CHAPTER III.6

THE THEORY OF FREE DIFFERENTIAL ALGEBRAS AND SOME APPLICATIONS

III.6.1 - Introduction

The aim of the present chapter is to emphasize the role played in supergravity theories by the algebraic concept of "free-differential algebra", which turns out to be a natural generalization of a Lie (super)-algebra. This notion is well established in mathematics and a complete account of the related theory is given by Sullivan. (See reference at the end of this part).

The previously discussed group-manifold scheme can accomodate simple supergravity in 4 and 5 dimensions but certainly not higher dimensional supergravities. This is so because when $D>5$ the supergravity multiplet contains, besides the vielbein $V^a_\mu(x)$, the gravitino $\psi_\mu(x)$ and the spin connection ω^{ab}_μ, which can be viewed as the potentials of the D-dimensional super Poincaré Lie Algebra $\mathbb{G}$, also some antisymmetric tensor fields. For example, in the D=11 theory the supergravity multiplet contains a 3-index photon $A_{\mu\nu\rho}(x)$ which, being a 3-form, cannot be identified as the potential of any group generator.

This situation raises the question whether the concept of Lie Algebra can be extended in such a way as to accomodate also forms of degree higher than one.

The answer is yes and the more general structure which does the job is the "Free Differential Algebra". We shall show that the entire programme of the group manifold can be almost trivially extended to the case where the Lie algebra $\mathbb{G}$ is replaced by a "Free Differential Algebra".

III.6.2 - The concept of free differential algebra

To introduce the concept of "Free Differential Algebra" we start from the dual formulation of the superalgebra via the Maurer-Cartan equation and we extend it to the case of p-forms with $p \geq 1$.

Consider an unspecified manifold M and a set of exterior forms $\{\theta^{A(p)}\}$, defined on M, labelled by the index A and by the degree p which may be different for different values of A.

Given this set we can write a "generalized Maurer-Cartan equation" of the following type:

$$d\theta^{A(p)} + \sum_{n=1}^{N} \frac{1}{n} C^{A(p)}_{B_1(p_1)\ldots B_n(p_n)} \theta^{B_1(p_1)} \wedge \ldots \wedge \theta^{B_n(p_n)} = 0 \qquad \text{(III.6.1)}$$

where $C^{A(p)}_{B_1(p_1)\ldots B_n(p_n)}$ are generalized structure constants with the same symmetry as induced by permuting the θ's in the wedge product. They are non-zero only if:

$$p + 1 = p_1 + p_2 + \ldots + p_n \,. \qquad \text{(III.6.2)}$$

Equation (III.6.1) is self-consistent only if $dd\theta^{A(p)}$ is identically zero. This implies:

$$0 = dd\theta^{A(p)} = -\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{1}{m}C^{A(p)}{}_{B_1(p_1)\ldots B_n(p_n)} \times$$
$$\times\, C^{B_1(p_1)}{}_{D_1(q_1)\ldots D_m(q_m)}\,\theta^{D_1(q_1)} \wedge \ldots \wedge \theta^{D_m(q_m)} \wedge \theta^{B_2(p_2)} \wedge \ldots \wedge$$
$$\wedge\, \theta^{B_n(p_n)} = 0\,. \qquad \text{(III.6.3)}$$

Whenever (III.6.3) holds true we say that Eq. (III.6.1) defines a "Free Differential Algebra" or a "Cartan Integrable System". The ordinary Lie Algebra case is retrieved when all the θ's have degree $p = 1$.

Indeed in this case Eqs. (III.6.1) and (III.6.3) reduce to

$$d\theta^A + \frac{1}{2}C^A{}_{BC}\,\theta^B \wedge \theta^C = 0 \qquad \text{(III.6.4)}$$

$$dd\theta^A = -\frac{1}{2}C^A{}_{BC}\,C^B{}_{DF}\,\theta^D \wedge \theta^F \wedge \theta^C = 0 \qquad \text{(III.6.5)}$$

that is, to the Maurer-Cartan equations and the Jacobi identities of an ordinary (super)-Lie Algebra.

III.6.3 - The structure of free differential algebras and some theorems

A complete classification of the free differential algebras, similar to the classification of semisimple Lie Algebras, is given in the mathematical literature by Sullivan.

It is very instructive to have a look at the most general form of a free differential algebra (F.D.A.) as it emerges from Sullivan's theorems. First we introduce the definition of "minimal algebra" which is one for which $C^{A(p)}{}_{B(p+1)} = 0$. This excludes the case where a $(p+1)$-form appears in the generalized Maurer-Cartan equation (III.6.1). For instance,

$$d\theta^{A(1)} = C^A{}_{BC}\,\theta^{B(1)} \wedge \theta^{C(1)} + \theta^{A(2)} \qquad \text{(III.6.6)}$$

does not define a minimal algebra. In a minimal algebra all non-differential terms are products of at least two elements of the algebra,

so that all forms appearing in the expansion of $d\theta^{A(p)}$ have at most degree p, the degree $(p+1)$ being ruled out.

On the other hand a "contractible algebra" is one where the only form appearing in the expansion of $d\theta^{A(p)}$ has degree $p+1$, namely:

$$d\theta^{A(p)} = \theta^{A(p+1)} \Rightarrow d\theta^{A(p+1)} = 0\,. \qquad \text{(III.6.7)}$$

A contractible algebra has a trivial structure. The basis $\{\theta^{A(p)}\}$ can be subdivided into two subsets $\{\xi^{A(p)}\}$ and $\{\omega^{B(p+1)}\}$ where A spans a subset of the values taken by B, so that

$$d\omega^{B(p+1)} = 0 \qquad \text{(III.6.8)}$$

for all values B and

$$d\xi^{A(p)} = \omega^{A(p+1)}\,. \qquad \text{(III.6.9)}$$

In words this means that $\{\omega^{B(p+1)}\}$ are all closed forms and $\{\omega^{A(p+1)}\}$ are those among the $\{\omega^{B(p+1)}\}$ which are also exact, being the derivatives of the $\{\xi^{A(p)}\}$. Denoting by M^k and C^k, respectively, the minimal and contractible algebras generated by all p-forms with $p \leq k$ we can write

$$d\,C^k \subset C^{k+1} \qquad \text{(III.6.10)}$$

for the contractible algebra, and

$$d\,M^k \subset M^k \wedge M^k \qquad \text{(III.6.11)}$$

for the minimal one.

Now Sullivan's fundamental theorem tells the following:

Theorem 1: The most general Free Differential Algebra is the direct sum of a contractible algebra with a minimal algebra.

Therefore we just need to study the structure of a minimal algebra.

The decomposition of any F.D.A. into its minimal and contractible parts:

$$\mathscr{A} = C \oplus M \qquad \text{(III.6.12)}$$

can be obtained via an iterative redefinition of the generators which we do not show here.

A second theorem states that the most general minimal algebra is obtained by means of the following iterative procedure.

Theorem 2: Iterative construction of a Minimal Free Differential Algebra:

a) First consider a finite dimensional ordinary Lie Algebra $\mathbb{G}$.

It is described by N 1-forms $\{\sigma^A\}$ satisfying:

$$d\sigma^A + \frac{1}{2} C^A_{\ .BC}\, \sigma^B \wedge \sigma^C = 0 \,. \qquad \text{(III.6.13)}$$

b) Next consider the spectrum of finite dimensional irreducible representations of $\mathbb{G}$. For each representation $D^{(n)}$ (n-labels the representation) we will have a matrix $D^{(n)}\,(T_A)^i_{\ .j}$ satisfying the commutation relations of $\mathbb{G}$. By means of this matrix we can introduce a $\mathbb{G}$-covariant derivative:

$$\nabla^{(n)} = (\mathbb{1})^i_{\ .j}\, d + \sigma^A \wedge D^{(n)}(T_A)^i_{\ .j} \qquad \text{(III.6.14)}$$

acting on objects which have an index j in the chosen representation space. Equation (III.6.13) guarantees that the operator $\nabla^{(n)}$ satisfies

$$\nabla^{(n)}\, \nabla^{(n)} = 0 \,. \qquad \text{(III.6.15)}$$

c) Now consider the polynomials of the following type:

$$\Omega^i_{(n,p)} = \Omega^i_{A_1 \ldots A_p}\, \sigma^{A_1} \wedge \ldots \wedge \sigma^{A_p} \qquad \text{(III.6.16)}$$

where i runs in the irrep $D^{(n)}$ and $A_1 \ldots A_p$ run in the adjoint representation of $\mathbb{G}$. $\Omega^i_{A_1 \ldots A_p}$ are constants.

