we gain a new equation of motion, namely the variation δA_{+}^{abc} which yields

$$F_{-}^{abc} = 0 \qquad \text{(III.7.106)}$$

after projecting on six vielbeins. Devising a suitable supersymmetry transformation for A_{+}^{abc} one, can cancel the F_{-} terms which previously did not cancel and in this way the action $\mathcal{A}' = \mathcal{A} + \Delta\mathcal{A}$ becomes supersymmetric.

CHAPTER III.8

D=11 SUPERGRAVITY

III.8.1 - Introduction

Since the beginning of Supergravity it was realized that its framework naturally leads to the idea of a multidimensional space-time with $D = 4+n$ dimensions. This is so because the Lagrangian can be constructed only in certain dimensions and has specific properties depending on D: in particular various arguments, already advocated in Part II, indicate that only $D \leq 11$ is allowed. Therefore the D=11 case is of special interest since, in such a field theory, the number of space-time dimensions is not a "fitted" parameter, rather it has an intrinsic justification (it is the maximum one allowed by local supersymmetry). On the other hand higher space-time dimensions is not a new idea. Since the classical work of Kaluza-Klein* it is known that gravity on a higher dimensional manifold M_D which splits into

* Th. Kaluza, Sitzungsber. Prens. Akad. Wis. Berlin, Kl (1921) 966; O. Klein, Z. Phys. 37 (1926) 895.

$M_D = M_4 \otimes M_n$ (where M_4 is Minkowski or de Sitter space and M_n is some compact "internal" manifold of dimension n) leads to an "effective" theory in 4-dimensions containing:

i) gravity

ii) the Yang-Mills fields of G, G being the symmetry group of the internal space M_n

iii) extra scalar fields determined by the properties of M_n.

What is remarkable and new about D=11 supergravity is that the splitting $M_{11} = M_4 \otimes M_7$ occurs spontaneously through the existence of special solutions where M_4 is anti de Sitter space and M_7 is any 7-dimensional Einstein space.

Spontaneous compactification of the D=11 theory is the subject of Part V; here we just mentioned it in order to stress the special importance of the D=11 supergravity theory, whose detailed study is presented in this Chapter.

The field content of the D=11 supersymmetry algebra is given by the "spin 2" vielbein field V^a_μ, the "spin 3/2" Majorana gravitino field ψ_μ and by the "spin 1" antisymmetric tensor field $A_{\mu\nu\rho}$. The on-shell bosonic and fermionic degrees of these fields match exactly. Indeed using formula (III.5.12) with D=11 we have

$$V^a_\mu \quad : \quad \frac{D(D-3)}{2} = 44 \tag{III.8.1}$$

$$A_{\mu\nu\rho} \quad : \quad \frac{(D-2)(D-3)(D-4)}{3!} = 84 \tag{III.8.2}$$

$$\psi_\mu \quad : \quad 2^{[D/2]}\frac{D-3}{2} = 128 \;. \tag{III.8.3}$$

The presence of the antisymmetric 3-tensor $A_{\mu\nu\rho}$ in the multiplet means that the corresponding D=11 gauge theory must be based on a F.D.A. rather than on a Lie Algebra. Actually the need of enlarging the rheonomy framework from Lie Algebras to F.D.A.'s was first realized in the context of the present theory.

III.8.2 - Free differential algebra of D=11 supergravity

We apply the iterative procedure explained in Chapter III.6 to the construction of the relevant F.D.A. The (super)-Lie algebra we start with is of course the super Poincaré algebra in D=11.

In D=11, as in any dimension, the super Poincaré algebra (that is the minimal grading of Eqs. (I.3.177) with just one odd spinorial generator Q_α) is obtained by adjoining to (I.3.177) the anticommutator

$$\{Q_\alpha, Q_\beta\} = i(C\,\Gamma^a)_{\alpha\beta}\, P_a \;. \tag{III.8.4}$$

The corresponding Maurer-Cartan equations read as follows:

$$R^{ab} \equiv d\omega^{ab} - \omega^{ac} \wedge \omega_c^{\ b} = 0 \tag{III.8.5a}$$

$$R^a \equiv \mathcal{D}V^a - \frac{i}{2}\bar\psi \wedge \Gamma^a\psi = 0 \tag{III.8.5b}$$

$$\rho \equiv \mathcal{D}\psi = 0 \;. \tag{III.8.5c}$$

They look exactly the same as in the $\overline{Osp(4/1)}$ case, (see Eqs. (III.3.5)), the only difference being that now the Latin indices a,b,c,... run from 0 to 10 and ψ is a 32 dimensional Majorana spinor. We use the same notations utilized throughout the book: we just recall that ω^{ab} is the SO(1,10) spin connection 1-form (dual to the Lorentz generator D_{ab}) and V^a and ψ^α are the left invariant 1-forms dual to $\tilde D_a$ and $\tilde D_\alpha$ respectively, i.e. to non left-invariant vector fields corresponding to the translation and to the supersymmetry generators. $\mathcal{D}V^a$ and $\mathcal{D}\psi$ denote the covariant Lorentz derivatives, namely:

$$\mathcal{D}V^a = dV^a - \omega^{ab} \wedge V_b \tag{III.8.6}$$

$$\mathcal{D}\psi = d\psi - \frac{1}{4}\,\omega^{ab}\,\Gamma_{ab}\psi \;. \tag{III.8.7}$$

Γ_a, Γ_{ab} are the usual 1-index and 2-index γ-matrices in D=11 whose definitions and properties are given in Chapter II.7.

We stress that we cannot start from the anti de Sitter version of the above super-Lie algebra (III.8.6) since, as discussed in the introduction, the supersymmetry algebra representation given by V^a_μ, ψ_μ and $A_{\mu\nu\rho}$ exists only for the Poincaré supergroup and not for its anti de Sitter extension, which in the present case would be Osp(32/1).

Using now (III.8.6) as a starting point we follow the iterative construction of the F.D.A. explained in the previous chapter. According to Theorem 2 of Sect. III.6.3 we investigate the Chevalley cohomology classes of the super Lie algebra (III.8.6). Considering the trivial representation $D^{(0)}$ in which $\nabla^{(n=0)}$ coincides with the normal d, one finds that in this representation there is a cohomology class of order four, namely:

$$\Omega(V,\omega,\psi) = \frac{1}{2}\bar{\psi} \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b \; . \qquad \text{(III.8.8)}$$

Indeed:

$$d\Omega = d(\frac{1}{2}\bar{\psi} \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b) = \mathcal{D}(\frac{1}{2}\bar{\psi} \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b) =$$

$$= \frac{i}{2}\bar{\psi} \wedge \Gamma^{ab}\psi \wedge \bar{\psi} \wedge \Gamma_a\psi \wedge V_b = 0 \qquad \text{(III.8.9)}$$

where we used Eqs. (III.8.5b) and the Fierz identity

$$\bar{\psi} \wedge \Gamma^{ab}\psi \wedge \bar{\psi} \wedge \Gamma_a\psi = 0 \qquad \text{(III.8.10)}$$

whose origin has been explained in Sect. (II.8.7). The existence of a non-trivial element of the cohomology class of order 4 enables us to extend the super Lie algebra (III.8.5) to a F.D.A. by introducing a 3-form A fulfilling the following equation:

$$dA - \Omega(V,\psi,\omega) = 0 \; . \qquad \text{(III.8.11)}$$

Equations (III.8.5) and Eqs. (III.8.8,11) together realize the following F.D.A.:

$$d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b = 0 \qquad \text{(III.8.12a)}$$

$$\mathcal{D}V^a - \frac{i}{2}\bar{\psi} \wedge \Gamma^a\psi = 0 \qquad \text{(III.8.12b)}$$

$$\mathcal{D}\psi = 0 \qquad \text{(III.8.12c)}$$

$$dA - \frac{1}{2}\bar{\psi} \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b = 0 \; . \qquad \text{(III.8.12d)}$$

Since the multiplet of fields appearing in (III.8.12) is the same as the physical multiplet, we can base our construction of D=11 supergravity on this F.D.A. By gauging the F.D.A. we obtain the definition of the "curvatures", according to the procedure explained in Sect. III.6.4 (*):

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b \qquad \text{(III.8.13a)}$$

$$R^a = \mathcal{D}V^a - \frac{i}{2}\bar{\psi} \wedge \Gamma^a\psi \qquad \text{(III.8.13b)}$$

$$\rho = \mathcal{D}\psi \qquad \text{(III.8.13c)}$$

$$R^{\square} = dA - \frac{1}{2}\bar{\psi} \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b \qquad \text{(III.8.13d)}$$

and by differentiation we find the "generalized Bianchi identities":

$$\mathcal{D}R^{ab} = 0 \qquad \text{(III.8.14a)}$$

$$\mathcal{D}R^a + R^{ab} \wedge V_b - i\,\bar{\psi} \wedge \Gamma^a\rho = 0 \qquad \text{(III.8.14b)}$$

$$\mathcal{D}\rho + \frac{1}{4}\Gamma_{ab}R^{ab} \wedge \psi = 0 \qquad \text{(III.8.14c)}$$

(*) According to our conventions we denote in the same way the fields satisfying (III.8.12) (left-invariant forms) and those appearing in (III.8.13) (soft forms ≡ non left-invariant).

$$dR^{\square} - \bar{\psi} \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b + \bar{\psi} \wedge \Gamma_{ab}\psi \wedge R^a \wedge V^b = 0 \qquad \text{(III.8.14d)}$$

III.8.3 - Extended F.D.A. and the introduction of a 6-form

At this point we could stop in our study of the F.D.A. since Eqs. (III.8.13-14) should give the right algebraic starting point for the construction of the theory. This is the case at least if we want to base our approach on the action principle as we will see later on.

Actually it is in any case valuable and instructive to investigate whether another iteration in the procedure of Sect. III.6.3 could yield a further extension of the F.D.A. (III.8.12) and, if so, what is its physical meaning.

Starting from (III.8.12) one can check that the 7-form:

$$\Omega'(V,\psi,\omega,A) = \frac{i}{2}\,\bar{\psi} \wedge \Gamma^{a_1\dots a_5}\psi \wedge V_{a_1} \wedge \dots \wedge V_{a_5} + \frac{15}{2}\,\bar{\psi} \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b \wedge A \qquad \text{(III.8.15)}$$

is an element of the Chevalley cohomology class of the F.D.A. (III.8.12). Indeed:

$$\begin{aligned} d\Omega' = &-\frac{5}{4}\,\bar{\psi} \wedge \Gamma^{a_1\dots a_4 m}\psi \wedge V_{a_1} \wedge \dots \wedge V_{a_4} \wedge \bar{\psi} \wedge \Gamma_m\psi - \\ &- i\,15\,\bar{\psi} \wedge \Gamma^{am}\psi \wedge V_a \wedge \bar{\psi} \wedge \Gamma_m\psi \wedge A + \\ &+ \frac{15}{4}\,\bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \wedge \bar{\psi} \wedge \Gamma_{\ell m}\psi \wedge V^{\ell} \wedge V^m . \end{aligned} \qquad \text{(III.8.16)}$$

This expression is easily seen to be identically zero if one uses the Fierz identities:

$$\bar{\psi} \wedge \Gamma^{a_1\dots a_4 m}\psi \wedge \bar{\psi} \wedge \Gamma_m\psi = 3\,\bar{\psi} \wedge \Gamma^{[a_1a_2}\psi \wedge \bar{\psi} \wedge \Gamma^{a_3a_4]}\psi \qquad \text{(III.8.17a)}$$

$$\bar{\psi} \wedge \Gamma^{ab}\psi \wedge \bar{\psi} \wedge \Gamma_b\psi = 0 \qquad \text{(III.8.17b)}$$

which easily follow from Table (II.8.XI).

Therefore we can enlarge the previous F.D.A. by introducing a 6-form B fulfilling the equation below:

$$dB - \frac{i}{2}\,\bar{\psi} \wedge \Gamma^{a_1\dots a_5}\psi \wedge V_{a_1} \wedge \dots \wedge V_{a_5} - \frac{15}{2}\,\bar{\psi} \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b \wedge A = 0 . \qquad \text{(III.8.18)}$$

If we add Eq. (III.8.18) to Eqs. (III.8.12) we obtain a new enlarged F.D.A. Using Table II.8.XI and the theorems of Sect. III.6.3 it may be easily checked that no further extension is possible. Therefore we are led to the conclusion that the F.D.A. given by the Eqs. (III.8.12) ⊕ (III.8.18) is the maximal F.D.A. extending the D=11 Poincaré supergroup.

