits_{x:X} B(x)}$$

$$(\Pi\text{-elimination}) \; \frac{\Gamma \vdash f : \prod\limits_{x:X} B(x) \qquad \Gamma \vdash a : X}{\Gamma \vdash f(a) : B[a/x]}$$

$$(\Pi\text{-computation}) \; \frac{\Gamma, x : X \vdash b(x) : B \qquad \Gamma \vdash a : X}{\Gamma \vdash (\lambda x.b(x))(a) = b(a) : B[a/x]} \; (\beta\text{-rule})$$

$$(\Pi\text{-computation}) \; \frac{\Gamma \vdash f : \prod\limits_{x:X} B(x)}{\Gamma \vdash \lambda x.f(x) = f : \prod\limits_{x:X} B(x)} \; (\eta\text{-rule})$$

Moreover, the above rules must preserve the judgmental equality in the following way:

$$\frac{\Gamma \vdash X = X' \quad \mathsf{type} \qquad \Gamma, x : X \vdash B(x) = B'(x) \quad \mathsf{type}}{\Gamma \vdash \prod_{x:X} B(x) = \prod_{x:X'} B'(x) \quad \mathsf{type}}$$

and when $x' : X$ is a fresh variable:

$$\frac{\Gamma, x : X \vdash B(x) \quad \mathsf{type}}{\Gamma \vdash \prod_{x:X} B(x) = \prod_{x':X} B(x') \quad \mathsf{type}}$$

$$\frac{\Gamma, x : X \vdash b(x) = b'(x) : B(x)}{\Gamma \vdash \lambda x.b(x) = \lambda x.b'(x) : \prod_{x:X} B(x)}$$

$$\frac{\Gamma \vdash f = f' : \prod_{x:X} B(x) \qquad \Gamma \vdash a : X}{\Gamma \vdash f(a) = f'(a) : B[a/x]}.$$

$\to$-type. When $x : X \vdash B$ is a type which does not depend on X, i.e. there are no occurrences of x in B, the Π-type $\prod_{x:X} B(x)$ is denoted with

$$X \to B.$$

Intuitively, this is the type of the functions between X and a fixed codomain B.

We now recall that some types can be described in an equivalent formulation as *inductive types*. This means that we specify

- the *constructors* of the type, that may be more than one or anyone,
- the *induction principle*: which explicates how to build a dependent type over the inductive type,
- the *computation rules*.

In order to do that, we use the Π-type. The two formulations are equivalent, and for the following type formers, we will recall both the formulations.

Σ-type. Given a type X and a dependent type $x : X \vdash B(x)$ we can form the type of the pairs $(x, b(x))$ where $b(x) : B(x)$ in the following way.

$$(\textstyle\sum\text{-formation})\ \frac{\Gamma \vdash X \quad \mathsf{type} \qquad \Gamma, x : X \vdash B(x) \quad \mathsf{type}}{\Gamma \vdash \sum_{x:X} B(x) \quad \mathsf{type}}$$

$$(\textstyle\sum\text{-introduction})\ \frac{\Gamma, x : X \vdash b(x) : B(x) \qquad \Gamma, \vdash a : X}{\Gamma \vdash (a, b(a)) : \sum_{x:X} B}$$

$$(\textstyle\sum\text{-elimination})\ \frac{\Gamma \vdash t : \sum_{x:X} B(x)}{\Gamma \vdash \pi_2(t) : B[\pi_1(t)/x]} \qquad \frac{\Gamma \vdash t : \sum_{x:X} B(x)}{\Gamma \vdash \pi_2(t) : B[\pi_1(t)/x]}$$

$$(\textstyle\sum\text{-computation})\ \frac{\Gamma, x : X \vdash b(x) : B(x) \qquad \Gamma \vdash a : X}{\Gamma \vdash \pi_1(a, b(a)) = a : X}\ \beta\text{-rule}$$

$$(\textstyle\sum\text{-computation})\ \frac{\Gamma, x : X \vdash b(x) : B(x) \qquad \Gamma \vdash a : X}{\Gamma \vdash \pi_2(a, b(a)) = b(a) : B[a/x]}\ \beta\text{-rule}$$