The above objects are named Chevalley cochains. A cochain $\Omega^i_{(n,p)}$ is called a cocyle if it is covariantly closed:

$$\nabla^{(n)}\, \Omega^i_{(n,p)} = 0 \,. \qquad \text{(III.6.17)}$$

On the other hand a cochain $\Omega^i_{(n,p)}$ is named coboundary if we can find a cochain $\tilde{\Omega}^i_{(n,p-1)}$ in the same representation such that

$$\Omega^i_{(n,p)} = \nabla^{(n)}\, \tilde{\Omega}^i_{(n,p-1)} \,. \qquad \text{(III.6.18)}$$

Of great interest are the cocyles which are not coboundaries. They are the representatives of the Chevalley cohomology classes of the chosen Lie Algebra $\mathbb{G}$. They are an intrinsic property of $\mathbb{G}$ and depend both on the degree p and the representation $D^{(n)}$

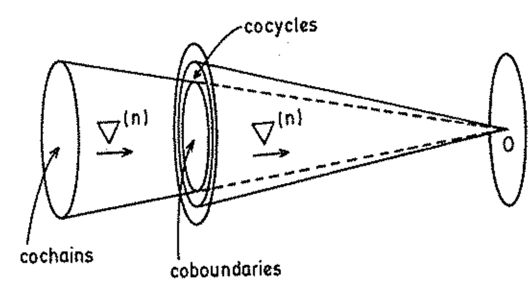

Fig. III.6.1.

d) Each cohomology class $\hat{\Omega}_{(n,p)}$ corresponds to a possible extension of (III.6.13) to a non trivial differential algebra. Indeed for each cocyle $\hat{\Omega}^i_{(n,p)}$ we can introduce a new form $A^i_{(n,p-1)}$ and write the generalized Maurer-Cartan equations:

$$d\sigma^A + \frac{1}{2} C^A_{\ BC}\, \sigma^B \wedge \sigma^C = 0 \qquad \text{(III.6.19a)}$$

$$\nabla^{(n)} A^{i}_{(n,p-1)} + \hat{\Omega}^{i}_{(n,p)} = 0 \qquad \text{(III.6.19b)}$$

a') At this juncture, we can start the whole procedure once again using (III.6.19) as starting point. Namely, keeping the definition of $\nabla^{(n)}$ fixed, we can consider the cochains constructed on the forms $\{\sigma^A, A^i\}$:

$$\Omega^{i'}_{(n,p)}(\sigma,A) = \Omega^{i}_{A_1 \ldots A_r\, i_1 \ldots i_s} \times$$
$$\times\, \sigma^{A_1} \wedge \ldots \wedge \sigma^{A_r} \wedge A^{i_1}_{(n_1,p_1)} \wedge \ldots \wedge A^{i_s}_{(n_s,p_s)} \qquad \text{(III.6.20)}$$

where

$$p = r + \sum_{j=1}^{s} p_j \qquad \text{(III.6.21)}$$

and i runs in the (n) representation; $A_1 \ldots A_n$ run in the adjoint representation and i_j run in the n_j-representation respectively. Equations (III.6.19) guarantee that the operator $\nabla^{(n)}$ is still closed $[\nabla^{(n)2} = 0]$ also in the larger space of the $\Omega^{i'}$ cochains. Hence we can still talk about cocycles, coboundaries and cohomology classes of the new structure (III.6.19). Each cohomology class $\hat{\Omega}^{i'}_{(n,p)}$ corresponds to a new further extension of the differential Algebra. It suffices to introduce new $A'^{i}_{(n,p-1)}$ forms and write:

$$d\sigma^A + \frac{1}{2} C^A{}_{BC}\, \sigma^B \wedge \sigma^C = 0 \qquad \text{(III.6.22)}$$

$$\nabla^{(n)} A^{i}_{(n,p-1)} + \hat{\Omega}^{i}_{(n,p)}(\sigma) = 0 \qquad \text{(III.6.23)}$$

$$\nabla^{(n)} A'^{i}_{(n,p-1)} + \hat{\Omega}'^{i}_{(n,p)}(\sigma,A) = 0\,. \qquad \text{(III.6.24)}$$

In this way one obtains, iteratively, the most general free differential algebra which contains a given ordinary superalgebra $\mathbb{G}$.

The above theorem is of outstanding relevance for the geometrical foundations of supergravity. Indeed it tells us that all the particles appearing in the spectrum, multi-index photons included, have a geometrical origin and are so to say "predicted" by the geometry of rigid superspace, namely the super Poincaré Lie Algebra.

The important point is that a Lie Algebra determines its possible extensions through the mechanism of Chevalley cohomology.

As an illustration we now give a few pedagogical examples:

Example 1: Consider a 1-form $\theta^{(1)}$ and a 2-form $\theta^{(2)}$. The most general nontrivial F.D.A. we can write with these ingredients is:

$$d\theta^{(1)} = 0 \qquad d\theta^{(2)} = \theta^{(2)} \wedge \theta^{(1)}\,. \qquad \text{(III.6.25)}$$

Indeed $d\theta^{(1)}$ cannot be $\theta^{(2)}$ unless $d\theta^{(2)} = 0$ because $d^2\theta^{(1)}$ has to be zero. On the other hand (III.6.25) are consistent since

$$d^2\theta^{(1)} = 0 \qquad \text{(III.6.26)}$$

$$d^2\theta^{(2)} = d\theta^{(2)} \wedge \theta^{(1)} + \theta^{(2)} \wedge d\theta^{(1)} =$$
$$= \theta^{(2)} \wedge \theta^{(1)} \wedge \theta^{(1)} = 0\,. \qquad \text{(III.6.27)}$$

The system (III.6.25) is in fact equivalent to an ordinary group. Indeed we can set

$$\theta^{(1)} = \sigma^1\,; \qquad \theta^{(2)} = \sigma^1 \wedge \sigma^2 \qquad \text{(III.6.28)}$$

where

$$d\sigma^1 = 0 \qquad d\sigma^2 = \sigma^1 \wedge \sigma^2 \qquad \text{(III.6.29)}$$

and then (III.6.25) follows from (III.6.29). The corresponding algebra has the form $[t_1,t_1] = [t_2,t_2] = 0$, $[t_1,t_2] = -2t_2$.

Example 2: Consider a bosonic 1-form b, a bosonic 2-form A and a fermionic 1-form ψ. Since ψ is fermionic it commutes with itself and with A but it anticommutes with b.

We can easily check that the following F.D.A. is consistent:

$$dA = A \wedge b + \psi \wedge \psi \wedge b; \quad db = 0; \quad d\psi = b \wedge \psi \ . \tag{III.6.30}$$

Just as in the previous example, (III.6.30) is equivalent to an ordinary supergroup. Indeed, introducing a new bosonic 1-form t we can write

$$A = \beta_1 \, t \wedge b + \beta_2 \, \psi \wedge \psi \tag{III.6.31}$$

$$dt = \gamma_1 \, b \wedge t + \gamma_2 \, \psi \wedge \psi \tag{III.6.32a}$$

$$db = 0 \tag{III.6.32b}$$

$$d\psi = b \wedge \psi \tag{III.6.32c}$$

where $\beta_1, \beta_2, \gamma_1, \gamma_2$ are numerical coefficients, to be fixed by requiring that (III.6.32) be consistent and (III.6.30) follow from (III.6.31-32). The only consistency requirement which constrains the parameters is $d^2t = 0$ which yields $(\gamma_1 - 2)\gamma_2 = 0$. Substitution of (III.6.31) into (III.6.30) yields $\beta_1\gamma_2 + \beta_2 = 1$. One can now choose different values for the parameters that still remain free, obtaining different groups.

Example 3: As a less trivial case we start from the $\overline{Osp(4/1)}$ group on which D=4, N=1 supergravity is based:

$$R^{ab} \equiv d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} = 0 \tag{III.6.33a}$$

$$R^a \equiv dV^a - \omega^{ab} \wedge V_b - \frac{i}{2}\bar\psi \wedge \gamma^a\psi = 0 \tag{III.6.33b}$$

$$\rho \equiv \mathcal{D}\psi \equiv d\psi - \frac{1}{4}\omega^{ab} \wedge \gamma_{ab}\psi = 0 \ . \tag{III.6.33c}$$

One notices that within $\overline{Osp(4/1)}$ the following 4-form

$$\Omega = \frac{i}{2}\bar\psi \wedge \gamma_m\psi \wedge V^m \tag{III.6.34}$$

is a representative of a non trivial cohomology class with respect to the identity representation. Indeed, if D=4, we have:

$$\nabla\Omega \equiv d\Omega = \frac{i}{2}\, 2\mathcal{D}\bar\psi \wedge \gamma_m\psi \wedge V^m + \left(\frac{i}{2}\right)^2 \bar\psi \wedge \gamma_m\psi \wedge \bar\psi \wedge \gamma^m\psi =$$

$$= -\frac{1}{4}\bar\psi \wedge \gamma^m\psi \wedge \bar\psi \wedge \gamma_m\psi = 0 \tag{III.6.35}$$

where we have used the Fierz identity (III.2.21). On the other hand one can easily be convinced that there is no way of producing Ω as the result of a derivative of a 3-form contained among the $\overline{Osp(4/1)}$ cochains. Therefore Ω is closed but not exact. Correspondingly it can be used as the non-derivative part in a generalized Maurer-Cartan equation for a 2-form, say T, and the $\overline{Osp(4/1)}$ Maurer-Cartan equations (III.6.33) can be extended to the following F.D.A.:

$$d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b = 0 \tag{III.6.36a}$$

$$\mathcal{D}V^a - \frac{i}{2}\bar\psi \wedge \gamma^a\psi = 0 \tag{III.6.36b}$$

$$\mathcal{D}\psi = 0 \tag{III.6.36c}$$

$$dT - \frac{i}{2}\bar\psi \wedge \gamma_m\psi \wedge V^m = 0 \ . \tag{III.6.36d}$$

In discussing Sullivan's theorems we have seen the rôle played by Chevalley cohomology. Now we note that Chevalley cohomology of a Lie Algebra can be further specialized to the concept of relative cohomology of a Lie Algebra $\mathbb{G}$ with respect to a subalgebra $\mathbb{H}$.