Introducing the curvature $\hat{R}^{\otimes}$ associated to the field B we have to supplement Eqs. (III.8.13) and (III.8.14) with the following new equations:

$$\hat{R}^{\otimes} = dB - \frac{i}{2}\,\bar{\psi} \wedge \Gamma_{a_1\dots a_5}\psi \wedge V^{a_1} \wedge \dots \wedge V^{a_5} - \frac{15}{2}\,\bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \wedge A \qquad \text{(III.8.19a)}$$

$$\begin{aligned} d\hat{R}^{\otimes} &- i\,\bar{\psi} \wedge \Gamma_{a_1\dots a_5}\rho \wedge V^{a_1} \wedge \dots \wedge V^{a_5} + \\ &+ i\,\frac{5}{2}\,\bar{\psi} \wedge \Gamma_{a_1\dots a_5}\psi \wedge R^{a_1} \wedge V^{a_2} \wedge \dots \wedge V^{a_5} - \\ &- 15\,\bar{\psi} \wedge \Gamma^{ab}\rho \wedge A \wedge V_a \wedge V_b + \frac{15}{2}\,\bar{\psi} \wedge \Gamma_{ab}\psi \wedge R^{\square} \wedge V^a \wedge V^b \\ &- 15\,\bar{\psi} \wedge \Gamma^{ab}\psi \wedge A \wedge R_a \wedge V_b = 0 . \end{aligned} \qquad \text{(III.8.19b)}$$

One may wonder whether the new field B can be given a physical interpretation despite the fact that the D=11 supersymmetry algebra does not contain it. The answer is that the field B can enter the theory if, like the spin connection ω^{ab}, it is not an independent field.

In particular, calling $B_{\mu_1 \ldots \mu_6}$ the space-time components of the 6-form B and $F_{\mu_1 \ldots \mu_7} \equiv \partial_{[\mu_1} B_{\mu_2 \ldots \mu_7]}$ its curl, it is attractive to assume that $F_{\mu_1 \ldots \mu_7}$ is related to the curl of $A_{\mu\nu\rho}$ by a duality relation:

$$F_{\mu_1 \ldots \mu_7} = \partial_{[\mu_1} B_{\mu_2 \ldots \mu_7]} \; . \tag{III.8.20}$$

In the following we shall see that this is indeed the case and that the formulation of the theory using the F.D.A. (III.8.6) ⊕ (III.8.8) ⊕ (III.8.16) allows us to gain more insight into the geometrical structure of D=11 supergravity.

III.8.4 - The gauging of F.D.A. revisited

In Sect. III.6.4 we discussed the gauging of an F.D.A. in a way quite analogous to the gauging of an ordinary Lie (super)-algebra, introducing the soft p-forms $\Pi^{A(p)}$ and defining the extended notion of curvatures $R^{A(p+1)}$ as the deviation from zero of the l.h.s. of the rigid F.D.A.

A physical theory is then constructed by regarding the soft forms $\Pi^{A(p)}$ as dynamical variables (the "Yang-Mills potentials"), requiring that the flat space $\{R^{A(p+1)} \equiv 0\}$ be a solution, and imposing the principle of rheonomy.

This procedure, for which several applications have been discussed (see Sects. III.6.5-7 and Chapter III.7), is evidently the natural extension of what one does in gauging a Lie algebra, but it is too naive for the following reason: if we introduce a new set of forms $\{\tilde{\theta}^{A(p)}\}$, defined in the following way:

$$\tilde{\theta}^{A(p)} = \Pi^{A(p)} \tag{III.8.21}$$

$$\tilde{\theta}^{A(p+1)} = R^{A(p+1)} \tag{III.8.22}$$

we see that we can reinterpret the definition of the curvatures (III.6.47) plus the Bianchi identity (III.6.48) as a single F.D.A. for a larger set of rigid forms $\{\tilde{\theta}^{A(p)}\}$. It follows that the concept of free differential algebra is already large enough to accommodate both the rigid fields and the curvatures and that the definition of these latter as deviation from zero of the Maurer-Cartan equations is ambiguous. A distinction between potentials and curvatures is on the other hand vital for any application to physics and we need a consistent one. Fortunately it is provided in an intrinsic fashion by Sullivan's structural Theorem 1 of Sect. III.6.3. In fact from the decomposition (III.6.12) we see that every F.D.A. has a unique minimal algebra M. It is this latter which plays the same role as the (rigid) Lie algebra in the gauging of groups and which describes the symmetry of the vacuum. Indeed the second structural theorem by Sullivan, in the same section, shows that M always contains an ordinary Lie sub-algebra (or super algebra), $L \subset M$, all the other generators of M being essentially determined by the Chevalley cohomology of the Lie algebra L. It follows that the generators of $\mathcal{A}$ which do not sit in M, and which therefore we call the contractible generators, are to be identified with the curvatures.

Hence we propose the following identification between mathematical and physical concepts.

Mathematics		Physics
Contractible algebra C	↔	Bianchi identities
Contractible generators	↔	Curvatures
Minimal algebra M	↔	Symmetry of vacuum (rigid algebra)
Minimal generators	↔	Yang-Mills potentials gauging vacuum symmetry

The practical outcome of this discussion is that we always have to start from a minimal algebra but, at the moment of gauging, we do not have to call curvature the deviation from zero of the minimal algebra equations. Rather, we have to be more subtle and allow the appearance of contractible generators wherever they can be introduced.

Let us exemplify these ideas with the gauging of the D=11 F.D.A. given by Eqs. (III.8.13) ⊕ (III.8.19a). In the previous section the curvatures $\{R^{ab}, R^a, \rho, R^\square, R^\otimes\}$, were introduced according to the old point of view, namely as the deviation from zero of the l.h.s. of the minimal F.D.A. (III.8.12) ⊕ (III.8.18). Following our new ideas, the curvatures, namely the contractible generators, can also appear multiplied by minimal generators if this is allowed by the symmetries of the minimal algebra. In our example there are two symmetries which we need to respect; one is SO(1,10) Lorentz invariance, the other is the global scale invariance of Eqs. (III.8.13,14,19) under the following replacements:

$$\omega^{ab} \to \omega^{ab} \quad ; \quad V^a \to wV^a \quad ; \quad \psi \to w^{1/2}\psi$$
$$A \to w^3 A \quad ; \quad B \to w^6 B \,. \tag{III.8.23}$$

This implies that the contractible generators $\{R^{ab}, R^a, \rho, R^\square, R^\otimes\}$ will have the same scaling weights as their corresponding minimal generators and can be placed in the algebra only where their scaling weight allows them to stay. Taking this into account we find that the most general "decontraction of the minimal algebra" is given by the same equations (III.8.13), but as far as (III.8.19a) is concerned we may have the more general equation

$$R^\otimes = dB - \frac{i}{2}\bar\psi \wedge \Gamma_{a_1 - a_5}\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_5} -$$
$$- \frac{15}{2}\bar\psi \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b \wedge A - \alpha R^\square \wedge A \equiv \hat R^\otimes - \alpha R^\square \wedge A \tag{III.8.24}$$

where α is a free parameter. The contractible generators R^{ab}, R^a, ρ, $R^\square$ and $R^\otimes$, on the other hand, satisfy the contractible algebra equations obtained by d-differentiation of Eqs. (III.8.13) and (III.8.24). These latter would be named Bianchi identities according to the old point of view. We write them symbolically as follows:

$$dR^{ab} + \ldots = 0 \quad ; \quad dR^a + \ldots = 0 \quad ; \quad d\rho + \ldots = 0$$
$$dR^\square + \ldots = 0 \quad ; \quad dR^\otimes + \ldots = 0 \tag{III.8.25}$$

their complete expression being given below in Eqs. (III.8.32f-j) after determination of the parameter α. According to the new point of view the new F.D.A. is nicely separated into two sets, corresponding to the splitting into a minimal algebra (Eqs. (III.8.5,8,16)) and a contractible algebra or Bianchi identities (Eqs. (III.8.25)). The whole difference between the old and the new point of view is the freedom in the choice of α. In the old picture we automatically set $\alpha = 0$ (Eq. (III.8.16)); now we want a better criterion to fix this number and therefore to choose among the infinity of F.D.A. which correspond to the same minimal model. This criterion is provided by the following observation: besides SO(1,10) and scale invariance (III.8.23) the minimal algebra does also possess a third symmetry property expressed by the invariance of Eqs. (III.8.5,8,18) under the following "gauge transformation"

$$A \to A + d\varphi^\square \tag{III.8.26}$$

$$B \to B + d\varphi^\otimes + \frac{15}{2}\varphi^\square \wedge \bar\psi \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \tag{III.8.27}$$

where $\varphi^\square$ is any 2-form and $\varphi^\otimes$ is any 5-form. We would like to promote this gauge invariance of the minimal algebra to an invariance of the decontracted algebra (III.8.13) ⊕ (III.8.24). In order to obtain it we modify Eq. (III.8.27) in the only possible way which is consistent with the scaling weights (III.8.23) and which reduces to (III.8.27) when all the contractible generators (curvatures) are set equal to zero; namely:

$$B \to B + d\varphi^\otimes + \frac{15}{2}\varphi^\square \wedge \bar\psi \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b + \beta\varphi^\square \wedge R^\square \,. \tag{III.8.28}$$

Next we require that the contractible generators, that is the curvatures R^{ab}, R^a, ρ, $R^{\square}$, $R^{\otimes}$, be invariant under the transformations (III.8.26) and (III.8.28). For R^{ab}, R^a, ρ, $R^{\square}$ this is trivially true while for $R^{\otimes}$ we obtain

$$\delta R^{\otimes} = d\delta B - \frac{15}{2}\bar{\psi}\wedge\Gamma^{ab}\psi\wedge V_a\wedge V_b\wedge\delta A - \alpha R^{\square}\wedge\delta A = 0\,. \quad \text{(III.8.29)}$$

Using Eqs. (III.8.26) and (III.8.28) one finds:

$$\begin{aligned}\delta R^{\otimes} = {} & (\beta-15)\varphi^{\square}\wedge\bar{\psi}\wedge\Gamma_{ab}\rho\wedge V^a\wedge V^b - \\ & -(\beta-15)\varphi^{\square}\wedge\bar{\psi}\wedge\Gamma_{ab}\psi\wedge R^a\wedge V^b + \\ & +(\beta-\alpha)d\varphi^{\square}\wedge R^{\square}\,. \end{aligned} \quad \text{(III.8.30)}$$

Therefore $R^{\otimes}$ is gauge invariant if and only if

$$\alpha = \beta = 15\,. \quad \text{(III.8.31)}$$

We conclude that there is a unique decontraction of the minimal F.D.A. which preserves all of its symmetries, given by the following set of equations:

$$R^{ab} = d\omega^{ab} - \omega^a{}_c\wedge\omega^{cb} \quad \text{(III.8.32a)}$$

$$R^a = dV^a - \omega^{ab}\wedge V_b - \frac{i}{2}\bar{\psi}\wedge\Gamma^a\psi = \mathscr{D}V^a - \frac{i}{2}\bar{\psi}\wedge\Gamma^a\psi \quad \text{(III.8.32b)}$$

$$\rho = d\psi - \frac{1}{4}\omega^{ab}\,\Gamma_{ab}\psi = \mathscr{D}\psi \quad \text{(III.8.32c)}$$

$$R^{\square} = dA - \frac{1}{2}\bar{\psi}\wedge\Gamma^{ab}\psi\wedge V_a\wedge V_b \quad \text{(III.8.32d)}$$

$$\begin{aligned}R^{\otimes} = {} & dB - \frac{i}{2}\bar{\psi}\wedge\Gamma_{a_1\ldots a_5}\psi\wedge V^{a_1}\wedge\ldots\wedge V^{a_5} - \\ & -\frac{15}{2}\bar{\psi}\wedge\Gamma_{ab}\psi\wedge V^a\wedge V^b\wedge A - 15\,R^{\square}\wedge A \end{aligned} \quad \text{(III.8.32e)}$$

$$\mathscr{D}R^{ab} = 0 \quad \text{(III.8.32f)}$$

$$\mathscr{D}R^a + R^{ab}\wedge V_b - i\,\bar{\psi}\wedge\Gamma^a\rho = 0 \quad \text{(III.8.32g)}$$

$$\mathscr{D}\rho + \frac{1}{4}\Gamma_{ab}\psi\wedge R^{ab} = 0 \quad \text{(III.8.32h)}$$

$$dR^{\square} - \bar{\psi}\wedge\Gamma_{ab}\rho\wedge V^a\wedge V^b + \bar{\psi}\wedge\Gamma_{ab}\psi\wedge R^a\wedge V^b = 0 \quad \text{(III.8.32i)}$$

$$\begin{aligned}dR^{\otimes} & - i\,\bar{\psi}\wedge\Gamma_{a_1\ldots a_5}\rho\wedge V^{a_1}\wedge\ldots\wedge V^{a_5} - \\ & -\frac{5}{2}i\,\bar{\psi}\wedge\Gamma_{a_1\ldots a_5}\psi\wedge R^{a_1}\wedge V^{a_2}\wedge\ldots\wedge V^{a_5} - \\ & -15\,\bar{\psi}\wedge\Gamma_{ab}\psi\wedge R^{\square}\wedge V^a\wedge V^b - 15\,R^{\square}\wedge R^{\square} = 0\,. \end{aligned} \quad \text{(III.8.32j)}$$

From the physical point of view Eqs. (III.8.32a-e) can be taken as the definitions of the curvatures, while Eqs. (III.8.32f-j) may be reinterpreted as the Bianchi identities. The latter are obtained from the former by d-differentiation.

We observe that the gauging given in the previous section, namely Eqs. (III.8.13,19a), is not correct if we are to preserve the symmetries of the minimal algebra including Eqs. (III.8.26) and (III.8.27). Eqs. (III.8.19a) and (III.8.18b) have to be replaced by the new equations (III.8.32e) and (III.8.32j).

Having fixed the F.D.A. we now proceed to the construction of the physical theory, i.e. of the action, of the supersymmetry transformations and of the space-time field equations.