Sometimes, we shall denote the first projection simply with π. The above rules must preserve the judgmental equality as in the Π-type. Equivalently, we can introduce the $\sum$-type through the constructor

$$(-,-) : \prod_{x:X}(B(x) \to \sum_{x':X} B(x')).$$

The induction principle asserts for every dependent type $p : \sum_{x:X} B(x) \vdash C(p)$ the existence of a term

$$\mathsf{Ind}_\Sigma : \prod_{x:X}\prod_{y:B(x)} C(x,y) \to \prod_{p:\sum_{x:X} B(x)} C(p)$$

which satisfies the computation rule

$$\mathsf{Ind}_\Sigma(f, (x, y)) = f(x, y).$$

$\times$-type. When $x : X \vdash B$ is a type which does not depend on X, i.e. there are no occurrences of x in B, the Σ-type $\sum_{x:X} B(x)$ is denoted with

$$X \times B.$$

Intuitively, this is the type of the pairs of elements of X and B. In set theory, this corresponds to the product of two sets. Using the induction principle of the $\times$-type we obtain the *Currying* operator

$$\mathsf{cur} : (X \to (Y \to Z)) \to ((X \times Y) \to Z).$$

Th λ abstraction provides also an inverse operator. Intuitively, this is a typical construction in type theory which expresses multi-variable functions as functions of functions and vice versa.

+-type. Given two types X and Y in a common context, we can build the *sum* type $X + Y$. Intuitively, this type is the disjoint union of the types X and Y.

$$(+\text{-formation})\ \frac{\Gamma \vdash X \quad \mathsf{type} \qquad \Gamma \vdash Y \quad \mathsf{type}}{\Gamma \vdash X + Y \quad \mathsf{type}}$$

$$(+\text{-introduction})\ \frac{\Gamma \vdash x : X \qquad \Gamma \vdash X + Y \quad \mathsf{type}}{\Gamma \vdash \mathsf{inl}(x) : X + Y}$$

$$(+\text{-introduction})\ \frac{\Gamma \vdash y : Y \qquad \Gamma \vdash X + Y \quad \mathsf{type}}{\Gamma \vdash \mathsf{inr}(y) : X + Y}$$

$$(+\text{-el.})\ \frac{\Gamma, z : X+Y \vdash C(z) \quad \mathsf{type} \qquad \Gamma, x : X \vdash d(x) : C(\mathsf{inl}(x)) \qquad \Gamma, y : Y \vdash e(y) : C(\mathsf{inr}(y))}{\Gamma, z : X+Y \vdash \mathsf{Ind}_+(z,d,e) : C(z)}$$

$$(+\text{-comp.})\ \frac{\Gamma, z : X+Y \vdash C(z) \quad \mathsf{type} \qquad \Gamma, x : X \vdash d(x) : C(\mathsf{inl}(x)) \qquad \Gamma, y : Y \vdash e(y) : C(\mathsf{inr}(y))}{\Gamma, x : X \vdash \mathsf{Ind}_+(\mathsf{inl}(x),d,e) = d(x) : C(\mathsf{inl}(x))}$$

$$(+\text{-comp.})\ \frac{\Gamma, z : X+Y \vdash C(z) \quad \mathsf{type} \qquad \Gamma, x : X \vdash d(x) : C(\mathsf{inl}(x)) \qquad \Gamma, y : Y \vdash e(y) : C(\mathsf{inr}(y))}{\Gamma, y : Y \vdash \mathsf{Ind}_+(\mathsf{inr}(y),d,e) = e(y) : C(\mathsf{inr}(y))}$$