It is actually this more restrictive cohomology which is most relevant to supergravity and lies at the basis of the rheonomy approach. It is easily introduced. In our previous definition the cochains $\Omega^{(1)}$

were polynomials in all the $\mathbb{G}$ one-forms $\{\sigma^A\}$. If we restrict $\Omega^{(1)}$ to be polynomials in the $\mathbb{K}$ components of $\{\sigma^A\}$ where

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K} \qquad \text{(III.6.37)}$$

is the splitting of $\mathbb{G}$ into a subalgebra $\mathbb{H}$ and a subspace $\mathbb{K}$ and if we also demand that the coefficients $\Omega^i_{A_1 \ldots A_p}$ be invariant tensors of $\mathbb{H}$, repeating, for the rest, the same construction as before, we obtain the relative cohomology groups $H^{(p)}$ (G, H, D).

The following theorem holds:

Chevalley-Eilenberg Theorem 1: If $\mathbb{G}$ is semisimple and D is irreducible (and non trivial), then for all choices of $\mathbb{H}$ and p we have

$$H^{(p)}\,(G,\, H,\, D) = 0\ . \qquad \text{(III.6.38)}$$

If a representation D is not irreducible (hence fully reducible) then the covariant derivative operator ∇ splits into the sum

$$\nabla = \Sigma_n\, \nabla^{(n)} \qquad \text{(III.6.39)}$$

where $\nabla^{(n)}$ are the covariant derivatives of the irreducible components of D. A p-form in the D-representation is also a sum $\omega^i = \Sigma\, \omega^i_{(n)}$ and for each irreducible component $\omega^i_{(n)}$ the above theorem can be applied. We have therefore the:

Corollary: For $\mathbb{G}$ semisimple and D a fully reducible representation not containing the identity representation

$$H^{(p)}(G,\, H,\, D) = 0\ . \qquad \text{(III.6.40)}$$

This means that a semisimple algebra can have cohomology classes, and hence free differential algebra extensions, only in the trivial representation $D = \mathbb{1}$. Moreover, the theorem shows that non trivial cohomologies are mainly to be found in nonsemisimple algebras.

In fact we have the following theorem:

Chevalley-Eilenberg Theorem 2: If $\mathbb{G}$ is semisimple and D is the identity representation, then there are no nontrivial 1-form and 2-form cohomology classes.

There is, however, always a non trivial 3-form cohomology class, namely:

$$\Omega = C_{ABC}\, \sigma^A \wedge \sigma^B \wedge \sigma^C \qquad \text{(III.6.41)}$$

where C_{ABC} are the structure constants with all the indices lowered. This means that for $\mathbb{G}$ semisimple every closed 1-form or 2-form is exact.

As an application of this theorem we notice that the 3-form Ω of Eq. (III.6.34) is closed only for the nonsemisimple group $\overline{\mathrm{Osp}(4/1)}$. Indeed if we were to use the semisimple version Osp(4/1), then from Eq. (III.3.189c), taken at zero curvature one has:

$$\mathcal{D}\psi = i\, \bar{e}\, \gamma_a \psi \wedge V^a \qquad \text{(III.6.42)}$$

and therefore

$$d\Omega \neq 0 \qquad \text{(III.6.43)}$$

according to the theorem.

As another application we may consider the extension of the (semisimple) group SO(3) defined by

$$d\sigma^i + \frac{1}{2}\, \varepsilon^{ijk}\, \sigma_j \wedge \sigma_k = 0\ . \qquad \text{(III.6.44)}$$

This algebra can be extended by means of a 2-form A whose derivative is

$$dA = \frac{1}{3}\, \varepsilon^{ijk}\, \sigma_i \wedge \sigma_j \wedge \sigma_k\ . \qquad \text{(III.6.45)}$$

Equations (III.6.44) and (III.6.45) are integrable since

$$\omega = \varepsilon_{ijk}\, \sigma^i \wedge \sigma^j \wedge \sigma^k \qquad \text{(III.6.46)}$$

is the only non trivial cohomology class of order three predicted by Chevalley's second theorem.

All the examples of F.D.A. given so far do not have direct applications to supergravity theories. In the remaining sections of this chapter we study a non trivial example of F.D.A. which is relevant for the formulation of a D=4 supergravity theory, namely the Sohnius-West model also called the new minimal formulation of N=1 off-shell supergravity.

This example will be the topic of Sect. III.6.5 and III.6.6 and will provide an example of supergravity theory with auxiliary fields whose action is off-shell supersymmetric. In Sect. III.6.7 we give the building principles of supergravity theories in their final form.

The role of F.D.A.'s in higher dimensional supergravities will be discussed in the next two chapters for the D=6 and D=11 theories and in Chapter IX of Part VI for the D=10 theory.

III.6.4 - Gauging of the free differential algebras and the building rules revisited

Physical applications of the F.D.A.'s require a generalization of the concepts of soft 1-forms and curvatures introduced for the gauging of the Maurer-Cartan equations (see Eqs. (III.3.130-134)). It will be evident that all the concepts advocated in the geometrical construction of supergravity actions based on Maurer-Cartan equations can be straightforwardly extended to theories based on F.D.A.'s. Therefore it suffices to give a sketch of this generalization.

First we introduce the soft forms $\pi^{A(p)}$: they are in one-to-one correspondence to the rigid forms $\theta^{A(p)}$, but do not fulfill the generalized Maurer-Cartan equations (III.6.1). The deviation from zero of the r.h.s. of (III.6.1) defines what we call the curvature of the set $\pi^{A(p)}$:

$$R^{A(p+1)} = d\pi^{A(p)} + \sum_{n=1}^{N} \frac{1}{n}\, C^{A(p)}_{B_1(p_1)\ldots B_n(p_n)}\, \pi^{B_1(p_1)} \wedge \ldots \wedge \pi^{B_n(p_n)} . \qquad \text{(III.6.47)}$$

Obviously the $\pi^{A(p)}$'s can be viewed as the "Yang-Mills potentials" of the soft F.D.A. and the $R^{A(p+1)}$ are their field-strengths, in the same way as the 1-forms μ^A are the ordinary Yang-Mills potentials of a (super) group and the $R^A(\mu)$ are their related curvatures. On the base space, i.e. on space-time, $\pi^{A(p)}$ and $R^{A(p+1)}$ can be equivalently considered as graded-antisymmetric tensor fields of degree p or $p+1$ respectively. If we apply the d-operator to both sides of Eq. (III.6.47) we obtain a differential identity on the $R^{A(p+1)}$ which we can call <u>generalized Bianchi identity</u>:

$$\nabla R^{A(p+1)} \underset{\text{def}}{\equiv} dR^{A(p+1)} + \sum_{n=1}^{N} C^{A(p)}_{B_1(p_1)\ldots B_n(p_n)} \times R^{B_1(p+1)} \wedge \pi^{B_2(p_2)} \wedge \ldots \wedge \pi^{B_n(p_n)} = 0 . \qquad \text{(III.6.48)}$$

In complete analogy to what one does in ordinary group theory we say that the left hand side of (III.6.48) defines the covariant derivative ∇ of an adjoint set of p+1-forms; that is if $H^{A(p+1)}$ is a set of p+1-forms, the combination:

$$\nabla H^{A(p+1)} = dH^{A(p+1)} + \sum_{n=1}^{N} C^{A(p)}_{B_1(p_1)\ldots B_n(p_n)} \times H^{B_1(p_1+1)} \wedge \pi^{B_2(p_2)} \wedge \ldots \wedge \pi^{B_n(p_n)} \qquad \text{(III.6.49)}$$

will be named the adjoint covariant derivative of $H^{A(p+1)}$. With this definition the Bianchi identity (III.6.48) just states that the covariant derivative of the curvature set $R^{A(p+1)}$ is zero just as it happens for ordinary supergroups.

Let us now assume that we have a multiplet $\nu_{A(D-p-1)}$ of forms whose degree is the complement of the degree of $H^{A(p+1)}$ with respect to some fixed number D. We say that $\{\nu_{A(D-p-1)}\}$ is a coadjoint set of (D-p-1)-forms if:

$$I = H^{A(p+1)} \wedge \nu_{A(D-p-1)} \tag{III.6.50}$$

is an invariant, meaning that the covariant derivative of I coincides with its ordinary exterior derivative:

$$\nabla I = \nabla H^{A(p+1)} \wedge \nu_{A(D-p-1)} + (-1)^{p+1} H^{A(p+1)} \wedge \nabla\nu_{A(D-p-1)} =$$
$$= dI = dH^{A(p+1)} \wedge \nu_{A(D-p-1)} + (-1)^{p+1} H^{A(p+1)} \wedge d\nu_{A(D-p-1)} \,. \tag{III.6.51}$$

Equation (III.6.51) provides the definition of coadjoint covariant derivative. Indeed, in order for (III.6.51) to be true, we must have

$$\nabla\nu_{A(D-p-1)} = d\nu_{A(D-p-1)} - (-1)^{p+1} \sum_{n=1}^{N} C^{B_1(p_1)}_{A(p)B_2(p_2)\ldots B_n(p_n)} \times$$
$$\times \pi^{B_2(p_2)} \wedge \ldots \wedge \pi^{B_n(p_n)} \wedge \nu_{B_1(D-p-1)} \tag{III.6.52}$$

where

$$p_1 + 1 = p + p_2 + p_3 + \ldots + p_n \,. \tag{III.6.53}$$

Let us now consider the construction of geometrical actions based on F.D.A.'s. The modifications to the building rules established in Sect. III.3.9 are almost obvious.