III.8.5 - Constructing the theory from Bianchi identities

In this section we implement the principle of rheonomy directly on the Bianchi identities (Eqs. (III.8.32f-j)) to obtain in the most direct way the rheonomic parametrization of the curvatures (III.8.32a-e) and the equations of motion of the physical fields on the 11-dimensional space-time. The method has been explained in detail in Sect. III.3.12

and has already been applied to the construction of the D=4, N=1 and N=2 theories and, in part, to the construction of D=6, N=1 supergravity.

Our problem consists in finding the most general parametrization of the curvatures (III.8.32a-e) which respects:

i) the principle of rheonomy

ii) the symmetries of the F.D.A. (III.8.32) namely:
   a) SO(1,10) gauge invariance;
   b) the gauge invariance of the fields A and B expressed by Eqs. (III.8.26) and (III.8.27);
   c) the rigid scale invariance under the substitution (III.8.23).

Moreover we will assume the kinematical constraint:

$$R^a = 0 \qquad \text{(III.8.33)}$$

already discussed in Sect. III.3.12: the same remarks apply also here. (We observe that in the D=5 and D=6 theories (Chapters III.5 and III.7) the space-time components of the torsion $R^a_{|bc}$ were not zero because of the existence of a 3-index field to which $R^a_{|bc}$ could be equated. In the present case, however, there is no such space time field so that $R^a_{|bc}$ must be zero. In particular this excludes the possibility of obtaining a strongly geometrical theory by means of the "torsion mechanism" which works in the D=5 and D=6 theories.)

The most general parametrization of the curvatures fulfilling the above listed requirements is given by the following ansatz:

$$R^a = 0 \qquad \text{(III.8.34)}$$

$$R^{\square} = F_{a_1\ldots a_4} V^{a_1} \wedge \ldots \wedge V^{a_4} \qquad \text{(III.8.35)}$$

$$R^{\otimes} = G_{a_1\ldots a_7} V^{a_1} \wedge \ldots \wedge V^{a_7} \qquad \text{(III.8.36)}$$

$$\rho = \rho_{ab} V^a \wedge V^b + H_a\psi \wedge V^a \qquad \text{(III.8.37)}$$

$$R^{ab} = R^{ab}{}_{mn} V^m \wedge V^n + \bar{\theta}^{ab}{}_c\psi \wedge V^c + \bar{\psi} \wedge K^{ab}\psi \, . \qquad \text{(III.8.38)}$$

where, as usual H_a, K_{ab} are matrices in spinor space and $\theta^{ab}{}_c$ is a spinor tensor, their explicit form being given below. Let us briefly justify the ansatz. The way to arrive at Eqs. (III.8.34-38) is, besides rheonomy, a scale argument. Indeed from the scaling behaviour

$$[R^{ab}] = [w^0] \qquad \text{(III.8.39a)}$$

$$[R^a] = [w] \qquad \text{(III.8.39b)}$$

$$[\rho] = [w^{1/2}] \qquad \text{(III.8.39c)}$$

$$[R^{\square}] = [w^3] \qquad \text{(III.8.39d)}$$

$$[R^{\otimes}] = [w^6] \qquad \text{(III.8.39e)}$$

which follows from (III.8.23) we find that the space-time fields, or inner components, scale as follows:

$$[F_{a_1\ldots a_4}] = [w^{-1}] \qquad \text{(III.8.40a)}$$

$$[G_{a_1\ldots a_7}] = [w^{-1}] \qquad \text{(III.8.40b)}$$

$$[\rho_{ab}] = [w^{-3/2}] \qquad \text{(III.8.40c)}$$

$$[R^{ab}{}_{cd}] = [w^{-2}] \qquad \text{(III.8.40d)}$$

$$[R^a{}_{bc}] = [w^{-1}] \qquad \text{(III.8.40e)}$$

Therefore taking into account rheonomy and SO(1,10)-covariance we see that:

a) In terms of the space-time fields (III.8.40) it is impossible to assign to ρ a ψ-ψ component scaling as $[w^{-1/2}]$.

b) $R^{\square}$ and $R^{\otimes}$ cannot have rheonomic outer components except possibly for a constant component of $R^{\square}$ in the $\psi\psi VV$ sector of the type

$$R^{\square}_{(\psi\psi VV)} \equiv a\,\bar\psi \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \,. \tag{III.8.41}$$

However this parametrization of $R^{\square}$ would be equivalent to a renormalization of the coefficient of $\bar\psi \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b$ in the definition of $R^{\square}$, Eq. (III.8.32d). Hence we can set $a = 0$.

Therefore the only outer components which ought to be determined in terms of the inner components (III.8.40) are the bosonic quantities H_a and K^{ab} appearing in (III.8.37-38) and the spinor quantity $\theta^{ab}{}_c$ appearing in (III.8.38). Since from (III.8.40)

$$[K^{ab}] \equiv [H_a] = [w^{-1}] \tag{III.8.42a}$$

$$[\theta^{ab}{}_c] = [w^{-3/2}] \tag{III.8.42b}$$

we see that the bosonic components K^{ab} and H_a can be given a non-zero value in terms of $F_{a_1\ldots a_4}$ and $G_{a_1\ldots a_7}$ while the spinor component $\theta^{ab}{}_c$ can be expressed in terms of ρ_{ab}.

Our ansatz (III.8.34-38) is then completely justified.

The next step is to implement the Bianchi identities (III.8.32f-j) on the general parametrization (III.8.34-38). According to the general discussion given in Sect. III.3.12 in this way we obtain both the space-time field equations and the supersymmetry transformations for the physical fields. Indeed once the parametrization of the curvatures is determined, the Lie derivative in the outer directions is also determined; moreover the field equations of $F_{a_1\ldots a_4}$, $G_{a_1\ldots a_7}$, ρ_{ab} and R^{ab} will follow as differential constraints imposed on the physical fields by compatibility of the rheonomic parametrization with the Bianchi identities.

The quickest way to determine the outer components is to look first at the $\psi\psi VVV$ content of the Bianchi identity (III.8.32i); since $R^a = 0$, (III.8.32i) reduces to

$$dR^{\square} - \bar\psi \wedge \Gamma^{ab}\rho \wedge V_a \wedge V_b = 0 \,. \tag{III.8.43}$$

Using the parametrizations (III.8.35) and (III.8.37) we see that at the $\psi\psi VVV$ level Eq. (III.8.43) becomes:

$$2i\, F_{a_1\ldots a_4}\bar\psi \wedge \Gamma^{a_1}\psi \wedge V^{a_2} \wedge V^{a_3} \wedge V^{a_4} - \bar\psi \wedge \Gamma_{a_1a_2} H_{a_3}\psi \wedge V^{a_1} \wedge V^{a_2} \wedge V^{a_3} = 0 \,. \tag{III.8.44}$$

Since H_a is a matrix in spinor space it can be expanded in the complete set of 32×32 Γ-matrices: Γ_a, Γ_{ab}, Γ_{abc}, Γ_{abcdf}. Taking into account that we must cancel the first term on the l.h.s. of (III.8.44) one can avoid to write down the most general expression for H_a; the proper ansatz is

$$H_a = a\, F_{ab_1b_2b_3}\Gamma^{b_1b_2b_3} + b\,\Gamma_{ab_1\ldots b_4}F^{b_1\ldots b_4} \,. \tag{III.8.45}$$

In fact $\Gamma_{a_1a_2}H_a$ can contain the 1-index matrix Γ_a only if H_a contains the odd matrices Γ_a, Γ_{abc}, Γ_{abcdf}; however we cannot construct a Lorentz vector from Γ_a and $F_{a_1\ldots a_4}$.

Introducing (III.8.45) into the second term of (III.8.44) and taking into account that

$$\bar\psi \wedge \psi = \bar\psi \wedge \Gamma_{abc}\psi = \bar\psi \wedge \Gamma_{abcd}\psi = 0 \tag{III.8.46}$$

since C, $C\Gamma^{abc}$ and $C\Gamma^{abcdf}$ are antisymmetric matrices, one easily finds:

$$(a + 8b)\, F_{ab_1b_2b_3}\bar\psi \wedge \Gamma^{a_1a_2b_1b_2b_3}\psi\, V^a \wedge V^{a_1} \wedge V^{a_2} = 0 \tag{III.8.47}$$

$$(2i-6a)\, F_{a_1\ldots a_4}\bar{\psi}\wedge\Gamma^{a_1}\psi\wedge V^{a_2}\wedge V^{a_3}\wedge V^{a_4}=0 \qquad \text{(III.8.48)}$$

that is,

$$a=\frac{i}{3} \quad ; \quad b=-\frac{i}{24} \, . \qquad \text{(III.8.49)}$$

At this point we can utilize the result just obtained in order to analyze the $2\psi-6V$ content of the Bianchi identity (III.8.32j). One finds:

$$\begin{aligned}&\frac{7}{2}\, i\, G_{a_1\ldots a_7}\bar{\psi}\wedge\Gamma^{a_1}\psi\wedge V^{a_2}\wedge\ldots\wedge V^{a_7}+\\&+\frac{1}{3}\bar{\psi}\wedge\Gamma^{a_1\ldots a_5}\,(F^{ab_1b_2b_3}{}_{b_1b_2b_3}-\\&-\frac{1}{8}F_{b_1\ldots b_4}\Gamma^{ab_1\ldots b_4})\psi\wedge V_a\wedge V_{a_1}\wedge\ldots\wedge V_{a_5}-\\&-15\,\bar{\psi}\wedge\Gamma_{ab}\psi\wedge F_{b_1\ldots b_4}V^a\wedge V^b\wedge V^{b_1}\wedge\ldots\wedge V^{b_4}=0\end{aligned} \qquad \text{(III.8.50)}$$

where we used Eqs. (III.8.37,45,49) and the relation

$$\mathcal{D}V^a=\frac{i}{2}\bar{\psi}\wedge\Gamma^a\psi \qquad \text{(III.8.51)}$$

which follows from (III.8.32b) at $R^a=0$.

Reducing the products of Γ-matrices, and using the identities (III.8.46), one finds three equations corresponding to the separate vanishing of the coefficients of the currents $\bar{\psi}\wedge\Gamma^{a_1a_2}\psi$, $\bar{\psi}\wedge\Gamma^{a_1\ldots a_6}\psi$ and $\bar{\psi}\wedge\Gamma^a\psi$.

The first two equations are identically satisfied while for the coefficient of $\bar{\psi}\wedge\Gamma^a\psi$ one finds that it is zero only if the fields $G_{a_1\ldots a_7}$ and $F_{a_1\ldots a_4}$ are dual to each other, namely if:

$$G_{a_1\ldots a_7}=\frac{1}{84}\,\varepsilon_{a_1\ldots a_7\, b_1\ldots b_4}F^{b_1\ldots b_4}\, . \qquad \text{(III.8.52)}$$

Equation (III.8.52) is an (algebraic) constraint between the space-time components of $R^{\otimes}$ and $R^{\square}$; it tells us that the field B is actually redundant in the description of the physical propagation on the D=11 space-time M_{11}. This is exactly what we expected according to the discussion of the previous section.

Introducing Eq. (III.8.52) into the space-time projection of the same Bianchi (III.8.32j) (8V-projection) one easily finds the further constraint:

$$\mathcal{D}_m F^{ma_1a_2a_3}+\frac{1}{96}F^{b_1\ldots b_4}F^{c_1\ldots c_4}\varepsilon_{b_1\ldots b_4\, c_1\ldots c_4a_1a_2a_3}=0\, . \qquad \text{(III.8.53)}$$

This constraint, being differential rather than algebraic, is a true propagation equation for the 3-form A on space-time. We call it the "Maxwell equation" for $A_{a_1a_2a_3}$. We notice the presence of the characteristic non-linear term F F which is not present in D=4 supergravities. An analogous term is present in D=5, see Table III.5.VI. It is a simple exercise to verify that the use of the equations (III.8.13,14) defining the smaller (gauged) F.D.A. <u>without</u> the 6-form B give the propagation equation (III.8.53) only as a spinorial derivative* of the fermionic equation. Thus the use of the extended algebra containing the 6-form B allows a more direct derivation of III.8.53.

To complete the solution of the Bianchi identities we still have to determine the tensorial structures $\theta^{ab}{}_c$, K^{ab} and to retrieve the space time equations of V^a_μ and ψ_μ. The procedure is quite similar to that used in the D=4 cases, so we just sketch it.