The above rules must preserve the judgmental equality. Equivalently, we can introduce the $+$-type $X+Y$ of two types X and Y through the constructors

$$\mathsf{inl} : X \to X+Y$$
$$\mathsf{inr} : Y \to X+Y.$$

The induction principle asserts for every dependent type $z : X+Y \vdash C(x)$ the existence of a term

$$\mathsf{Ind}_+ : \prod_{x:X} C(\mathsf{inl(x)}) \to \prod_{y:Y} C(\mathsf{inr(y)}) \to \prod_{z:X+Y} C(z)$$

which satisfies the computation rules

$$\mathsf{Ind}_+(d,e,\mathsf{inl}(x)) = d(x)$$
$$\mathsf{Ind}_+(d,e,\mathsf{inr}(y)) = e(y).$$

Empty type. This is the type without elements, which plays the role of the *false* predicate in the type system. This type corresponds to the empty set in set theory and will be denoted with **0**.

$$(\mathbf{0}\text{-formation})\ \frac{}{\mathbf{0} \quad \mathsf{type}}$$

$$(\mathbf{0}\text{-elimination})\ \frac{\Gamma \vdash a : \mathbf{0} \qquad \Gamma, x : X \vdash C(x) \quad \mathsf{type}}{\Gamma, x : X \vdash r_0(a) : C(x)}\ .$$

The empty type has no introduction or computation rule. The elimination rule implies that from a term of the type **0** we can obtain a term of any type. This rule resembles the *ex falso quodlibet* priciple. Equivalently, we can formulate the following induction principle for every dependent type $x : \mathbf{0} \vdash C(x)$

$$\mathsf{Ind}_0 : \prod_{x:\mathbf{0}} C(x).$$

Unit type. This type plays the role of the *true* predicate and it has just a canonical inhabitant. In set theory, it corresponds to the one element set.

$$(\mathbf{1}\text{-formation})\ \frac{}{\mathbf{1} \quad \mathsf{type}}$$

$$(\mathbf{1}\text{-introduction})\ \frac{}{* : \mathbf{1}}$$

$$(\mathbf{1}\text{-elimination})\ \frac{\Gamma \vdash a : \mathbf{1} \qquad \Gamma, x : \mathbf{1} \vdash C(x) \quad \mathsf{type} \qquad \Gamma \vdash b : C(*)}{\Gamma \vdash r_1(a, b) : C(a)}$$

$$(\mathbf{1}\text{-computation})\ \frac{\Gamma, x : \mathbf{1} \vdash C(x) \quad \mathsf{type} \qquad \Gamma \vdash b : C(*)}{\Gamma \vdash r_1(*, b) = b : C(*)}$$

Equivalently, we can introduce the type **1** through the constructor

$$* : \mathbf{1}.$$

The induction principle asserts that for every dependent type $x : \mathbf{1} \vdash C(x)$ there exists a term of the type

$$\mathsf{Ind}_1 : C(*) \to \prod_{x:\mathbf{1}} C(x)$$

which satisfies the computation rule

$$\mathsf{Ind}_1(p, x) = p.$$

Two elements type. This is the type with two canonical elements, also called the *Booleans* type. It plays the role of the type of the two truth values *true* and *false*. In set theory, it corresponds to the set with two elements.

$$(\mathbf{2}\text{-formation})\ \frac{}{\mathbf{2} \quad \mathsf{type}}$$

$$(\mathbf{2}\text{-introduction})\ \frac{}{0_2 : \mathbf{2}} \qquad \frac{}{1_2 : \mathbf{2}}$$

$$(\mathbf{2}\text{-el.})\ \frac{\Gamma \vdash a : \mathbf{2} \qquad \Gamma, x : \mathbf{2} \vdash C(x) \quad \mathsf{type} \qquad \Gamma \vdash b_0 : C(0_2) \qquad \Gamma \vdash b_1 : C(1_2)}{\Gamma \vdash r_2(a, b_0, b_1) : C(a)}$$

$$(\mathbf{2}\text{-comp.})\ \frac{\Gamma, x : \mathbf{2} \vdash C(x) \quad \mathsf{type} \qquad \Gamma \vdash b_0 : C(0_2) \qquad \Gamma \vdash b_1 : C(1_2)}{\Gamma \vdash r_2(0_2, b_0, b_1) = b_0 : C(0_2)}$$

$$(\mathbf{2}\text{-comp.})\ \frac{\Gamma, x : \mathbf{2} \vdash C(x) \quad \mathsf{type} \qquad \Gamma \vdash b_0 : C(0_2) \qquad \Gamma \vdash b_1 : C(1_2)}{\Gamma \vdash r_2(1_2, b_0, b_1) = b_1 : C(1_2)}$$