The proper generalization of rule A is the following: an action is geometrical when it is constructed in terms of the soft forms $\pi^{A(p)}$ and of their curvatures without use of the Hodge duality on forms. Furthermore the action usually contains also multiplets of 0-forms F^I, and can be written as follows:

$$\mathscr{A} = \int_{M^D \subset M} [\Lambda + R^{A(p+1)} \wedge \nu_{A(D-p-1)} + R^{A(p+1)} \wedge R^{B(q+1)} \wedge \nu_{AB(D-p-q-2)} +$$
$$+ \ldots] + \int_{M^D} \mathscr{L}_{KIN}(F^I, R^A, \pi^A) \,. \tag{III.6.54}$$

Here $M^D \subset M$ is a floating D-dimensional hypersurface and M is the supermanifold on which the soft forms $\pi^{A(p)}$ are defined. Λ, ν_A and ν_{AB} are polynomials in the $\pi^{A(p)}$'s of degree D, D-p-1, D-p-q-2 respectively. Notice that (III.6.54) is completely analogous to (III.3.148) and to its generalization Eq. (III.4.36). As we discussed in Sect. III.3.9 and examplified in the N=2, D=4 theory and also in the coupling of gravity to matter fields, the extra term $\mathscr{L}_{KIN}$ appearing in (III.6.54) contains multiplets of 0-forms F^I which are necessary for the construction of the kinetic term of fields whose spin is ≤ 1. In this way one obtains the most general action which is still geometrical, since it does not contain the duality operator.

As far as rule B of Sect. III.3.9 is concerned, we require that when the F.D.A. is H-gauge invariant, H being a Lie subgroup of the given F.D.A., then the corresponding action should also be H-gauge-invariant. The corresponding requirements on the general term of (III.6.54) are exactly the same as in the case of an ordinary supergroup.

The condition to be satisfied by the action (III.6.54) in order to admit the vacuum solution, rule D, is also the straightforward generalization of condition (III.3.158) (or of its counterpart in presence of 0-forms Eq. (III.4.48)), namely:

$$\begin{cases} \dfrac{\delta\Lambda}{\delta\pi^{A(p)}} + \nabla\nu_{A(D-p-1)} = 0 \\ \dfrac{\delta\mathscr{L}_{KIN}}{\delta F^I} = 0 \end{cases} \quad \text{at} \quad \begin{cases} R^{A(p+1)} = 0 & \text{(III.6.55a)} \\ F^I = 0 & \text{(III.6.55b)} \end{cases}$$

Equations (III.6.55a) are the variational equations derived from (III.6.54) evaluated at vanishing curvatures in absence of 0-forms. When the 0-form fields are also present we must add the conditions that the 0-form variational equations be satisfied at both $R^{A(p+1)}$ and F^I equal to zero. Indeed, as we pointed out several times, the F^I's are to be identified in second order formalism with the field strengths of physical fields (see Chapter I.5 and Sect. III.4.3).

The rheonomy principle E can also be extended in a straightforward manner. We demand that the parametrization of the generalized curvatures, given by the equations of motion, must be such that the outer components, that is the components along at least one ψ, must be expressible in terms of the physical field-strengths on the space-time M^D.

Finally the good scaling behavior, building rule C, can be extended also to the general case since, as we are going to see in the sequel, the F.D.A. and its related Bianchi identities possess a rigid scale invariance.

In the next section we study a first example of a physical theory based on a F.D.A.

III.6.5 - The Sohnius-West model (New minimal N=1 supergravity): the on-shell formulation

We begin by introducing a slight generalization of the F.D.A. given in Eqs. (III.6.36). Let us start from the D=4 super Poincaré Algebra with a chiral charge whose associated 1-form potential is denoted by A. The ordinary Maurer-Cartan equations corresponding to this superalgebra are:

$$d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b = 0 \qquad \text{(III.6.56a)}$$

$$\mathcal{D}V^a - \frac{i}{2}\bar{\psi} \wedge \gamma^a\psi = 0 \qquad \text{(III.6.56b)}$$

$$\mathcal{D}\psi - \frac{i}{2}\gamma_5\psi \wedge A = 0 \qquad \text{(III.6.56c)}$$

$$dA = 0 \qquad \text{(III.6.56d)}$$

The 2-form:

$$\Omega = \frac{i}{2}\bar{\psi} \wedge \gamma_m\psi \wedge V^m \qquad \text{(III.6.57)}$$

introduced in Eq. (III.6.34), is closed also within the new group $Osp(4/1) \times U(1)$, since Ω is a U(1) scalar. Indeed:

$$d\Omega = -\frac{i}{2}\,2\,\bar{\psi} \wedge \gamma^m \mathcal{D}\psi \wedge V_m + \left(\frac{i}{2}\right)^2 \bar{\psi} \wedge \gamma_m\psi \wedge \bar{\psi} \wedge \gamma^m\psi$$

$$= \frac{1}{2}\bar{\psi} \wedge \gamma_5\gamma_m\psi \wedge A \wedge V^m - \frac{1}{2}\bar{\psi} \wedge \gamma^m\psi \wedge \bar{\psi} \wedge \gamma_m\psi \equiv 0 \qquad \text{(III.6.58)}$$

since both terms are identically zero.

Therefore introducing the 2-form T we may add to Eqs. (III.6.56) the further equation (III.6.36d) namely:

$$R^{\otimes} \equiv dT - \frac{i}{2}\bar{\psi} \wedge \gamma^m\psi \wedge V_m = 0 \qquad \text{(III.6.59)}$$

Equations (III.6.56) plus (III.6.59) define the F.D.A. underlying the Sohnius-West model, also called the new minimal N=1 supergravity.

According to the discussion of the previous section we now gauge the F.D.A. just introduced.

Using the same symbols $(\omega^{ab}, V^a, \psi, A, T)$ for the gauged fields as for the left-invariant ones, we define the following set of generalized curvatures:

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} \equiv \mathcal{R}^{ab} \qquad \text{(III.6.60a)}$$

$$R^a = \mathcal{D}V^a - \frac{i}{2}\bar{\psi} \wedge \gamma^a\psi \qquad \text{(III.6.60b)}$$

$$\rho = \mathcal{D}\psi - \frac{i}{2}\gamma_5\psi \wedge A \qquad \text{(III.6.60c)}$$

$$R^{\square} = dA \qquad \text{(III.6.60d)}$$

$$R^{\otimes} = dT - \frac{i}{2}\bar{\psi} \wedge \gamma_m\psi \wedge V^m \qquad \text{(III.6.60e)}$$

Notice that R^{ab}, R^a, ρ and $R^{\square}$ are 2-forms while $R^{\otimes}$ is a 3-form.

By d-differentiation of both sides of the structural equations (III.6.60) one obtains the generalized Bianchi identities

$$\mathscr{D} R^{ab} = 0 \qquad \text{(III.6.61a)}$$

$$\mathscr{D} R^{a} + R^{ab} \wedge V_b - i\,\bar{\psi} \wedge \gamma^a \rho = 0 \qquad \text{(III.6.61b)}$$

$$\mathscr{D} \rho + \frac{i}{2} \gamma_5 \rho \wedge A + \frac{1}{4} R^{ab} \wedge \gamma_{ab}\psi - \frac{1}{2} i\gamma_5 \psi \wedge R^{\square} = 0 \qquad \text{(III.6.61c)}$$

$$d\, R^{\square} = 0 \qquad \text{(III.6.61d)}$$

$$d\, R^{\otimes} - i\,\bar{\psi} \wedge \gamma_m \rho \wedge V^m + \frac{i}{2} \bar{\psi} \wedge \gamma_m \psi \wedge R^m = 0 \qquad \text{(III.6.61e)}$$

The soft F.D.A. equations (III.6.60) are an extension of the ordinary soft $\overline{Osp(4/1)}$ structural equations. Hence we know that V^a_μ, ω^{ab}_μ and ψ_μ are liable to represent the vielbein, the spin connection and the gravitino field of D=4 N=1 supergravity. Furthermore, as we are going to see in due course, the two extra fields A_μ and $T_{\mu\nu}$ will play the rôle of auxiliary fields. Namely they will be non-propagating fields which are zero on-shell, but whose presence is necessary for the off-shell closure of the supersymmetry algebra under which the Lagrangian is invariant. Thus they play in N=1 D=4 supergravity the same rôle played in the Wess-Zumino model by F and G (see Chapter II.6).

The interest of the Sohnius-West model based on (III.6.60) is thus twofold: on one hand it is an example of the gauging (≡ softening) of a F.D.A.; on the other hand it is also a geometric theory with auxiliary fields which has, therefore, a closed off-shell algebra of transformations. This will provide us with the first explicit example of off-shell rheonomy. Moreover it will cast new light on the interpretation of geometrical actions.

Since our goal is that of finding an off-shell formulation of supergravity, what we need is a parametrization of the curvatures satisfying the following conditions:

i) consistency with the Bianchi identities (III.6.61), without implying the space-time equations. This guarantees off-shell closure of the algebra.

ii) rheonomy, that is, all the outer components of the curvatures are given in terms of the inner ones.

iii) consistency with the variational equations associated to the action.

Fulfillment of these conditions defines off-shell rheonomy.