From the ψVV projection of (III.8.32g) we get:

$$\bar{\theta}^{ab}{}_c\psi\wedge V_b\wedge V^c-i\,\bar{\psi}\wedge\Gamma^a\rho_{bc}V^b\wedge V^c=0\, . \qquad \text{(III.8.54)}$$

This equation is exactly analogous to (III.3.216) (the only difference being that the spinor space is 32-dimensional), so that it has exactly the same solution:

* By spinorial derivative we mean the 1-ψ component of the exterior differential d.

$$\Theta_{ab|c} = i(\Gamma_a \rho_{bc} - \Gamma_b \rho_{ac} - \Gamma_c \rho_{ab}) . \qquad \text{(III.8.55)}$$

Considering now the $\psi\psi V$ projection of the torsion-Bianchi (III.8.32g), using the explicit form of H_a, one easily finds:

$$\bar\psi \wedge K^{ab}\psi \wedge V_b - \frac{1}{3} (\bar\psi \wedge \Gamma^a \Gamma^{a_1a_2a_3}\psi F_{a_1a_2a_3b} + \frac{1}{8} \bar\psi \wedge \Gamma^a \Gamma^{a_1\dots a_4 b}\psi F_{a_1\dots a_4}) \wedge V^b = 0 \qquad \text{(III.8.56)}$$

that is, after simple Γ-algebra:

$$\bar\psi \wedge K^{ab}\psi - \frac{1}{3} (3\,\bar\psi \wedge \Gamma_{mn}\psi F^{abmn} + \frac{1}{8} \bar\psi \wedge \Gamma^{aba_1\dots a_4}\psi F_{a_1\dots a_4}) = 0 \qquad \text{(III.8.57)}$$

and therefore

$$K^{ab} = \Gamma_{mn} F^{mnab} + \frac{1}{24} \Gamma^{aba_1\dots a_4} F_{a_1\dots a_4} . \qquad \text{(III.8.58)}$$

Thus the rheonomic parametrization of the curvatures is completely determined. To prove the consistency of our solution one must also check all the outer projections of the Bianchi identities. In so doing one finds that some of them simply reproduce the previous results, while the other ones are satisfied only if ρ^{ab} and $R^{ab}{}_{cd}$ satisfy the space-time differential constraints that we can identify as the space-time field equations of V^a_μ and ψ_μ; namely one finds:

$$R^{am}{}_{bm} - \frac{1}{2} \delta^a_b R^{mn}{}_{mn} = 3 F^{ac_1c_2c_3} F_{bc_1c_2c_3} - \frac{3}{8} \delta^a_b F^{c_1\dots c_4} F_{c_1\dots c_4} \qquad \text{(III.8.59a)}$$

$$\Gamma^{abc} \rho_{bc} = 0 . \qquad \text{(III.8.59b)}$$

The derivation of these equations from the analysis of Bianchi identities follows the same pattern as in the case of D=4 N=1 supergravity which was illustrated in Sect. III.3.12 (see Eqs. III.3.220 and following). Indeed also in the present case Eq. (III.8.59b) is derived from the $\psi\psi V$ sector of the gravitino-Bianchi (III.8.32h); one finds:

$$i\,\rho_{ab}\,\bar\psi \wedge \Gamma^a\psi \wedge V^b - \psi \wedge \bar\psi \,(\frac{i}{3} \Gamma^{b_1b_2b_3} L_{ab_1b_2b_3} - \frac{i}{24} \Gamma_{ab_1\dots b_4} L_{b_1b_2b_3b_4}) + \frac{1}{4} \Gamma_{ab}\psi \wedge \bar\psi\, \Theta^{ab}{}_c \wedge V^c = 0 \qquad \text{(III.8.60a)}$$

$L_{a_1\dots a_4}$ being defined by

$$\mathcal{D} F_{a_1\dots a_4} = \mathcal{D}_{[\ell} F_{a_1\dots a_4]} V^\ell + L_{a_1\dots a_4}\psi \qquad \text{(III.8.60b)}$$

We see that the only new feature with respect to Eq. (III.3.220) is the presence of $L_{a_1\dots a_4}$, the spinor derivative of $F_{a_1\dots a_4}$. This however can be immediately computed from the $4V - 1\psi$ vector of the Bianchi Eq. (III.8.32i); one finds:

$$L_{a_1\dots a_4} = \Gamma_{[a_1a_2} \rho_{a_3a_4]} .$$

Inserting in (III.8.60a) one obtains after some Γ-matrix algebra Eq. (III.8.59b). We leave as an exercise to derive (III.8.59a) by a supersymmetry transformation of Eq. (III.8.59b). A quicker way to derive them is of course from the variational principle once we have constructed the Lagrangian. We turn our attention to this problem.

III.8.4 - The action of D=11 supergravity

In the previous section we showed that the on-shell D=11 supergravity theory is completely determined by the D=11 F.D.A. supplemented with the requirement of rheonomy. As far as most of the applications of D=11 supergravity are concerned, in particular the search of compactifying solutions, which will be the main object of the next part, this is enough. It is however important for many purposes, including the study of the quantum properties of the theory, to have an action from which the previously determined parametrization of the curvatures and space-time propagation equations are derived as superspace variational equations.

We proceed by applying the building rules of Sects. III.3.9 and III.6.4. Assuming that no 0-form is present we consider the most general "strongly geometrical" Lagrangian according to the general formula (III.3.148). We have:

$$\mathcal{A} = \int_{M_{11} \subset R^{11/32}} (\Lambda + R^A \wedge \nu_A + R^A \wedge R^B \wedge \nu_{AB}) \ . \qquad \text{(III.8.61)}$$

The "adjoint" index $A \equiv (ab, a, \alpha, \square, \otimes)$ runs over the set of indices labelling the curvatures (III.8.32a-32e); Λ, ν_A and ν_{AB} are polynomials in the minimal generators of the F.D.A., namely $\{\omega^{ab}, V^a, \psi, A, B\}$, whose form-degree is 11, 9 and 7 respectively. They sit in the scalar, coadjoint and coadjoint $\otimes$ coadjoint representation respectively. (What is meant by adjoint and coadjoint in the context of a F.D.A. has been explained in Sect. III.6.4). Finally the integration is performed over an 11-dimensional hypersurface floating in the superspace $R^{11/32}$ whose cotangent basis is spanned by the set $\{V^a, \psi\}$. We assume factorization of the variables associated to the ω^{ab}, A, and B-directions. This we can do since the F.D.A. is invariant under the combined gauge invariance SO(1,10) $\otimes$ the gauge invariance (III.8.26,27). According to the building rule B), this invariance has to be preserved in the construction of the action. (It was indeed preserved in the analysis of the Bianchi identities of previous section).

In principle one expects having to add to $\mathcal{A}$ a piece containing an extra 0-form field $F_{a_1 \ldots a_4}$ in order to be able to write the kinetic term for the propagation of the physical $A_{a_1 a_2 a_3}$ field (in analogy to what has been done, for example, in D=4, N=2 supergravity to generate the kinetic term of the Maxwell 1-form A). For the moment, however, we restrict ourselves to the consideration of the strongly geometrical Lagrangian without any addition of extra terms. The reason is the following: in principle it might be possible to generate the kinetic term for the 3-form A using only the strongly geometrical Lagrangian (III.8.61) in analogy to what happens in D=5 and D=6. Indeed, among the possible terms contained in $R^A \wedge R^B \wedge \nu_{AB}$ we might write

$$R^{\otimes} \wedge R^{\square} \ . \qquad \text{(III.8.62)}$$

The term (III.8.62) is gauge invariant under the SO(1,10) gauge transformations and under the gauge transformations (III.8.26,27). Furthermore it scales as $[w^9]$, the scale power of the Einstein term. Since, as we know from the analysis of the previous section, the space-time components of $R^{\otimes}$ and $R^{\square}$ are dual to each other (see Eq. (III.8.52)), the term (III.8.62) gives rise to the kinetic term

$$\text{const} \times F_{a_1 \ldots a_4} F^{a_1 \ldots a_4} \qquad \text{(III.8.63)}$$

once the field $G_{a_1 \ldots a_7}$ has been eliminated by use of Eq. (III.8.52).

Unfortunately this beautiful mechanism does not work since, as it will be shown in the following, the field B and its curvature enter the Lagrangian (III.8.61) only through a total derivative. Therefore the 6-form B does not contribute to the equations of motion and the duality relation (III.8.52) cannot be retrieved from the variational principle.

We begin by determining the linear part of the Lagrangian (III.8.61). We can write:

$$\mathcal{L}^{(\text{linear})} = \Lambda + R^A \wedge \nu_A \equiv \Lambda + R^{ab} \wedge \nu_{ab} + R^a \wedge \nu_a +$$
$$+ \bar{\rho} \wedge \eta + R^{\square} \wedge \nu_{\square} + R^{\otimes} \wedge \nu_{\otimes} \ . \qquad \text{(III.8.64)}$$

Since the l.h.s. of Eq. (III.8.32c) contains the curvature $R^{\square}$, the last two terms of Eq. (III.8.64) can also be rewritten as:

$$R^{\square} \wedge \nu_{\square} + R^{\otimes} \wedge \nu_{\otimes} = R^{\square} \wedge \hat{\nu}_{\square} + \hat{R}^{\otimes} \wedge \nu_{\otimes} \qquad \text{(III.8.65)}$$

where

$$\hat{\nu}_{\square} = \nu_{\square} - 15\, \nu_{\otimes} \wedge A \qquad \text{(III.8.66)}$$

and where $\hat{R}^{\otimes}$ has been defined in Eq. (III.8.19a).

From the scaling properties of the curvatures (III.8.39) and from the fact that the Einstein term

$$R^{ab} \wedge V^{c_1} \wedge \dots \wedge V^{c_9}\, \varepsilon_{abc_1 \dots c_9} \qquad \text{(III.8.67)}$$

scales as $[w^9]$ we see that the polynomials ν_A must scale as follows:

$$\begin{aligned} [\nu_{ab}] &= [w^9] \quad ; \quad [\Lambda] = [w^9] \\ [\nu_a] &= [w^8] \quad ; \quad [\hat{\nu}_{\square}] = [\nu_{\square}] = [w^6] \\ [n] &= [w^{8+\frac{1}{2}}] \quad ; \quad [\nu_{\otimes}] = [w^4] \; . \end{aligned} \qquad \text{(III.8.68)}$$

Taking into account (III.8.68) and the SO(1,10) gauge invariance, the proper ansatz for the polynomials Λ, ν_A is given by:

$$\begin{aligned} \Lambda = {} & a\bar{\psi} \wedge \Gamma^{a_1 a_2}\psi \wedge \bar{\psi} \wedge \Gamma^{a_3 a_4}\psi \wedge V^{a_5} \wedge \dots \wedge V^{a_{11}}\, \varepsilon_{a_1 \dots a_{11}} + \\ & + b\bar{\psi} \wedge \Gamma^{a_1 a_2}\psi \wedge \bar{\psi} \wedge \Gamma^{a_3 a_4}\psi \wedge V_{a_1} \wedge \dots \wedge V_{a_4} \wedge A \end{aligned} \qquad \text{(III.8.69a)}$$

$$\nu_{ab} = -\frac{1}{9}\, \varepsilon_{abc_1 \dots c_9}\, V^{c_1} \wedge \dots \wedge V^{c_9} \qquad \text{(III.8.69b)}$$

$$\begin{aligned} \nu_a = {} & i\,\beta_1 V_a \wedge \bar{\psi} \wedge \Gamma^{c_1 \dots c_5}\psi \wedge V^{c_6} \wedge \dots \wedge V^{c_{11}}\, \varepsilon_{c_1 \dots c_{11}} + \\ & + \beta_2 \bar{\psi} \wedge \Gamma^a \psi \wedge V_a \wedge B + i\,\beta_3 \bar{\psi} \wedge \Gamma_{ac_1 \dots c_4}\psi \wedge V^{c_1} \wedge \dots \wedge V^{c_4} \wedge A \end{aligned} \qquad \text{(III.8.69c)}$$

$$\begin{aligned} n = {} & h_1\, \Gamma_{c_1 \dots c_8}\psi \wedge V^{c_1} \wedge \dots \wedge V^{c_8} + h_2\, \Gamma_{ab}\psi \wedge V^a \wedge V^b \wedge B + \\ & + i\, h_3\, \Gamma_{c_1 \dots c_5}\psi \wedge V^{c_1} \wedge \dots \wedge V^{c_5} \wedge A \end{aligned} \qquad \text{(III.8.69d)}$$

$$\begin{aligned} \hat{\nu}_{\square} = {} & i\, k_1 \bar{\psi} \wedge \Gamma_{a_1 \dots a_5}\psi \wedge V^{a_1} \wedge \dots \wedge V^{a_5} + \\ & + k_2 \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \wedge A \end{aligned} \qquad \text{(III.8.69e)}$$

$$\nu_{\otimes} = k_3 \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \, . \qquad \text{(III.8.69f)}$$

Actually there are more possible 4-ψ structures which could enter the general expression for Λ; it is easy to verify that using the 4-ψ identities given in Table II.8.XI all of them may be reduced to those introduced in Eq. (III.8.69a).