Equivalently, we can introduce the type **2** through the constructors

$$0_2 : \mathbf{2} \qquad 1_2 : \mathbf{2}.$$

The induction principle asserts that for every dependent type $x : \mathbf{2} \vdash C(x)$ there exists a term of the type

$$\mathsf{Ind}_2 : C(0_2) \to (C(1_2) \to \prod_{x:\mathbf{2}} C(x))$$

which satisfies the computation rules

$$\mathsf{Ind}_2(p_0, p_1, 0_2) = p_0$$
$$\mathsf{Ind}_2(p_0, p_1, 1_2) = p_1.$$

Similarly, it is possible to define the type with arbitrary n elements. However, we stop at the case two and proceed with the type of the natural numbers.

Natural numbers type.

$$(\mathbb{N}\text{-formation}) \; \frac{}{\mathbb{N} \quad \mathsf{type}}$$

$$(\mathbb{N}\text{-introduction}) \; \frac{}{0_N : \mathbb{N}} \qquad \frac{x : \mathbb{N}}{\mathsf{succ}(x) : \mathbb{N}}$$

$$(\mathbb{N}\text{-el.}) \; \frac{a : \mathbb{N} \qquad x : \mathbb{N} \vdash C(x) \quad \mathsf{type} \qquad b : C(0_N) \qquad x : \mathbb{N}, y : C(x) \vdash e(x,y) : C(\mathsf{succ}(x))}{x : \mathbb{N}, y : C(x) \vdash r_N(a, b, e(x,y)) : C(a)}$$

$$(\mathbb{N}\text{-comp.}) \; \frac{x : \mathbb{N} \vdash C(x) \quad \mathsf{type} \qquad b : C(0_N) \qquad x : \mathbb{N}, y : C(x) \vdash e(x,y) : C(\mathsf{succ}(x))}{x : \mathbb{N}, y : C(x) \vdash r_N(0_N, b, e(x,y)) = b : C(0_N)}$$

$$(\mathbb{N}\text{-comp.}) \; \frac{a : \mathbb{N} \qquad x : \mathbb{N} \vdash C(x) \quad \mathsf{type} \qquad b : C(0_N) \qquad x : \mathbb{N}, y : C(x) \vdash e(x,y) : C(\mathsf{succ}(x))}{x : \mathbb{N}, y : C(x) \vdash r_N(\mathsf{succ}(a), b, e(x,y)) = e(a, r_N(a, b, e(x,y))) : C(\mathsf{succ}(a))}$$

Equivalently, we can introduce the type $\mathbb{N}$ through the constructors

$$0_N : \mathbb{N} \qquad \mathsf{succ} : \mathbb{N} \to \mathbb{N}.$$

The induction principle asserts that for every dependent type $x : \mathbb{N} \vdash C(x)$ there exists a term of the type

$$\mathsf{Ind}_N : C(0_N) \to ((\prod_{x:\mathbb{N}} C(x) \to C(\mathsf{succ}(n))) \to \prod_{x:\mathbb{N}} C(x))$$

which satisfies the computation rules

$$\begin{aligned} \mathsf{Ind}_N(p_0, p_s, 0_N) &= 0_N \\ \mathsf{Ind}_N(p_0, p_s, \mathsf{succ}(x)) &= p_s(x, \mathsf{Ind}_N(p_0, p_s, x)). \end{aligned}$$

Identity type. This is the type of the system which internalizes the equality predicate.