In order to decide whether the condition iii) holds, we must of course find an action. Using the building rules for geometrical actions established in Sect. III.3.9 and generalized in the previous section for F.D.A. one obtains:

$$\mathscr{A} = \int_{M^4 \subset M} (R^{ab} \wedge V^c \wedge V^d \,\varepsilon_{abcd} + 4\,\bar{\psi} \wedge \gamma_5 \gamma_a \rho \wedge V^a - 4\, R^{\square} \wedge T)\,. \qquad \text{(III.6.62)}$$

This action can be easily retrieved by observing that the only other terms that we can add to the action (III.3.51) of N=1, D=4 supergravity must be consistent with the SO(1,3) × U(1) gauge invariance of Eqs. (III.6.60-61) and have the same scaling power $[w^2]$. Now Eqs. (III.6.60-61) imply that fields and curvatures must scale as follows:

$$\omega^{ab} \to \omega^{ab} \quad ; \quad R^{ab} \to R^{ab} \qquad \text{(III.6.63a)}$$

$$V^a \to wV^a \quad ; \quad R^a \to wR^a \qquad \text{(III.6.63b)}$$

$$\psi \to w^{1/2}\psi \quad ; \quad \rho \to w^{1/2}\rho \qquad \text{(III.6.63c)}$$

$$A \to A \quad ; \quad R^{\square} \to R^{\square} \qquad \text{(III.6.63d)}$$

$$T \to w^2 T \quad ; \quad R^{\otimes} \to w^2 R^{\otimes} \qquad \text{(III.6.63e)}$$

The only geometrical term fulfilling the two requirements is easily seen to be: $aR^{\square} \wedge T$. Its coefficient can be found by imposing the vacuum condition on the equation obtained by δA variation. Then one finds immediately: $a = -4$. This justifies Eq. (III.6.62). The equations of motion derived from the action (III.6.62) are:

$$\delta\omega^{ab} : 2\,\varepsilon_{abcd}\, R^c \wedge V^d = 0 \tag{III.6.64a}$$

$$\delta V^d : 2\,\varepsilon_{abcd}\, R^{ab} \wedge V^c + 4\,\bar\psi \wedge \gamma_5\gamma_d\rho = 0 \tag{III.6.64b}$$

$$\delta\psi : 8\,\gamma_5\gamma_m\rho \wedge V^m - 4\,\gamma_5\gamma_m\psi \wedge R^m = 0 \tag{III.6.64c}$$

$$\delta A : R^{\otimes} = 0 \tag{III.6.64d}$$

$$\delta T : R^{\square} = 0 \tag{III.6.64e}$$

The first three equations are the same as in N=1 D=4 supergravity (see Eqs. (III.3.52)) and lead to the same space time equations and rheonomic conditions.

Equations (III.6.64d,e), however, say that both the A and T fields have a vanishing curvature in superspace. Therefore the action (III.6.64) describes a theory which is not essentially different from the usual N=1 D=4 theory, and the algebra of supersymmetry transformations leaving (III.6.64) invariant does not close off-shell. This will be apparent from our subsequent developments. Hence for the purpose of arriving to an off-shell supersymmetric theory we extend the previous action by adding a kinetic term for $T_{\mu\nu}$. This we do by introducing a 0-form f_a according to the procedure explained in Chapter I.5 and in the construction of the D=4 N=2 theory, Sect. III.4.3. Following these latter examples we supplement (III.6.62) with

$$\mathscr{A}_{KIN} = \int_{M^4 \subset M} (\alpha\, f_a\, R^{\otimes} \wedge V^a + \beta\, f_\alpha f^\alpha \varepsilon_{ijk\ell}\, V^i \wedge V^j \wedge V^k \wedge V^\ell) \tag{III.6.65}$$

where f^a is an SO(1,3)-vector valued 0-form which we want to identify with the VVV components of $R^{\otimes}$. Indeed setting

$$R^{\otimes} = f^i\, V^j \wedge V^k \wedge V^\ell\, \varepsilon_{ijk\ell} \tag{III.6.66}$$

by δf^a variation of (III.6.65) we find:

$$\alpha\, f^i\, \varepsilon_{ijk\ell}\, \varepsilon^{jk\ell a} + 2\,\beta\, f^a\, \varepsilon_{ijk\ell}\, \varepsilon^{ijk\ell} = 0 \tag{III.6.67}$$

which implies $\beta = \alpha/8$. Therefore we arrive at the following final form of the action with 0-forms:

$$\begin{aligned}\mathscr{A} + \mathscr{A}_{KIN} = \int_{M^4 \subset M} &\{R^{ab} \wedge V^c \wedge V^d\, \varepsilon_{abcd} + 4\,\bar\psi \wedge \gamma_5\gamma_a\rho \wedge V^a \\ &- 4\,R^{\square} \wedge T + \alpha(f^a\, R^{\otimes} \wedge V_a \\ &+ \frac{1}{8}\, f_a f^a\, \varepsilon_{ijk\ell}\, V^i \wedge V^j \wedge V^k \wedge V^\ell)\}\end{aligned} \tag{III.6.68}$$

The new equations of motion are:

$$\delta\omega^{ab} : 2\,\varepsilon_{abcd}\, R^c \wedge V^d = 0 \tag{III.6.69a}$$

$$\delta f^a : \alpha\, R^{\otimes} = \alpha\, f^i\, V^j \wedge V^k \wedge V^\ell\, \varepsilon_{ijk\ell} \tag{III.6.69b}$$

$$\begin{aligned}\delta V^a : 2\,R^{k\ell} \wedge V^n\, \varepsilon_{ak\ell n} - 4\,\bar\psi \wedge \gamma_5\gamma_a\rho \\ - \alpha\, f^a\, R^{\otimes} + \frac{\alpha}{2}\, f_m f^m\, \varepsilon_{aijk}\, V^i \wedge V^j \wedge V^k = 0\end{aligned} \tag{III.6.69c}$$

$$\begin{aligned}\delta\psi : 8\,\gamma_5\gamma_m\rho \wedge V^m - i\,\alpha\,\gamma_m\psi \wedge V^m \wedge f_a\, V^a \\ - 4\,\gamma_5\gamma_m\psi \wedge R^m = 0\end{aligned} \tag{III.6.69d}$$

$$\begin{aligned}\delta T : -4\,R^{\square} + \alpha\, V^m \wedge \mathscr{D} f_m - \frac{i\alpha}{2}\, f_m \bar\psi \wedge \gamma^m\psi \\ - \alpha\, f_m\, R^m = 0\end{aligned} \tag{III.6.69e}$$

$$\delta A : R^{\otimes} = 0 \tag{III.6.69f}$$

Considering f^a as a 0-form field strength, according to the discussion following (III.4.48) and (III.6.65), we see that Eqs. (III.6.69) satisfy the vacuum condition, namely they admit the solution $R^{ab} = R^a = R^{\otimes} = R^{\square} = f^a = 0$ independently of the value of the parameter α.

Next we observe that the on-shell content of the extended action (III.6.68) is identical to that given by action (III.6.62).

Indeed Eqs. (III.6.69b) and (III.6.69f) imply $f^a = 0$ so that the set of equations (III.6.69) is equivalent to the set (III.6.64). Therefore, recalling Eqs. (III.3.74) and (III.3.62), we find the following rheonomic parametrization of the curvatures:

$$R^a = R^\square = R^\otimes = 0 \qquad \text{(III.6.70a)}$$

$$\rho = \rho_{ab}\, V^a \wedge V^b \qquad \text{(III.6.70b)}$$

$$R^{ab} = R^{ab}{}_{cd}\, V^c \wedge V^d + \bar{\theta}^{ab}{}_c\, \psi \wedge V^c \qquad \text{(III.6.70c)}$$

where $\bar{\theta}^{ab}{}_c$ is given in Eq. (III.3.72), and the tensor $R^{ab}{}_{.mn}$ and the tensor spinor ρ_{mn} satisfy the propagation equations:

$$R^{am}{}_{bm} - \frac{1}{2}\delta^a{}_b\, R^{mn}{}_{mn} = 0 \qquad \text{(III.6.70d)}$$

$$\gamma^m \rho_{mn} = 0 \qquad \text{(III.6.70e)}$$

Hence rheonomy if fulfilled because every non-vanishing outer component of the curvature is expressed in terms of the inner ones $R^{ab}{}_{.mn}$, ρ_{mn}. This implies that every solution of the restriction of Eqs. (III.6.70d, e) to the 4-dimensional space-time surface can be uniquely extended to a solution on the whole superspace manifold. Notice that the above result is independent of α. If we perform a superspace general coordinate transformation on this extended solution and then we restrict it once again to the x-manifold we obtain a new solution of the restricted equations. Hence rheonomy guarantees that the x-space field equations have an algebra of symmetries with as many generators as there are forms in the differential algebra (III.6.60).

III.6.6 - The Sohnius-West model: off-shell extensions

The important point to realize is that Eqs. (III.6.70) allow the extension to the whole superspace only of those initial data which in x-space satisfy the differential equations (III.6.70d,e). That it must be so follows from the fact that on-shell the theory is not essentially different from the N=1 D=4 supergravity of Chapter III.3 since both $R^\square$ and $R^\otimes$, the curvatures associated to the new fields, are zero on shell.

Therefore if we insert Eqs. (III.6.70a,b,c) into the Bianchi identities (III.6.61) we obtain, as in Sect. III.3.6, that they are satisfied only upon use of the space-time equations (III.6.70d,e).

In the present case, however, we are more ambitious and we would like a recipe to extend not only solutions of the space-time equations, but also arbitrary off-shell configurations. The reason is that eventually we want to do quantum physics and we need supersymmetry of the action functional and not only of the equations of motion.

In order to do that we have to figure out whether we can write a rheonomic parametrization of the curvatures (III.6.60) which is softer than that given by the superspace equations of motion. By softer we mean the following: the new parametrization should contain as a particular case the on-shell parametrization (III.6.70a,b,c), but in such a way that the Bianchi identities should not imply the space-time equations (III.6.70d,e). Contrary to what happens in pure supergravity, this program can be solved in the present case since the extra fields $T_{\mu\nu}$ and A_μ provide a more general rheonomic parametrization of the curvatures.