Applying the same criteria of correct $[w^9]$ scaling and SO(1,10) gauge invariance to the quadratic part of Eq. (III.8.61) we easily see that just two terms are allowed:

$$R^A \wedge R^B \wedge \nu_{AB} = \gamma_1\, R^{\square} \wedge R^{\square} \wedge A + \gamma\, R^{\square} \wedge R^{\otimes} \; . \qquad \text{(III.8.70)}$$

Actually there are some redundancies in the ansatz (III.8.69). Indeed, some of the so far undetermined coefficients appearing in (III.8.69) can be set equal to zero by adding to the Lagrangian a total derivative. We proceed as in the 5-dimensional case (see Sect. III.5.3, Eqs. (III.5.51-54)). Let us consider the possible Lorentz invariant 10-forms whose scaling weight is $[w^9]$: there are just two of them, namely:

$$\phi_1 = i\, \bar{\psi} \wedge \Gamma^{a_1 \dots a_5}\psi \wedge V_{a_1} \wedge V_{a_5} \wedge A \qquad \text{(III.8.71a)}$$

$$\phi_2 = \bar{\psi} \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b \wedge B\,. \qquad \text{(III.8.71b)}$$

By differentiating the linear combination $\alpha_1\phi_1 + \alpha_2\phi_2$ we obtain:

$$\begin{aligned} d(\alpha_1\phi_1 + \alpha_2\phi_2) = & -2i\,\alpha_1\bar{\psi} \wedge \Gamma_{a_1\ldots a_5}\rho \wedge V^{a_1} \wedge \ldots \wedge V^{a_5} \wedge A + \\ & + 5i\,\alpha_1\bar{\psi} \wedge \Gamma_{a_1\ldots a_5}\psi \wedge R^{a_1} \wedge V^{a_2} \wedge \ldots \wedge V^{a_5} \wedge A - \\ & - i\,\alpha_1 R^{\square} \wedge \bar{\psi} \wedge \Gamma_{a_1\ldots a_5}\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_5} - \\ & - \frac{1}{112}(\alpha_2 - \alpha_1)\varepsilon_{a_1\ldots a_7\, b_1\ldots b_4}\bar{\psi} \wedge \Gamma^{b_1b_2}\psi \wedge \bar{\psi} \wedge \\ & \quad \wedge \Gamma^{b_3b_4}\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_7} - \\ & - 2\,\alpha_2\bar{\psi} \wedge \Gamma_{ab}\rho \wedge V^a \wedge V^b \wedge B + \\ & + 2\,\alpha_2\bar{\psi} \wedge \Gamma_{ab}\psi \wedge R^a \wedge V^b \wedge B + \\ & + \alpha_2\,\hat{R}^{\otimes} \wedge \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b + (15i\,\alpha_1 + \\ & + \frac{15}{2}\alpha_2)\,\bar{\psi} \wedge \Gamma_{a_1a_2}\psi \wedge \bar{\psi} \wedge \Gamma_{a_3a_4}\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_4} \wedge A \end{aligned} \qquad \text{(III.8.72)}$$

where we have used the definition (III.8.32a-e), the relation $R^{\otimes} = \hat{R}^{\otimes} - 15\,R^{\square} \wedge A$ and the Fierz identities

$$\bar{\psi} \wedge \Gamma^{a_1\ldots a_4 m}\psi \wedge \bar{\psi} \wedge \Gamma_m\psi = 3\,\bar{\psi} \wedge \Gamma^{[a_1a_2}\psi \wedge \bar{\psi} \wedge \Gamma^{a_3a_4]}\psi \qquad \text{(III.8.73a)}$$

$$\bar{\psi} \wedge \Gamma^{am}\psi \wedge \bar{\psi} \wedge \Gamma_m\psi = 0 \qquad \text{(III.8.73b)}$$

$$\begin{aligned} & \bar{\psi} \wedge \Gamma_{[a_1\ldots a_5}\psi \wedge \bar{\psi} \wedge \Gamma_{a_6a_7]}\psi = \\ & = \frac{i}{56}\varepsilon_{a_1\ldots a_7\, b_1\ldots b_4}\bar{\psi} \wedge \Gamma^{b_1b_2}\psi \wedge \bar{\psi}\,\Gamma^{b_3b_4}\psi \end{aligned} \qquad \text{(III.8.73c)}$$

which are simple consequences of the relations given in Table II.8.XI.

By adding the total derivative (III.8.72) to the Lagrangian (III.8.61) and using the arbitrariness of α_1 and α_2 we can cancel any two of the terms in the linear part of the Lagrangian whose coefficients are a, b, k_1, k_2, k_3, h_2, h_3, β_2, β_3. We make the (arbitrary) choice:

$$\beta_2 = \beta_3 = 0 \qquad \text{(III.8.74)}$$

in the ansatz (III.8.69).

Next we implement the building rule B, namely the gauge invariance of the action under the transformation (III.8.26,27).

To achieve this it is convenient to have the linear part of the Lagrangian (III.8.64) written in terms of $R^{\otimes}$ rather than of $\hat{R}^{\otimes}$ since the former is gauge invariant while the latter is not. Performing the variation (III.8.26,27) one finds:

$$\begin{aligned} \delta\mathscr{A} = \int\Big\{ & \big| b\bar{\psi} \wedge \Gamma_{a_1a_2}\psi \wedge \bar{\psi} \wedge \Gamma_{a_3a_4}\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_4} + \\ & + k_2\,R^{\square} \wedge \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b + \\ & + \gamma_1\,R^{\square} \wedge R^{\square} + i\,h_3\bar{\rho} \wedge \Gamma_{c_1\ldots c_5}\psi \wedge V^{c_1} \wedge \ldots \wedge V^{c_5} + \\ & + 15\,k_3\,R^{\square} \wedge \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b\big| \wedge d\,\varphi^{\square} + \\ & + h_2\,\Gamma_{ab}\psi \wedge V^a \wedge V^b(d\,\varphi^{\otimes} + \frac{15}{2}\varphi^{\square} \wedge \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b + \\ & + 15\,\varphi^{\square} \wedge R^{\square})\Big\} = 0\,. \end{aligned} \qquad \text{(III.8.75)}$$

After partial integration of $d\varphi^{\square}$ and $d\varphi^{\otimes}$ one finds that $\delta\mathscr{A}$ is zero if

$$h_2 = h_3 = 0 \qquad \text{(III.8.76a)}$$

$$4b = \gamma_1 = 15\,k_3\,. \qquad \text{(III.8.76b)}$$

The "vacuum condition" (building rule D) is sufficient to fix the coefficients of the linear part of the Lagrangian so far undetermined. We can express it either in the form (III.3.158):

$$\frac{\delta\Lambda}{\delta\mu^{A}} + \nabla\nu_{A} = 0 \quad (\text{at } R^{A}=0) \tag{III.8.77}$$

the set of the μ_A's being $\mu^{A} \equiv (\omega^{ab}, V^{a}, \psi, A, B)$, or, what amounts to the same thing, by writing down the equations of motion and keeping only the terms of 0th-order in the curvatures. Using the former prescription we must first compute the covariant derivative of the coadjoint multiplet ν_A according to Eq. (III.6.52). The easiest way to arrive at its explicit form is to use $H^{A} \equiv R^{A}$ in (III.6.52); indeed from:

$$\nabla(R^{A} \wedge \nu_{A}) = d(R^{A} \wedge \nu_{A}) \tag{III.8.78}$$

using the Bianchi identities (III.8.32f-j) and equating the coefficients of the curvatures in both sides, one easily finds:

$$\nabla\nu_{ab} = \mathscr{D}\nu_{ab} + V_{[a} \wedge \nu_{b]} + \frac{1}{4}\bar{\psi} \wedge \Gamma_{ab}{}^{n} \tag{III.8.79a}$$

$$\begin{aligned}\nabla\nu_{a} = {} & \mathscr{D}\nu_{a} - \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^{b} \wedge \nu_{\Box} - \\ & - \frac{5}{2} i\, \bar{\psi} \wedge \Gamma_{ab_1\dots b_4}\psi \wedge V^{b_1} \wedge \dots \wedge V^{b_4} \wedge \nu_{\otimes} - \\ & - 15\, \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^{b} \wedge A \wedge \nu_{\otimes}\end{aligned} \tag{III.8.79b}$$

$$\begin{aligned}\nabla n = {} & \mathscr{D} n - i\, \Gamma^{a}\psi \wedge \nu_{a} - \Gamma_{a_1 a_2}\psi \wedge V^{a_1} \wedge V^{a_2} \wedge \nu_{\Box} - \\ & - i\, \Gamma^{a_1\dots a_5}\psi \wedge V_{a_1} \wedge \dots \wedge V_{a_5} \wedge \nu_{\otimes} - \\ & - 15\, \Gamma_{ab}\psi \wedge V^{a} \wedge V^{b} \wedge A \wedge \nu_{\otimes}\end{aligned} \tag{III.8.79c}$$

$$\nabla\nu_{\otimes} = d\,\nu_{\otimes} \tag{III.8.79d}$$

$$\nabla\nu_{\Box} = d\nu_{\Box} - \frac{15}{2}\bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^{a} \wedge V^{b} \wedge \nu_{\otimes} \ . \tag{III.8.79e}$$

Substituting in the last formulae the explicit form of the multiplet ν_A given by Eqs. (III.8.69) (after the terms in h_2, h_3, β_2, β_3 have been deleted according to Eqs. (III.8.74) and (III.8.75a)) one easily arrives at the following set of equations for the existence of the solution $R^{A} = 0$:

$$\frac{\delta\Lambda}{\delta\omega^{ab}} + \nabla\nu_{ab} = 0 \quad (R^{A}=0):$$

$$\begin{aligned}\Rightarrow \quad & -\frac{i}{2}\varepsilon_{abc_1\dots c_9}\bar{\psi} \wedge \Gamma^{c_1}\psi \wedge V^{c_2} \wedge \dots \wedge V^{c_9} + \\ & + i\, \beta_1 \bar{\psi} \wedge \Gamma^{a_1\dots a_5}\psi \wedge V^{a_6} \wedge \dots \wedge V^{a_{11}} \wedge V^{a} \wedge V^{b}\, \varepsilon_{a_1\dots a_{11}} + \\ & + \frac{1}{4} h_1 \bar{\psi} \wedge \Gamma_{ab}\Gamma_{a_1\dots a_8}\psi \wedge V^{a_1} \wedge \dots \wedge V^{a_8} = 0\end{aligned} \tag{III.8.80}$$

$$\frac{\delta\Lambda}{\delta V^{a}} + \nabla\nu_{a} = 0 \quad (R^{A}=0):$$

$$\begin{aligned}\Rightarrow \quad & -\frac{1}{2}\beta_1 \bar{\psi} \wedge \Gamma^{a_1\dots a_5}\psi \wedge \bar{\psi} \wedge \Gamma_{a}\psi \wedge V^{a_6} \wedge \dots \wedge V^{a_{11}}\, \varepsilon_{a_1\dots a_{11}} + \\ & + 3\, \beta_1 \bar{\psi} \wedge \Gamma^{a_1\dots a_5}\psi \wedge \bar{\psi} \wedge \Gamma^{a_6}\psi \wedge V_{a} \wedge V^{a_7} \wedge \dots \wedge V^{a_{11}}\, \varepsilon_{a_1\dots a_{11}} - \\ & - k_3 \bar{\psi} \wedge \Gamma_{c_1 c_2}\psi \wedge \Big(\frac{5}{2} i\, \bar{\psi} \wedge \Gamma_{aa_1\dots a_4}\psi \wedge V^{a_1} \wedge \dots \wedge V^{a_4} + \\ & + 15\, \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^{b} \wedge A\Big) \wedge V^{c_1} \wedge V^{c_2} - \\ & - \big(i\, k_1 \bar{\psi} \wedge \Gamma_{a_1\dots a_5}\psi \wedge V^{a_1} \wedge \dots \wedge V^{a_5} + \\ & + k_2 \bar{\psi} \wedge \Gamma_{c_1 c_2}\psi \wedge A \wedge V^{c_1} \wedge V^{c_2}\big)\, \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^{b} + \\ & + 7a\, \varepsilon_{aa_1\dots a_{10}}\bar{\psi} \wedge \Gamma^{a_1 a_2}\psi \wedge \bar{\psi} \wedge \Gamma^{a_3 a_4}\psi \wedge V^{a_5} \wedge \dots \wedge V^{a_{10}} + \\ & + 4b\bar{\psi} \wedge \Gamma^{aa_1}\psi \wedge \bar{\psi} \wedge \Gamma^{a_2 a_3}\psi \wedge V_{a_1} \wedge V_{a_2} \wedge V_{a_3} \wedge A = 0\end{aligned} \tag{III.8.81}$$

$$\frac{\delta\Lambda}{\delta\bar{\psi}} + \nabla\eta = 0 \quad (R^A = 0):$$

$$\Rightarrow \quad 4a\, \Gamma^{a_1a_2}\psi \wedge \bar{\psi} \wedge \Gamma^{a_3a_4}\psi \wedge V^{a_5} \wedge \ldots \wedge V^{a_{11}}\, \varepsilon_{a_1\ldots a_{11}} +$$
$$+ 4b\, \Gamma^{a_1a_2}\psi \wedge \bar{\psi} \wedge \Gamma^{a_3a_4}\psi \wedge V_{a_1} \wedge \ldots \wedge V_{a_4} \wedge A -$$
$$- 4i\, h_1\, \Gamma_{a_1\ldots a_8}\bar{\psi} \wedge \Gamma^{a_1}\psi \wedge V^{a_2} \wedge \ldots \wedge V^{a_8} +$$
$$+ \beta_1\, \Gamma_a\psi \wedge \bar{\psi} \wedge \Gamma^{a_1\ldots a_5}\psi \wedge V^a \wedge V^{a_6} \wedge \ldots \wedge V^{a_{11}}\, \varepsilon_{a_1\ldots a_{11}} -$$
$$- ik_3\, \Gamma_{a_1\ldots a_5}\psi \wedge \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_5} \wedge V^a \wedge V^b -$$
$$- 15\, k_3\, \Gamma_{a_1a_2}\psi \wedge \bar{\psi} \wedge \Gamma_{a_3a_4}\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_4} \wedge A -$$
$$- \Gamma_{a_1a_2}\psi \wedge (ik_1\bar{\psi} \wedge \Gamma_{b_1\ldots b_5}\psi \wedge V^{b_1} \wedge \ldots \wedge V^{b_5} +$$
$$+ k_2\bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \wedge A) \wedge V^{a_1} \wedge V^{a_2} = 0 \qquad \text{(III.8.82)}$$

$$\frac{\delta\Lambda}{\delta A} + \nabla\nu_{\square} = 0 \quad (R^A = 0):$$

$$\Rightarrow \quad b\, \bar{\psi} \wedge \Gamma^{a_1a_2}\psi \wedge \bar{\psi} \wedge \Gamma^{a_3a_4}\psi \wedge V_{a_1} \wedge \ldots \wedge V_{a_4} -$$
$$- \frac{5}{2} k_1 \bar{\psi} \wedge \Gamma_{a_1\ldots a_4 m}\psi \wedge \bar{\psi} \wedge \Gamma^m\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_4} +$$
$$+ \frac{1}{2}(k_2 - 15)\bar{\psi} \wedge \Gamma_{a_1a_2}\psi \wedge \bar{\psi} \wedge \Gamma_{a_3a_4}\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_4} = 0$$
$$\text{(III.8.83)}$$

$$\frac{\delta\Lambda}{\delta B} + \nabla\nu_{\otimes} = 0 \quad (R^A = 0):$$

$$\Rightarrow \quad i\, k_3\bar{\psi} \wedge \Gamma_{ab}\psi \wedge \bar{\psi} \wedge \Gamma^a\psi \wedge V^b = 0 \ . \qquad \text{(III.8.84)}$$