$$(\mathsf{Id}\text{-form.}) \; \frac{\Gamma \vdash X \quad \mathsf{type} \qquad \Gamma \vdash a : X \qquad \Gamma \vdash b : X}{\Gamma \vdash \mathsf{Id}_X(a,b) \quad \mathsf{type}}$$

$$(\mathsf{Id}\text{-intro.}) \; \frac{\Gamma \vdash X \quad \mathsf{type} \qquad \Gamma \vdash a : X}{\Gamma \vdash \mathsf{refl}_a : \mathsf{Id}_X(a,a)}$$

(Id-elimination)

$$\frac{\Gamma, x : X, y : X, p : \mathsf{Id}_X(x,y) \vdash C(x,y,p) \quad \mathsf{type} \qquad \Gamma, x : X \vdash d(x) : C(x, x, \mathsf{refl}(x))}{\Gamma, x : X, y : X, p : \mathsf{Id}_X(x,y) \vdash J(d(x), p) : C(x,y,p)}$$

(Id-computation)

$$\frac{\Gamma, x : X, y : X, p : \mathsf{Id}_X(x,y) \vdash C(x,y,p) \quad \mathsf{type} \qquad \Gamma, x : X \vdash d(x) : C(x, x, \mathsf{refl}(x))}{\Gamma, x : X \vdash J(d(x), \mathsf{refl}(x)) = d(x) : C(x, x, \mathsf{refl}(x))}$$

Equivalently, we can present the identity type of a type X through the constructor

$$\mathsf{refl}_x : \mathsf{Id}_X(x, x)$$

for every term $x : X$. The induction principle also called *path-induction* asserts that for any term $a : X$ and dependent type $x : X, p : \mathsf{Id}_X(a, x) \vdash C(x, p)$ there is a term of the type

$$\mathsf{pathind}_a : C(a, \mathsf{refl}_a) \to \prod_{x:X} \prod_{p:\mathsf{Id}_X(a,x)} C(x, p)$$

which satisfies the computation rule

$$\mathsf{pathind}_a(p, a, \mathsf{refl}_a) = p.$$

The identity type we presented is called *intensional* and it is the one we considered for the results obtained in this thesis. The identity type is called *extensional* if the following rule is assumed

$$\text{(Reflection rule)}\ \frac{x, y : X \vdash p : \mathsf{Id}_X(x, y)}{x = y}$$

Universe. An element of this type is itself a type. The need of assuming a universe are several and as shown by Smith in [Smi88], a universe is necessary in order to have a type theory in which the Peano's fourth axiom holds. There are two principal formulations, one à la Tarski and one à la Russell. We start from the first one which assumes the existence of a dependent type $\mathcal{T}$ over the universe.

$$(\mathcal{U}\text{-formation})\ \frac{}{\mathcal{U} \quad \mathsf{type}}$$

$$(\mathcal{U}\text{-introduction})\ \frac{x : \mathcal{U}}{\mathcal{T}(x) \quad \mathsf{type}}$$

The types of the universe $\mathcal{U}$ are called *small types* and if $\mathcal{T}(x)$ is a type, a dependent small type on it is a term of

$$\mathcal{T}(x) \to \mathcal{U}.$$

We now require that the small types are closed under the type constructors introduced so far.

- (Closure for Π-type) There exists a function

  $$\tilde{\Pi} : \prod_{x:U} (\mathcal{T}(x) \to \mathcal{U}) \to \mathcal{U}$$

  such that

  $$\mathcal{T}(\tilde{\Pi}(x, b)) = \prod_{y:\mathcal{T}(x)} \mathcal{T}(b(y)).$$

- (Closure for Σ-type) There exists a function

  $$\tilde{\Sigma} : \prod_{x:U} (\mathcal{T}(x) \to \mathcal{U}) \to \mathcal{U}$$

  such that

  $$\mathcal{T}(\tilde{\Sigma}(x, b)) = \sum_{y:\mathcal{T}(x)} \mathcal{T}(b(y)).$$