To determine it we write down the most general rheonomic parametrization which is compatible with the $SO(1,3) \times U(1)$ gauge invariance and the scaling law (III.6.63). In other words we use the building rules of supergravity in connection with the Bianchi identities as explained in Sect. III.3.12. Moreover we impose from the beginning the two constraints:

$$R^a = 0 \qquad \text{(III.6.71a)}$$

$$R^{\otimes} = f^a V^b \wedge V^c \wedge V^d \epsilon_{abcd} \qquad \text{(III.6.71b)}$$

These in fact are kinematical equations of motion which allow the elimination of the first order fields ω_μ^{ab} and f^a.(*) What we want to remove are the space-time equations of the physical propagating fields, namely: $(V_\mu^a, \psi_\mu, A_\mu, T_{\mu\nu})$.

Taking Eqs. (III.6.71) into account it is easy to write down the following rheonomic (and hopefully off-shell) ansatz for the curvatures:

$$R^a = 0 \qquad \text{(III.6.72a)}$$

$$R^{\otimes} = f^i V^j \wedge V^k \wedge V^\ell \epsilon_{ijk\ell} \qquad \text{(III.6.72b)}$$

$$R^{ab} = R^{ab}{}_{mn} V^m \wedge V^n + (2 i \bar{\rho}_c{}^{[a} \gamma^{b]} - i \bar{\rho}^{ab} \gamma_c)\psi \wedge V^c$$
$$- i c_1 \epsilon^{abcd} \bar{\psi} \wedge \gamma_d \psi f_c \qquad \text{(III.6.72c)}$$

$$\rho = \rho_{ab} V^a \wedge V^b + a i \gamma_5 \psi \wedge f_a V^a - c_2 i \gamma_5 \gamma_{mn} \psi \wedge V^m f^n \qquad \text{(III.6.72d)}$$

$$R^{\square} = F_{ab} V^a \wedge V^b + \bar{\psi} \chi_m \wedge V^m + i c_3 \bar{\psi} \wedge \gamma_m \psi f^m \qquad \text{(III.6.72e)}$$

We point out a few peculiarities. First of all we have given to the ψV component of R^{ab} the same form that was obtained in Eq. (III.3.219a) when we solved the Bianchi identities of the D=4, N=1 theory. This form, called $\overset{(II)}{\theta}{}^{ab}_c$ in Table III.3.I, was shown to be on shell equivalent to the one used in (III.6.70c), found by solving the space-time equations of motion ($\overset{(I)}{\theta}{}^{ab}_c$ in Table III.3.I). It is quite natural to use $\overset{(II)}{\theta}{}^{ab}_c$ in our ansatz since our goal is to find a parametrization of the curavtures which satisfies the Bianchi identities <u>without the use of the equations of motion</u>.

(*) (III.6.71a) fixes also the vielbein supersymmetry transformation law (see the comment after Eq. (III.3.208)).

Secondly the spinor χ_m with scaling behaviour $[w^{-3/2}]$, entering the definition of the ψV component of $R^{\square}$, must be made out of ρ_{ab} by rheonomy.

Thirdly χ_m itself must be proportional to the l.h.s. of the gravitino equation (III.6.70e). Indeed the given ansatz (III.6.72) must reduce to the on-shell parametrization (III.6.70a,b,c) if we supplement (III.6.72) with the equations of motion of the propagating fields. Now Eqs. (III.6.70) imply $f^a = 0$; hence, in order to retrieve Eqs. (III.6.70), the aforementioned proportionality between χ_m and $\gamma^n \rho_{mn}$ must hold.

We are therefore left with the computation of a, c_1, c_2, c_3 and of the explicit form of the spinor χ by working out the Bianchi identities. Let us rewrite Eqs. (III.6.61) using the first order constraints (III.6.71):

$$\mathscr{D} R^{ab} = 0 \qquad \text{(III.6.73a)}$$

$$R^{ab} \wedge V_b - i \bar{\psi} \wedge \gamma^a \rho = 0 \qquad \text{(III.6.73b)}$$

$$\mathscr{D}\rho + \frac{i}{2}\gamma_5 \rho \wedge A + \frac{1}{4}\gamma_{ab} R^{ab} \wedge \psi - \frac{1}{2} i \gamma_5 \psi \wedge R^{\square} = 0 \qquad \text{(III.6.73c)}$$

$$\mathscr{D}(f_a V_b \wedge V_c \wedge V_d \epsilon^{abcd}) - i \bar{\psi} \wedge \gamma^a \rho \wedge V_a = 0 \qquad \text{(III.6.73d)}$$

$$d R^{\square} = 0 \qquad \text{(III.6.73e)}$$

Let us first consider Eq. (III.6.73b); the ψVV component is identically satisfied by the parametrization (III.6.72c); at the $\psi\psi V$ level one easily finds

$$c_1 = c_2 \qquad \text{(III.6.74)}$$

Next we examine the $\psi\psi VV$ projection of (III.6.73d); one finds:

$$\frac{3i}{2} f^a \bar{\psi} \wedge \gamma^b \psi \wedge V^c \wedge V^d \epsilon_{abcd} + i \bar{\psi} \wedge \gamma^{[a} H^{b]} \psi \wedge V_a \wedge V_b = 0 \qquad \text{(III.6.75)}$$

where we have set

$$H^b = i\,a\gamma_5\, f^b - i\, c_2\,\gamma_5\,\gamma_{ab}\, f^b\ . \qquad \text{(III.6.76)}$$

It easily follows:

$$c_1 = c_2 = 3/2\ . \qquad \text{(III.6.77)}$$

The ψVVV projection of the same equation gives on the other hand:

$$\psi \wedge \Xi_a \wedge V_b \wedge V_c \wedge V_d\, \varepsilon^{abcd} - i\,\bar\psi \wedge \gamma_m\, \rho_{ab}\, V^a \wedge V^b \wedge V^m = 0 \qquad \text{(III.6.78)}$$

where Ξ_a is ψ-component of $\mathscr{D} f_a$:

$$\mathscr{D} f_a = \mathscr{D}_m\, f_a\, V^m + \bar\psi\, \Xi_a\ . \qquad \text{(III.6.79)}$$

Equation (III.6.78) implies:

$$\Xi^a = -\frac{i}{3!}\,\varepsilon^{abcd}\,\bar\psi \wedge \gamma_b\, \rho_{cd} \qquad \text{(III.6.80)}$$

In this way we have solved Eqs. (III.6.73b) and (III.6.73d).

Next we consider the gravitino Bianchi equation (III.6.73c). At the $\psi\psi\psi$ level we have:

$$\frac{1}{2}\,a\, f_m\,\gamma_5\,\psi \wedge \bar\psi \wedge \gamma^m\,\psi + i\,\frac{3}{8}\,\gamma_{ab}\,\psi \wedge \bar\psi \wedge \gamma_c\,\psi\, f_d\, \varepsilon^{abcd} + \frac{1}{2}\, c_3\,\gamma_5\,\psi \wedge \bar\psi \wedge \gamma^m\,\psi\, f_m = 0 \qquad \text{(III.6.81)}$$

This equation can be analyzed into its irreducible constituents using Table II.8.V.

Annihilating the coefficient of $\Xi^{(12)}$ yields

$$c_3 = 3 - a\ . \qquad \text{(III.6.82)}$$

Finally we examine the $\psi\psi V$ projection of Eq. (III.6.73c); we find:

$$i\,\rho_{ab}\,\bar\psi \wedge \gamma^a\psi \wedge V^b - i\,a\gamma_5\,\psi \wedge \bar\psi\,\Xi_a \wedge V^a - \frac{3i}{2}\,\gamma_5\gamma_{ab}\,\psi \wedge \bar\psi\,\Xi^a \wedge V^b - \frac{i}{4}\,\gamma^{rs}\,\psi \wedge \bar\psi(2\,\gamma_r\,\rho_{sb} + \gamma_b\,\rho_{rs}) \wedge V^b - \frac{i}{2}\,\gamma_5\,\psi \wedge \bar\psi\,\chi_a \wedge V^a = 0 \qquad \text{(III.6.83)}$$

where Ξ_b is given by Eq. (III.6.80). Decomposing $\psi \wedge \bar\psi$ according to Eq. (II.8.V) we obtain two equations corresponding to the currents $\bar\psi\,\gamma^m\psi$ and $\bar\psi\,\gamma^{mn}\psi$. After some γ-matrix algebra one finds the following common solution:

$$\chi^a = 2\left(\frac{ia}{3!}\,\varepsilon^{abcd}\,\gamma_b\,\rho_{cd} - \gamma_5\,\gamma_m\,\rho^{ma}\right) \qquad \text{(III.6.84)}$$

The two terms on the r.h.s. of this equation are both proportional to the l.h.s. of the gravitino equation (see Eqs. (III.2.62c) and (III.2.100b)).

Therefore the spinor χ_m is indeed proportional to the l.h.s. of the gravitino equation as anticipated.

At this point the parametrization of the curvatures has been completely found with the exception of the parameter a, which is still undetermined. The only remaining place where it could be determined is in the $\psi\psi\psi$-projection of the gravitino Bianchi identity (III.6.73c). However in this projection the two terms containing the a parameter cancel identically leaving an equation among the ρ_{ab} components which is identically satisfied. However this apparent indeterminacy is spurious since the presence of the parameter a just amounts to a redefinition of the gauge field A. Indeed we see that the term $i\,a\gamma_5\,\psi f^a \wedge V_a$ appearing in the r.h.s. of Eq. (III.6.72d) can be reabsorbed in the $SO(1,3)\otimes U(1)$ covariant derivative $f \equiv \nabla\psi$ by redefining $A' = A + 2a\, f^a\, V_a$. Thus, fixing the value of a is equivalent to fixing a particular definition of A. In the following we shall keep this freedom for later convenience.

