The final result of our calculation is:

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\psi_c = - i\,\bar{\varepsilon}_1\,\gamma^{\ell}\,\psi_{\ell c} + \frac{1}{4}\,\bar{\varepsilon}_1\,\gamma^{\ell}\,\varepsilon_2\,A_{\ell cs}\,E^s + \frac{1}{8}\,\bar{\varepsilon}_1\,\gamma^{\ell m}\,\varepsilon_2\,B_{\ell mcs}\,E^s \qquad \text{(III.2.A18)}$$

which is Eq. (III.2.70a) of the text.

CHAPTER III.3

SUPERGRAVITY IN SUPERSPACE AND THE RHEONOMY PRINCIPLE

III.3.1 - From space-time to superspace

Let us briefly review the logic of the previous chapter. Our aim was local, rather than global, supersymmetry invariance. This required the introduction of the spin 3/2 field ψ_μ which "gauges" the supersymmetry charge Q. Hence the problem of constructing N=1 supergravity, that is the "gauge" action of the N=1 supersymmetry algebra, was turned into the problem of coupling the Rarita Schwinger field to Einstein gravity. Utilizing first order formalism, namely treating the spin connection ω_μ^{ab} as an independent field, the above problem has a unique solution, up to a one parameter freedom, which can be reabsorbed into the normalization of ψ_μ.

The solution is given by the action (III.2.18) and the free parameter is a. It remained to be seen that (III.2.18) is indeed invariant against suitable supersymmetry transformations. Fixing a=4, we showed that the resulting action (III.2.54) is invariant under the transformations (III.2.55,56,61) or (III.2.62) in second order or first order formalism, respectively.

Our study of these transformations made manifest that they are not gauge transformations of the super Poincaré algebra. Equations (III.2.55,56,61) reduce to a true representation of the N=1 super Poincaré algebra only in their linearized form and on the free fields satisfying the linearized field equations.

At this point, the structural difference between Yang-Mills theories and supergravity theories should be clear to the reader.

The action of the former theories is fixed by the requirement of invariance against a set of local transformations which is given a priori (the gauge transformations). In the latter theories the set of local transformations against which the action should be invariant is not known a priori and is actually engineered in such a way as to befit the action.

A classical way out of this vicious circle is provided by the time honoured but very cumbersome method of the Noether coupling. In this approach the Lagrangian and the supersymmetry transformation rules which befit it are determined at the same time. One starts from a linearized action invariant under global supersymmetry transformations and uses a recursive procedure in the gravitational coupling constant $\kappa = \sqrt{4\pi G}$ in order to determine the needed additions to the Lagrangian and to the transformation rules.

The reason why in the previous chapter we were able to avoid such a lengthy construction is purely accidental and limited to the N=1 case. We were just lucky that the coupling of a spin 3/2 field to gravity is a problem with a unique and easy solution.

We cannot expect such an easy life in the case of higher N or higher dimensional supergravities.

This is why we are now going to reconsider the theory we just derived from a different and more formal viewpoint. Our aim is that of devising a better and more algorithmic setup for the construction of all supergravity theories. The key point in this programme is obviously a more satisfactory understanding of the supersymmetry transformation

rules (III.2.55,56,61 or 62c). This is achieved, as in the case of rigid supersymmetry (see Part Two), through the concepts of superspace and superfields.

We plan to extend, in an appropriate way, our space-time fields V^a_μ, ψ_μ, ω^{ab}_μ to fields (actually 1-forms) defined over superspace. In this way we shall be able to reinterpret the supersymmetry transformations (III.2.62) as superspace Lie derivatives. This is fully analogous to what we did in Chapter II.6 (Section II.6.3) for the rigid Wess-Zumino multiplet.

The principle of rheonomy, already discussed in Chapter II.6, makes the extension from space-time to superspace uniquely defined and consequently allows for a geometric interpretation of the supersymmetry rules. To summarize this principle in one sentence: we demand the θ-dependence of every superfield to be determined by the x-dependence of all the superfields in our stock.

When the principle of rheonomy is exploited to get rid of unwanted degrees of freedom we can identify supergravity with the geometric theory of superspace in the same way Einstein gravity is the geometric theory of space-time.

Indeed the pair of 1-forms (V^a,ψ) once extended to superspace can be viewed as a single object E^A, the supervielbein, namely a local cotangent frame on $M^{4/4}$. More generally $\mu^A \equiv (\omega^{ab},V^a,\psi)$ constitute an intrinsic reference frame in the cotangent plane to the soft super Poincaré group.

III.3.2 - Geometry of superspace

As stated, our aim is to show that supergravity can be naturally interpreted as a theory in superspace or, better, as the theory of superspace.

In order to arrive at this interpretation we reconsider the superspace structural equations given in Chapter II.3.

We start with the $\overline{Osp(4/1)}$ Maurer-Cartan equations defined in (II.3.27) in terms of the left invariant 1-forms:

$$\overset{o}{\mu} = \overset{o}{\mu}^{A} T_A \equiv \frac{1}{2} \overset{o}{\omega}^{ab} J_{ab} + \overset{o}{V}^{a} P_a + \overset{o}{\bar\psi} Q \tag{III.3.1}$$

where $T_A \equiv (J_{ab}, P_a, Q)$ are the generators of $\overline{Osp(4/1)}$ (the super Poincaré group) and $\overset{o}{\mu}^{A} = (\overset{o}{\omega}^{ab}, \overset{o}{V}^{a}, \overset{o}{\psi})$ is defined on the rigid group manifold $\overline{Osp(4/1)}$. Rigid superspace is defined by the structural equations (II.3.27) obtained by restricting the Maurer-Cartan equations to the coset space

$$\mathbb{R}^{4/4} \equiv \overline{Osp(4,1)}/SO(1,3) . \tag{III.3.2}$$

We assume that the fields $\overset{o}{\mu}^{A}$ transform as gauge fields under Lorentz transformations. This means that $\overline{Osp(1/4)}$ has been given the structure of a fiber bundle P with $\mathbb{R}^{4/4}$ as base space and SO(1,3) as fiber:

$$\overline{Osp(4,1)} \equiv P(\mathbb{R}^{4/4}, SO(1,3)) . \tag{III.3.3}$$

The soft superspace, $M^{4/4}$, is defined by the new 1-forms

$$\mu = \mu^{A} T_A \equiv \frac{1}{2} \omega^{ab} J_{ab} + V^{a} P_a + \bar\psi Q \tag{III.3.4}$$

which are not left invariant. Then the structural equations of the soft superspace define the curvatures as the deviation of the l.h.s. of the Maurer-Cartan equations from zero, and are given by Eqs. (II.3.27) with $(R^{ab}, R^{a}, \rho) \neq 0$. Let us rewrite them here for convenience:

$$R^{ab} = d\omega^{ab} - \omega^{a}{}_{c} \wedge \omega^{cb} \equiv \mathscr{R}^{ab} \tag{III.3.5a}$$

$$R^{a} = \mathscr{D} V^{a} - \frac{i}{2} \bar\psi \wedge \gamma^{a} \psi \tag{III.3.5b}$$

$$\rho = \mathscr{D}\psi . \tag{III.3.5c}$$

or in a compact notation

$$R^{A} = d\mu^{A} + \frac{1}{2} C^{A}{}_{BC}\, \mu^{B} \wedge \mu^{C} \tag{III.3.6}$$

where

$$R^{A} \equiv (R^{ab}, R^{a}, \rho) . \tag{III.3.7}$$

According to our approach to the study of local geometry given in Chapter I.3 (V^{a}, ψ) represent a basis of 1-forms at each point P of the cotangent plane to $M^{4/4}$: $T^{*}_{P}(M^{4/4})$. They constitute the so-called supervielbein basis. The spin connection ω^{ab} is the gauge connection of the Lorentz group which is the structural group acting on $T^{*}_{P}(M^{4/4})$.

Alternatively one may start with the soft group manifold $\tilde{G} \equiv \widetilde{Osp(4/1)}$ formally described by the same equations (III.3.5) where now, however, the fields μ^{A} depend on the coordinates of the whole $\tilde{G}$. In this case it is natural to interpret the triplet $\mu^{A} \equiv (\omega^{ab}, V^{a}, \psi)$ as a local (super)-vielbein frame spanning a cotangent frame on $T^{*}_{P}(\tilde{G})$ (see the discussion in section I.3.7). Imposing on $\tilde{G}$ the fiber bundle structure (SO(1,3)-horizontality of R^{A})

$$\tilde{G} \equiv \tilde{G}(M^{4/4}, SO(1,3)) \tag{III.3.8}$$

one retrieves the structural equations defining $M^{4/4}$.

d-differentiation of both sides of (III.3.5) gives the Bianchi identities (II.3.77) which once more we rewrite here for completeness:

$$\mathscr{D} R^{ab} = 0 \tag{III.3.9a}$$

$$\mathscr{D} R^{a} + R^{ab} \wedge V_b - i\, \bar\psi \wedge \gamma^{a} \rho = 0 \tag{III.3.9b}$$

$$\mathscr{D}\rho + \frac{1}{4} R^{ab} \wedge \gamma_{ab}\, \psi = 0 . \tag{III.3.9c}$$

Let us now derive the G-gauge transformations of $\mu^{A} \equiv (\omega^{ab}, V^{a}, \psi)$, where $G = \overline{Osp(1/4)}$. The quickest way to write them explicitly is to recall that the Bianchi identities (III.3.9) are equivalent to the statement that the G-covariant derivative of the adjoint multiplet of the curvatures R^{A} is zero

$$\nabla^{(G)} R^A \equiv d R^A + C^A{}_{BC}\, \mu^B \wedge R^C = 0 \qquad (III.3.10)$$

where the operator ∇ has been defined in (I.3.126-127). A G-gauge transformation of the field μ^A is given by the G-covariant derivative of ε^A where ε^A is a parameter in the adjoint of representation of G:

$$\delta_\varepsilon^{(gauge)}\, \mu^A = \nabla \varepsilon^A \,. \qquad (III.3.11)$$

In our case $G = \overline{Osp(1/4)}$ and $\varepsilon^A \equiv (\varepsilon^{ab}, \varepsilon^a, \varepsilon^\alpha)$. The Lorentz content of the ∇ derivative, when acting on the adjoint multiplet, can be read off directly from the explicit form of the Bianchi identities (III.3.9). We obtain:

$$\delta_\varepsilon^{(gauge)}\, \omega^{ab} = (\nabla\varepsilon)^{ab} \equiv \mathscr{D}\varepsilon^{ab} \qquad (III.3.12a)$$

$$\delta_\varepsilon^{(gauge)}\, V^a = (\nabla\varepsilon)^a \equiv \mathscr{D}\varepsilon^a + \varepsilon^{ab} V_b - i\,\bar\psi \gamma^a \varepsilon \qquad (III.3.12b)$$

$$\delta_\varepsilon^{(gauge)}\, \psi = \nabla\varepsilon \equiv \mathscr{D}\varepsilon + \frac{1}{4}\varepsilon^{ab}\gamma_{ab}\psi \,. \qquad (III.3.12c)$$

Here ∇ and $\mathscr{D}$ represent the $\overline{Osp(4/1)}$ and SO(1,3) covariant derivatives respectively. In particular, if $\varepsilon^A \equiv (0, 0, \varepsilon^\alpha)$ we get the explicit form of a <u>gauge supersymmetry transformation</u>:

$$\delta_\varepsilon\, \omega^{ab} = 0 \qquad (III.3.13a)$$

$$\delta_\varepsilon\, V^a = -\, i\,\bar\psi\gamma^a\varepsilon \qquad (III.3.13b)$$

$$\delta_\varepsilon\, \psi = \mathscr{D}\varepsilon \,. \qquad (III.3.13c)$$

Setting instead:

$$\varepsilon^A = (\varepsilon^{ab}, 0, 0) \quad \text{or} \quad \varepsilon^A = (0, \varepsilon^a, 0)$$

yields the form of a Lorentz or of a translation gauge transformation respectively:

$$\delta\omega^{ab} = \mathscr{D}\varepsilon^{ab} \qquad (III.3.14a)$$

$$\delta\, V^a = \varepsilon^{ab} V_b \qquad (III.3.14b)$$

$$\delta\, \psi = \frac{1}{4}\varepsilon^{ab}\gamma_{ab}\psi \qquad (III.3.14c)$$

$$\delta\omega^{ab} = 0 \qquad (III.3.15a)$$

$$\delta\, V^a = \mathscr{D}\varepsilon^a \qquad (III.3.15b)$$

$$\delta\, \psi = 0 \,. \qquad (III.3.15c)$$

Let us also write down the transformation law of μ^A under (infinitesimal) <u>diffeomorphisms</u>. It will be of the utmost importance in the following for the interpretation of supersymmetry. It is better to work with soft 1-forms μ^A on $\tilde G$, without imposing a priori the fiber bundle structure (III.3.8). This allows a unified description of the SO(1,3) gauge transformation (III.3.12) and of the superspace diffeomorphisms.

Let

$$\varepsilon = \frac{1}{2}\varepsilon^{ab}\tilde D_{ab} + \varepsilon^a \tilde D_a + \bar\varepsilon \tilde D \equiv \varepsilon^A \tilde D_A \qquad (III.3.16)$$

be a general tangent vector on $\tilde G$ with $\tilde D_A$ dual to μ^B

$$\mu^B(\tilde D_A) = \delta^B_A \,. \qquad (III.3.17)$$

Here and in the following we denote by $D_A(\tilde D_A)$ the tangent vector on the soft group manifold dual to the (non) left-invariant 1-forms $\sigma^A(\sigma^A)$, according to the nomenclature introduced in Part II. The symbols $T_A \equiv (J_{ab}, P_a, Q_\alpha)$ will be reserved to the abstract Lie algebra generators; (when thought as vector fields they are left-invariant and $D_A \equiv T_A$). Explicitly

$$\omega^{ab}(\tilde{D}_{cd}) = \delta^{ab}_{cd} \quad ; \quad \omega^{ab}(\tilde{D}_c) = \omega^{ab}(\tilde{D}_\alpha) = 0 \tag{III.3.18a}$$

$$V^a(\tilde{D}_b) = \delta^a_b \quad ; \quad V^a(\tilde{D}_{bc}) = V^a(\tilde{D}_\alpha) = 0 \tag{III.3.18b}$$

$$\psi^\alpha(\tilde{D}_\beta) = \delta^\alpha_\beta \quad ; \quad \psi^\alpha(\tilde{D}_{ab}) = \psi^\alpha(\tilde{D}_a) = 0 \,. \tag{III.3.18c}$$

An infinitesimal diffeomorphism on μ^A is given by the Lie derivative (see (I.1.227)):

$$\delta^{(\mathrm{diff.})}_\varepsilon \mu^A \equiv \ell_\varepsilon \mu^A = (\varepsilon\rfloor d + d\, \varepsilon\rfloor)\mu^A \,. \tag{III.3.19}$$

Alternatively, using Eq. (I.3.136), we may write

$$\delta^{(\mathrm{diff.})}_\varepsilon \mu^A \equiv \ell_\varepsilon \mu^A = (\nabla\varepsilon)^A + \varepsilon\rfloor R^A \equiv (\nabla\varepsilon)^A + 2\,\varepsilon^B R^A{}_{BC}\,\mu^C \tag{III.3.20}$$

and by making the Lorentz content explicit we find

$$\ell_\varepsilon \omega^{ab} = (\nabla\varepsilon)^{ab} + \varepsilon\rfloor R^{ab} \tag{III.3.21a}$$

$$\ell_\varepsilon V^a = (\nabla\varepsilon)^a + \varepsilon\rfloor R^a \tag{III.3.21b}$$

$$\ell_\varepsilon \psi = \nabla\varepsilon + \varepsilon\rfloor \rho \,. \tag{III.3.21c}$$

Let us now impose the SO(1,3)-horizontality condition on the R^A's

$$\tilde{D}_{ab}\rfloor R^A = 0 \tag{III.3.22}$$

so that $\tilde{G}$ assumes the fiber bundle structure (III.3.8). Taking $\varepsilon = \varepsilon^{ab}\tilde{D}_{ab}$ Eq. (III.3.20) becomes:

$$\ell_\varepsilon \mu^A = (\nabla\varepsilon)^A \tag{III.3.23}$$

that is, we obtain the SO(1,3) gauge transformation (III.3.11) and $\tilde{D}_{ab} \equiv D_{ab}$.

On the other hand if $\varepsilon = \varepsilon^a \tilde{D}_a + \bar{\varepsilon}\tilde{D}$ Eqs. (III.3.21) describe a diffeomorphism in superspace $M^{4/4}$ which cannot be interpreted as a pure Q- and P_a-gauge transformation (III.3.13-14), unless we also impose the further horizontality constraints $\tilde{D}_a\rfloor R^A = \tilde{D}_\alpha\rfloor R^A = 0$. However if these conditions were to be imposed, the fields μ^A would have a trivial (factorized) dependence on the superspace coordinates (x^μ, θ^α) and the soft (super)-coset $\tilde{G}/SO(1,3) = M^{4/4}$ would reduce to the rigid superspace $G/SO(1,3) \equiv \mathbb{R}^{4/4}$.

Therefore in the construction of a physical theory we need non-vanishing curvature-terms in the r.h.s. of (III.3.21). In this way the fields μ^A can exhibit a non trivial ($\equiv$ dynamical) dependence on their arguments.

III.3.3 - The rheonomy principle

In order to relate the previous formal apparatus describing (local) superspace geometry with supergravity theory we make the fundamental assumption that <u>the fields ω^{ab}, V^a and ψ introduced in previous chapter for the space-time description of supergravity are the same fields entering the structural equations (III.3.5)</u>.

The problem with this identification is that the soft 1-forms $\mu^A \equiv (\omega^{ab}, V^a, \psi)$ are defined on $M^{4/4}$ (or $\tilde{G}$) while the fields used in Chapter III.2 are defined only on space-time M^4.

If supergravity theory were invariant under the gauge supersymmetry transformations (III.3.13), then one could factorize the θ-dependence of the superfield $\mu^A(x,\theta)$ through a Q-gauge transformation in the same way as, starting with the 1-forms $\mu^A(x,\theta,\eta)$, defined on the whole $\tilde{G}$, one can factorize the dependence on the SO(1,3) parameters η^{ab} through a Lorentz gauge transformation.

However we know from the discussion of D=4 N=1 supergravity on space-time that, while $\mu^A \equiv (\omega^{ab}, V^a, \psi)$ undergoes gauge transformation under the action of the Lorentz subgroup, the transformation properties under the Q_α and P_a generators are definitely non-gauge. (Notice that the non gauge invariance under Q_α implies the non gauge invariance under P_a because of the relation $\{\bar{Q}_\alpha, \bar{Q}_\beta\} = i(C\gamma^a)_{\alpha\beta} P_a$). The fact that the supersymmetries (and the translations) are not gauge transformations implies, as we have already observed at the end of the previous section, that $\mu^A \equiv (V^a, \omega^{ab}, \psi)$ must have a non trivial dependence on the θ (and x)-coordinates of the superspace $M^{4/4}$.