By separately setting to zero the different independent structures appearing in Eqs. (III.8.80-84) one finds several conditions on the coefficients a, b, h_1, β_1, k_1, k_2, k_3. The actual computation is

somewhat cumbersome since we must use several times the basic properties of the 11-dimensional Γ-matrices given in Chapter II.7; of particular use is the Γ-matrices product expansion formula, Eq. (II.7.37), the dualization formula

$$\varepsilon_{a_1\ldots a_n\, b_1\ldots b_m}\Gamma^{b_1\ldots b_m} = (-1)^{\frac{m(m-1)}{2}}(-i)m!\,\Gamma_{a_1\ldots a_n} \quad (m+n=11) \qquad \text{(III.8.85)}$$

and the symmetry properties of the Γ-matrices (see Table II.7.III)), implying that only Γ^a, Γ^{ab}, $\Gamma^{a_1\ldots a_5}$ and their dual give rise to non-vanishing 2ψ-currents $\bar{\psi} \wedge \Gamma^{(n)}\psi$.

Moreover, except for (III.8.80), the above equations involve 3ψ and 4ψ products which in general may have linear dependences among themselves (Fierz identities). Therefore we also need to make repeated use of the formulas given in Table II.8.VI in order to annihilate only the coefficients of the independent 3ψ- or 4ψ-structures. Let us now sketch the derivation of the resulting relations for each of the equations (III.8.80-84).

From (III.8.80), after reduction of the structure $\bar{\psi} \wedge \Gamma_{ab}\Gamma_{a_1\ldots a_8}\psi$, one finds two independent structures:

$$\varepsilon_{abc_1\ldots c_9}\bar{\psi} \wedge \Gamma^{c_1}\psi \wedge V^{c_2} \wedge \ldots \wedge V^{c_9} \qquad \text{(III.8.86a)}$$

$$V^a \wedge V^b \wedge \bar{\psi} \wedge \Gamma^{c_1\ldots c_5}\psi \wedge V^{c_6} \wedge \ldots \wedge V^{c_{11}}\,\varepsilon_{c_1\ldots c_{11}} \ . \qquad \text{(III.8.86b)}$$

Setting their coefficients equal to zero one obtains:

$$h_1 = 2 \qquad \text{(III.8.87a)}$$

$$\beta_1 = 14/5!\ h_1 \ . \qquad \text{(III.8.87b)}$$

In Eq. (III.8.81) it is worth to express all the 4ψ-structures in terms of the irreducible basis $X^{(330)}_{a_1\ldots a_4}$, $X^{(462)}_{f_1\ldots f_5}$ and $X^{(4920)}_{f_1\ldots f_5 a}$ and systematically use the 4ψ relations given in Table II.8.XI.

One finds that the irreducible structures entering Eq. (III.8.81) are

$$X^{(330)}_{ac_1c_2c_3}\, V^{c_1} \wedge V^{c_2} \wedge V^{c_3} \wedge A \qquad \text{(III.8.88a)}$$

$$X^{(330)}_{f_1\ldots f_4}\, V_{f_5} \wedge \ldots \wedge V_{f_{10}}\, \varepsilon^{af_1\ldots f_{10}} \qquad \text{(III.8.88b)}$$

$$X^{(462)}_{f_1\ldots f_5} \wedge V^{f_1} \wedge \ldots \wedge V^{f_5} \wedge V^a \qquad \text{(III.8.88c)}$$

$$X^{(4290)}_{f_1\ldots f_5\, a}\, V_{f_6} \wedge \ldots \wedge V_{f_{11}}\, \varepsilon^{f_1\ldots f_{11}} \qquad \text{(III.8.88d)}$$

and, correspondingly one finds the following relations:

$$4b - k_2 - 15\, k_3 = 0 \qquad \text{(III.8.89a)}$$

$$7a - \frac{15}{14}\, \beta_1 + \frac{1}{56}\, k_1 + \frac{5}{112}\, k_3 = 0 \qquad \text{(III.8.89b)}$$

$$-5\, k_3 + 5\, k_1 + 6!\, \frac{5}{2}\, \beta_1 = 0 \qquad \text{(III.8.89c)}$$

$$-\frac{\beta_1}{2} - \frac{5}{3600}\, k_1 + \frac{5}{3600}\, k_3 = 0\,. \qquad \text{(III.8.89d)}$$

In Eq. (III.8.82), (the gravitino field equation at zero curvature) one must analogously use the irreducible basis of the 3ψ-structures given by the Ξ's 3ψ-forms of the Table II.8.XI. For example, the terms with coefficient h_1 can be reduced in the following way:

$$\Gamma_{a_1\ldots a_8}\psi \wedge \bar\psi \wedge \Gamma^{a_1}\psi = \Gamma_{a_1\ldots a_8}\, (\Xi^{a_1}_{(320)} + \frac{1}{11}\, \Gamma^{a_1}\, \Xi_{(32)}) =$$

$$= (-\,\Gamma_{a_2\ldots a_8}\, \Gamma_{a_1} + 7\, \delta^{a_1}_{[a_2}\, \Gamma_{a_3\ldots a_8]})(\Xi^{a_1}_{(320)} + \frac{1}{11}\, \Gamma^{a_1}\, \Xi_{(32)}) =$$

$$= 7\, \Gamma_{[a_2\ldots a_7}\, \Xi^{(320)}_{a_8]} - \Gamma_{a_2\ldots a_8}\, \Xi^{(32)} + \frac{7}{11}\, \Gamma_{a_2\ldots a_8}\, \Xi^{(32)} =$$

$$= 7\, \Gamma_{[a_2\ldots a_7}\, \Xi^{(320)}_{a_8]} - \frac{4}{11}\, \Gamma_{a_2\ldots a_8}\, \Xi^{(32)}\,. \qquad \text{(III.8.90)}$$

(In passing from the second to the third line we made use of the irreducibility constraint $\Gamma^a\, \Xi^{320}_a = 0$).

Analogously the 3-ψ structure with the β_1-coefficient can be decomposed as follows

$$\Gamma^a\psi \wedge \bar\psi \wedge \Gamma_{a_1\ldots a_5}\psi = \Gamma^a\, \Xi^{(4224)}_{a_1\ldots a_5} + 2(\Gamma_{a|a_1a_2a_3} +$$

$$+ 3\, \delta^a_{[a_1}\, \Gamma_{a_2a_3})\Xi^{(1408)}_{a_4a_5]} + \frac{5}{9}\, (\Gamma^a_{\cdot|a_1\ldots a_4} +$$

$$+ 4\, \delta^a_{[a_1}\, \Gamma_{a_2a_3a_4})\Xi^{(320)}_{a_5]} - \frac{1}{77}\, (\Gamma^a_{\cdot a_1\ldots a_5} +$$

$$+ 5\, \delta^a_{[a_1}\, \Gamma_{a_2\ldots a_5]})\Xi^{(32)}\,. \qquad \text{(III.8.91)}$$

Notice that after multiplication by the vielbein factor in the same term, $V^a \wedge V^{a_6} \wedge \ldots \wedge V^{a_{11}}\, \varepsilon_{a_1\ldots a_{11}}$, the contribution from the irrep <u>4224</u> disappears. Indeed

$$\Gamma^a\, \Xi^{(4224)}_{a_1\ldots a_5} \wedge V_a \wedge V_{a_6} \wedge \ldots \wedge V_{a_{11}}\, \varepsilon^{a_1\ldots a_{11}} =$$

$$= \Gamma^a\, \Xi^{(4224)}_{a_1\ldots a_5}\, \varepsilon_{aa_6\ldots a_{11}\, c_1\ldots c_4}\, \varepsilon^{a_1\ldots a_{11}} \wedge$$

$$\wedge\, (7!4!\, \varepsilon^{c_1\ldots c_4\, b_1\ldots b_6}\, V_{b_1} \wedge \ldots \wedge V_{b_6}) =$$

$$= 6!5!\, \Gamma^a\, \Xi^{(4224)}_{a_1\ldots a_5}\, \delta^{a_1\ldots a_5}_{ac_1\ldots c_4}\, (7!4!\, \varepsilon^{c_1\ldots c_4\, b_1\ldots b_6}\, \times$$

$$\times\, V_{b_1} \wedge \ldots \wedge V_{b_6}) \equiv 0 \qquad \text{(III.8.92)}$$

since the antisymmetrized 5-indexed δ-symbol forces the contraction of a with a_i $(i=1,\dots,5)$ thus giving zero by the irreducibility of the $\Xi_{a_1\dots a_5}$.

The term with the k_1-coefficient can be decomposed by analogous manipulations as in Eq. (III.8.91) and we again find that the irrep 4224 does not contribute.

The remaining structures in (III.8.82) are all computable using the formula

$$\Gamma_{[c_1\dots c_n}\psi \wedge \bar\psi \wedge \Gamma_{a_1a_2]}\psi = \Gamma_{[c_1\dots c_n}\Xi^{(1408)}_{a_1a_2]} - \frac{2}{9}\Gamma_{c_1\dots c_n[a_1}\Xi^{(320)}_{a_2]} + \frac{1}{11}\Gamma_{c_1\dots c_n\, a_1a_2}\Xi^{(32)} \qquad \text{(III.8.93)}$$

which immediately follows from Table II.8.XI.

Introducing all the decompositions thus found into Eq. (III.8.82), after considerable Γ-matrix algebra and tensor calculus one finds that the independent structures to be annihilated are

$$\Gamma_{a_1a_2}\Xi^{(1408)}_{a_3a_4} \wedge V^{a_1} \wedge \dots \wedge V^{a_4} \wedge A \qquad \text{(III.8.94a)}$$

$$\Gamma_{a_1a_2a_3}\Xi^{(320)}_{a_4} \wedge V^{a_1} \wedge \dots \wedge V^{a_4} \wedge A \qquad \text{(III.8.94b)}$$

$$\Gamma_{a_1\dots a_4}\Xi^{(32)} \wedge V^{a_1} \wedge \dots \wedge V^{a_4} \wedge A \qquad \text{(III.8.94c)}$$

$$\Gamma_{[a_1\dots a_5}\Xi^{(1408)}_{a_6a_7]} \wedge V^{a_1} \wedge \dots \wedge V^{a_7} \qquad \text{(III.8.94d)}$$

$$\Gamma_{a_1\dots a_6}\Xi^{(320)}_{a_7} \wedge V^{a_1} \wedge \dots \wedge V^{a_7} \qquad \text{(III.8.94e)}$$

$$\Gamma_{a_1\dots a_7}\Xi^{(32)} \wedge V^{a_1} \wedge \dots \wedge V^{a_7} \qquad \text{(III.8.94f)}$$

The coefficients of the first three structures turn out to be proportional; therefore we get the single equation

$$4b - k_2 - 15\,k_3 = 0\,. \qquad \text{(III.8.95)}$$

Setting to zero the coefficients of (III.8.94d,e,f) one respectively finds:

$$24.5!\,k_1 + (6!)^2\,\beta_2 + 7!4!.4a = 0 \qquad \text{(III.8.96a)}$$

$$-\,28\,h_1 - \frac{5}{9}k_1 + \frac{2}{9}k_3 + 80\,\beta_1 - \frac{112}{3}a = 0 \qquad \text{(III.8.96b)}$$

$$-\,112\,h_1 - k_1 + 7\,k_3 - 120\,\beta_1 + 672\,a = 0\,. \qquad \text{(III.8.96c)}$$

Equations (III.8.95,96) exhaust the content of Eq. (III.8.82).