Thus we arrive at the conclusion that the parametrization of the curvatures in Eqs. (III.6.72), with c_1, c_2, c_3 and χ_m given by

Eqs. (III.6.74,77,82) and (III.6.84) respectively, is consistent with the Bianchi's and does not imply the space-time equations of the physical fields. Therefore, using Eqs. (III.6.72) into the definition of the superspace Lie derivative:

$$\ell_\varepsilon \mu^A = \underline{\varepsilon} \rfloor d\mu^A + d\, \underline{\varepsilon} \rfloor \mu^A = (\nabla\varepsilon)^A + \underline{\varepsilon} \rfloor R^A \qquad \text{(III.6.85)}$$

where $\mu^A \equiv \{V^a, \psi, \omega^{ab}, A, T\}$, we obtain an off-shell closed algebra of supersymmetry transformations.

It is useful, at this point, to compare the previously found parametrization of the curvatures with the variational equations (III.6.70).

We have already observed that the use of the whole content of Eqs. (III.6.69) imply $f_a = 0$, and hence $R^a = R^\square = R^\otimes = 0$ so that we are on-shell. If however we disregard Eq. (III.6.69f), and we consider Eqs. (III.6.69a) and (III.6.69d), using the ψV component of ρ as given by the Bianchi's, from Eq. (III.6.69d) we would obtain:

$$8\,\gamma_5\gamma_a(i\, a\, \gamma_5\, \psi\, f^b\, V_b - \frac{3i}{2}\, \gamma_5\gamma_{rs}\, \psi \wedge V^r f^s) \wedge V^a$$
$$- i\,\alpha\,\gamma_a\, \psi \wedge V^a \wedge V^b f_b = 0 \qquad \text{(III.6.86)}$$

This equation is inconsistent unless one has $f^a = 0$, since the second term inside the brackets has no counterpart. This shows the mechanism of rheonomy: the on-shell parametrization of the curvatures, which is determined by the variational equations, is consistent with that given by the Bianchi identities, but requires the use of the space-time propagation equations; in this case: $f_a = 0$.

Now we come to the invariance of the action functional under the transformations (III.6.85). The fact that (III.6.85) close an algebra of off-shell transformations for the physical fields does not mean, of course, that this should also be a closed symmetry algebra of the action (III.6.68). We have given in Chapter III.3 the condition under which an action functional of a certain 4-form over some 4-dimensional surface M_4 immersed in superspace is invariant under a closed set of supersymmetry transformations. There we found that this happens if and only if the Lagrangian is a closed form, namely: $d\mathscr{L} = 0$.

Therefore our next task is to consider the d-derivative of the Lagrangian (III.6.68). Using the Bianchi identities (III.6.73) in second order formalism (namely with the two kinematical constraints (III.6.71) included) one obtains:

$$d\mathscr{L} = 4\,\bar\rho \wedge \gamma_5\gamma_a\, \rho \wedge V^a - 4\, R^\square \wedge R^\otimes$$
$$+ \alpha\, \mathscr{D} f_a \wedge R^\otimes \wedge V^a + i\,\alpha\, f_a\, \bar\psi \wedge \gamma_b\, \rho\, V^b \wedge V^a$$
$$- \frac{i\alpha}{2} f^a R^\otimes \wedge \bar\psi \wedge \gamma_a\, \psi + \frac{\alpha}{4}\, \mathscr{D} f_a \wedge f^a\, \varepsilon_{ijk\ell}\, V^i \wedge V^j \wedge V^k \wedge V^\ell$$
$$+ \frac{i}{4}\,\alpha\, f^a f_a\, \varepsilon_{ijk\ell}\, \bar\psi \wedge \gamma^i \psi \wedge V^j \wedge V^k \wedge V^\ell \qquad \text{(III.6.87)}$$

The first term on the r.h.s. is the exterior derivative of the N=1 D=4 supergravity Lagrangian evaluated at $R^a = 0$, (see Eq. (III.3.116)), the others are new.

Equation (III.6.87) has two relevant projections, namely $\psi\psi VVV$ and $\psi VVVV$.

From the $\psi VVVV$-projection one finds, after use of the parametrization (III.6.72) and some simple algebra, that all the terms cancel except the following one:

$$i(16a - 12 - \alpha)\, \varepsilon^{abcd} f_a\, \bar\psi \wedge \gamma_b\, \rho_{cd} = 0 \qquad \text{(III.6.88)}$$

Therefore we find the first condition:

$$16a - 12 - \alpha = 0 \qquad \text{(III.6.89)}$$

Next we consider the $\psi\psi VVV$-projection. One has

$$4\,\bar\psi\wedge\tilde H_m\gamma_5\gamma_a H_n\psi\wedge V^n\wedge V^m\wedge V^a -$$

$$-\,4i(3-a)\,\bar\psi\wedge\gamma^m\psi\wedge f_m f^a V^b\wedge V^c\wedge V^d\,\varepsilon_{abcd}\,+$$

$$+\,i\,\alpha\,f_a\bar\psi\wedge\gamma_b H_m\psi\wedge V^m\wedge V^b\wedge V^a -$$

$$-\,\frac{i\alpha}{2}f^a f^i\wedge V^j\wedge V^k\wedge V^\ell\wedge\varepsilon_{ijk\ell}\,\bar\psi\wedge\gamma_a\psi$$

$$+\,\frac{i\alpha}{4}\,f^2\,\varepsilon_{ijk\ell}\,\bar\psi\wedge\gamma^i\psi\wedge V^j\wedge V^k\wedge V^\ell = 0 \qquad \text{(III.6.90)}$$

where we have set according to (III.6.76,77):

$$H_m = i\,a\,f_m\gamma_5 - \frac{3}{2}i\,\gamma_5\gamma_{mn}f^n \qquad \text{(III.6.91a)}$$

$$\tilde H_m = \gamma_0 H^\dagger\gamma_0\,. \qquad \text{(III.6.91b)}$$

After some tensor algebra one arrives at the following equations:

$$-\,\mathscr{F}^t - \left(\frac{3}{2}\,i\,\alpha\,f^2\,\delta^t_m + 24i(3-a)f^t f_m\right)\bar\psi\wedge\gamma^m\psi = 0 \qquad \text{(III.6.92)}$$

where

$$\mathscr{F}^t = 4\,\bar\psi\wedge\tilde H_m\gamma_5\gamma_a H_n\,\psi\,\varepsilon^{atmn}\,. \qquad \text{(III.6.93)}$$

Straightforward manipulations on (III.6.93) give the result:

$$\mathscr{F}^t = \{i(24a-18)(f^2\delta^t_m - f^t f_m) + 54i\,f^t f_m\}\bar\psi\wedge\gamma^m\psi\,. \qquad \text{(III.6.94)}$$

Therefore (III.6.92) takes the final form:

$$i\left(\frac{3\alpha}{2} - 24a + 18\right)f^2\bar\psi\wedge\gamma^t\psi = 0 \qquad \text{(III.6.95)}$$

We see that the relation between a and α implied by Eq. (III.6.85) is the same as (III.6.89). Furthermore $d\mathscr{L}$ vanishes identically along any 5-form with more than 2ψ-fields. We conclude that the action (III.6.68) is invariant against an off-shell closed algebra of supersymmetry transformations, provided the parameter α and a entering the Lagrangian (III.6.68) and the curvatures (III.6.72) satisfy the relation (III.6.89). The explicit supersymmetry transformations are given in Table III.6.I at the end of this chapter: they are easily found as already explained by the combined use of the Lie derivative formula (III.6.85)

$$\ell_\varepsilon\, V^a = (\nabla\varepsilon)^a + \varepsilon\rfloor R^a \qquad \text{(III.6.96a)}$$

$$\ell_\varepsilon\,\psi = (\nabla\varepsilon) + \varepsilon\rfloor\rho \qquad \text{(III.6.96b)}$$

$$\ell_\varepsilon\, A = (\nabla\varepsilon)^\square + \varepsilon\rfloor R^\square \qquad \text{(III.6.96c)}$$

$$\ell_\varepsilon\, T = (\nabla\varepsilon)^\otimes + \varepsilon\rfloor R^\otimes \qquad \text{(III.6.96d)}$$

$$\ell_\varepsilon\,\omega^{ab} = (\nabla\varepsilon)^{ab} + \varepsilon\rfloor R^{ab} \qquad \text{(III.6.96e)}$$

and the parametrization (III.6.72) of the curvatures. As usual $\varepsilon\equiv\bar\varepsilon^\alpha\,\tilde D_\alpha$ and the covariant derivative $(\nabla\varepsilon)^A\equiv(\nabla\varepsilon^{ab},\nabla\varepsilon^a,\nabla\varepsilon,\nabla\varepsilon^\square,\nabla\varepsilon^\otimes)$ can be easily detained from the explicit form of the Bianchi identities $\nabla R^A = 0$ (see the remark following Eq. (III.3.11)).

As a final remark we note that once the value of a has been fixed by a particular choice of A, α is also fixed. If $a=\frac{3}{4}$, then $\alpha=0$ and the action takes the very simple form (III.6.62). It is a strongly geometrical action since all the non-geometrical terms involving 0-forms have disappeared.

We summarize our discussion as follows:

a) The Sohnius-West theory is the local theory of an appropriate free differential algebra.

b) Requiring the independence of the action from the specific choice of the space-time section of superspace, namely off-shell supersymmetry invariance, fixes the action completely. In practice this is done by imposing the condition $d\mathscr{L}=0$.