The important point coming out of this discussion is that the identification of the soft 1-forms (III.3.4) with the fields of supergravity requires that the space-time fields $V^a_\mu(x)$, $\psi^\alpha_\mu(x)$, $\omega^{ab}_\mu(x)$ considered in the previous chapter be interpreted as the space-time boundary values of the superspace superfields $\mu^A = \mu^A(x,\theta)$; more precisely the 1-forms

$$V^a = V^a_\mu(x)\ dx \tag{III.3.24a}$$

$$\psi = \psi_\mu(x)\ dx^\mu \tag{III.3.24b}$$

$$\omega^{ab} = \omega^{ab}_\mu(x)\ dx^\mu \tag{III.3.24c}$$

are the boundary value at $\theta^\alpha = 0$ of the restriction, on the bosonic cotangent plane, of the corresponding 1-forms in superspace; namely:

$$V^a(x) = V^a(x,\theta)\Big|_{\substack{\theta=0\\ d\theta=0}} \tag{III.3.25a}$$

$$\psi(x) = \psi(x,\theta)\Big|_{\substack{\theta=0\\ d\theta=0}} \tag{III.3.25b}$$

$$\omega^{ab}(x) = \omega^{ab}(x,\theta)\Big|_{\substack{\theta=0\\ d\theta=0}} \tag{III.3.25c}$$

where(*)

$$V^a(x,\theta) = V^a_\mu(x,\theta)\ dx^\mu + V^a_{\bar{\alpha}}(x,\theta)\ d\theta^{\bar{\alpha}} \tag{III.3.26a}$$

$$\psi(x,\theta) = \psi_\mu(x,\theta)\ dx^\mu + \psi_{\bar{\alpha}}(x,\theta)\ d\theta^{\bar{\alpha}} \tag{III.3.26b}$$

$$\omega^{ab}(x,\theta) = \omega^{ab}_\mu(x,\theta)\ dx^\mu + \omega^{ab}_{\bar{\alpha}}(x,\theta)\ d\theta^{\bar{\alpha}}\ . \tag{III.3.26c}$$

To give a precise meaning to the identification of the space-time field of supergravity with the 1-forms satisfying Eqs. (III.3.5), we have to specify the mapping which extends the purely space-time configurations described by (III.3.25) to configurations on the whole superspace.

The extension mapping

$$\text{rh:}\begin{cases} V^a(x) \to V^a(x,\theta) & \text{(III.3.27a)}\\ \psi(x) \to \psi(x,\theta) & \text{(III.3.27b)}\\ \omega^{ab}(x) \to \omega^{ab}(x,\theta) & \text{(III.3.27c)}\end{cases}$$

(*) Since we make scarce use of the spinorial coordinate indices we will denote them in this section by a lower case Greek letter with a bar while the unbarred Greek indices will be reserved to describe intrinsic fermionic indices.

is called the rheonomic extension mapping according to the same nomenclature used in the rigid supersymmetric case. The knowledge of this mapping is essential in order to interpret the theory based on the superspace fields (III.3.26) as a space-time theory. Indeed the introduction of the superfields $\mu^A(x,\theta) \equiv (V^a(x,\theta),\ \psi(x,\theta),\ \omega^{ab}(x,\theta))$ increases the number of the physical degrees of freedom, since each component in the θ-expansion of the superfield represents an a priori new x-space field.

In order to have the same physical content of the space-time theory we must be able to determine all the fields contained in the θ-expansion of the superfield $\mu^A(x,\theta)$, and all its $d\theta$-components, in terms of its space-time restriction $\mu^A_\mu(x,0)dx^\mu$. This is what the knowledge of the rheonomic mapping (III.3.27) amounts to.

Let us now explain what are the conditions to be imposed on the μ^A's in order that the mapping (III.3.27) be completely determined.

Following the same procedure given in the rigid case (Chapter II.6), we consider the diffeomorphic mapping generated by the Lie derivative ℓ_ε (III.3.20) with $\varepsilon = \varepsilon^{\bar\alpha}\,\partial/\partial\theta^{\bar\alpha}$.

In this case (III.3.20) becomes:

$$\mu^A(x,\theta+\delta\theta) = \mu^A(x,\theta) + \ell_\varepsilon\,\mu^A(x,\theta) = \mu^A(x,\theta) + (\nabla\varepsilon)^A + \varepsilon\rfloor R^A =$$

$$= \mu^A(x,\theta) + (\nabla\varepsilon)^A + 2\,\bar\varepsilon^{\bar\alpha}\,R^A_{\bar\alpha L}\,dZ^L \qquad \text{(III.3.28)}$$

where $dZ^L \equiv (d\theta^{\bar\alpha}, dx^\mu)$. In general, the mapping (III.3.28) is not rheonomic since, in order to compute $\mu^A(\mu,\ \theta+\delta\theta)$ from, say, $\theta=\theta_0\equiv 0$, $d\theta=0$, we need a complete set of Cauchy data: not only the $\theta=0$ space-time configuration $\mu^A_\mu(x,0)$ but also the "normal derivatives" $\partial/\partial\theta^\alpha\ \mu^A_\mu(x,\theta)|_{\theta=0}$.

Equivalently, one may substitute the normal derivatives with $R^A_{\bar\alpha L}$.

The last statement is justified by the following identity:

$$\frac{\partial}{\partial\theta^{\bar\alpha}}\,\mu^A(x,\theta)\Big|_{\theta=d\theta=0} = \frac{\partial}{\partial\theta^{\bar\alpha}}\,\mu^A_\mu(x,0)\,dx^\mu = \partial/\partial\theta^{\bar\alpha}\rfloor\,d\mu^A\Big|_{\theta=d\theta=0}$$

$$\equiv \partial/\partial\theta^{\bar\alpha}\rfloor\left[(R^A_{LM} - \frac{1}{2}\,C^A_{BC}\,\mu^B_L\,\mu^C_M)dZ^L \wedge dZ^M\right]_{\theta=d\theta=0}$$

$$= 2(R^A_{\bar\alpha M} - \frac{1}{2}\,C^A_{BC}\,\mu^B_{\bar\alpha}\,\mu^C_M)dZ^M\Big|_{\theta=d\theta=0} =$$

$$= 2(R^A_{\bar\alpha\nu}(x,0) - \frac{1}{2}\,C^A_{BC}\,\mu^B_{\bar\alpha}(x,0)\,\mu^C_\nu(x,0))\,dx^\nu \qquad \text{(III.3.29)}$$

where we have used Eq. (III.3.6).

The knowledge of $\mu^A(x,0)$ and $\partial/\partial\theta^{\bar\alpha}\,\mu^A(x,0)$ is thus equivalent to the knowledge of $\mu^A_\mu(x,0)$ and $R^A_{\bar\alpha L}(x,0)$.

The concept of rheonomy can now be introduced as follows. Let us assume that the "outer" components $R^A_{\bar\alpha L}$ can be expressed algebraically in terms of the purely space-time or inner components $R^A_{\mu\nu}$:

$$R^A_{\bar\alpha L} = C^{A|\mu\nu}_{\bar\alpha L|B}\,R^B_{\mu\nu} \qquad \text{(III.3.30)}$$

where the $C^{A|\mu\nu}_{\bar\alpha L|B}$ are constant tensors and, according to our conventions, μ,ν are space-time (bosonic) coordinate indices, $\bar\alpha$ is a spinorial index associated to the $\theta^{\bar\alpha}$ coordinate and $L\equiv(\bar\alpha,\mu)$; A and B are super-Lie algebra indices.

Then, recalling that

$$R^A_{\mu\nu} = \partial_{[\mu}\,\mu^A_{\nu]} + \frac{1}{2}\,C^A_{BC}\,\mu^B_{[\mu}\,\mu^C_{\nu]} \qquad \text{(III.3.31)}$$

we recognize that when (III.3.30) holds the knowledge of a purely space-time configuration: $\{\mu^A_\mu(x,0);\ \partial_\mu\mu^A_\nu(x,0)\}$ determines in a complete way the extension mapping (III.3.27). Indeed, inserting (III.3.30) into (III.3.28) we find

$$\delta\mu^A(x,\theta) = (\nabla\varepsilon)^A + 2\,\varepsilon^{\bar\alpha}\,C^{A|\mu\nu}_{\bar\alpha L|B}\,R^B_{\mu\nu}(x,0)\,dZ^L\ . \qquad \text{(III.3.32)}$$

Therefore the complete θ-dependence of the superfield $\mu^A(x,\theta)$ can be recovered starting from the initial purely space-time $(\theta=0)$ configuration. In other words $\mu^A_\mu(x,0)$ and the space-time tangent derivatives $\partial_\mu \mu^A_\nu(x,0)$ (or equivalently $\mu^A_\mu(x,0)$ and $R^A_{\mu\nu}(x,0)$, by (III.3.29)) constitute a complete set of Cauchy data on M^4 once (III.3.30) is satisfied. Indeed the space-time normal derivatives $\partial/\partial\theta\ \mu^A(x,0)$ are expressible in terms of $\mu^A(x,0)$ and $\partial_{[\mu}\mu^A_{\nu]}(x,0)$ via Eqs. (III.3.30) and (III.3.29). In Fig. III.3.I we have depicted this fact.

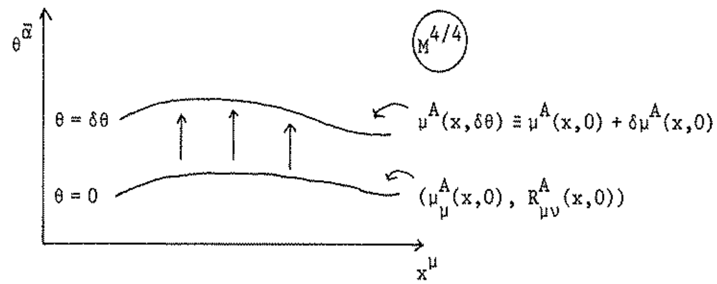

Fig. III.3.I

The set of constraints (III.3.30) which relate the inner $R^A_{\mu\nu}$ and the outer $R^A_{\bar{\alpha}L}$ components of the curvatures R^A are named <u>rheonomic constraints</u>. The property expressed by (III.3.30) will be referred to as "<u>rheonomy</u>" and a theory admitting a set of rheonomic constraints is likewise named rheonomic. What we have shown is that the physical content of a superspace rheonomic theory is completely determined by means of a purely space-time description.

Alternatively, if we regard the Lie derivative as the generator of the functional change of μ^A at the same coordinate point:

$$\ell_\varepsilon \mu^A = \mu^{A'}(x,0) - \mu^A(x,0) \tag{III.3.33}$$

then the rheonomic mapping (III.3.32) can be rewritten as follows:

$$\delta\mu^A(x,0) = (\nabla\varepsilon)^A + 2\,\varepsilon^{\bar{\alpha}}\, C^{A|\mu\nu}_{\bar{\alpha}L|B}\, R^B_{\mu\nu}(x,0)\, dZ^L\ . \tag{III.3.34}$$

Written in this form <u>the rheonomic mapping maps a space-time configuration into a new space-time configuration</u>. (See Fig. III.3.II). In particular if the theory described by the μ^A-fields (the action, or its equations of motion) is invariant under superspace diffeomorphisms, then it can be restricted to space-time and (III.3.34) will appear as a symmetry transformation of the space-time theory.

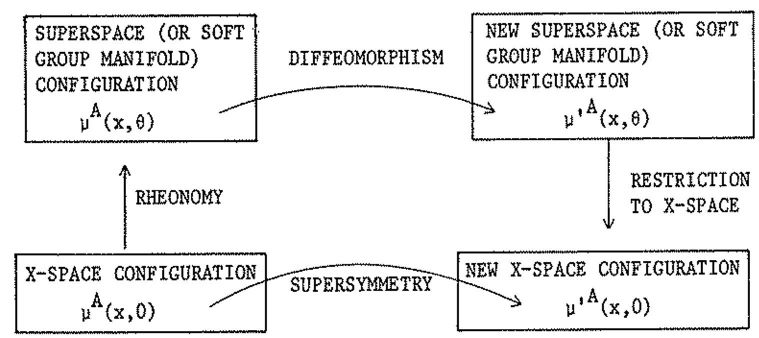

Fig. III.3.II

Since $\varepsilon^{\bar{\alpha}}$ is a spinorial parameter, the rheonomic mapping realized on space-time field configurations will be identified as a <u>supersymmetry transformation</u>. In the following we shall need the expression of rheonomic constraints using intrinsic components of the curvatures; it is clear that the property of rheonomy does not depend on the particular basis chosen for the 1-forms: the coordinate basis used above $\{d\theta^{\bar{\alpha}}, dx^\mu\}$ and the anholonomic supervielbein basis $\{V^a, \psi^\alpha\}$ are equally viable. Indeed we can write:

$$R^A_{\bar{\alpha}L} = R^A_{BC}\, \mu^B_{\bar{\alpha}}\, \mu^L \equiv$$

$$\equiv R^A_{ab}\, V^a_{\bar{\alpha}}\, V^b_L + 2\, R^A_{\alpha\beta}\, V^a_{\bar{\alpha}}\, \psi^\beta_L + R^A_{\beta\gamma}\, \psi^\beta_{\bar{\alpha}}\, \psi^\gamma_L \tag{III.3.35a}$$

$$R^A_{\mu\nu} = R^A_{BC}\, \mu^B_\mu\, \mu^C_\nu \equiv$$

$$\equiv R^A_{ab}\, V^a_\mu\, V^b_\nu + 2\, R^A_{a\beta}\, V^a_\mu\, \psi^\beta_\nu + R^A_{\beta\gamma}\, \psi^\beta_\mu\, \psi^\gamma_\nu \tag{III.3.35b}$$

so that an algebraic relation of the type (III.3.30) among the holonomic outer and inner components $R^A_{\bar{\alpha}L}$ and $R^A_{\mu\nu}$ implies an analogous relation among the intrinsic components $R^A_{\alpha C}$ and R^A_{mn}. Explicitly we can write

$$R^A_{\alpha C} = C'^{A|mn}_{\alpha C|B} \, R^B_{mn} \tag{III.3.36}$$

where the C''s are constant anholonomic tensors (they are in fact the C's appearing in (III.3.30) evaluated in the intrinsic basis).

In this way we can rewrite the supersymmetry transformations as follows

$$\begin{aligned} \ell_\varepsilon \, \mu^A(x,0) &= (\nabla\varepsilon)^A + \varepsilon \rfloor R^A(x,0) \\ &= (\nabla\varepsilon)^A + 2\,\bar{\varepsilon}^\alpha \, R^A_{\alpha C}(x,0)\, \mu^C \\ &= (\nabla\varepsilon)^A + 2\,\bar{\varepsilon}^\alpha \, C^{A|mn}_{\alpha C|B} \, R^B_{mn}(x,0)\, \mu^C \end{aligned} \tag{III.3.37}$$

where we used $\varepsilon = \bar{\varepsilon}\tilde{D}$, $\mu^A(\tilde{D}_\alpha) = \delta^A_\alpha$ and $R^A \equiv R^A_{BC}\, \mu^B \wedge \mu^C$.

It is in this intrinsic form that the supersymmetry transformations will appear in supergravity theories.

One might wonder whether the transformations (III.3.37) (or (III.3.32)) close an algebra. At the first glance one could think that this should be the case, since (III.3.37) are Lie derivatives and as such they should obey the algebra (I.1.239). Indeed if we do not impose (or derive) any rheonomic constraint of the type (III.3.36) then the Lie algebra (I.1.239) does certainly close.

The existence of rheonomic constraints, however, changes the situation. Demanding the closure of the Lie derivatives is equivalent to demanding the integrability of the rheonomic constraints and this imposes new constraints on the inner (≡ space-time) components of the curvatures.

To understand this point one observes that the Lie derivatives close an algebra

$$[\ell_{\varepsilon_1}, \ell_{\varepsilon_2}] = \ell_{[\varepsilon_1, \varepsilon_2]} \tag{III.3.38}$$

provided they are consistently defined, namely, provided the operator used in their definitions is a true exterior derivative: $d^2 = 0$. If one uses the equivalent form

$$\ell_\varepsilon \, \mu^A = (\nabla\varepsilon)^A + \varepsilon \rfloor R^A \tag{III.3.39}$$

checking that $d^2 = 0$ amounts to checking that the Bianchi identities are satisfied by the curvatures R^A.

Now, in presence of the constraints (III.3.36) the Bianchi identities loose the character of identities and become integrability equations for the constraints. Since the rheonomic constraints express each outer component $R^A_{\alpha C}$ in terms of the inner ones R^A_{mn} then the Bianchi-integrability equations are equations among the space-time components of the curvatures which must be valid everywhere in superspace and in particular on the restriction to the space-time hypersurface.

Hence we reach the conclusion that the supersymmetry transformations (III.3.37) close an algebra only if the space-time curvatures R^A_{mn} satisfy certain integrability equations given by the Bianchi identities.

From a physical point of view these equations cannot be anything else than the space-time equations of motion of the theory.(*) Any different equation of motion would be inconsistent with the Bianchi identities.

Summarizing, in a rheonomic theory we expect that the supersymmetry transformations (III.3.37) close an algebra only on the on-shell configurations of the fields $\mu^A(x,0)$. We will see an explicit

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(*) Of course, this is true in the absence of auxiliary fields (these are indeed introduced to obtain an off-shell closure of the supersymmetry algebra, see Chapters II.6 and III.6).

example of this mechanism in the next section. From the rheonomic lifting point of view, Eq. (III.3.32), this state of affairs means that we can lift to superspace only those configurations which are solutions of the x-space field equations. Arbitrary configurations of the x-space fields cannot be lifted to superspace.

At this point we can enlarge our scheme by observing that the horizontality constraint (III.3.22)

$$\underline{\tilde{D}_{ab}|}R^A = 0 \Leftrightarrow R^A_{(ab)C} = 0 \tag{III.3.40}$$

which allows for the restriction of the soft 1-forms μ^A defined on the whole soft $\tilde{G}$ to the base space $M^{4/4} \equiv \tilde{G}/H$, can also be thought of as a "rheonomic" constraint relating the outer $\mathbb{H}$-components of the curvatures to the other superspace components

$$R^A_{\mathbb{H},C} = C^{A|DF}_{\mathbb{H}C|B} R^B_{DF} = 0 . \tag{III.3.41}$$

This case is degenerate since all the constants $C^{\cdots}_{\cdots}$'s are zero. (Here $\mathbb{H}$ and $\{C,D,F\}$ are indices in the tangent plane to H and $\tilde{G}/H$ respectively). In this way one can think of $\mu^A(x,0)$ as a 1-form which has been restricted first to superspace, by imposing the horizontality constraints (III.3.40) and then to the physical space-time by imposing the rheonomic constraints (III.3.36). The rheonomic mapping and the $\mathbb{H}$-Lorentz transformations reconstruct the full dependence of $\mu^A = \mu^A(x,\theta,\eta)$ on the θ^α and η^{ab} variables respectively.