- (Closure for +-type) There exists a function
$$\tilde{+} : \mathcal{U} \to (\mathcal{U} \to \mathcal{U})$$
such that
$$\mathcal{T}(\tilde{+}(x, y)) = \mathcal{T}(x) + \mathcal{T}(y).$$
- (Closure for **0,1,2** and $\mathbb{N}$) There exist terms $0_u, 1_u, 2_u$ and N_u of $\mathcal{U}$ such that
$$\mathcal{T}(0_u) = \mathbf{0} \quad \mathcal{T}(1_u) = \mathbf{1} \quad \mathcal{T}(2_u) = \mathbf{2} \quad \mathcal{T}(N_u) = \mathbb{N}.$$
- (Closure for identity type) There exists a function
$$\tilde{I} : \prod_{x:\mathcal{U}} \mathcal{T}(x) \to (\mathcal{T}(x) \to \mathcal{U})$$
such that for every $x_1, x_2 : \mathcal{T}(x)$ it holds
$$\mathcal{T}(\tilde{I}(x, x_1, x_2)) = \mathsf{Id}_{\mathcal{T}(x)}(x_1, x_2).$$

The description à la Russell of the universe type is more informal and we simply write
$$(\mathcal{U}\text{-introduction}) \ \frac{X : \mathcal{U}}{X \quad \mathsf{type}}$$
without assuming the dependent type $\mathcal{T}$ and the superscript $\sim$ in the type constructor preservation. This notation could be unclear but it is more practical. When it does not create confusion we shall adopt this notation for the small types.

Remark B.0.1. The type theory $\mathcal{ML}$ introduced in Chapter 1 is given by the rules we have listed up to here.

Functional extensionality. This axiom assumes that, for every type X and dependent type $x : X \vdash B(x)$ and for any pair of elements $f, g : \prod_{x:X} B(x)$, there exists a function
$$\mathsf{funext} : \prod_{x:X} \mathsf{Id}_{\mathsf{B(x)}}(f(x), g(x)) \to \mathsf{Id}_{\prod_{x:X} B(x)}(f, g).$$
Intuitively, the axiom expresses the property that to functions which have the same values are equal. The opposite is always true thanks to an application of path induction.

Transport. This operation is a useful tool to connect elements of a dependent type. Given a type $x : X \vdash B(x)$ and an element $b(x) : B(X)$, we can transport this element into the fibers $B(y)$ of the element y such that there is a path $p : \mathsf{Id}_X(x, y)$. This is obtained through the path induction which defines a term of
$$\mathsf{tr}_B : \prod_{x,y:X} \mathsf{Id}_X(x, y) \to (B(x) \to B(y)).$$
We will usually denote $\mathsf{tr}_B(p, b(x))$ with $p^*(b) : B(y)$.

One may wonder if, for $f : \prod_{x:X} B(x)$ and $p : Id_X(x, y)$, the two values $f(x)$ and $f(y)$ are provably equal. Indeed they are. In order to do that, one first transports $f(x) : B(x)$ along p, then compares that term with $f(y)$ in $B(y)$ as path induction induces a term of
$$\mathsf{adp}_f(p) : \mathsf{Id}_{B(y)}(p^* f(x), f(y)).$$

Curry-Howard correspondence. We end this chapter summarizing the correspondence between the logical objects and the type and set theoretic objects. The first two columns summarize the Curry-Howard correspondence between logic and type theory.

Logic	**Type theory**	**Set theory**
X formula	X type	X set
B formula in context $\bar{x}$	$\Gamma \vdash B$ type	$\{B_\gamma\}_{\gamma \in \Gamma}$ family of sets
a proof of X	$t : X$	$x \in X$
$\perp$	$\mathbf{0}$	$\emptyset$
$\top$	$\mathbf{1}$	$\{*\}$
$X \wedge Y$	$X \times Y$ type	$X \times Y$
$X \vee Y$	$X + Y$ type	$X \dot{\cup} Y$
$X \implies Y$	$X \to Y$ type	Y^X
$(\exists x : X) B(x)$	$\sum_{x:A} B(x)$	$(\dot{\bigcup} B_a)_{a \in A}$
$(\forall x : X) B(x)$	$\prod_{x:A} B(x)$	$(\prod_{a \in A} B_a)$

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