As far as the Eq. (III.8.83) is concerned it is sufficient to use the fourth 4ψ-Fierz identity of Table II.8.XI to reduce all the terms to the single structure $X^{(330)}_{a_1\dots a_4}\,V^{a_1}\wedge\dots\wedge V^{a_4}$; one immediately obtains:

$$k_2 - 15(k_1 + k_3) + 2b = 0\,. \qquad \text{(III.8.97)}$$

Finally Eq. (III.8.84) is trivial since $\bar\psi \wedge \Gamma_{ab}\psi \wedge \bar\psi \wedge \Gamma^a\psi = 0$ is an immediate yield of the second 4ψ-relation of Table II.8.XI.

Equations (III.8.87), (III.8.89), (III.8.95,96), (III.8.97) and the two relations previously found from gauge invariance namely (III.8.75a,b), constitute altogether a system of 12 linear equations for the 8 unknowns a, b, β_2, h_1, k_1, k_2, k_3, γ_1. Far from being over-determined, the resulting system is in fact undetermined. Actually one easily finds that one can solve the system by expressing 7 of the unknowns in terms of the remaining one. Choosing $k_3 \equiv k$ as the free parameter the solution is:

$$a = \frac{1}{4}\left(1 - \frac{1}{28}k\right) \quad ; \quad b = -15\left(14 - \frac{k}{2}\right)$$

$$\beta_1 = 7/30 \quad ; \quad h_1 = 2$$

$$k_1 = -84 + k \quad ; \quad k_2 = -840\left(1 - \frac{1}{56}k\right)$$

$$\gamma_1 = -840 + 30k \quad . \tag{III.8.98}$$

By substitution of (III.8.98), (III.8.74), and (III.8.75a) into the ansatz (III.8.69) and using the relation (III.8.24) one obtains the following explicit form of the action (III.8.61):

$$\mathscr{A}(k,\gamma) = \int \mathscr{L}(k,\gamma) \tag{III.8.99a}$$

$$\begin{aligned}\mathscr{L}(k,\gamma) = &-\frac{1}{9} R^{a_1 a_2} \wedge V^{a_3} \wedge \ldots \wedge V^{a_{11}} \varepsilon_{a_1 \ldots a_{11}} + \\ &+ \frac{7}{30} i\, R^a \wedge V_a \wedge \bar{\psi} \wedge \Gamma^{b_1 \ldots b_5}\psi \wedge V^{b_6} \wedge \ldots \wedge V^{b_{11}} \varepsilon_{b_1 \ldots b_{11}} + \\ &+ k\, R^{\otimes} \wedge \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b + \\ &+ i(k - 84) R^{\square} \wedge \bar{\psi} \wedge \Gamma_{a_1 \ldots a_5}\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_5} + \\ &+ (30k - 840) R^{\square} \wedge \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \wedge A + \\ &+ 2\bar{\rho} \wedge \Gamma_{c_1 \ldots c_8}\psi \wedge V^{c_1} \wedge \ldots \wedge V^{c_8} + \\ &+ \frac{1}{4}\left(1 - \frac{1}{28}k\right)\bar{\psi} \wedge \Gamma^{a_1 a_2}\psi \wedge \bar{\psi} \wedge \Gamma^{a_3 a_4}\psi \wedge \\ &\quad \wedge V^{a_5} \wedge \ldots \wedge V^{a_{11}} \varepsilon_{a_1 \ldots a_{11}} + \\ &+ 15\left(\frac{k}{2} - 14\right)\bar{\psi} \wedge \Gamma^{a_1 a_2}\psi \wedge \bar{\psi} \wedge \Gamma^{a_3 a_4}\psi \wedge \\ &\quad \wedge V_{a_1} \wedge \ldots \wedge V_{a_4} \wedge A \\ &+ (30k - 840) R^{\square} \wedge R^{\square} \wedge A + \gamma\, R^{\square} \wedge R^{\otimes} \quad . \end{aligned} \tag{III.8.99b}$$

The Lagrangian $\mathscr{L}(k,\gamma)$ contains still two undetermined parameters, k and γ.

III.8.7 - The completion of the action and the equations of motion

Now we show that the action (III.8.99) actually does not depend on the parameters γ, k.

Indeed, let us consider the equations of motion of ω^{ab} and B derived from (III.8.99). We easily get:

Torsion equation: ($\delta\omega^{ab}$-variation)

$$R^{a_1} \wedge V^{a_2} \wedge \ldots \wedge V^{a_9} \varepsilon_{ab\, a_1 \ldots a_9} = 0 \tag{III.8.100}$$

B-field equation:

$$(\gamma - 2k)(\bar{\psi} \wedge \Gamma_{ab}\rho \wedge V^a \wedge V^b - \bar{\psi} \wedge \Gamma_{ab}\psi \wedge R^a \wedge V^b) = 0 \, . \tag{III.8.101}$$

Taking the 9-V projection of (III.8.100) and with the same kind of manipulations as those utilized to obtain (I.4.29) from (I.4.28) one finds: $R^a{}_{mn} = 0$. Since the $V\psi$ and $\psi\psi$ component are obviously zero we get:

$$R^a = 0 \, . \tag{III.8.102}$$

Inserting (III.8.102) into (III.8.101) one finds that if $\gamma \neq 2k$ then Eq. (III.8.101), projected onto the $\psi VVVV$ sector, implies:

$$\Gamma_{[ab}\, \rho_{cd]} = 0 \, . \tag{III.8.103}$$

Together with the space-time gravitino equation $\Gamma^{abc}\rho_{bc} = 0$ which is derived below, Eq. (III.8.103) implies

$$\rho_{ab} = 0 \qquad \text{(III.8.104)}$$

which would mean a trivial theory.

Hence we are forced to set:

$$\gamma = 2k \; . \qquad \text{(III.8.105)}$$

At this point collecting all the k-dependent terms of the Lagrangian (III.8.99) one finds that they sum up to an exact form. Indeed, considering the exact form: $2k\, d(A \wedge dB)$ one obtains:

$$2k\, d(A \wedge dB) = 2k\, dA \wedge dB =$$

$$= 2k(R^{\square} + \frac{1}{2}\bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b) \wedge (R^{\otimes} + \frac{i}{2}\bar{\psi} \wedge \Gamma_{a_1 \ldots a_5}\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_5} + \frac{15}{2}\bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \wedge A + 15 R^{\square} \wedge A) \qquad \text{(III.8.106)}$$

where the definitions (III.8.32d,e) have been used.

Expanding the wedge product of the quantities between brackets one reproduces exactly all the k-dependent terms of the Lagrangian (III.8.99). Hence the value of $k \equiv 2\gamma$ is unessential and we can put $k = 0$.

This in turn implies that the field B which appears through its curvature $R^{\otimes}$ only in the total derivative (III.8.106) drops out of our Lagrangian. We therefore arrive at the conclusion that the duality relation between the space-time components of $R^{\otimes}$ and $R^{\square}$, that we found from the Bianchi identities, Eq. (III.8.52), cannot be retrieved from the variational principle. Hence our hope to reconstruct a kinetic term for the fields A by means of the term $R^{\otimes} \wedge R^{\square}$ (see discussion following Eq. (III.8.62)), must be abandoned. On the other hand the Lagrangian (III.8.99) with $2k = \gamma \equiv 0$ is now lacking the kinetic term for the A field and is therefore bound, as it stands, to describe a trivial theory. This may be easily ascertained by considering the A equation. After δA variation of the action (III.8.99) (with $k = \gamma = 0$)

one easily finds that the only term which survives the projection on 8V is

$$- 2520\, R^{\square} \wedge R^{\square} \qquad \text{(III.8.107)}$$

since all the other terms contain at least one ψ-field. Hence the 8V-projection of the A-equation gives

$$R^{\square}_{a_1 \ldots a_4} R^{\square}_{b_1 \ldots b_4} \varepsilon^{a_1 \ldots a_4 b_1 \ldots b_4 c_1 c_2 c_3} = 0 \qquad \text{(III.8.108)}$$

which, instead of being the propagation equation for A, is an algebraic constraint implying $R^{\square}_{a_1 \ldots a_4} = 0$.

We have thus to resort to the explicit addition of a kinetic action for A by introducing a 0-form.

In complete analogy with the procedure adopted for N=2, D=4 supergravity and in the matter coupled gravity theories, we introduce the following action:

$$\mathscr{A} = \int \mathscr{L}(m,n) \qquad \text{(III.8.109a)}$$

$$\mathscr{L}(m,n) = \mathscr{L}_0\,(\gamma = k = 0) + \mathscr{L}'(m,n) \qquad \text{(III.8.109b)}$$

where $\mathscr{L}_0$ is given by (III.8.99b) and $\mathscr{L}'(m,n)$ is given by:

$$\mathscr{L}'(m,n) = m\, F^{a_1 \ldots a_4} R^{\square} \wedge V^{a_5} \wedge \ldots \wedge V^{a_{11}} \varepsilon_{a_1 \ldots a_{11}} + n\, F_{a_1 \ldots a_4} F^{a_1 \ldots a_4} V^{c_1} \wedge \ldots \wedge V^{c_{11}} \varepsilon_{c_1 \ldots c_{11}} \; . \qquad \text{(III.8.110)}$$

$F^{a_1 \ldots a_4}$ being a 4-index skew-symmetric 0-form and m, n two numerical parameters. Equation (III.8.110) corresponds to the 1st-order formulation of the Maxwell Lagrangian for A.

Next we show that, provided m and n take specific values, the Lagrangian (III.8.109) is non-trivial and describes a rheonomic theory.

The equations of motion of the action (III.8.99) ⊗ (III.8.109) are the following ones:

Torsion equation (variation in ω^{ab}):

$$\varepsilon_{abc_1\dots c_9} R^{c_1} \wedge V^{c_2} \wedge \dots \wedge V^{c_9} = 0 . \qquad \text{(III.8.111)}$$

First Maxwell equation ($F_{a_1\dots a_4}$ variation):

$$m R^\square \wedge V^{a_5} \wedge \dots \wedge V^{a_{11}} \varepsilon_{a_1\dots a_{11}} + 2n F_{a_1\dots a_4} V^{c_1} \wedge \dots \wedge V^{c_{11}} \varepsilon_{c_1\dots c_{11}} = 0 . \qquad \text{(III.8.112)}$$

Second Maxwell equation (variation in A):

$$168i \bar\psi \wedge \Gamma_{a_1\dots a_5}\rho \wedge V^{a_1} \wedge \dots \wedge V^{a_5} - 2520 \bar\psi \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \wedge R^\square - 2520 R^\square \wedge R^\square + m \mathcal{D} F_{a_1\dots a_4} \wedge V_{a_5} \wedge \dots \wedge V_{a_{11}} \varepsilon^{a_1\dots a_{11}} + \frac{7}{2} mi F_{a_1\dots a_4} \bar\psi \wedge \Gamma_{a_5}\psi \wedge V_{a_6} \wedge \dots \wedge V_{a_{11}} \varepsilon^{a_1\dots a_{11}} = 0 . \qquad \text{(III.8.113)}$$

Gravitino equation (variation in $\bar\psi$):

$$4 \Gamma_{a_1\dots a_8}\rho \wedge V^{a_1} \wedge \dots \wedge V^{a_8} - 168i \Gamma_{a_1\dots a_5}\psi \wedge V^{a_1} \wedge \dots \wedge V^{a_5} \wedge R^\square - m \Gamma_{ab}\psi \wedge V^a \wedge V^b \wedge F_{c_1\dots c_4} V_{c_5} \wedge \dots \wedge V_{c_{11}} \varepsilon^{c_1\dots c_{11}} = 0 . \qquad \text{(III.8.114)}$$

Einstein equation (variation in V_r):

$$- R^{a_1 a_2} \wedge V^{a_3} \wedge \dots \wedge V^{a_{10}} \varepsilon_{a_1\dots a_{10} r} + \frac{7}{15} i R_r \wedge \bar\psi \wedge \Gamma_{b_1\dots b_5}\psi \wedge V_{b_6} \wedge \dots \wedge V_{b_{11}} \varepsilon^{b_1\dots b_{11}} + \frac{7}{5} i R_a \wedge V^a \wedge \bar\psi \wedge \Gamma_{b_1\dots b_5}\psi \wedge V_{b_6} \wedge \dots \wedge V_{b_{10}} \varepsilon^{b_1\dots b_{10} r} + \frac{7}{15} i V_r \wedge \bar\psi \wedge \Gamma_{b_1\dots b_5}\psi \wedge V_{b_6} \wedge \dots \wedge V_{b_{11}} \varepsilon^{b_1\dots b_{11}} - \frac{7}{5} i \bar\psi \wedge \Gamma_{b_1\dots b_5}\psi \wedge V_r R_{b_6} \wedge V_{b_7} \wedge \dots \wedge V_{b_{11}} \varepsilon^{b_1\dots b_{11}} - 420 i R^\square \wedge \bar\psi \wedge \Gamma^{a_1\dots a_4 r}\psi \wedge V_{a_1} \wedge \dots \wedge V_{a_4} + 16 \bar\rho \wedge \Gamma_{c_1\dots c_7}\psi \wedge V^{c_1} \wedge \dots \wedge V^{c_7} + 11n F_{a_1\dots a_4} F^{a_1\dots a_4} V^{c_1} \wedge \dots \wedge V^{c_{10}} \varepsilon_{c_1\dots c_{10} r} + 7m F_{a_1\dots a_4} V_{a_5} \wedge \dots \wedge V_{a_{10}} \wedge R^\square \varepsilon^{a_1\dots a_{10} r} - m F_{a_1\dots a_4} \bar\psi \wedge \Gamma_{rb}\psi \wedge V^b \wedge V_{a_5} \wedge \dots \wedge V_{a_{11}} \varepsilon^{a_1\dots a_{11}} = 0 . \qquad \text{(III.8.115)}$$

(In deriving the previous equations one may take advantage of the fact that all the terms of 0-order in the curvatures can be omitted since the "vacuum condition" assures that they must sum up to zero).