The Lie derivative formula

$$\ell_\varepsilon\, \mu^A(x,\theta,\eta) = (\nabla\varepsilon)^A + \underline{\varepsilon|}R^A \tag{III.3.42}$$

with $\varepsilon = \varepsilon^{ab}\tilde{D}_{ab} + \varepsilon^a \tilde{D}_a + \bar{\varepsilon}\tilde{D}$ supplemented with the horizontality and the rheonomic constraints gives, in one stroke, the Lorentz gauge transformations $(\varepsilon^a = \varepsilon^\alpha = 0)$:

$$\ell_{(\varepsilon^{ab} D_{ab})}\, \mu^A = (\nabla\varepsilon)^A \tag{III.3.43}$$

the general diffeomorphisms on space-time $(\varepsilon^{ab} = \varepsilon^\alpha = 0)$

$$\ell_{(\varepsilon^a \tilde{D}_a)}\, \mu^A = (\nabla\varepsilon)^A + \varepsilon^a R^A_{aC}\, \mu^C \tag{III.3.44}$$

and the supersymmetry transformations $(\varepsilon^{ab} = \varepsilon^a = 0)$

$$\ell_{(\bar{\varepsilon}\tilde{D})}\, \mu^A = (\nabla\varepsilon)^A + 2\, \bar{\varepsilon}^\alpha\, C^{A|mn}_{\alpha C|B}\, R^B_{mn}\, \mu^C . \tag{III.3.45}$$

We previously pointed out that the closure of the supersymmetry algebra in general requires further constraints on the space-time components R^A_{mn}, and that these constraints are to be identified with the space-time equations of motion. On the other hand the closure of the gauge transformations (III.3.43) and of the space-time diffeomorphisms (III.3.44) does not give further constraints.

This is the main difference between supersymmetry and all the other symmetries so far considered in physical theories.

In the above discussion we had in mind the D=4 super Poincaré group $\overline{Osp(1,4)}$, which is the basis of the simplest supergravity theory.

Our procedures, however, can be easily generalized to any super-group in any space-time dimension D. The superspace will be given by $\tilde{G}/H$, G being the supergroup and H the factorized subgroup which should always contain the Lorentz group $SO(1,D-1)$.

Hence we can summarize the results of our discussion in the following statements:

a) Starting with a set of 1-forms on the coset $\tilde{G}/H$ (or, more generally, on $\tilde{G}$) the theory can be interpreted as a space-time theory on M^D if it is rheonomic, that is if the curvatures satisfy the constraints (III.3.36) (and, on $\tilde{G}$, also the horizontality constraints (III.3.40)).

b) Suppose the theory invariant under superspace- (or $\tilde{G}$-)diffeomorphisms. Then, besides Lorentz and space-time coordinate invariance, we also have invariance under a set of supersymmetry transformations.

These transformations close an algebra only on the on-shell configurations of the physical fields.

In the next two sections we will examplify the above discussion in the case of N=1 D=4 supergravity.

We conclude this section by remarking that the concept of rheonomy in superspace has an interesting analogy with the concept of analyticity. This was already pointed out in the rigid supersymmetry case: the analogy remains valid also in our more general context of a soft superspace $M^{4/4}$.

Indeed according to Eqs. (III.3.30) and (III.3.32) the rheonomy constraints (III.3.30) are equivalent to constraints between $\partial/\partial x^\mu$ (inner) and $\partial/\partial\theta^\alpha$ (outer) derivatives of $\mu^A(x,0)$. This is analogous to the Cauchy-Riemann equations for an analytic function $f(x+iy)$. Considering the correspondence:

$$x \rightarrow x^\mu$$
$$y \rightarrow \theta^\alpha$$
$$f(x,y) \rightarrow \mu^A(x^\mu,\theta^\alpha)$$

we see that there is a nice analogy between rheonomic constraints in superspace and the Cauchy-Riemann equations relating the x- and y-derivatives of a function in the complex plane.

Furthermore, just as the analyticity of a function allows for its determination in the whole complex plane, once its boundary value on any line (say $y=0$) is given, in the same way we have found that rheonomy allows the reconstruction of the superfield potential $\mu^A(x,\theta)$ (via the extension mapping (III.3.27)) from its boundary value (say $\theta=0$).

We are thus led to regard the rheonomy of μ^A as a kind of analyticity in superspace.

III.3.4 - An extended action principle

It follows from the above general discussion that, if a superspace formulation of supergravity exists, it must be a rheonomic one. Indeed a rheonomic theory in superspace can be restricted to space-time and, vice versa, a purely space-time theory can be extended to superspace if it is rheonomic. Since supergravity on space-time does exist (it was derived in Chapter III.2) then the corresponding superspace theory should be rheonomic.

The natural starting point for the construction of such a theory is an action principle defined in superspace (or in the whole $\tilde{G}$). The essential requirement is that the variational principle should encompass, as equations of motion, both the proper space-time equations and the rheonomic constraints necessary for a consistent reduction to space-time.

The approach we shall follow in constructing such a variational principle in superspace is the so-called "group manifold approach". It was so named since one usually starts with fields which are 1-forms defined over a soft group manifold $\tilde{G}$; this is not, however, the relevant point. As we stressed many times, one could equally well and more simply start with fields which are defined over superspace, i.e. over the coset of the soft group manifold modulo the factorized gauge group, usually taken to be SO(1,3).

What is really important in this approach is the fact that the rheonomic constraints together with the space-time equations can be derived as sectors of a single set of exterior field equations in superspace. For this to be possible it is necessary to use a suitable extended variational principle.

We begin by observing that the x-space and superspace formulations have complementary and, apparently, mutually exclusive advantages: in x-space one never has to worry about unwanted degrees of freedom and the Lagrangian, without further ado, makes sense as a conventional field theory, yet the supersymmetry transformations are awkward things deprived of a good interpretation. In superspace, on the other hand, the transformation rules are given "a priori" as diffeomorphisms, but the Lagrangian is not found in a simple and clean way and there are problems in giving the x-surface a privileged role.

Here we would like to find a formulation which encompasses the two traditional ones. More precisely, we would like a formulation in terms of superfields where, however, the Lagrangian is integrated on a 4-dimensional surface. This apparently difficult programme can be carried through in a very simple way, at the price of using a generalized action principle. This generalization was already introduced for pure gravity on the soft Poincaré group (see Chapter I.4) and for the Wess-Zumino model and super Yang-Mills theory on rigid superspace (see Chapter II.9).

Let us recall the main idea: usually one is accustomed to actions of the type

$$S = \int_{\Omega} \mathscr{L}(\phi) \tag{III.3.46}$$

where Ω is a manifold of dimension n and $\mathscr{L}$ is a scalar density, namely a form of maximum degree ($\mathscr{L}$ = n-form) constructed with the fields $\phi_i = \phi_i(x)$ which are p-forms on the space Ω. In this way the action $S = S[\phi]$ is a functional of the field configurations only. A natural generalization of equation (I.6) is obtained if we consider as Lagrangian $\mathscr{L}(\phi)$ a form of degree $D < N$ ($\mathscr{L}$ = D-form). In this case we can write:

$$S = S[\phi, M_0] = \int_{M_D} \mathscr{L}_D(\phi) \tag{III.3.47}$$

where M_D is a D-dimensional submanifold of Ω ($M_D \subset \Omega$). In this way the action becomes a functional both of the field configurations $\phi(x)$ and of the surface M_D embedded in Ω. We can then consider the classical equations of motion obtained by demanding that S be minimal, both with respect to variations of the fields and of the surface M_D. These equations are rather complicated and may be non local if $\mathscr{L}_D$ is chosen at random. Suppose, however, that the fields $\{\phi_i\}$ are a set

of exterior forms of various degrees p and that the Lagrangian is obtained from $\{\phi_i\}$ using only the diffeomorphism-invariant operations of exterior algebra, namely the exterior derivative $d: \phi \to d\phi$ and the wedge product: $\wedge: (\phi_1, \phi_2) \to \phi_1 \wedge \phi_2$. If these conditions hold, then any deformation of the surface M_D can be compensated by a diffeomorphism of the fields $\{\phi_i\}$. This has the very simple implication that the complete set of variational equations associated to the action (III.3.47) is given by the usual equations of motion obtained by varying (III.3.47) in the fields <u>on a fixed surface</u>:

$$\frac{\delta \mathscr{L}}{\delta \phi_i} = 0 \tag{III.3.48}$$

with the proviso that these exterior algebra equations hold not only on M_D but <u>on the whole Ω-space</u>.

Using the generalized action principle we were able to derive in Chapter I.4, as a rather academic example, the Lorentz gauge transformations of the fields from the $\mathbb{H}$-horizontality of the curvatures on the (soft) Poincaré manifold ISO(1,3). For rigid supersymmetric theories, however, (Wess-Zumino and super Yang-Mills Lagrangians) the generalized action principle was instead used to derive the <u>rheonomic constraints</u>, generating the supersymmetry transformations as Lie derivatives in superspace. In that case we started for simplicity with fields on the rigid superspace $\overline{Osp(1/4)}/SO(1,3)$. (The SO(1,3) factorization could have been obtained by considering fields on the whole group manifold $\overline{Osp(1/4)}$).

We will see that these mechanisms work also in the case of supergravity. The rheonomic constraints (and, if we like, also the SO(1,3)-horizontality constraints) generating supersymmetry (and SO(1,3) gauge) transformations will be obtained from the generalized action principle on superspace (or, more generally, on the soft group manifold).

As already stressed some lines above, we can go through these derivations only if the action (III.3.47) is "geometrical". By this term we mean that it is constructed out of exterior forms, using the diffeomorphic invariant operations d (exterior derivative) and $\wedge$ (wedge product). In particular we exclude the Hodge duality operator *. There are two good reasons for this exclusion. The first is the lack of a satisfactory extension of the * operation from bosonic manifolds to supermanifolds. The second is that the Hodge dual implicitly involves the notion of a "metric" and of the dimensionality of the base space.

An equation on exterior forms where the Hodge operator appears makes sense only if we specify the manifold on which it holds. On the other hand an equation on exterior forms involving only the d and $\wedge$ operations makes sense on any manifold and it is our privilege to extend it smoothly from a smaller to a larger one. This is precisely what we need in a rheonomic theory. The field equations in these theories are exterior form equations. Restricted to the x-space they yield the usual equations of motion while extended to the whole superspace they give, as an extra bonus, the rheonomic constraints on the superspace curvatures.

One may wonder how this miracle can happen. The answer is that the "correct" rheonomic constraints and the "correct" field equations are very close relatives. They are both implied by the Bianchi identities of the supergroup (or free differential algebra) and are, in a sense, an intrinsic property of the supergroup. It is not surprising, therefore, that the action which gives the "correct" field equations provides also the "correct" rheonomic constraints upon extension of these field equations to the whole superspace. We will come back to this issue later on.

We just emphasize that from this viewpoint supersymmetry behaves in the same way as the ordinary gauge symmetries. Indeed the "interactions", namely the terms in the Lagrangian, are fully determined by the chosen symmetry principle.

As a final remark we notice that the exclusion of the Hodge dual operator necessarily implies the use of first-order formalism for all the fields: the derivative ϕ_a of a scalar field ϕ or the field strength F_{mn} of a spin one field A will be introduced as independent

objects and will be independently varied in the action. Such a procedure should already be familiar to the reader from Chapters I.5, II.6 and II.9.

III.3.5 - D=4, N=1 supergravity and rheonomy

At this point we should derive a suitable rheonomic superspace Lagrangian for N=1, D=4 supergravity.

A set of rules for such constructions will be given in section III.3.7. Here we take a different approach. Regarding the Lagrangian as God-given we just work out the consequences of its field equations when they are implemented on the whole superspace.

Actually, anticipating the results of section III.3.8, we have that the x-space Lagrangian given in Eq. (III.2.18) can be equally well used as superspace rheonomic Lagrangian. It just suffices to extend (III.2.18) to superspace via the mapping

$$\mu^A(x,0)\big|_{d\theta=0} \rightarrow \mu^A(x,\theta) \qquad \text{(III.3.49)}$$

where the fields $\mu^A = (V^a, \omega^{ab}, \psi)$ are the soft 1-forms satisfying the structural equations (III.3.5). We stress that this extension is possible because the action (III.2.18) is already in first order form, the Hodge dual being already excluded. Hence there is no obstruction to the rheonomic extension mapping

$$\mathscr{L}_{\text{space-time}}\,(\mu^A(x,0)\big|_{d\theta=0}) \rightarrow \mathscr{L}_{\text{superspace}}\,(\mu^A(x,\theta)). \qquad \text{(III.3.50)}$$

The Lagrangian being obtained via this simple procedure we now show that the use of the extended variational principle yields, besides the usual equations of motion, also the rheonomic constraints.

Let us consider once more Eq. (III.2.18) where ω^{ab}, V^a, ψ are now the soft 1-forms of $\overline{Osp(4/1)}$ and $R^A = (R^{ab}, R^a, \rho)$ the corresponding curvatures (see Eq. (III.3.5)). The extended action is:

$$\mathscr{A}^{D=4,\,N=1}_{\text{extended}} = \int_{M_4 \subset \tilde{G}} [R^{ab} \wedge V^c \wedge V^d\, \varepsilon_{abcd} + 4\, \bar{\psi} \wedge \gamma_5 \gamma_a \rho \wedge V^a] \qquad \text{(III.3.51)}$$

where M_4 is a bosonic hypersurface floating in $\widetilde{Osp(1/4)}$. Since the Lagrangian is "geometrical" in the sense previously discussed, the equations of motion have the same form as in the space-time approach (III.2.22, 24, 25). The only difference is that now they hold on the whole $\widetilde{Osp(1/4)}$ manifold. For completeness let us rewrite these equations:

$$2\,\varepsilon_{abcd}\, R^c \wedge V^d = 0 \tag{III.3.52a}$$

$$2\,\varepsilon_{abcd}\, R^{ab} \wedge V^c + 4\,\bar\psi \wedge \gamma_5\,\gamma_d\,\rho = 0 \tag{III.3.52b}$$

$$8\,\gamma_5\,\gamma_m\,\rho \wedge V^m - 4\,\gamma_5\,\gamma_m\,\psi \wedge R^m = 0\,. \tag{III.3.52c}$$

In order to work out the content of these 3-form equations on $\tilde G$ we expand the curvatures R^A on the intrinsic 2-form basis on $\tilde G$:

$$R^A = R^A_{BC}\,\mu^B \wedge \mu^C \tag{III.3.53}$$

where

$$\mu^A \wedge \mu^B \equiv \{V^a \wedge V^b;\ V^a \wedge \omega^{bc};\ \omega^{ab} \wedge \omega^{cd};\ V^a \wedge \psi^\alpha;\ \omega^{ab} \wedge \psi^\alpha;\ \psi^\alpha \wedge \psi^\beta\}\,. \tag{III.3.54}$$

Then we must impose the vanishing of the coefficients of all the independent 3-form monomials on $\tilde G$.

Let us first consider the 3-forms which contain at least one spin connection ω^{ab}, that is the projection along monomials of the type

$$\begin{array}{ll} \omega\wedge\omega\wedge\omega & \omega\wedge V\wedge V \\ \omega\wedge\omega\wedge V & \omega\wedge V\wedge\psi \\ \omega\wedge\omega\wedge\psi & \omega\wedge\psi\wedge\psi\,. \end{array} \tag{III.3.55}$$

It is clear that since no bare ω^{ab}-field is present in the Lagrangian (and therefore in the equations of motion (III.3.52)), these projections always have coefficients containing at least one ω^{ab}-component of the curvatures, namely: $R^A(\tilde D_{ab}, \tilde D_B)$. Hence one obtains:

$$R^A(\tilde D_{ab}, \tilde D_B) = 0 \Leftrightarrow R^A_{(ab),B} = 0 \qquad B \equiv \{ab, a, \alpha\} \tag{III.3.56}$$

This is fully analogous to what happens in pure gravity (see Eq. (I.4.52)).

The result just obtained, namely the SO(1,3)-factorization, is a kinematical constraint which is a straightforward consequence of the SO(1,3)-gauge invariance of (III.3.41) forbidding the presence of the bare ω^{ab} field in the Lagrangian. As such it is not of physical relevance. We can just start with an SO(1,3)-factorized set of fields and curvatures in $M^{4/4} = \widetilde{Osp(4,1)}/SO(1,3)$ obeying the gauge properties (III.3.12).

Then, instead of implementing the action principle on the whole ISO(1,3), we can take $M^{4/4}$ as the embedding space for M^4, namely

$$\mathscr{A}^{D=4,\ N=1}_{\text{extended}} = \int_{M_4 \subset M^{4/4}} \mathscr{L}\,. \tag{III.3.57}$$

The embedding equations of motion (III.3.52) are then to be thought as restricted to $M^{4/4}$ and we should analyze them only with respect to the supervielbein set of 2-forms:

$$\{V^a \wedge V^b;\ V^a \wedge \psi^\alpha;\ \psi^\alpha \wedge \psi^\beta\} \tag{III.3.58}$$

the SO(1,3)-gauge invariance of the μ^A's being guaranteed a priori by the fiber-bundle structure of $\tilde G$.

The reason why we discussed the possibility of obtaining the horizontality Eqs. (III.3.56) and hence the fiber bundle structure $\tilde G = \tilde G(\tilde G/SO(1,3),\ SO(1,3))$ from the action principle extended to the whole $\tilde G$ is that, as already stressed in the previous section, Eqs. (III.3.56) on $\tilde G$ are quite analogous to the rheonomic conditions we are going to obtain from the superspace analysis of the equations of motion. The rheonomic conditions are responsible for the supersymmetry of the theory. Furthermore the possibility of starting directly from $\tilde{G}$ justifies the denomination of (soft) group manifold approach given to our action principle and is satisfactory from a purely formal point of view.

In the following chapters, when examining higher D-dimensional supergravity theories, unless explicitly stated, we shall always start with fields already defined on superspace, the coset of the appropriate soft super group $\tilde{G}$ modulo SO(1,D-1) (or modulo a suitable extension thereof).

Let us now proceed to the superspace analysis of the 3-form equations (III.3.52).

Let us expand R^A in the intrinsic (super)-vielbein basis (V^a, ψ) of $M^{4/4}$

$$R^A = R^A_{ab}\, V^a \wedge V^b + 2\, R^A_{\alpha a}\, \psi^\alpha \wedge V^a + R^A_{\alpha\beta}\, \psi^\alpha \wedge \psi^\beta \,. \qquad \text{(III.3.59)}$$

Notice that $R^A_{\alpha a}$ is a spinor if A is a bosonic index $(R^A \equiv R^{ab}, R^a)$ and a matrix if A is also spinorial $(R^A \equiv \rho)$. Vice versa $R^A_{\alpha\beta}$ is a matrix for A bosonic and a spinor-valued matrix for A fermionic.

Therefore we rewrite (III.3.59) as follows:

$$R^{ab} = R^{ab}_{cd}\, V^c \wedge V^d + \bar{\theta}^{ab}_c\, \psi \wedge V^c + \bar{\psi} \wedge K^{ab}\, \psi \qquad \text{(III.3.60a)}$$

$$R^a = R^a_{bc}\, V^b \wedge V^c + \bar{\theta}^a_c\, \psi \wedge V^c + \bar{\psi} \wedge K^a\, \psi \qquad \text{(III.3.60b)}$$

$$\rho = \rho_{ab}\, V^a \wedge V^b + H_c \psi \wedge V^c + \Omega_{\alpha\beta}\, \psi^\alpha \wedge \psi^\beta \qquad \text{(III.3.60c)}$$

where $\theta^{ab|c}$, $\theta^{a|c}$ are spinor-tensors, $K^{ab} = -K^{ba}$, K^a and H^c are 4×4 matrices in spinor space and $\Omega^{\alpha\beta}$ is a Majorana-spinor-valued 4×4 matrix in the same space.