III.6.7 - The building rules in their final form

The discussion of the last section has put into evidence the differences existing between the theories with auxiliary fields like the Sohnius-West model and the theories without auxiliary fields. In the latter case, the action determined by the building rules is not, in general, invariant against a closed algebra, but only its equations of motion are.

In the purely space-time approach to supergravity (Noether method) the action is usually determined using the concept of an algebra of transformations which closes only on-shell, that is, an open algebra of transformations. Actually this requirement is equivalent to all the building rules of the geometrical approach including also the last. There are cases where the concept of invariance of the space-time action is convenient also in the geometric approach. This happens when the action contains terms which are purely space-time, that is proportional to D vielbeins (D is the space-time dimension) and which have no geometrical origin, that is, they do not involve the gauge fields of the theory. Usually such terms are 4-fermion terms arising in the matter coupled supergravity theories or are functionals of the scalar fields (see Part IV). It is evident that the coefficients in front of these terms cannot be determined by the vacuum condition, since they do not contain the gauge fields, nor by the rheonomy principle, since they appear only in the V....V projection of the superspace equations of motion, namely in the spacetime propagation equations.

It is true that a complete analysis of the Bianchi identities of the theory would determine the space-time equations of motion as integrability conditions of the rheonomic conditions; however this would be a very cumbersome and lengthy way to find the undetermined parameters.

A much easier way, instead, is to resort to supersymmetry invariance of the action, which, in our language, is implemented by $\varepsilon \rfloor d\mathcal{L} = 0$ (see Section III.3.8 and in particular Eqs. (III.3.117-118)). Explicit examples of this procedure will be given in Part IV.

Since the discussion given in Sect. III.3.9 we have extended and slightly modified the building rules for constructing the action of supergravity theories. Specifically we have introduced modifications of the building rules A and D of Chapter III.4, we have extended rules A-E to the case of Free Differential Algebras, and we have also discussed the role of supersymmetry invariance of the action and the meaning of the condition $d\mathcal{L} = 0$. It is worthwhile at this point to give a short summary of the building rules in their final form.

i) A supergravity theory in D-dimensions is based on a Free Differential Algebra; the physical fields and their field strengths are described by the soft forms and curvatures of the given F.D.A.

Rule A: The Lagrangian $\mathcal{L}$ of the theory is geometrical, in the sense of being a D-form constructed using only p-forms, d-exterior derivatives and exterior products. It does not contain the Hodge duality operator but will contain in general multiplets of 0-forms for the construction of kinetic terms in first order formalism. The action is obtained by integrating $\mathcal{L}$ on a D-dimensional surface $M_D \subset M$, M being the manifold on which the p-forms are defined:

$$\mathcal{A} = \int_{M_D \subset M} \mathcal{L} \quad .$$

The equations of motion are obtained by varying $\mathcal{A}$ according to the extended action principle; they are valid on the whole M.

Rule B: $\mathcal{A}$ must be H-invariant, H being the invariance subgroup of the F.D.A.

Rule C: Each term of the Lagrangian must scale homogeneously as $[w^{D-2}]$ under the rigid rescaling which leaves the F.D.A. and its generalized Bianchi identities invariant. ($[w^{D-2}]$ is the scaling behavior of the Einstein term).

Rule D: The equations of motion must be satisfied by the vacuum configuration defined by the conditions

$$R^A = 0 \quad ; \quad F^I = 0$$

R^A being the curvatures of the F.D.A. and F^I the multiplets of 0-forms.

Rule E: The superspace equations of motion must admit non-trivial rheonomic solutions besides the trivial one $R^A = F^I = 0$.

Rule F: The constraints on the inner curvature components implied by the Bianchi identities by the outer sectors of the variational equations (that is by the rheonomic parametrization of the curvatures) must coincide with the inner sectors of the same equations. This requirement guarantees the self consistency, as differential form equations, of the equations of motion. (No new information on the physical field is gained by extending the equations from space-time to superspace).

Furthermore this principle is necessary for a complete equivalence between the geometric approach and the Noether approach.

A violation of this rule will be discussed in constructing the Lagrangian of D=6 supergravity.

These principles suffice for the geometric construction of the supergravity theories without auxiliary fields. In the case of theories with auxiliary fields, namely in the case where the action should be invariant against a closed supersymmetry algebra, we must add the further rule.

Rule G: The Lagrangian is a closed form in superspace: $d\mathcal{L} = 0$.

Finally we note that in theories where there appear terms proportional to the maximum set of vielbeins $\epsilon_{a_1 \ldots a} V^{a_1} \wedge \ldots \wedge V^{a}$ their

coefficients cannot be fixed by rules A-E). They could be fixed by rule F or resorting to the study of the Bianchi identities. In practice however it is convenient to fix these remaining coefficients simply by requiring supersymmetry invariance of the action $\epsilon| d\mathcal{L} = 0$. Indeed the algebra being already determined by the previous steps we can just focus on these space-time terms and cancel their variation by suitable choice of the coefficients. How this works will be seen in Part IV.

TABLE III.6.I

Summary of the Sohnius-West new minimal model (N=1, D=4 Supergravity with auxiliary fields).

A) Gauge multiplet: $(\omega^{ab}{}_{\mu}, V^a{}_{\mu}, \psi_\mu, A_\mu, T_{\mu\nu})$

Super Poincaré gauge fields: (ω^{ab}, V^a, ψ)

Auxiliary fields: { A: U(1)-gauge connection; T: 2-form }

B) Free Differential Algebra Curvatures

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b$$

$$R^a = \mathcal{D} V^a - \frac{i}{2} \bar{\psi} \wedge \gamma^a \psi$$

$$\rho = \mathcal{D}\psi - \frac{i}{2} \gamma_5 \psi \wedge A$$

$$R^{\square} = dA$$

$$R^{\otimes} = dT - \frac{i}{2} \bar{\psi} \wedge \gamma_a \psi \wedge V^a$$

C) Generalized Bianchi identities

$$\mathcal{D} R^{ab} = 0$$

$$\mathcal{D} R^a + R^{ab} \wedge V_b - i \bar{\psi} \wedge \gamma^a \rho = 0$$

$$\mathcal{D}\rho + \frac{i}{2} \gamma_5 \rho \wedge A + \frac{1}{4} R^{ab} \wedge \gamma_{ab}\psi - \frac{i}{2} \gamma_5 \psi \wedge R^{\square} = 0$$

$$d R^{\square} = 0$$

$$d R^{\otimes} - i \bar{\psi} \wedge \gamma_a \rho \wedge V^a + \frac{i}{2} \bar{\psi} \wedge \gamma_a \psi \wedge R^a = 0$$

D) Off-shell parametrization of the curvatures

$$R^a = 0$$

$$R^{\otimes} = f^i \, V^j \wedge V^k \wedge V^\ell \, \varepsilon_{ijk\ell}$$

$$R^{ab} = R^{ab}{}_{mn} \, V^m \wedge V^n + (2i \, \bar{\rho}^{c[a} \, \gamma^{b]} - i \, \bar{\rho}^{ab} \, \gamma_c)\psi \wedge V^c$$

$$- i \, \frac{3}{2} \, \varepsilon^{abcd} \, \bar{\psi} \wedge \gamma_d \, \psi \, f_c$$

$$R^{\square} = F_{ab} \, V^a \wedge V^b + \bar{\psi} \, \frac{ia}{3} \, \varepsilon^{abcd} \, \gamma_b \rho_{cd} - 2 \, \gamma_5 \gamma_m \rho^{ma}) \wedge V_a$$

$$+ i(3-a)\bar{\psi} \wedge \gamma^a \, \psi \, f_a$$

$$\rho = \rho_{ab} \, V^a \wedge V^b + i \, a \gamma_5 \psi \wedge f^a \wedge V_a - \frac{3}{2} \, i \gamma_5 \gamma_{mn} \psi \wedge V^m f^n$$

(a free).

E) Supersymmetry transformations leaving the action invariant and closing an off-shell algebra (2nd-order formalism)

$$\delta_\varepsilon \, V^a = - i \, \bar{\psi} \, \gamma^a \varepsilon$$

$$\delta_\varepsilon \, \psi = \mathscr{D} \, \varepsilon + \frac{i}{2} \, \gamma_5 \, A \varepsilon + a \, i \, \gamma_5 \, \varepsilon \, f^a \, V_a - \frac{3}{2} \, i \, \gamma_5 \gamma_{mn} \, \varepsilon \, V^m f^n$$

$$\delta_\varepsilon \, A = \bar{\varepsilon} \, (\frac{ia}{3} \, \varepsilon^{abcd} \, \gamma_b \rho_{cd} - 2 \, \gamma_5 \gamma_m \rho^{ma}) V_a$$

$$\delta_\varepsilon \, T = i \, \bar{\psi} \gamma_a \varepsilon \wedge V^a$$

$$\delta_\varepsilon \, \omega^{ab} = \text{chain rule}$$

F) Invariant Action

$$\mathscr{A} = \int_{M_4 \subset M} \mathscr{L}$$

where:

$$\mathscr{L} = R^{ab} \wedge V^c \wedge V^d \, \varepsilon_{abcd} + 4 \, \bar{\psi} \wedge \gamma_5 \gamma_a \rho \wedge V^a - 4 \, R^{\square} \wedge T$$

$$+ (16a - 12)(f^a \, R^{\otimes} \wedge V_a + \frac{1}{8} \, f_a f^a \, \varepsilon_{ijk\ell} \, V^i \wedge V^j \wedge V^k \wedge V^\ell)$$

and

$$M = \widetilde{Osp(4/1)}/SO(1,3) \otimes U(1).$$