As we have seen before, Eq. (III.8.111) (coinciding with (III.8.100)) implies Eq. (III.8.102), namely $R^a = 0$.

Therefore the supertorsion vanishes on-shell just as in N=1 and N=2 4-dimensional supergravities. Equation (III.8.102) can be solved for the connection $\omega^{ab}{}_\mu$ as a functional of V^a_μ and ψ_μ. Explicitly:

$$\mathscr{D}_m F^{mc_1c_2c_3} + \frac{1}{96}\varepsilon^{c_1c_2c_3a_1\ldots a_8} F_{a_1\ldots a_4} F_{a_5\ldots a_8} = 0 \qquad \text{(III.8.133)}$$

which coincides of course with Eq. (III.8.53).

Finally the 10-V projection of the gravitino equation (III.8.114) yields

$$4\,\Gamma_{a_1\ldots a_8}\,\rho_{ab}\,V^a \wedge V^b \wedge V^{a_1} \wedge \ldots \wedge V^{a_8} = 0$$

$$\Rightarrow 4\,\Gamma_{a_1\ldots a_8}\,\rho_{ab}\,\varepsilon^{aba_1\ldots a_8 p} = 0\,. \qquad \text{(III.8.134)}$$

Using the dualization formula (III.8.85) one finds

$$\Gamma^{abc}\rho_{bc} = 0\,. \qquad \text{(III.8.135)}$$

In Table III.8.I we have collected the results of this section. We notice that from the action point of view the theory can be thought of as based on the restricted F.D.A. given by Eqs. (III.8.5) ⊕ (III.8.11) without consideration of the successive extension by means of the field B.

Indeed the construction of the Lagrangian starting from the restricted F.D.A. would have been exactly the same, the only difference being that the coefficients k, γ appearing in (III.8.99) would have been set equal to zero from the beginning and that the gauge invariance to be implemented would have been restricted to the transformation (III.8.26). Therefore we have inserted in Table (III.8.I) the restricted F.D.A. as basic starting point of the theory. At this point one may ask why one prefers to construct the action starting from the more general F.D.A. (III.8.32). The answer is the following. In this way we have shown that the field B cannot enter the action so that there exists no formulation of D=11 supergravity in terms only of the B-field. Moreover the discussion of Sect. III.8.4 has shown the naturalness of the general F.D.A. (III.8.32) for the formulation of the on-shell theory without the action principle. Indeed it is more difficult to show how the propagation

equation (III.8.53) for the 3-form A can be retrieved in the restricted framework of the algebra (III.8.17) by merely working out the Bianchi identities (III.8.18), since one has to perform a supersymmetry transformation of the gravitino equation to deduce it. The conclusion is that if we want to work out the classical theory from an action principle both the F.D.A.'s (III.8.13,14) and (III.8.32) work equally well, the latter formulation is more suitable if we want to be able to deduce in an easy way the curvature parametrizations and all the equations of motion by working out the constraints implied through the Bianchi identities by rheonomy, gauge invariance and homogeneous scaling behaviour.

Actually nothing prevents us from writing down the action of the theory by letting the 6-form B to appear in it; it suffices to add the total derivative (III.8.106) with a value of k different from zero.

For example if one chooses $k=28$ the Lagrangian (III.8.99b) takes the following form:

$$\begin{aligned}
\mathscr{L} = & -\frac{1}{9} R^{a_1a_2} \wedge V^{a_3} \wedge \ldots \wedge V^{a_{11}}\,\varepsilon_{a_1\ldots a_{11}} + \\
& + 2\,\bar\rho \wedge \Gamma_{c_1\ldots c_8}\psi \wedge V^{c_1} \wedge \ldots \wedge V^{c_8} + \\
& + \frac{7}{30}\, i\, R^m \wedge V_m \wedge \bar\psi \wedge \Gamma_{a_1\ldots a_5}\psi \wedge V_{a_6} \wedge \ldots \wedge V_{a_{11}}\,\varepsilon^{a_1\ldots a_{11}} + \\
& + 28\, R^{\otimes} \wedge \bar\psi \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b - \\
& - 56\, i\, R^{\square} \wedge \bar\psi \wedge \Gamma_{a_1\ldots a_5}\psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_5} + \\
& + 56\, R^{\otimes} \wedge R^{\square} - 2\, F_{a_1\ldots a_4} R^{\square} \wedge V_{a_5} \wedge \ldots \wedge V_{a_{11}}\,\varepsilon^{a_1\ldots a_{11}} + \\
& + \frac{1}{165} F_{a_1\ldots a_4} F^{a_1\ldots a_4}\,\varepsilon_{b_1\ldots b_{11}}\, V^{b_1} \wedge \ldots \wedge V^{b_{11}}
\end{aligned} \qquad \text{(III.8.136)}$$

which is somewhat simpler than (III.8.99b) since it contains one term less. Of course if we write the action in this form we are relying on the maximally extended F.D.A. (III.8.32); this point of view is interesting because it tells us that the dynamical equations of a certain smaller multiplet of gauge fields are actually contained in the Bianchi identities of a larger gauge algebra. This implies that:

i) The curl of some gauge field not appearing in the Lagrangian is dual to the curl of the physical field appearing in the Lagrangian.

ii) The Lagrangian can be written in the most elegant and economical way using the curvatures of the full algebra, rather than the subalgebra corresponding to the physical fields.

iii) The action is invariant under the full gauge algebra although some gauge fields appear only through total derivatives.

TABLE III.8.I

Summary of D=11 Supergravity

Free Differential Algebra and Bianchi identities

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b \qquad \mathscr{D} R^{ab} = 0$$

$$R^a = \mathscr{D} V^a - \frac{i}{2}\bar\psi \wedge \Gamma^a \psi \qquad \mathscr{D} R^a + R^{ab} \wedge V_b - i\bar\psi \wedge \Gamma^a \rho = 0$$

$$\rho = \mathscr{D}\psi \qquad \mathscr{D}\rho + \frac{1}{4}\Gamma_{ab} R^{ab} \wedge \psi = 0$$

$$R^{\square} = dA - \frac{1}{2}\bar\psi \wedge \Gamma^{ab}\psi \wedge V^a \wedge V^b \qquad dR^{\square} - \bar\psi \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b + \bar\psi \wedge \Gamma_{ab}\psi \wedge R^a \wedge V^b = 0$$

TABLE III.8.I continued

Geometric action

$$\mathscr{A} = \int_{M_{11} \subset R^{11/32}} \mathscr{L}$$

where:

$$\mathscr{L} = -\frac{1}{9} R^{a_1 a_2} \wedge V^{a_3} \wedge \dots \wedge V^{a_{11}} \varepsilon_{a_1 \dots a_{11}} + 2\bar\rho \wedge \Gamma_{c_1 \dots c_8}\psi \wedge V^{c_1} \wedge \dots \wedge V^{c_8}$$

$$+ \frac{7}{30} i R^a \wedge V_a \wedge \bar\psi \wedge \Gamma^{b_1 \dots b_5}\psi \wedge V^{b_6} \wedge \dots \wedge V^{b_{11}} \varepsilon_{b_1 \dots b_{11}}$$

$$- i\, 84\, R^{\square} \wedge \bar\psi \wedge \Gamma_{a_1 \dots a_5}\psi \wedge V^{a_1} \wedge \dots \wedge V^{a_5} - 840\, R \wedge \bar\psi \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \wedge A$$

$$+ \frac{1}{4}\bar\psi \wedge \Gamma^{a_1 a_2}\psi \wedge \bar\psi \wedge \Gamma^{a_3 a_4}\psi \wedge V^{a_5} \wedge \dots \wedge V^{a_{11}} \varepsilon_{a_1 \dots a_{11}}$$

$$- 210\, \bar\psi \wedge \Gamma^{a_1 a_2}\psi \wedge \bar\psi \wedge \Gamma^{a_3 a_4}\psi \wedge V_{a_1} \wedge \dots \wedge V_{a_4} \wedge A - 840\, R^{\square} \wedge R^{\square} \wedge A$$

$$+ 2\, F^{a_1 \dots a_4} R^{\square} \wedge V^{a_5} \wedge \dots \wedge V^{a_{11}} \varepsilon_{a_1 \dots a_{11}}$$

$$- 165\, F_{a_1 \dots a_4} F^{a_1 \dots a_4} V^{c_1} \wedge \dots \wedge V^{c_{11}} \varepsilon_{c_1 \dots c_{11}}$$

and $R^{11/32} \equiv$ eleven dimensional superspace.

On-shell solution for the curvatures

$$R^a = 0$$

$$R^{\square} = F_{a_1 \dots a_4} V^{a_1} \wedge \dots \wedge V^{a_4}$$

$$\rho = \rho_{ab} V^a \wedge V^b + \frac{i}{3}\left(\Gamma^{b_1 b_2 b_3} F_{ab_1 b_2 b_3} - \frac{1}{8}\Gamma_{ab_1 b_2 b_3 b_4} F^{b_1 b_2 b_3 b_4}\right)\psi \wedge V^a$$

TABLE III.8.I continued

On-shell solution for the curvatures (cont'd)

$$R^{ab} = R^{ab}{}_{mn}\, V^m \wedge V^n + \bar{\theta}^{ab}{}_c \psi \wedge V^c + \bar{\psi} \wedge \Gamma_{mn} \psi\, F^{mnab} + \frac{1}{24} \bar{\psi} \wedge \Gamma^{aba_1 \dots a_4} \psi\, F_{a_1 \dots a_4}$$

$$\theta^{ab}{}_c = \begin{cases} \overset{II}{\theta}{}^{ab}{}_c = i(\Gamma_a\, \rho_{bc} - \Gamma_b\, \rho_{ca} + \Gamma_c\, \rho_{ba}) & \text{(from Bianchi identities)} \\ \overset{I}{\theta}{}^{ab}{}_c = i\left(\frac{1}{2}\, \Gamma^{abcmn} - \frac{2}{9}\, \Gamma^{mn[a}\, \eta^{b]c} + 2\, \Gamma^{ab[m}\, \eta^{n]c}\right)\rho_{mn} & \text{(from variational principle)} \end{cases}$$

$$\left(\overset{I}{\theta}{}^{ab}{}_c = \overset{II}{\theta}{}^{ab}{}_c \text{ on shell} \right)$$

Propagation equations

$$R^{am}{}_{bm} - \frac{1}{2}\, \delta^a{}_b\, R^{mn}{}_{mn} = 3\, F^{ac_1c_2c_3}\, F_{bc_1c_2c_3} + \frac{3}{8}\, \delta^a{}_b\, F_{c_1 \dots c_4}\, F^{c_1 \dots c_4}$$

$$\Gamma^{abc}\, \rho_{bc} = 0$$

$$\mathscr{D}^m\, F_{ma_1a_2a_3} + \frac{1}{96}\, F^{b_1b_2b_3b_4}\, F^{c_1c_2c_3c_4}\, \varepsilon_{b_1 \dots b_4\, c_1 \dots c_4\, a_1a_2a_3} = 0$$

First order supersymmetry transformations

$$\delta V^a = i\, \bar{\psi}\, \Gamma^a \varepsilon$$

$$\delta\psi = \mathscr{D}\varepsilon + \frac{i}{3}\left(\Gamma^{b_1b_2b_3}\, F_{ab_1b_2b_3} - \frac{1}{8}\, \Gamma_{ab_1b_2b_3b_4}\, F^{b_1b_2b_3b_4}\right)\varepsilon V^a$$

$$\delta A = \bar{\varepsilon}\, \Gamma_{ab} \psi \wedge V^a \wedge V^b$$

$$\delta\omega^{ab} = \bar{\overset{I}{\theta}}{}^{ab}{}_c\, \varepsilon V^c + 2\, \bar{\varepsilon}\, \Gamma_{mn} \psi\, F^{mnab} + \frac{1}{12}\, \bar{\varepsilon}\, \Gamma^{aba_1 \dots a_4} \psi\, F_{a_1 \dots a_4}$$

Historical Remarks and References for Part Three

Supergravity was discovered in 1976 by Daniel Freedman, Sergio Ferrara and Peter van Nieuwenhuizen who used second order formalism for the gravitational field [32].

Very shortly after came the paper by Stanley Deser and Bruno Zumino where supergravity was derived utilizing first order formalism [20].

In both Refs. [32] and [20] the Noether coupling method (also named component approach) was used to derive the Lagrangian and the transformation rules of N=1, D=4 supergravity. De Sitter supergravity was first obtained in Ref. [37].

The construction of the other pure supergravity models is the yield of the years 1976-1980.

N=2 supergravity is due to S. Ferrara and P. van Nieuwenhuizen [26] (see Chapter III.4).

The relation between the cosmological constant and the gauge coupling constant, discussed at the end of Chapter III.3.4, was given first in Refs. [11] and [27]. The simple N=3 theory, not treated here, was obtained by Freedman and by Ferrara, Scherk and Zumino, see Refs. [31,25].