Let us first consider the $\psi\psi\psi$ projection of the equations (III.3.52). Equation (III.3.52a) yields $0 = 0$, while (III.3.52b) and (III.3.52c) immediately imply:

$$K^a = \Omega_{\alpha\beta} \equiv 0 \,. \qquad \text{(III.3.61)}$$

We now consider the projection $\psi\psi V$. Equation (III.3.52a) is fulfilled using (III.3.61). Equation (III.3.52b) gives a relation between K^{ab} and H_c, namely:

$$2\, \bar{\psi} \wedge K^{ab}\, \psi \wedge V^c\, \varepsilon_{abcd} + 4\, \bar{\psi} \wedge \gamma_5\, \gamma_d\, H_c\, \psi \wedge V^c = 0 \,. \qquad \text{(III.3.62)}$$

Equation (III.3.52c) implies

$$\bar{\theta}^a_c = 0 \qquad \text{(III.3.63)}$$

where (III.3.61) has been used.

Coming now to the projection ψVV we find that Eq. (III.3.52a) is again satisfied using (III.3.61). Equation (III.3.52b) gives a relation between $\theta^{ab|c}$ and ρ_{ab}:

$$2\, \bar{\theta}^{ab}_m\, \psi \wedge V^m \wedge V^c\, \varepsilon_{abcd} + 4\, \bar{\psi} \wedge \gamma_5\, \gamma_d\, \rho_{ab}\, V^a \wedge V^b = 0 \,. \qquad \text{(III.3.64)}$$

Equation (III.3.52c) relates H_c to R^a_{bc}:

$$\gamma_5\, \gamma_m\, H_c \wedge V^c \wedge V^m - 4\, \gamma_5\, \gamma_m\, \psi \wedge R^m_{ab}\, V^a \wedge V^b = 0 \,. \qquad \text{(III.3.65)}$$

Finally the VVV-projection gives the propagation equations (III.2.28, 34) for the space-time components R^{ab}_{cd}, R^a_{bc}, and ρ_{ab} in the same way as in Chapter III.2 (Eqs. III.2.28, 33, 34).

For completeness we rewrite them here

$$R^a_{mn} = 0 \qquad \text{(III.3.66a)}$$

$$8\, \gamma_5\, \gamma_{ar}\, \psi_s\, \varepsilon^{arst} = 0 \qquad \text{(III.3.66b)}$$

$$8(R^{m\cdot}_{\ n\cdot} - \frac{1}{2}\, \delta^m_n\, R^{\cdot\cdot}_{\cdot\cdot}) = 0 \,. \qquad \text{(III.3.66c)}$$

These equations are valid on the whole $M^{4/4}$ (that is on any slice $\theta = \text{const}$ of $M^{4/4}$), in particular at $\theta = 0$.

It is now easy to determine the outer components of the curvatures. Using (III.3.66a) in Eq. (III.3.65) one finds

$$H_c = 0 \tag{III.3.67a}$$

and this, inserted into Eq. (III.3.62) implies

$$K^{ab} = 0 \ . \tag{III.3.67b}$$

Therefore the only non trivial relation among the curvature components is Eq. (III.3.64).

Let us solve it explicitly. We set

$$V^a \wedge V^b = -\frac{1}{4}\,\varepsilon^{abcd}\,\varepsilon_{cdrs}\,V^r \wedge V^s \equiv \varepsilon^{abcd}\,\Omega_{cd} \tag{III.3.68}$$

where Ω_{cd} is a 2-form defined by (III.3.68). We obtain:

$$(\bar{\Theta}_f^{ab}\,\varepsilon^{fcpq}\,\varepsilon_{abcd} + 2\,\bar{\rho}_{ab}\,\gamma_5\,\gamma_d\,\varepsilon^{abpq})\psi \wedge \Omega_{pq} = 0 \ . \tag{III.3.69}$$

It follows that:

$$\bar{\Theta}^{pq}{}_d - 2\,\delta_d^{[q}\,\bar{\Theta}^{p]m}{}_m + \varepsilon^{pqrs}\,\bar{\rho}_{rs}\,\gamma_5\,\gamma_d = 0 \ . \tag{III.3.70}$$

Contracting q and d one obtains

$$\bar{\Theta}^{pm}{}_m = \frac{1}{2}\,\varepsilon^{pqrs}\,\bar{\rho}_{rs}\,\gamma_5\,\gamma_q \ . \tag{III.3.71}$$

Substituting into (III.3.70) one finally finds:

$$\bar{\Theta}^{pq}{}_d = -\,\varepsilon^{pqrs}\,\bar{\rho}_{rs}\,\gamma_5\,\gamma_d - \delta_d^{[p}\,\varepsilon^{q]mst}\,\bar{\rho}_{st}\,\gamma_5\,\gamma_m \ . \tag{III.3.72}$$

Let us summarize the results of our analysis of the superspace equations of motion. First of all their outer projections have provided a set of algebraic equations among the curvatures components:

$$K^{ab} = K^a = \Omega_{\alpha\beta} = \Theta_c^a = H_c = 0 \tag{III.3.73a}$$

$$\bar{\Theta}_c^{ab} = -\,\varepsilon^{abrs}\,\rho_{rs}\,\gamma_5\,\gamma_c - \delta_c^{[a}\,\varepsilon^{b]mst}\,\bar{\rho}_{st}\,\gamma_5\,\gamma_m \ . \tag{III.3.73b}$$

Equivalently we can say that the parametrization of the curvatures along the intrinsic basis (V^a, ψ) is given by:

$$R^{ab} = R^{ab}_{cd}\,V^c \wedge V^d + (-\,\varepsilon^{abrs}\,\bar{\rho}_{rs}\,\gamma_5\,\gamma_c - \delta_c^{[a}\,\varepsilon^{b]mst}\,\bar{\rho}_{st}\,\gamma_5\,\gamma_m)\psi \wedge V^c \tag{III.3.74a}$$

$$R^a = R^a_{mn}\,V^m \wedge V^n \tag{III.3.74b}$$

$$\rho = \rho_{ab}\,V^a \wedge V^b \tag{III.3.74c}$$

where R^{ab}_{cd}, R^a_{mn} and ρ_{ab} satisfy the inner equations (III.3.66).

Equations (III.3.73) fit the general definition of rheonomic constraints as given in Eq. (III.3.30). Indeed we see that all the outer components of the curvatures defined in (III.3.60) namely:

$$R^A(\tilde{D}, \tilde{D}_B) \equiv \{K^{ab}, K^a, \Omega_{\alpha\beta}, \Theta_c^{ab}, \Theta_c^a, H_c\} \tag{III.3.75}$$

are given in terms of the inner (or space-time) components

$$R^A(\tilde{D}_c, \tilde{D}_d) \equiv \{R^{ab}_{cd}, R^a_{cd}, \rho_{cd}\} \tag{III.3.76}$$

($\tilde{D}_B$ is a generic tangent vector dual to μ^A and $\tilde{D}$ is the supersymmetry charge dual to ψ).

Actually in the present case the rheonomy property is trivial for all the outer components (III.3.75) except $\theta^{ab|c}$. This is the only one to be non zero and it is expressed by Eq. (III.3.73b) in terms of the inner curvature ρ_{ab}. (We note that if also $\theta^{ab|c}$ were zero then we would get a complete factorization of the θ^{α}-coordinates and hence also of the x^{μ}-coordinates, thus obtaining a trivial theory).

Secondly, the space-time projection of the superspace equations (III.3.52) gives the differential equations (III.3.66). Since the theory is rheonomic they can be restricted to space-time $(\theta^a = 0)$ and therefore they coincide with those found in the purely space-time approach (see Eqs. (III.2.34)). It is then clear that the physical content of the theory derived from our superspace Lagrangian coincides with that of the purely space-time theory of Chapter III.2. In the next sections we shall gain a deeper understanding of the theory relying on the superspace formulation.

III.3.6 - Rheonomic constraints and Bianchi identities

It is clear that the rheonomic differential constraints (III.3.73-74) must satisfy integrability conditions. Expressed in terms of the curvatures R^A the integrability condition $d^2 = 0$ is simply given by the Bianchi identities (III.3.9). Now we want to show that, as was anticipated in Sect. III.3.4, the integrability conditions of the rheonomic constraints (III.3.73-74) are the space-time field equations namely the equations (III.3.66). To show this we simply insert the rheonomic parametrization of the curvatures (III.3.60) (using also (III.3.73a)) into the Bianchi identities (III.3.9). We obtain

$$\mathscr{D}(R^{ab}_{mn}\, V^m \wedge V^n) + \mathscr{D}(\bar\theta^{ab}_c\, \psi \wedge V^c) = 0 \qquad \text{(III.3.77a)}$$

$$R^{ab}_{mn}\, V^m \wedge V^n \wedge V^b + \bar\theta^{ab}_c\, \psi \wedge V^c \wedge V^b - i\, \bar\psi \wedge \gamma^a \rho_{mn}\, V^m \wedge V^n = 0 \qquad \text{(III.3.77b)}$$

$$\mathscr{D}(\rho_{ab}\, V^a \wedge V^b) + \frac{1}{4}\gamma_{ab}\, \psi \wedge (R^{ab}_{mn}\, V^m \wedge V^n + \bar\theta^{ab}_c\, \psi \wedge V^c) = 0 . \qquad \text{(III.3.77c)}$$

Let us examine the ψVV content of the torsion-Bianchi, Eq. (III.3.77b). Using $R^a_{mn} = 0$ and the rheonomy constraint (III.3.64) we find that (III.3.77b) reduces to an algebraic equation:

$$\gamma_5\, \gamma_{[n}\, \tilde\rho_{m]a} + \frac{1}{2}\gamma_s\, \gamma^t\, \rho_{t[m}\, \delta_{n]a} + i\, \gamma_a\, \rho_{mn} = 0 \qquad \text{(III.3.78)}$$

where we have set

$$\tilde\rho_{ab} = \varepsilon_{abcd}\, \rho_{cd} . \qquad \text{(III.3.79)}$$

The second term in (III.3.78) is proportional to the l.h.s., E_p, of the gravitino field equation (III.3.66b). Indeed

$$E_p \equiv \gamma_5\, \gamma_r\, \varepsilon_p{}^{rst}\, \rho_{st} \equiv \gamma_5\, \gamma^r\, \tilde\rho_{pr} . \qquad \text{(III.3.80)}$$

Moreover, utilizing the following identity (whose proof we postpone to the end of this section):

$$\rho_{ab} + \frac{i}{2}\gamma_5\, \tilde\rho_{ab} = -\frac{i}{4}\gamma^m\, \gamma_{ab}\, E_m \qquad \text{(III.3.81)}$$

we can express ρ_{ab} in term of $\tilde\rho_{ab}$ and E_m; hence Eq. (III.3.78) becomes:

$$\frac{1}{2}\gamma_5\, \gamma_n\, \tilde\rho_{ma} - \frac{1}{2}\gamma_m\, \tilde\rho_{na} + \frac{1}{2}\gamma_a\, \gamma_5\, \tilde\rho_{mn} = \frac{1}{2} E_{[m}\, \delta_{n]a} + \gamma_a\, \gamma^p\, \gamma_{mn}\, E_p . \qquad \text{(III.3.82)}$$

Now the l.h.s. of (III.3.82) can be rewritten as follows:

$$\frac{1}{2}\gamma_5\, \gamma_n\, \tilde\rho_{ma} + \frac{1}{2}\gamma_5\, \gamma_m\, \tilde\rho_{an} + \frac{1}{2}\gamma_5\, \gamma_a\, \tilde\rho_{nm} = -\frac{3!}{2}\gamma_5\, \gamma_{[a}\, \tilde\rho_{mn]} = 3!\, i\, \gamma_{[a}\, \rho_{mn]} - \frac{3}{2}\gamma_a\, \gamma^s\, \gamma_{mn}\, E_s = i\, \gamma_5\, \varepsilon_{amnp}\, E^p - \frac{3}{2}\gamma_a\, \gamma_s\, \gamma_{mn}\, E^s \qquad \text{(III.3.83)}$$

where again we made use of (III.3.81) and of the dual of (III.3.80).

Hence the (ψVV) torsion-Bianchi becomes:

$$i\,\gamma_5\,\varepsilon^{amnp}\,E_p - \frac{3}{2}\gamma^a\gamma^p\gamma^{mn}E_p - \frac{1}{2}E^{[m}\delta^{n]a} + $$
$$+ \frac{1}{4}\gamma^a\gamma^p\gamma^{mn}E_p = 0\,. \qquad \text{(III.3.84)}$$

It is immediate to see that (III.3.84) implies

$$E_p = 0 \qquad \text{(III.3.85)}$$

that is the space-time gravitino equation.

To get the space-time Einstein equation we consider the Riemann-Bianchi identity (III.3.77a). Explicitly we have

$$(\mathcal{D}R^{ab}_{mn})V^m \wedge V^n + R^{ab}_{mn}\,i\,\bar\psi \wedge \gamma^m\psi \wedge V^n + \bar\rho\,\Theta^{ab}_n \wedge V^n -$$
$$- \bar\psi \wedge \mathcal{D}\Theta^{ab}_n - \frac{i}{2}\bar\psi\,\Theta^{ab}_n \wedge \bar\psi \wedge \gamma^n\psi = 0 \qquad \text{(III.3.86)}$$

where we used

$$\mathcal{D}V^a = \frac{i}{2}\bar\psi \wedge \gamma^a\psi \qquad \text{(III.3.87)}$$

since $R^a = 0$.

$\mathcal{D}R^{ab}_{mn}$ and $\mathcal{D}\Theta^{ab}|_n$ must be further decomposed along the superspace basis (V^a, ψ).

If we consider the $\psi\psi$V content of (III.3.86) we obtain

$$R^{ab}_{mn}\,i\,\bar\psi \wedge \gamma^m\psi \wedge V^n + \varepsilon^{abrs}\,\bar\psi \wedge \gamma_5\gamma_n B_{rs}\psi$$
$$+ \varepsilon^{trs[a}\,\bar\psi \wedge \gamma_5\gamma_t B_{rs}\psi\,\delta^{b]}_n = 0 \qquad \text{(III.3.88)}$$

where we used the explicit form of $\Theta^{ab}|_c$ and we defined B_{rs} as the ψ-component of $\mathcal{D}\rho_{rs}$

$$\mathcal{D}\rho_{rs} = \mathcal{D}_\ell\rho_{rs}V^\ell + B_{rs}\psi\,. \qquad \text{(III.3.89)}$$

On the other hand the explicit value of B_{rs} can be obtained from the ψVV projection of the gravitino-Bianchi identity (III.3.77c):

$$(\mathcal{D}\rho_{mn}) \wedge V^m \wedge V^n + i\,\rho_{mn}\bar\psi \wedge \gamma^m\psi \wedge V^n +$$
$$+ \frac{1}{4}\gamma_{ab}\psi \wedge R^{ab}_{mn}V^m \wedge V^n = 0 \qquad \text{(III.3.90)}$$

from which it follows (ψVV-sector):

$$B_{mn} = -\frac{1}{4}\gamma_{ab}R^{ab}_{mn}\,. \qquad \text{(III.3.91)}$$

Inserting this value in Eq. (III.3.88) we have:

$$[R^{ab}_{\ell n} - \frac{1}{4}(\varepsilon^{abrs}\varepsilon_{npq\ell} + \varepsilon^{trs[a}\varepsilon_{tpq\ell}\delta^{b]}_n)R^{pq}_{rs}]\bar\psi \wedge \gamma^\ell\psi \wedge V^n = 0\,. \qquad \text{(III.3.92)}$$

We may now set to zero the quantity inside the brackets and perform the straightforward algebra. One arrives at:

$$4\,\delta^{[a}_{[\ell}R^{b]}{}_{n]} - \delta^{[a}_{\ell}R^{b]}{}_n - \frac{1}{2}\delta^{ab}_{\ell n}R = 0\,. \qquad \text{(III.3.93)}$$

where $R^a{}_b \equiv R^{an}{}_{bn}$ and $R \equiv R^{nm}{}_{nm}$.

Decomposing the second term into symmetric and antisymmetric parts we get the following two equations:

$$R^{[a}{}_\ell\,\delta^{b]}_n + R^{[a}{}_n\,\delta^{b]}_\ell = 0 \qquad \text{(III.3.94a)}$$

$$3\,\delta^{[a}_{[\ell}R^{b]}{}_{n]} - \frac{1}{2}R\,\delta^{ab}_{\ell n} = 0 \qquad \text{(III.3.94b)}$$

Contracting a with ℓ in (III.3.94a, b) we get, respectively

$$R_{ab} = R_{ba} \qquad \text{(III.3.95)}$$

and

$$\frac{3}{2}\,(R^b{}_n - \frac{1}{2}\,\delta^b_n\, R) = 0\ . \qquad \text{(III.3.96)}$$

Equation (III.3.95) tells us that the Ricci tensor is symmetric, while Eq. (III.3.96) is the wanted Einstein equation.

The other projections of the Bianchi identities (III.3.77) give either the space-time identities satisfied by the Riemann tensor

$$R^a{}_{[b|mn]} = 0 \qquad \text{(III.3.97)}$$

$$\mathcal{D}_{[\ell}\, R^{ab}{}_{mn]} = 0 \qquad \text{(III.3.98)}$$

or reproduce previous results.

Summarizing: the projection of the equations of motion in superspace along the various 3-forms monomials do not give independent equations. In particular the inner or space-time propagation equations of the physical fields V^a_μ, ψ_μ are just a consequence of the outer equations defining the rheonomic constraints of the theory.

Proof of the identity (III.3.81):

From (III.3.71) written in the two equivalent forms:

$$E_p = \gamma_5\, \gamma_r\, \epsilon_p{}^{rst}\, \rho_{st} = -\,i\,\gamma_p\,\gamma^{st}\,\rho_{st} + 2i\,\gamma^t\,\rho_{pt} \qquad \text{(III.3.99)}$$

one finds

$$\gamma^p\, E_p = -\,2i\,\gamma^{st}\,\rho_{st} \qquad \text{(III.3.100a)}$$

$$2\,E_\ell - \gamma_\ell\,\gamma^m\,E_m = 4i\,\gamma^t\,\rho_{\ell t} \qquad \text{(III.3.100b)}$$

$$\gamma^{ab}\,\gamma^m\,E_m = -\,2i\,\gamma^{ab}\,\gamma^{st}\,\rho_{st} \equiv -\,2i(i\,\gamma_5\,\tilde\rho_{ab} + 2\,\rho_{ab} - 4\,\gamma_{[b}\,\gamma^t\,\rho_{a]t})\ . \qquad \text{(III.3.101)}$$

Substituting the r.h.s. of (III.3.100b) into (III.3.101) after some γ-matrix rearrangements one gets the identity (III.3.81).

III.3.7 - On-shell supersymmetry

In the previous two sections we discussed the retrieving of the rheonomic constraints from the analysis of the outer equations of motion. We have seen that the rheonomic constraints imply the propagation equations of the space-time fields.

Let us now turn our attention to the main goal of the superspace approach to supergravity, namely the understanding of how the rheonomic constraints define the rheonomic mapping and, consequently, the space-time supersymmetry transformations rules. In this section we confine ourselves to the symmetries of the equations of motion, (on-shell supersymmetry), postponing to Sect. III.3.8 the discussion of the action invariance.

Let us consider the superspace equations of motion (III.3.52). Since they are written in terms of forms they are invariant under superspace diffeomorphisms generated by the Lie derivative $\ell_\varepsilon\ \mu^A$.

Hence the rheonomic mapping (III.3.32) is a symmetry of the superspace equations of motion. The explicit form of (III.3.37) is easily found utilizing the rheonomic parametrization of the curvatures, (III.3.74). Indeed from (III.3.74a, 72) and (III.3.66a) one easily obtains:

$$\varepsilon\rfloor R^{ab} = 2\,\varepsilon^r\,V^s\,R^{ab}_{rs} + 2\,\bar\theta^{ab}_c\,\psi\,\varepsilon^c + 2\,\bar\theta^{ab}_c\,\varepsilon\,V^c \qquad \text{(III.3.102a)}$$

$$\varepsilon\rfloor R^a = 0 \qquad \text{(III.3.102b)}$$

$$\varepsilon\rfloor \rho = 2\,\varepsilon^r\,\rho_{rs}\,V^s \qquad \text{(III.3.102c)}$$

where ε is a general tangent vector to $M^{4/4}$

$$\varepsilon = \bar\varepsilon\,\tilde D + \varepsilon^a\,\tilde D_a\ .$$

Since the equations of motion on $\tilde{G}$ imply the horizontality conditions $_{ab}|R^A = 0$ (III.3.56), the expressions (III.3.102) are also valid on $\tilde{G}$ if we take

$$\varepsilon = \varepsilon^A \tilde{D}_A = \varepsilon^{ab} D_{ab} + \bar{\varepsilon} \tilde{D} + \varepsilon^a \tilde{D}_a$$

and make use of both the rheonomic constraints (III.3.73-74) and the horizontality constraints (III.3.56). ($D_{ab} \equiv \tilde{D}_{ab}$)

Inserting (III.3.102) into the general expression (III.3.21) one finds:

$$\delta_\varepsilon \omega^{ab} = (\nabla\varepsilon)^{ab} + 2\, \varepsilon^r V^s R^{ab}_{rs} + 2\, \bar{\theta}^{ab}_c \psi \varepsilon^c + 2\, \bar{\theta}^{ab}_c \varepsilon V^c \qquad \text{(III.3.103a)}$$

$$\delta_\varepsilon V^a = (\nabla\varepsilon)^a \qquad \text{(III.3.103b)}$$

$$\delta_\varepsilon \psi = \nabla\varepsilon + 2\, \varepsilon^r \rho_{rs} V^s \qquad \text{(III.3.103c)}$$

where θ^{ab}_c is defined by Eq. (III.3.72).

Specializing ε we find the possible symmetries of the equations of motion:

i) If $\varepsilon = \varepsilon^{ab} D_{ab}$ we get (III.3.12a) namely an SO(1,3)-gauge transformation. Of course this is also a symmetry of the action.

ii) If $\varepsilon = \varepsilon^a \tilde{D}_a$ from (III.3.103) we get the anholonomized form of an infinitesimal diffeomorphism on space-time:

$$\delta_\varepsilon \omega^{ab} = 2\, \varepsilon^r V^s R^{ab}_{rs} + 2\, \bar{\theta}^{ab}_c \psi \varepsilon^c \qquad \text{(III.3.104a)}$$

$$\delta_\varepsilon V^a = \mathscr{D} \varepsilon^a \qquad \text{(III.3.104b)}$$

$$\delta\psi = 2\, \varepsilon^r \rho_{rs} V^s \qquad \text{(III.3.104c)}$$

where we used Eqs. (III.3.12) for the $\overline{Osp(1/4)}$ covariant derivative $(\nabla\varepsilon)^A$. Notice that no rheonomic constraint enters in (III.3.104). Moreover the transformations (III.3.95) are also a symmetry of the action (III.3.51). Indeed not only the Lagrangian, but also the integration surface is left invariant by the space-time diffeomorphisms, so that every item entering the action functional is invariant against this type of symmetry.

iii) Let us now consider a purely superspace transformation

$$\varepsilon \equiv \bar{\varepsilon} \tilde{D} \qquad \text{(III.3.105)}$$

so that the rheonomic constraints come into play. In this case Eqs. (III.3.103) become:

$$\delta_\varepsilon \omega^{ab} = (\nabla\varepsilon)^{ab} + 2\, \bar{\theta}^{ab}_c \varepsilon V^c \qquad \text{(III.3.106a)}$$

$$\delta_\varepsilon V^a = (\nabla\varepsilon)^a \qquad \text{(III.3.106b)}$$

$$\delta\psi = \nabla\varepsilon \qquad \text{(III.3.106c)}$$

and using (III.3.12) we get

$$\delta_\varepsilon \omega^{ab} = 2\, \bar{\theta}^{ab}_c \varepsilon V^c \qquad \text{(III.3.107a)}$$

$$\delta_\varepsilon V^a = i\, \bar{\varepsilon} \gamma^a \psi \qquad \text{(III.3.107b)}$$

$$\delta_\varepsilon \psi = \mathscr{D} \varepsilon \qquad \text{(III.3.107c)}$$

where θ^{ab}_c is defined in (III.3.72).

The idea that the rheonomic constraints generate a purely space-time symmetry of the theory is examplified in Eqs. (III.3.107). Indeed by considering the Lie derivative from the active point of view, namely as the functional change of μ^A at the same coordinate point, we see that (III.3.107) represents a supersymmetry transformation leaving the space-time equations of motion (III.3.66) invariant.

As we explained in Sect. III.3.4, checking the closure of the supersymmetry algebra is equivalent to checking the Bianchi identities in presence of the rheonomic constraints. In Sect. III.3.6, we showed that the equations of motion are implied by the Bianchi's. Hence we conclude that the supersymmetry transformations close an algebra on the on-shell configurations.

The present example verifies the general statements made in Sect. III.3.3.

III.3.8 - Action invariance and off-shell supersymmetry

The supersymmetry transformation laws (III.3.107) close an algebra only on the on-shell configurations.

Furthermore, for the time being, they are only symmetries of the field equations: indeed the action is not necessarily invariant against these transformations. It is then a natural question to ask whether the laws (III.3.107) can be modified, via suitable on-shell vanishing additions, so that they become genuine symmetries of the action functional.

To answer this question we take a more general viewpoint and we consider when a geometrical action like (III.3.47) is invariant with respect to general diffeomorphisms in superspace.

In our setup the invariance of the action does not coincide with the invariance (modulo total divergences) of the Lagrangian.

This is so because even if $\mathscr{L}$ remained invariant the integration volume M_D would anyhow change under a fermionic diffeomorphism. Indeed M_D is mapped into itself by an inner (space-time) diffeomorphism, but it is distorted into a neighbouring surface by any infinitesimal transformation in the θ-directions.

Therefore the bosonic coordinate transformations are always symmetries of the action, without further specifications, while the fermionic diffeomorphisms are symmetries of the action only if the action does not depend on the choice of the integration surface M_D.

It is easy to see when this happens. Using Stoke's theorem we can write:

$$\mathscr{A}(M_D + \delta M_D) - \mathscr{A}(M_D) = \int_\Omega d\mathscr{L} \qquad \text{(III.3.108)}$$

where $M_D + \delta M_D$ is the new integration hypersurface after the infinitesimal diffeomorphism and Ω is the (super) volume contained between the two surfaces, see Fig. III.3.III. (We are assuming that superspace is naturally compactified along the space-time directions).*

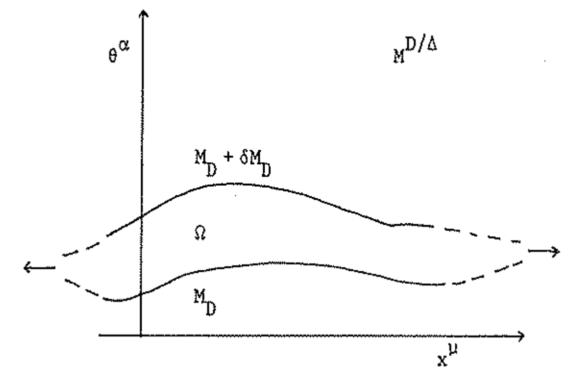

Fig. III.3.III

Hence the diffeomorphisms in superspace are an off-shell invariance of the general geometrical action (III.3.46) if:

$$d\mathscr{L} = 0 \qquad \text{(III.3.109)}$$

that is if $\mathscr{L}$ is a closed form in superspace. As an exercise let us check that the supergravity Lagrangian (III.3.51) is not a closed form. We easily find:

---

* In D-dimensions superspace $M^{D/\Delta}$ is the coset $\tilde{G}/H$, $\tilde{G}$ being the soft super Poincaré group in D dimensions, $H \equiv SO(1,D)$ and Δ = the number of fermionic generators of the Lie algebra $\mathbb{G}$.

$$d\mathcal{L} = \mathcal{D}R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} + 2\, R^{ab} \wedge \mathcal{D}V^c \wedge V^d \varepsilon_{abcd} +$$
$$+ 4\, \bar{\rho} \wedge \gamma_5\, \gamma_m\, \rho \wedge V^m - 4\, \bar{\psi} \wedge \gamma_5\, \gamma_m\, \mathcal{D}\rho \wedge V^m -$$
$$- 4\, \bar{\psi} \wedge \gamma_5\, \gamma_m\, \rho \wedge \mathcal{D}V^m \qquad \text{(III.3.110)}$$

(where we traded d for $\mathcal{D}$ when acting on a Lorentz invariant tensor contraction).

Using the definition (III.3.5b) and the Bianchi identities (III.3.9a, c) one finds:

$$d\mathcal{L} = 2\, R^{ab} \wedge R^c \wedge V^d \varepsilon_{abcd} + i\, R^{ab} \wedge \bar{\psi} \wedge \gamma^c\, \psi \wedge V^d \varepsilon_{abcd} +$$
$$+ 4\, \bar{\rho} \wedge \gamma_5\, \gamma_m\, \rho \wedge V^m - \bar{\psi} \wedge \gamma_5\, \gamma_m\, \gamma_{ab}\, \psi \wedge R^{ab} \wedge V^m -$$
$$- 4\, \bar{\psi} \wedge \gamma_5\, \gamma_m\, \rho \wedge R^m - 2i\, \bar{\psi} \wedge \gamma_5\, \gamma_m\, \rho \wedge \bar{\psi} \wedge \gamma^m\, \psi\, . \qquad \text{(III.3.111)}$$

Now, using

$$\gamma_5\, \gamma^m\, \gamma_{ab} = \gamma_5 (2\, \delta^m_{[a}\, \gamma_{b]} + i\, \varepsilon^m{}_{abc}\, \gamma_5\, \gamma^c) \qquad \text{(III.3.112)}$$

we get:

$$\bar{\psi} \wedge \gamma_5\, \gamma_m\, \gamma_{ab}\, \psi = i\, \varepsilon_{mabc}\, \bar{\psi} \wedge \gamma^c\, \psi\, . \qquad \text{(III.3.113)}$$

Here we used the fact that $\psi^t \wedge \psi$ is a symmetric tensor in the spinor indices while $C\gamma_5\gamma_a$ is an antisymmetric one, and therefore $\bar{\psi}\gamma_5\gamma_a\psi = 0$.

In the same way we find

$$\bar{\psi} \wedge \gamma_5\, \gamma_m\, \rho = \bar{\rho} \wedge \gamma_5\, \gamma_m\, \psi \qquad \text{(III.3.114)}$$

since a spinorial 1-form anticommutes with a spinorial 2-form and $C\gamma_5\gamma_m$ is antisymmetric.

By virtue of the Fierz identity (III.2.26) it follows that

$$\bar{\psi} \wedge \gamma_5\, \gamma_m\, \rho \wedge \bar{\psi} \wedge \gamma^m\, \psi = \bar{\rho} \wedge \gamma_5\, \gamma_m\, \psi \wedge \bar{\psi} \wedge \gamma^m\, \psi = 0\, . \qquad \text{(III.3.115)}$$

Hence (III.3.110) becomes:

$$d\mathcal{L} = 2\, R^{ab} \wedge R^c \wedge V^d \varepsilon_{abcd} + 4\, \bar{\rho} \wedge \gamma_5\, \gamma_m\, \rho \wedge V^m -$$
$$- 4\, \bar{\psi} \wedge \gamma_5\, \gamma_m\, \rho \wedge R^m\, . \qquad \text{(III.3.116)}$$

This result implies that the supersymmetry transformations, being outer diffeomorphisms supplemented with the rheonomy constraints, cannot be an off-shell invariance of the action. This confirms our previous statements, and is in line with the results of the purely space-time approach, where we found that the supersymmetry transformations close an algebra only on the on-shell fields. However, in the space-time approach of Chapter III.2, we found that even if supersymmetry does not close an off-shell algebra, nevertheless it is an invariance of the action.(*)

The invariance of the space-time Lagrangian under supersymmetry can be retrieved also in the present framework. From an active point of view the supersymmetry variation of the action is given in terms of the contraction $\underline{\varepsilon|} d\mathcal{L}$ along a spinorial tangent vector $\varepsilon = \bar{\varepsilon}\, \vec{D}$; indeed computing the Lie derivative ℓ_ε we find:

$$\delta_\varepsilon \mathcal{A} = \int_{M^D} \ell_\varepsilon \mathcal{L} = \int_{M^D} (\underline{\varepsilon|} d + d\, \underline{\varepsilon|})\, \mathcal{L}$$
$$= \int_{M^D} \underline{\varepsilon|} d\, \mathcal{L} \qquad \text{(III.3.117)}$$

since we can discard the total derivative term $d(\underline{\varepsilon|}\mathcal{L})$.

(*) Here we distinguish between the words "invariance" and "symmetry". A set of infinitesimal transformations constitute an "invariance" if they leave the action invariant; however they may not close an off-shell algebra. A set of transformations will instead be a "symmetry" if they both leave the action invariant and close an off-shell algebra.

Therefore the action is supersymmetry invariant if

$$\epsilon\rfloor d\mathscr{L} = 0 \,. \qquad \text{(III.3.118)}$$

Applying this condition to (III.3.116) we get:

$$\begin{aligned}\epsilon\rfloor d\mathscr{L} = {} & 2\, \epsilon\rfloor R^{ab} \wedge R^{c} \wedge V^{d}\, \epsilon_{abcd} + 2\, R^{ab} \wedge \epsilon\rfloor R^{c} \wedge V^{d}\, \epsilon_{abcd} + \\ & + 8\, \epsilon\rfloor\bar{\rho} \wedge \gamma_5\gamma_a\rho \wedge V^{a} - 4\, \bar{\epsilon}\, \gamma_5\gamma_m\rho \wedge R^{m} + \\ & + 4\, \bar{\psi} \wedge \gamma_5\gamma_m\, \epsilon\rfloor\rho \wedge R^{m} + 4\, \bar{\psi} \wedge \gamma\, \gamma\, \rho \wedge \epsilon\rfloor R^{m}\end{aligned} \qquad \text{(III.3.119)}$$

where we made use of the properties:

$$\epsilon\rfloor V^{a} = 0 \qquad \text{(III.3.120a)}$$

$$\epsilon\rfloor \psi = \epsilon \qquad \text{(III.3.120b)}$$

following from $\epsilon = \bar{\epsilon}\,\tilde{D}$.

From Eq. (III.3.119) we see that in general we can make the action invariant by imposing suitable constraints on the outer components R^{A} so that $\epsilon\rfloor d\mathscr{L} = 0$. Let us distinguish two cases:

a) suppose we work in first order formalism, that is we do not implement the (space-time) equation $R^{a}_{mn} = 0$; then $\epsilon\rfloor d\mathscr{L} = 0$ is achieved if

$$\epsilon\rfloor \rho = 0 \qquad \text{(III.3.121a)}$$

$$\epsilon\rfloor R^{a} = 0 \qquad \text{(III.3.121b)}$$

and if

$$2\, \epsilon\rfloor R^{ab} \wedge V^{c}\, \epsilon_{abcd} + 4\, \bar{\epsilon}\, \gamma_5\gamma_d\rho = 0 \,. \qquad \text{(III.3.122)}$$

Writing

$$\epsilon\rfloor R^{ab} = \bar{\epsilon}\, \Theta^{ab}_{c}\, V^{c} \qquad \text{(III.3.123)}$$

we see that the ψV projection of Eq. (III.3.122) coincides with Eq. (III.3.64), if ψ is replaced by ϵ. Therefore we get the same solution (III.3.72).

Moreover, using the parametrization (III.3.60), the two equations (III.3.121) imply

$$\Theta^{a}_{c} = K^{a} = H_{c} = \Omega_{\alpha\beta} = 0 \qquad \text{(III.3.124)}$$

and these equations, together with the solution (III.3.72), define the same rheonomic constraints already derived from the equations of motion (III.3.73-74). Hence we conclude that the action (III.3.51) is invariant, in first order formalism, against the same set of transformations which form a closed symmetry algebra for the equations of motion. In this way we have given the proof of Eqs. (III.2.62).

This conclusion is not so trivial as it might seem at first sight. Indeed in off-shell extending the on-shell transformation laws (III.3.107) there is an ambiguity which is proportional to the left-hand side of the equations of motion. What we have verified is that this ambiguity is zero in first order formalism.

Indeed, if we now consider second order formalism, implementing the equation $R^{a}_{mn} = 0$, we get:

b) $\epsilon\rfloor d\mathscr{L} = 0$ iff

$$\epsilon\rfloor R^{a} = 0 \qquad \text{(III.3.125a)}$$

$$\epsilon\rfloor \rho = 0 \,. \qquad \text{(III.3.125b)}$$

In this case we have

$$\epsilon\rfloor R^{ab} \equiv \delta_{\epsilon}\, \omega^{ab} = \text{chain rule} \qquad \text{(III.3.126)}$$

and $\delta_\varepsilon \omega^{ab}$ is given by the chain rule. It was explicitly computed in Eq. (III.2.61), and we rewrite it here for completeness:

$$\delta_\varepsilon \omega_{ab} \text{ (second-order)} = - i(\bar{\varepsilon} \gamma_m \rho_{ab} - 2 \bar{\varepsilon} \gamma_{[a} \rho_{b]m})V^m . \qquad \text{(III.3.127)}$$

Notice that we used the identification:

$$\rho_{ab} \equiv V^\mu_{[a} V^\nu_{b]} \mathscr{D}_\mu \psi_\nu . \qquad \text{(III.3.128)}$$

Comparing (III.3.127) with the first-order law (III.3.107a) we see that the two are different.

According to the previous discussion this difference must be proportional to the l.h.s. of an equation of motion since both are off-shell extensions of the same on-shell symmetry. (Actually the first order law coincides, as we have just seen, with the on-shell symmetry of the space-time field equations).

The verification of the last statement is an immediate consequence of the equations (III.3.80) and (III.3.81). Using these equations it is immediate to show that the difference between the two laws (III.3.107a) and (III.3.127) is given by:

$$(\delta_\varepsilon^{\text{first order}} - \delta_\varepsilon^{\text{second order}})\omega^{ab} = 2 \bar{\varepsilon} \gamma^{c[a} E^{b]} V_c + $$
$$+ i \varepsilon^{\ell sab} \bar{\varepsilon} \gamma_5 \gamma_{cs} E_\ell V^c \qquad \text{(III.3.129)}$$

where E^c is the l.h.s. of the gravitino space-time Eq. (III.3.66b); on-shell the two transformation laws coincide, as anticipated.

This concludes our discussion on the relation between the superspace formulation of D=4 N=1 supergravity and the space-time approach given in Chapter III.2.

III.3.9 - Building rules for supergravity Lagrangians

So far we have considered the superspace formulation of N=1, D=4 supergravity, taking the Lagrangian (III.3.51) as given.

It is time to see how this Lagrangian can be retrieved from a convenient set of general rules.

As we are going to see the building rules utilized in part I to derive the Einstein-Cartan Lagrangian of general relativity will suffice to our needs provided we add the requirement of rheonomy.

The reason is not hard to understand. Our geometrical treatment of N=1, D=4 supergravity relies on its being the "gauge" theory of the super Poincaré group $\overline{Osp(1,4)}$ in the same fashion as gravity is the "gauge" theory of the Poincaré group ISO(1,3).

Hence the same considerations used in pure gravity should apply to supergravity provided we make the replacements $ISO(1,3) \to \overline{Osp(1,4)}$ and $\mu^A \equiv (\omega^{ab}, V^a) \to \mu^A \equiv (\omega^{ab}, V^a, \psi)$. Furthermore we must add the requirement of rheonomy in order to obtain a theory interpretable on space-time. (It will turn out that in the case of D=4, N=1 supergravity the rheonomy requirement follows automatically from the other principles but this is only an accident).

Since all we are going to say can be applied to more general theories than N=1 supergravity, we will reformulate the building rules used in part I in a more general setting. We shall leave unspecified the super-Lie algebra we start from and the number D of space-time dimensions.(*)

This allows to give a formulation of the building rules easily adaptable to more general supergravity theories $(N > 1$ or $D > 4)$.

Let us then start with a super-Lie algebra $\mathbb{G}$:

$$[T_A, T_B] = C^C_{AB} T_C \qquad \text{(III.3.130)}$$

and recall the Maurer-Cartan equations (see Eqs. (II.3.27)):

$$d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C = 0 \qquad \text{(III.3.131)}$$

(*) For higher dimensional supergravities we will have to enlarge the concept of (super)-Lie algebra to that of free differential algebra; see Chapter III.6.

where σ^A are a set of left-invariant 1-forms dual to the left invariant generator T_A

$$\sigma^A(T_B) = \delta^A_B \,. \qquad \text{(III.3.132)}$$

Since we are going to discuss supergravity theories, we always assume that $\mathbb{G}$ contains the Poincaré or the anti de Sitter algebra in D dimensions, ISO(1,D-1) or SO(2,D-1) respectively.

Next we introduce the soft 1-forms $\mu = \mu^A T_A$ defined on the soft group manifold $\tilde{G}$ which are not left-invariant and we denote by $\tilde{D}_A$ the tangent vectors dual to μ^B (soft generators)

$$\mu^A(\tilde{D}_B) = \delta^A_B \,. \qquad \text{(III.3.133)}$$

By replacing σ^A with μ^A the l.h.s. of (III.3.131) is no longer zero and defines the curvatures of the μ^A's:

$$R^A = d\mu^A + \frac{1}{2} C^A_{BC}\, \mu^B \wedge \mu^C \,. \qquad \text{(III.3.134)}$$

Correspondingly the $\tilde{D}_A$ tangent vectors satisfy the soft algebra defined in (I.3.132).

The integrability condition $d^2 = 0$ on (III.3.134) gives rise to the Bianchi identities on the R^A:

$$\nabla R^A \equiv dR^A + C^A_{BC}\, \mu^B \wedge R^C = 0 \qquad \text{(III.3.135)}$$

where ∇ is the G-covariant derivative.

Furthermore we make also the assumption (always verified in the known theories) that $\mathbb{G}$ has the following structure:

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K} \qquad \text{(III.3.136)}$$

$$\mathbb{K} = \mathbb{I} \oplus \mathbb{D} \qquad \text{(III.3.137)}$$

where $\mathbb{H}$ is a Bose-subalgebra and $\mathbb{K}$ a subspace, further split in a Bose subspace $\mathbb{I}$ and a subspace $\mathbb{D}$ with the following properties:

$$[\mathbb{H},\mathbb{H}] \subset \mathbb{H}; \quad [\mathbb{H},\mathbb{I}] \subset \mathbb{I}; \quad [\mathbb{H},\mathbb{D}] \subset \mathbb{D} \qquad \text{(III.3.138)}$$

$$[\mathbb{I},\mathbb{I}] \subset \mathbb{H} \qquad \text{(III.3.139)}$$

$$[\mathbb{I},\mathbb{D}] \subset \mathbb{D} \qquad \text{(III.3.140)}$$

$$[\mathbb{D},\mathbb{D}\} \subset \mathbb{I} \oplus \mathbb{H} \equiv \mathbb{B} \subset \mathbb{G} \,. \qquad \text{(III.3.141)}$$

Equation (III.3.138) says that $\mathbb{H}$ is a subalgebra and that $\mathbb{I}$ and $\mathbb{D}$ span two of its representations. Equation (III.3.139) says that $\mathbb{B} = \mathbb{I} \oplus \mathbb{H}$ is a Bose subalgebra of $\mathbb{G}$ and that B/H is a symmetric space. Equation (III.3.140) says that $\mathbb{D}$ carries a representation not only of $\mathbb{H}$, rather of the full $\mathbb{B}$ and finally Eq. (III.3.141) says that also G/B is a symmetric space.

This splitting of $\mathbb{G}$ in a triple is the heart of the whole rheonomy framework and deserves a special nomenclature. It is the following one:

- $\mathbb{H}$ = <u>gauge subalgebra</u>
- $\mathbb{I}$ = <u>inner space</u> (Its dimension is equal to the number of space-time dimensions. dim $\mathbb{I}$ = D).
- $\mathbb{D}$ = <u>outer space</u>.

As far as the indices are concerned, A,B,... will run on $\mathbb{G}$, H, H', H" will run on $\mathbb{H}$ and K, K', K" will run on $\mathbb{K}$ being, therefore, split in the following way: K = {I,O} where I, I', I" run on $\mathbb{I}$ and O, O', O" on $\mathbb{D}$.

This same decomposition holds when we turn the Lie algebra $\mathbb{G}$ into a soft one: $\mathbb{G} \to \tilde{\mathbb{G}}$. Indeed the structure of the algebra generated by the soft generators $\tilde{D}_A$ is the same as that generated by the left-invariant ones D_A. Accordingly we maintain the same kind of notations and nomenclature also in the soft case. Therefore we can decompose (III.3.134) as follows:

$$R^H = d\mu^H + \frac{1}{2} C^H_{H'H''} \mu^{H'} \wedge \mu^{H''} \qquad \text{(III.3.142a)}$$

$$R^I = \mathscr{D}^{(H)} \mu^I + \frac{1}{2} C^I_{00'} \mu^0 \wedge \mu^{0'} \qquad \text{(III.3.142b)}$$

$$R^0 = \mathscr{D}^{(H)} \mu^0 + C^0_{I0'} \mu^I \wedge \mu^{0'} \qquad \text{(III.3.142c)}$$

where $\mathscr{D}^{(H)}$ is the H-covariant derivative.

Now the key observation is the following. If the fields and the field-strengths of the physical theory are to be interpreted geometrically in terms of the μ^A's and R^A's just introduced, then the equations derived from the action principle (and the action itself) cannot violate any of the symmetries and of the properties implied by the definition of the curvature, Eqs. (III.3.134), and by the Bianchi identities, Eqs. (III.3.135). Hence the Lagrangian must be constructed in a way compatible with these properties and these symmetries. Let us list them:

i) coordinate invariance: this is an obvious consequence of the fact that (III.3.134-135) are exterior form equations involving only the two diffeomorphic invariant operations, namely the exterior product and the exterior differentiation.

ii) The exclusion of the Hodge dual: this point was already discussed in previous sections. It is essential if we want to preserve our freedom of extending or restricting the manifold on which the fundamental field equations are to be enforced.

For example, if $H \subset G$ is a subgroup of G and (I.3.131, 134-135) are H-gauge invariant, then it is possible to interpret (III.3.134) as structure equations for the forms $\mu^A \equiv (\mu^H, \mu^K)$ defined on the coset manifold G/H. We may also restrict (III.3.134-135) to the space-time manifold (inner directions) and they will hold on this manifold as well.

iii) Rigid scale invariance: the decomposition (III.3.138-141) is invariant under the following rescalings of the generators

$$T_H \to T_H; \qquad T_I \to w^{-1} T_I; \qquad T_0 \to w^{-\frac{1}{2}} T_0 \qquad \text{(III.3.143)}$$

where $w \neq 0$ is the scaling factor. We denote by μ^H, μ^I, μ^0, the 1-forms dual to the soft generators $\tilde{D}_H$, $\tilde{D}_I$, $\tilde{D}_0$ and by R^H, R^I, R^0 their curvatures. Equations (III.3.134) and (III.3.135) are invariant under the substitutions

$$\mu^H \to \mu^H; \qquad \mu^I \to w\mu^I; \qquad \mu^0 \to w^{\frac{1}{2}} \mu^0 \qquad \text{(III.3.144a)}$$

$$R^H \to R^H; \qquad R^I \to wR^I; \qquad R^0 \to w^{\frac{1}{2}} R^0 . \qquad \text{(III.3.144b)}$$

For example, Eqs. (III.3.5) and (III.3.9) are invariant under the rigid rescaling:

$$\omega^{ab} \to \omega^{ab}; \qquad V^a \to wV^a; \qquad \psi \to w^{\frac{1}{2}} \psi \qquad \text{(III.3.145a)}$$

$$R^{ab} \to R^{ab}; \qquad R^a \to wR^a; \qquad \rho \to w^{\frac{1}{2}} \rho . \qquad \text{(III.3.145b)}$$

iv) H-gauge invariance. H is a subgroup of G which admits SO(1,D), i.e. the Lorentz group, as a factor: $H = SO(1,D) \otimes H'$. Equations (III.3.134-135) are gauge invariant under H as a consequence of the fact that $\mathbb{I}$ and $\mathbb{D}$ are both representations of $\mathbb{H}$, (Eqs. (III.3.139-140)).

Therefore we can decompose (III.3.134-135) in their H-content and the resulting set of equations is explicitly covariant with respect to H. For example in the gravity case $(G = ISO(1,3))$ and in the N=1 D=4 supergravity case $(G = \overline{Osp(1,4)})$ Eqs. (I.4.11), (I.4.12) and (III.3.5), (III.3.9) are covariant under SO(1,3) gauge-transformations since in both cases $H = SO(1,3)$ $(H' = \mathbb{1})$. We will see that in more complicated supergravities $H' \neq \mathbb{1}$ and it is usually given by a U(1)-group.

v) $R^A = 0$ satisfies Eqs. (III.3.134-135): indeed in this case they reduce to the Maurer-Cartan equations of G for the left-invariant forms σ^A and to the Jacobi identities for the structure constants, respectively.

The fundamental properties of the defining equations (III.3.134-135) being listed, we now require that the action and the field equations of the theory based on (III.3.134-135) should respect the symmetries and the properties expressed by i) - v). This leads to the following building rules:

A: GEOMETRICITY

By this we mean that the Lagrangian should be a D-form constructed out of the soft 1-forms μ^A (and, if necessary, also of some 0-forms defined on $\tilde{G}$) using only the diffeomorphic invariant operations d and $\wedge$, with the exclusion of the Hodge duality operator. Specifically we say that a theory is strongly geometrical if its Lagrangian can be constructed using only the μ^A 1-forms. Otherwise, when 0-forms are also needed, the theory is geometrical "tout-court".

Let us first comment about the possible appearance of 0-forms. The introduction of 0-form fields is obviously necessary if we want to couple supergravity multiplets to matter multiplets containing spin 0 or 1/2-fields and if the supergravity multiplet itself contains such fields (in D=4 this happens for $N > 2$). The study of supergravity Lagrangians containing spin zero and spin 1/2 fields will be the main goal of Part IV.

In this part we restrict our attention to Lagrangians containing gravitational multiplets lower bounded by spin one.

The extension of the building rules to more general cases will be discussed when needed.

We point out that the exclusion of the spin 1/2 and spin 0 fields from a fundamental higher dimensional Lagrangian does not exclude their appearance, through the Kaluza-Klein mechanism, in the lower dimensional effective theory after dimensional reduction. This will be discussed in detail in Part V.

It must be said that even in strongly geometrical theories, where all the fields are gauge potentials of a Lie algebra or of a free differential algebra (see Chapter III.6), 0-forms are most of the time needed to construct the kinetic term of the spin one fields, avoiding in this way the Hodge dual. The relevant procedure, which is just first order formalism for gauge fields, was extensively discussed in Chapter I.5. It relies on the introduction of a 0-form F_{ab} which, through its own equation of motion, is identified with the vielbein-vielbein component of the field strength

$$F \equiv dA + A \wedge A = F_{ab} V^a \wedge V^b + \text{more} .$$

As seen in Chapter II.9, while discussing super Yang-Mills theories, F_{ab} plays an important role also as Lagrange multiplier of the rheonomic constraints on F. Indeed the rheonomic parametrization of the curvature F is enforced as an equation of motion by the δF_{ab}-variation of the Lagrangian. This will continue to be true in the local supersymmetric case and also when F sits in the gravitational multiplet of curvatures rather than being part of the matter multiplets.

The only exceptions to this state of affairs are two theories, D=5, N=2 and D=6, N=1 supergravity where the "spin one" fields of the graviton multiplet acquire a kinetic term without the use of an additional 0-form.(*) In these cases the role usually played by F_{ab} is taken over by the torsion, that is by the non-metric part of the spin connection. In these theories the first order formulation of the gravitational sector already suffices to build up a kinetic term also for the spin one fields. Such a mechanism is very elegant and economical, but it is unfortunately limited to the two theories in question. Indeed it can be realized only under very special conditions. It will be discussed in detail in later chapters.

(*) Hence D=5, N=2 and D=6, N=1 are the only "strongly geometrical" theories besides D=4, N=1.

Anyhow, with a geometrical Lagrangian $\mathcal{L}$ (in the sense discussed above), the action is obtained by integrating $\mathcal{L}$ on a D-dimensional hypersurface M_D immersed in the soft manifold $\tilde{G}$, according to the extended action principle introduced in Sect. III.3.4.

$$\mathcal{A} = \int_{M^D \subset \tilde{G}} \mathcal{L} \ . \qquad \text{(III.3.146)}$$

Assuming for the rest of this section that the Lagrangian $\mathcal{L}$ is not only geometrical, but also strongly geometrical, we see that principle A implies that $\mathcal{L}$ should be a polynomial (in the exterior algebra sense) in the curvatures R^A.

Indeed since:

$$d\mu^A = R^A - \frac{1}{2} C^A_{BC}\, \mu^B \wedge \mu^C \qquad \text{(III.3.147a)}$$

$$dR^A = - C^A_{BC}\, \mu^B \wedge R^C \qquad \text{(III.3.147b)}$$

then in the absence of 0-forms the most general Lagrangian $\mathcal{L}(\mu^A, d\mu^A)$ can be expanded as follows:

$$\mathcal{L} = \Lambda^{(D)} + R^A \wedge \nu_A^{(D-2)} + R^A \wedge R^B \wedge \nu_{AB}^{(D-4)} +$$

$$+ R^A \wedge R^B \wedge R^C \wedge \nu_{ABC}^{(D-6)} + \ldots \qquad \text{(III.3.148)}$$

where the coefficients have the general form

$$\Lambda^{(D)} = C_{A_1 \ldots A_D}\, \mu^{A_1} \wedge \ldots \wedge \mu^{A_D} \qquad \text{(III.3.149a)}$$

$$\nu_A^{(D-2)} = C_{AA_1 \ldots A_{D-2}}\, \mu^{A_1} \wedge \ldots \wedge \mu^{A_{D-2}} \qquad \text{(III.3.149b)}$$

$$\nu_{AB}^{(D-4)} = C_{ABA_1 \ldots A_{D-4}}\, \mu^{A_1} \wedge \ldots \wedge \mu^{A_{D-4}} \qquad \text{(III.3.149c)}$$

. . . .

and the C's are constants.

Notice that the degree of the polynomial in R^A is at most $[D/2]$ since $\mathcal{L}$ is a D-form. We shall see that for ordinary supergravity theories $\mathcal{L}$ is always a polynomial of degree 2 in the R^A's, for any D. This is to be expected since, otherwise, one would obtain space-time propagation equations of order greater than 2 (and products of more than 4 ψ-fields at one point: for example $R^a \wedge R^b \wedge R^c$ would contain ψ^6-terms).

This rule will be violated by D=10 Anomaly Free Supergravity (AFS) which is discussed in Part VI.

The reason is that AFS, because of its relation to the superstring, is no longer a second order theory as all the other models discussed in this book, rather it is a higher derivative field theory. A lot of novel features do appear in this case whose discussion is postponed to Part VI.

Considering now the property iv) of the Eqs. (III.3.134-135) we establish the second building rule, namely:

B) H-GAUGE INVARIANCE

The Lagrangian in (III.3.146) must be H-invariant, H being a subgroup of G with SO(1,D) as a factor: $H = SO(1,D) \times H'$. (For $D = 2n$ we will restrict SO(1,D) to its connected part so that parity is conserved).

To implement principle B on the Lagrangian given by (III.3.148-149), we notice that each term in (III.3.148) must be an H-scalar; in particular the indices A, B, ... running in the G-adjoint representation must be saturated in a H-invariant way. Furthermore the polynomials Λ, ν_A, ν_{AB}, etc. defined by (III.3.149) must have the following properties:

a) the coefficients $C_{A_1 \ldots A_D}$, $C_{AA_1 \ldots A_{D-2}}, \ldots$ must be constructed using only H-invariant tensors;

b) Bare μ^H gauge fields can appear in the polynomials (III.3.149) only if their global coefficient in the Lagrangian is an H-covariant closed D-1 form. (The curvatures R^A may, of course, contain μ^H-fields since, by definition, they are H-gauge covariant objects).

Statement a) is obvious. To prove statement b) let us suppose, for simplicity, that (III.3.148) contains μ^H linearly; then we can split $\mathcal{L}$ as follows:

$$\mathcal{L} = \mathcal{L}_0 + \Gamma_H \wedge \mu^H \tag{III.3.150}$$

where $\mathcal{L}_0$ is the part that does not contain the bare μ^H field and $\Gamma_H\, \mu^H$ is the rest (Γ_H is a (D-1)-form). Using (III.3.20) and the horizontality condition (III.3.22), we have

$$\delta\mu^H = (\nabla\varepsilon)^H \tag{III.3.151}$$

with $\varepsilon = \varepsilon^H D_H$ an infinitesimal parameter, and we find the following gauge variation for $\mathcal{L}$:

$$\begin{aligned}\delta_{\varepsilon^{H'}}\mathcal{L} &= \delta_{\varepsilon^{H'}}\Gamma_H \wedge \mu^H + \Gamma_H \wedge \delta_{\varepsilon^{H'}}\mu^H \\ &= C^{H''}_{H'H}\,\mu^{H'}\,\Gamma_{H''} \wedge \mu^H + \Gamma_H \wedge \nabla\varepsilon^H \\ &= C^{H''}_{H'H}\,\mu^{H'}\,\Gamma_{H''} \wedge \mu^H + (-1)^{D-1}(\nabla(\Gamma_H\varepsilon^H) - \nabla\Gamma_H\varepsilon^H)\ .\end{aligned} \tag{III.3.152}$$

Since $\nabla(\Gamma_H\varepsilon^H) \equiv d(\Gamma_H\varepsilon^H)$ the action is invariant under the gauge variation if:

$$\{C^{H''}_{H'H}\,\varepsilon^{H'}\,\Gamma_{H''} \wedge \mu^H - (-1)^{D-1}\,\nabla\Gamma_H\,\varepsilon^H\} = 0\ . \tag{III.3.153}$$

Now, $\nabla\Gamma_H$ cannot contain the μ^H-connection as a factor since $\Gamma_{H'}$, and therefore $\nabla\Gamma_{H'}$, are good H-tensors. It follows that the two terms inside the curly bracket vanish separately. For the first term this would imply $\Gamma_H \equiv 0$ contrary to the hypothesis, and we conclude that no bare μ^H-connection can appear in the Lagrangian. This conclusion, however, fails when the first term is zero, that is when H contains an abelian subgroup (a U(1)-factor).

In this case, performing a U(1) gauge transformation of parameter $\varepsilon^\bullet$, Eq. (III.3.153) takes the following form

$$\nabla^{[U(1)]}\Gamma_\bullet = 0 \tag{III.3.154a}$$

or, equivalently:

$$d\Gamma_\bullet = 0\ . \tag{III.3.154b}$$

We will encounter this situation in the construction of $D > 4$ supergravity theories where such U(1)-closed forms do actually exist.

When rules A and B have been implemented generally one is left with several different terms corresponding to the possible Lorentz invariant D-forms one may construct out of the μ^H and R^A's, each being multiplied by an unknown constant coefficient. Usually one obtains a dramatic reduction of the number of the possible terms entering $\mathcal{L}$ by considering property iii). Let us state the third building rule:

C) HOMOGENEOUS SCALING LAW

Each term in the Lagrangian must scale homogeneously under the scaling law (III.3.144); in particular in D-dimensions each term must scale as $[w^{D-2}]$, the scale-weight of the Einstein term. Indeed the equations of motion on superspace give relations among the various curvature components which have to be independent of w. Otherwise they would be inconsistent with the Bianchi identities which are independent of w. (For example in D=4, N=1 supergravity both the inner equations (III.3.66) and the outer equations (III.3.73-74) (the rheonomic constraints) are independent of w). Now equations of motion independent of w require that all the terms in the Lagrangian should scale homogeneously under (III.3.144). In particular any supergravity Lagrangian contains the Einstein term (see Sect. III.5.1)

$$R^{ab} \wedge V^{c_1} \wedge \dots \wedge V^{c_{D-2}} \varepsilon_{abc_1 \dots c_{D-2}}$$

which scales as w^{d-2}, so that rule C is justified.

Now property v) of the Bianchi identities leads to the fourth building rule:

D) VACUUM EXISTENCE

The field equations on the G (or G/H) manifold should admit the solution $R^A = 0$ (the "vacuum"), and therefore be at least linear in the curvature 2-forms R^A.

In order to implement this requirement in the superspace equations of motion derived from the action (III.3.146), we use formulae (III.3.148-149).

Assuming for simplicity that the polynomial is of degree 2 in the R^A we find:

$$\frac{\delta \mathscr{L}}{\delta \mu^L} = \frac{\delta \Lambda}{\delta \mu^L} + \nabla \nu_L + \left(\frac{\delta \nu_B}{\delta \mu^L} + 2\, \nabla \nu_{LB}\right) \wedge R^B + \frac{\delta \nu_{AB}}{\delta \mu^L} \wedge R^A \wedge R^B = 0 \tag{III.3.155}$$

where we have used

$$\delta R^A = \nabla(\delta \mu^A) \tag{III.3.156}$$

so that, by partial integration:

$$\begin{aligned} \nu_A \wedge \delta R^A &= \nabla(\nu_A \wedge \delta\mu^A) + \nabla \nu_A \wedge \delta \mu^A \\ &= d(\nu_A \wedge \delta\mu^A) + \nabla \nu_A \wedge \delta \mu^A . \end{aligned} \tag{III.3.157}$$

From (III.3.155) we see that in order for $R^A = 0$ to be a solution of the field equations we must have

$$\frac{\delta \Lambda}{\delta \mu^A} + \nabla \nu_A = 0 \quad \text{at} \quad R^A = 0 . \tag{III.3.158}$$

One could ask whether principle D is not too restrictive. We may argue as follows. Suppose Eq. (III.3.158) is not verified. This means that $R^A = 0$ is not a solution. Then we have two cases: either there are no solutions at all or there is a solution of the type:

$$R^A = \frac{1}{2} F^A{}_{BC} \mu^B \wedge \mu^C \tag{III.3.159}$$

where $F^A{}_{.BC}$ are constants. Indeed the field equations are algebraic equations for $R^A{}_{.BC}$ (with constant coefficients) and therefore either they have no solution or have constant solutions. Bringing the r.h.s. of Eq. (III.3.159) to the l.h.s. we can say that the solution (III.3.159) is given by $\hat{R}^A = 0$, where $\hat{R}^A$ is the curvature of a new group whose structure constants are $C^A{}_{BC} - F^A{}_{BC}$:

$$\hat{R}^A = d\mu^A + \frac{1}{2} (C^A{}_{BC} - F^A{}_{BC}) \mu^B \wedge \mu^C . \tag{III.3.160}$$

This means that we could just start over again and rewrite our Lagrangian in terms of the curvatures of a $\hat{G}$ algebra and now $\hat{R}^A = 0$ would be a solution. Hence there is no loss of generality in asking that $R^A = 0$ should be a solution.

We observe now that, since one has already imposed rule C, the equations of motion cannot contain terms of zero order in the curvatures R^A. Therefore, by projecting on an intrinsic basis of (D-1)-forms on $\mathbb{K} \oplus \mathbb{H}$, Eqs. (III.3.155) become algebraic equations for the intrinsic components $R^A{}_{BC}$, where

$$R^A = R^A{}_{BC}\, \mu^B \wedge \mu^C \tag{III.3.161}$$

with no 0-order term. Then, according to the splitting (III.3.136-141) of the G superalgebra, the intrinsic components $R^A{}_{BC}$ are separated in the following groups:

$$\{R^A{}_{HH'};\ R^A{}_{HI};\ R^A{}_{HO}\} \equiv R^A{}_{HB} \tag{III.3.162}$$

$$\{R^A{}_{OO'};\ R^A{}_{OI}\} \equiv R^A{}_{OK} \tag{III.3.163}$$

$$R^A{}_{II'} \ . \tag{III.3.164}$$

The first group $R^A{}_{HB}$ is given the name of gauge components, the second group $R^A{}_{OK}$ is given the name of outer components, finally $R^A{}_{II'}$ are called the inner components. Then our fifth building rule can be stated as follows:

E) RHEONOMY (AND HORIZONTALITY)

We assume that the outer components (and gauge components) obey respectively the following equations of motion:

$$R^A{}_{OK} = C^{A|II'}_{OK|B}\ R^B{}_{II'} \tag{III.3.165}$$

$$(R^A{}_{HB} = 0) \ . \tag{III.3.166}$$

We have put the horizontality equations between brackets since they do not appear as field equations when we start with a $\tilde{G}$ which is given "a priori" the fiber bundle structure:

$$\tilde{G} = \tilde{G}(\tilde{G}/H,\ H) \ . \tag{III.3.167}$$

In this case we formulate the theory directly on the superspace $\tilde{G}/H$ and the horizontality conditions are part of the definition of μ^A and R^A on the coset $\tilde{G}/H$. The curvatures R^A therefore have only $\mathbb{K}$-components and Eq. (III.3.165) suffices. If, on the other hand, we choose to start with the fields defined on $\tilde{G}$ then (III.3.166) is required to be a field equation.

This, however, is always satisfied if we have already imposed rule B, namely the H-gauge invariance of the Lagrangian. Indeed it is easy to convince oneself that from a gauge invariant Lagrangian we cannot obtain field equations with bare H-connection μ^H. Hence, while projecting on the (D-1)-basis containing at least one μ^H we always get the horizontality conditions. Thus, when (III.3.166) holds, we can show that the dependence of the μ^A from the gauge parameter η is given by

$$\mu^H(\eta,x,\theta) = [g^{-1}(\eta)dg(\eta)]^H + g^{-1}(\eta)\mu^H(x,\theta)g(\eta) \tag{III.3.168a}$$

$$\mu^K(\eta,x,\theta) = [g^{-1}(\eta)]^K{}_{K'}\ \mu^{K'}(x,\theta) \tag{III.3.168b}$$

and therefore the dependence on the gauge parameters is factorized as it must be on the coset $\tilde{G}/H$ (see also Part I from (I.3.142) to (I.3.148)).

Hence the essential requirement in the rule E is the validity of Eq. (III.3.165) (where the C's are not all zero). We notice that Eq. (III.3.165) is obviously equivalent to a rheonomic parametrization of all the curvatures R^A in the superspace $\tilde{G}/H$, namely, (III.3.165) is equivalent to

$$R^A = R^A{}_{II'}V^I \wedge V^{I'} + C^{A|II'}_{OI''|B}\ R^B{}_{II'}\ V^{I''} \wedge \psi^0 + C^{A|II'}_{OO'|B}\ R^B{}_{II'}\ \psi^0 \wedge \psi^{0'} \tag{III.3.169}$$

where for concreteness we assumed that the inner directions are spanned by the vielbein $\mu^I \equiv V^I$, and that the outer directions are spanned by the gravitino's $\mu^0 \equiv \psi^0$ (0 being a spinor index).

We have already explained the meaning of the rheonomic conditions for the space time interpretation of N=1, D=4 supergravity. The same considerations apply here to the more general set-up we are discussing.

(It is sufficient to substitute $M^{4/4}$ with the general superspace $\tilde{G}/H$ and M^4 with a D-dimensional (bosonic) hypersurface M_D immersed in G/H, conventionally the $\theta^{\bar{\alpha}} = 0$ surface).

The important point of the discussion given in Sect. III.3.3 is that, when rheonomy (and H-horizontality) hold, the Lie derivative

$$\ell_\varepsilon \mu^A = (\nabla\varepsilon)^A + \underline{\varepsilon|} R^A \qquad \text{(III.3.170)}$$

provides a unified description of all the symmetry transformations, i.e. H-gauge transformations, M_D-diffeomorphisms and supersymmetry transformations.

Indeed setting

$$\varepsilon = \varepsilon^H D_H + \varepsilon^0 \tilde{D}_0 + \varepsilon^I \tilde{D}_I \qquad \text{(III.3.171)}$$

the three types of transformations are generated by the D_H, $\tilde{D}_I$ and $\tilde{D}_0$, respectively. Explicitly we obtain

$$\begin{aligned} \ell_\varepsilon \mu^A &= (\nabla\varepsilon)^A + 2\,\varepsilon^B \mu^C R^A{}_{BC} \\ &= (\nabla\varepsilon)^A + 2\,\varepsilon^I \mu^{I'} R^A{}_{II'} + 2\,\varepsilon^{[I''} \mu^{0]} C^{A|II'}_{0I''|B} R^B{}_{II'} \\ &\quad + 2\,\varepsilon^0 \mu^{0'} C^{A|II'}_{00'|B} R^B{}_{II'} \,. \end{aligned} \qquad \text{(III.3.172)}$$

Taking successively $\varepsilon = \varepsilon^H, \varepsilon^I, \varepsilon^0$ we find the H-gauge transformation

$$\ell_{\varepsilon^H} \mu^A = (\nabla\varepsilon^H)^A \qquad \text{(III.3.173)}$$

the M_D-diffeomorphisms

$$\begin{aligned} \ell_{\varepsilon^I} \mu^A &= (\nabla\varepsilon^I)^A + 2\,\varepsilon^I \mu^{I'} R^A{}_{II'} \\ &\quad + 2\,\varepsilon^{I''} \mu^0 C^{A|II'}_{0I''|B} R^B{}_{II'} \end{aligned} \qquad \text{(III.3.174)}$$

and the supersymmetry transformations:

$$\ell_{\varepsilon^0} \mu^A = (\nabla\varepsilon^0)^A + 2\,\varepsilon^0 \mu^I C^{A|II'}_{0I''|B} R^B{}_{II'} + 2\,\varepsilon^0 \mu^{0'} C^{A|II'}_{00'|B} R^B{}_{II'} \,. \qquad \text{(III.3.175)}$$

We stress that the knowledge of the rheonomic parametrization (III.3.169) is equivalent to giving the supersymmetry transformations of all the fields.

We also stress that the supersymmetry transformations (III.3.175) close an algebra only on the on-shell space-time configurations of the fields, namely on the $\mu^A_I{}^{(\text{on-shell})}(x,0)$. Henceforth these transformations are, in general a symmetry only of the equations of motion. Care is needed to promote them to symmetries of the action.

These subtleties have already been discussed in previous sections in the particular case of N=1, D=4 supergravity.

Let us recall the main argument. The Lie derivatives ℓ_ε close an algebra if $d^2 = 0$ or equivalently if $\nabla R^A = 0$. Since the curvatures components are given by the rheonomic parametrization (III.3.169), the Bianchi identities imply differential constraints on the inner components $R^A{}_{II'}$ entering (III.3.169). These differential constraints are, by definition, the space-time equations of motion of the theory.

Indeed the space-time equations derived from the Lagrangian must be consistent with the Bianchi identities and therefore must coincide with the differential constraints implied by the rheonomic parametrization of the curvatures.

Finally, let us comment on the meaning of our building rules.

The reader might have developed the impression that the <u>rules A, B, C, D, E</u>, are like the axioms of a mathematical theory defining the type of structures one likes to consider. If this were the case we would have the possibility of relaxing some of the axioms enlarging in this way our hunting ground. This does not happen here. For instance, the reader has certainly considered the question whether the last rule <u>(rheonomy)</u> can be relaxed and whether superspace theories not interpretable on space-time make sense. They don't. Actually they do not even exist. Indeed no example of a Lagrangian can be produced whose associated field equations leave some of the outer curvature components undetermined. If the Lagrangian is chosen at random then the field equations are simply inconsistent and no solution can be found except the trivial one where everything is zero. If the Lagrangian is consistent, namely if it yields a consistent set of field equations for the curvatures, then it is also the only rheonomic one.

The reason for this is simple. What we are actually trying to obtain is a locally supersymmetric Lagrangian for a multiplet of fields which constitute a representation of supersymmetry.

From the algebraic theory of superalgebra representations, discussed in Chapters II.4 and II.5 we know a priori which theories do exist and with which spectrum of fields. The only problem is to construct the unique supersymmetric Lagrangian. Relying on the geometrical interpretation of the supersymmetry transformations, closely related to the Bianchi identities, rules A, B, C, D, E provide, more than a set of axioms, a description of all the properties characterizing the unique supersymmetric Lagrangian. Hence they furnish a practical way of deriving this latter. Furthermore they naturally lead to a nice formulation of $\mathcal{L}$ in geometrical terms. What must be rejected is the idea that rules A, B, C, D, E could be regarded as independent requirements.

Taken together they are just synonymous to a single word: local supersymmetry. Taken separately they are meaningless.

In particular to complete the picture and avoid a possible source of paradoxes, an example of which will be discussed in Chapter III.7, one should add a sixth building rule which refines those listed above. It is the following:

F) COMPLETENESS OF THE FIELD EQUATIONS

We have seen that the variational equations associated to the Lagrangian give:

a) when enforced on space-time, the field equations of the physical fields

b) when extended to superspace, the rheonomic constraints.

We have also seen that the rheonomic constraints yield, through the Bianchi identities, relations on the physical fields identifiable as field equations. Obviously the two sets of equations of motion, those derived from the space-time restriction of the Lagrangian, and those implied by the rheonomic constraints via the Bianchi identities, should be consistent. This property has been assumed in stating our rheonomy principle. Actually we should be more precise. In fact the word consistency is still too loose as it allows the exceptional case where the space-time field equations are simply a subset of the field equations implied by the rheonomic constraints through the Bianchis. This is a very subtle global inconsistency of the superspace field equations, i.e. of the theory.

Indeed what can happen is that the system of variational equations associated to the Lagrangian has a larger set of solutions in x-space than in superspace. In other words there are configurations which solve the equations in x-space but admit no extension in the θ-directions.

The whole idea underlying rheonomy, namely the smooth extension from space-time to superspace, is disrupted in this case. It follows that the corresponding Lagrangian is not supersymmetric invariant, the non invariance being proportional to the missing field equations.

The problem of finding a Lagrangian for these theories is not hopeless as it might seem. Indeed what one has to do is just to add a convenient set of Lagrangian multipliers which enforce the missing field equations restoring, in this way, the sixth building rule which can be now stated as follows: "The x-space field equations must be complete. They must encompass all the statements on the inner curvatures implied by the rheonomic constraints via the Bianchi identities".

III.3.10 - Retrieving N=1, D=4 supergravity from the building rules

According to the previous discussion, the general form of the N=1, D=4 supergravity Lagrangian is the following:

$$\mathcal{L}^{(D=4,\ N=1)} = \Lambda + R^A \wedge \nu_A + R^A \wedge R^B \wedge \nu_{AB} \qquad \text{(III.3.177)}$$

where

$$\Lambda = C_{A_1A_2A_3A_4}\, \mu^{A_1} \wedge \mu^{A_2} \wedge \mu^{A_3} \wedge \mu^{A_4} \qquad \text{(III.3.178a)}$$

$$\nu_A = C_{AA_1A_2}\, \mu^{A_1} \wedge \mu^{A_2} \qquad \text{(III.3.178b)}$$

$$\nu_{AB} = \text{0-form} \qquad \text{(III.3.178c)}$$

where μ^A and R^A are the $\overline{Osp(4/1)}$-Lie algebra valued soft 1-forms and curvatures, respectively defined by (III.3.4) and (III.3.5). We now proceed exactly as in the pure gravity case (see Sect. I.4.3). First of all rule B, namely SO(1,3)-gauge invariance, implies that the possible quadratic terms entering (III.3.177) are the following ones

i) $R^{ab} \wedge R_{ab}$ (III.3.179a)

ii) $R^{ab} \wedge R^{cd}\, \varepsilon_{abcd}$ (III.3.179b)

iii) $\bar\rho \wedge \rho$ (III.3.179c)

iv) $\bar\rho \wedge \gamma_5\, \rho$ (III.3.179d)

v) $R^a \wedge R_a$. (III.3.179e)

In writing (III.3.179) we assume that the 0-forms ν_{AB} are effectively constants; if we allowed them to be functions on $M^{4/4}$ we would obtain a matter-coupled supergravity theory where the quadratic terms play an essential role. This is in fact what happens for supergravity in presence of higher curvature-terms in the Riemann tensor, relevant for their connection with string theory (see Part VI).

Exactly as in the gravity case (see Eqs. (I.4.63-64)), one shows that i) and ii) are total derivatives since the definition of R^{ab} for $\overline{Osp(4/1)}$ is the same as for ISO(1,3).

It is also easy to see that iii), iv) and v) can be reduced to linear terms in the curvatures by the addition of total derivatives. Actually iii) and iv) can also be eliminated "a priori" by the scaling argument embodied by rule C of the previous section. Let us observe that anyhow the Einstein term must be present in (III.3.177):

$$R^{ab} \wedge V^c \wedge V^d\, \varepsilon_{abcd} \qquad \text{(III.3.180)}$$

According to (III.3.145), it scales as w^2. On the other hand, iii) and iv) scale only as w and therefore we must drop them. (Notice that the same kind of argument eliminates also i) and ii) independently from the fact that they are total derivatives since both of them scale as w^0). Therefore we must only show that v) can be reduced to a linear term. Indeed we have:

$$\begin{aligned} R^a \wedge R_a &= (\mathcal{D} V^a - \tfrac{i}{2}\bar\psi \wedge \gamma^a\psi) \wedge R_a = \\ &= \mathcal{D}(V^a \wedge R_a) + V^a(-R^{ab} \wedge V_b + i\,\bar\psi \wedge \gamma^a\rho) - \\ &\quad - \tfrac{i}{2}\bar\psi \wedge \gamma^a\psi \wedge R_a \end{aligned} \qquad \text{(III.3.181)}$$

where we have used the definition (III.3.5b) and the Bianchi identity (III.3.9b). Equation (III.3.181) proves our statement since $\mathcal{D}(V^a \wedge R_a) \equiv d(V^a \wedge R_a)$.

Let us now consider the linear part of (III.3.177). Using SO(1,3)-gauge invariance the general form of (III.3.178) is the following:

$$\Lambda = \alpha_1 \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d + i\,\alpha_2 \varepsilon_{abcd} \bar\psi \wedge \gamma^{ab}\psi \wedge V^c \wedge V^d$$

$$+ i\,\alpha_3 \bar\psi \wedge \gamma^{ab}\psi \wedge V_a \wedge V_b \qquad \text{(III.3.182a)}$$

$$\nu_{ab} = \beta_1 \varepsilon_{abcd} V^c \wedge V^d + \beta_2 V_a \wedge V_b + i\,\beta_3 \bar\psi \wedge \gamma_{ab}\psi$$

$$+ i\,\beta_4 \varepsilon_{abcd} \bar\psi \wedge \gamma^{cd}\psi \qquad \text{(III.3.182b)}$$

$$\nu_a = i\,\gamma_1 \bar\psi \wedge \gamma_a\psi \qquad \text{(III.3.182c)}$$

$$\nu = \delta_1 \gamma_5 \gamma_a\psi \wedge V^a + i\,\delta_2 \gamma_a\psi \wedge V^a . \qquad \text{(III.3.182d)}$$

(Recall that because of the commuting property of $\psi^t \wedge \psi$ only the currents $\bar\psi \wedge \gamma^a\psi$ and $\bar\psi \wedge \gamma^{ab}\psi$ are different from zero).

Once again the argument of w^2-homogeneous scaling reduces dramatically the number of possible terms. Indeed using (III.3.145) one gets immediately:

$$\alpha_1 = \alpha_2 = \alpha_3 = \beta_3 = \beta_4 = 0 . \qquad \text{(III.3.183)}$$

Moreover the requirement of parity conservation kills the terms which do not transform as a pseudoscalar density like the Einstein term $R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} \equiv 4 R\sqrt{-g}\, d^4x$ (see also (I.4.78-82)).

Therefore we also set

$$\beta_2 = \gamma_1 = \delta_2 = 0 . \qquad \text{(III.3.184)}$$

In conclusion SO(1,3) and parity invariance plus the right scaling behaviour reduces (III.3.177) to the following simple form

$$\mathcal{L}^{D=4,\,N=1} = \beta_1 \varepsilon_{abcd} R^{ab} \wedge V^c \wedge V^d + \delta_1 \bar\psi \wedge \gamma_5 \gamma_m \rho \wedge V^m \qquad \text{(III.3.185)}$$

where only the ratio $\delta_1/\beta_1 \equiv a$ is to be fixed (an overall factor being of course unessential). The requirement that the "vacuum" configuration be a solution of the equations of motion on $M^{4/4}$ (or on $\widetilde{Osp(4/1)}$) fixes this last parameter.

Indeed upon variation of ω^{ab}, V^a and ψ we find:

$$2\mathcal{D}V^c \wedge V^d \varepsilon_{abcd} + \frac{1}{4} a \bar\psi \wedge \gamma_5 \gamma_d \gamma_{ab}\psi \wedge V^d = 0 \qquad \text{(III.3.186a)}$$

$$2 R^{ab} \wedge V^c \varepsilon_{abcd} + a \bar\psi \wedge \gamma_5 \gamma_d \rho = 0 \qquad \text{(III.3.186b)}$$

$$8 \gamma_5 \gamma_a \rho \wedge V^a - a \gamma_5 \gamma_a \psi \wedge R^a = 0 . \qquad \text{(III.3.186c)}$$

The vacuum configuration

$$\rho = R^a = R^{ab} = 0 \qquad \text{(III.3.187)}$$

is obviously a solution of equations (III.3.186b,c). Equation (III.3.186a), on the other hand, is the same as Eq. (III.2.20). In Chapter III.1 we found that (III.2.20) can be rewritten as

$$2 R^c \wedge V^d \varepsilon_{abcd} = 0 \qquad \text{(III.3.188)}$$

if $a = 4$ (see Eq. (III.2.22)). In this case it satisfies the vacuum requirement.

Thus we arrived at the now well-known Lagrangian of Eq. (III.3.51). Notice that we have not yet used the requirement of rheonomy. However it is a rheonomic Lagrangian as it was shown previously. In more complicated cases the requirement of rheonomy has to be imposed in order to determine all the coefficients as we are going to see in the next chapters.

III.3.11 - Extension to anti de Sitter supergravity

As a further application of our building rules we give now the explicit derivation of the anti de Sitter version of N=1 D=4 supergravity.

This construction requires only some minor modifications with respect to the case previously examined. Indeed the only difference lies in the fact that we must start with the Lie algebra of the uncontracted Osp(1/4) group, i.e. the graded extension of the anti de Sitter group, rather than with its contracted version $\overline{Osp(4/1)}$.

The Maurer-Cartan equations defining the Lie algebra of Osp(4/N) were given in Part II, Eqs. (II.3.27). Setting N=1 and letting ω^{ab}, V^a, ψ be the soft 1-forms we define the following Osp(4/1)-curvatures:

$$R^{ab} = \mathcal{R}^{ab} + 4\,\bar{e}^2\, V^a \wedge V^b + \bar{e}\,\bar{\psi} \wedge \gamma^{ab}\psi \qquad \text{(III.3.189a)}$$

$$R^a = \mathcal{D}V^a - \frac{i}{2}\bar{\psi} \wedge \gamma^a\psi \qquad \text{(III.3.189b)}$$

$$\rho = \mathcal{D}\psi - i\,\bar{e}\,\gamma_a\psi \wedge V^a \qquad \text{(III.3.189c)}$$

where, as usual

$$\mathcal{R}^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} \qquad \text{(III.3.190)}$$

and $\bar{e}$ is the rescaling parameter defined in (II.3.26). The Bianchi identities associated with the curvatures (III.3.179) were also given in (II.3.76) for a generic N. In the N=1 case we rewrite them here for completeness

$$\mathcal{D}R^a + R^{ab} \wedge V_b - i\,\bar{\psi} \wedge \gamma^a\rho = 0 \qquad \text{(III.3.191a)}$$

$$\mathcal{D}R^{ab} - 8\,\bar{e}^2\, R^a \wedge V^b + 2\,\bar{e}\,\bar{\psi} \wedge \gamma^{ab}\rho = 0 \qquad \text{(III.3.191b)}$$

$$\mathcal{D}\rho - i\,\bar{e}\,\gamma_a\psi \wedge R^a - \frac{1}{4}R^{ab} \wedge \gamma_{ab}\psi = 0\,. \qquad \text{(III.3.191c)}$$

We can now proceed as in the $\overline{Osp(4/1)}$ case except for one point: in applying the scale arguments we must pay attention to the fact that Eqs. (III.3.189) and (III.3.191) are invariant with respect to the rescalings (III.3.145) provided we also scale the parameter $\bar{e}$ as follows:

$$\bar{e} \to w^{-1}\,\bar{e}\,. \qquad \text{(III.3.192)}$$

Hence in constructing the Lagrangian we can also use positive powers of $\bar{e}$ to get terms with scaling power 2. Negative powers of $\bar{e}$ are forbidden since the Lagrangian must admit a finite limit $\bar{e}\to 0$ which is nothing else but the previous super Poincaré Lagrangian. Keeping this in mind we find again that, since $\bar{e}^n$ lowers the scaling behaviour, all the possible quadratic terms can be omitted except $R^a \wedge R_a$. Moreover, identity (III.3.181) holds true also in this case since the equations (III.3.189b) and (III.3.191a) are the same for both Osp(4/1) and $\overline{Osp(4/1)}$.

Thus we are left again with a linear Lagrangian whose general coefficients are given by Eqs. (III.3.182).

The difference with respect to the $\overline{Osp(4/1)}$ case comes at this point. Since the three terms in the expression for Λ have a too high scaling behaviour we may lower it by multiplication with a suitable positive power of $\bar{e}$.

We do it by the following substitution

$$\alpha_1 \to \bar{e}^2\,\alpha_1' \qquad \text{(III.3.193a)}$$

$$\alpha_2 \to \bar{e}\,\alpha_2' \qquad \text{(III.3.193b)}$$

$$\alpha_3 \to \bar{e}\,\alpha_3' \qquad \text{(III.3.193c)}$$

so that all the terms in Λ do now scale as w^2. The scaling argument now implies only:

$$\beta_3 = \beta_4 = 0 \qquad \text{(III.3.194)}$$

since we cannot increase the scaling powers of the corresponding terms by positive powers of $\bar{e}$. The requirement of parity conservation furthermore gives again

$$\beta_2 = \gamma_1 = \delta_2 = 0; \qquad \alpha_3 = 0 \,. \tag{III.3.195}$$

Therefore we are left with the following action:

$$\mathscr{A}^{D=4,\ N=1}_{\text{de Sitter}} = \int_{M^4 \subset M^{4/4}} \mathscr{L} \tag{III.3.196a}$$

$$\mathscr{L} = \varepsilon_{abcd} R^{ab} \wedge V^c \wedge V^d + \delta_1 \bar{\rho} \wedge \gamma_5 \gamma_a \psi \wedge V^a$$
$$+ \alpha_1' \bar{e}^2 \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d +$$
$$+ i \alpha_2' \bar{e} \varepsilon_{abcd} \bar{\psi} \wedge \gamma^{ab} \psi \wedge V^c \wedge V^d \tag{III.3.196b}$$

where we have set $\beta_1 = 1$.

The three coefficients entering (III.3.196) can be fixed again by the requirement that the "vacuum"

$$R^{ab} = R^a = \rho = 0 \tag{III.3.197}$$

be a solution of the superspace (or $\widetilde{Osp}(4/1)$) field equations derived from (III.3.196). Let us observe, anyhow, that the definition of the "vacuum" in the Osp(4/1)-case is different from the contracted case, $\overline{Osp(4/1)}$, since the definitions of the curvatures are different (compare (III.3.189) and (III.3.5)). Indeed we have seen in Chapter II.3, that an anti de Sitter vacuum corresponds to a Riemannian space with constant curvature given by (II.3.55b).

Let us now vary the fields ω^{ab}, V^a, ψ. We find respectively

$$2 \varepsilon_{abcd} \mathscr{D} V^c \wedge V^d + \frac{1}{4} \delta_1 \bar{\psi} \wedge \gamma_5 \gamma_d \gamma_{ab} \psi \wedge V^d = 0 \tag{III.3.198a}$$

$$2 R^{ab} \wedge \varepsilon_{abcd} V^c + 16 \bar{e}^2 \varepsilon_{abcd} V^a \wedge V^b \wedge V^c +$$
$$+ \delta_1 \bar{\psi} \wedge \gamma_5 \gamma_d \rho + 2 i \bar{e} \delta_1 \bar{\psi} \wedge \gamma_5 \gamma_{ad} \psi \wedge V^a +$$

$$+ 4 \bar{e}^2 \alpha_1' \varepsilon_{abcd} V^a \wedge V^b \wedge V^c +$$
$$+ 2 i \alpha_2' \bar{e} \varepsilon_{abcd} \bar{\psi} \wedge \gamma^{ab} \psi \wedge V^c = 0 \tag{III.3.198b}$$

$$2 \bar{e} \varepsilon_{abcd} \gamma^{ab} \psi \wedge V^c \wedge V^d + 2 \delta_1 \gamma_5 \gamma_a \rho \wedge V^a +$$
$$+ i \bar{e} \delta_1 \gamma_5 \gamma_{ad} \psi \wedge V^a \wedge V^d +$$
$$+ 2 i \alpha_2' \bar{e} \varepsilon_{abcd} \gamma^{ab} \psi \wedge V^c \wedge V^d = 0 \quad . \tag{III.3.198c}$$

Notice that, apart from the terms in α_1' and α_2', there are other contributions in $\bar{e}$ and $\bar{e}^2$ coming from the variation of V^a and ψ inside the definition of the Osp(4/1) curvatures (III.3.189). We have also used (III.3.189b,c) in expressing V^a and ψ in terms of R^a and ρ, and the Fierz identity (III.2.26).

Equation (III.3.189a) is the same as in the $\overline{Osp(4/1)}$ case and therefore implies once more $\delta_1 = 4$.

Imposing the vanishing of the terms which do not contain the curvatures in (III.3.198b,c) and using the γ-matrix relation:

$$\gamma_5 \gamma_{ab} = -\frac{i}{2} \varepsilon_{abcd} \gamma^{cd} \tag{III.3.199}$$

from (III.3.198b) we obtain

$$16 \bar{e}^2 + 4 \bar{e}^2 \alpha_1' = 0 \tag{III.3.200a}$$

$$\delta_1 + 2 i \alpha_2' = 0 \tag{III.3.200b}$$

that is $\alpha_1' = -4$, $\alpha_2' = 2i$. Equation (III.3.198c) yields

$$2 \bar{e} + \frac{1}{2} \bar{e} \delta_1 + 2 i \alpha_2' = 0 \tag{III.3.201}$$

which is identically satisfied by the previous values. Thus we have completely determined the Lagrangian:

$$\mathcal{L} = R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} + 4 \bar{\psi} \wedge \gamma_5 \gamma_a \rho \wedge V^a$$

$$- 4 \bar{e}^2 \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d -$$

$$- 2 \bar{e} \varepsilon_{abcd} \bar{\psi} \wedge \gamma^{ab} \psi \wedge V^c \wedge V^d . \qquad \text{(III.3.202)}$$

We notice that using the above values of the coefficients α'_1, α'_2, the equations (III.3.198) become:

$$2 \varepsilon_{abcd} R^c \wedge V^d = 0 \qquad \text{(III.3.203a)}$$

$$2 \varepsilon_{abcd} R^{ab} \wedge V^c + 4 \bar{\psi} \wedge \gamma_5 \gamma_d \rho = 0 \qquad \text{(III.3.203b)}$$

$$8 \gamma_5 \gamma_a \rho \wedge V^a - 4 \gamma_5 \gamma_a \psi \wedge R^a = 0 \qquad \text{(III.3.203c)}$$

and these are formally identical with the equations of motion (III.3.52) of the contracted case, apart from the different definition of the curvature R^{ab} and ρ.

The discussion of factorization and rheonomic symmetry developed in Sect. III.3.5 applies, therefore, to the Osp(4/1)-theory without changes. In particular from (III.3.203) one derives the rheonomic parametrization of the curvatures:

$$R^a = 0 \qquad \text{(III.3.204a)}$$

$$R^{ab} = R^{ab}{}_{cd} V^c \wedge V^d + \bar{\theta}^{ab}{}_c \psi \wedge V^c \qquad \text{(III.3.204b)}$$

$$\rho = \rho_{mn} V^m \wedge V^n \qquad \text{(III.3.204c)}$$

where the inner components $R^{ab}{}_{cd}$ and ρ_{mn} obey, respectively, the Einstein and Rarita-Schwinger propagation equations

$$\varepsilon^{abcd} \gamma_5 \gamma_b \rho_{cd} = 0 \qquad \text{(III.3.205a)}$$

$$R^{am}{}_{bm} - \frac{1}{2} \delta^a{}_b R^{mn}{}_{mn} = 0 \qquad \text{(III.3.205b)}$$

and the outer component $\theta^{ab}{}_{|c}$ is given by the same rheonomic constraint (III.3.72).

Therefore the on-shell supersymmetry transformations are formally given by Eqs. (III.3.106), the only difference being that now $(\nabla\varepsilon)^A$ represents an Osp(4/1)-covariant derivative. Its explicit form can be read off directly from the Osp(4/1) Bianchi identities, according to the discussion leading to (III.3.12).

We obtain

$$\nabla\varepsilon^{ab} = \mathcal{D}\varepsilon^{ab} - 8 \bar{e}^2 \varepsilon^{[a} V^{b]} + 2 \bar{e} \bar{\psi} \gamma^{ab} \varepsilon \qquad \text{(III.3.206a)}$$

$$\nabla\varepsilon^a = \mathcal{D}\varepsilon^a + \varepsilon^{ab} V_b - i \bar{\psi} \gamma^a \varepsilon \qquad \text{(III.3.206b)}$$

$$\nabla\varepsilon = \mathcal{D}\varepsilon - i \bar{e} \gamma_a \varepsilon V^a - \frac{1}{4} \varepsilon^{ab} \gamma_{ab} \psi . \qquad \text{(III.3.206c)}$$

Restricting ourselves to supersymmetry transformations $(\varepsilon^{ab} = \varepsilon^a = 0)$ and inserting (III.3.206) and (III.3.204) into Eqs. (III.3.103) we find the on-shell first order supersymmetry transformations of anti de Sitter supergravity:

$$\delta\omega^{ab} = 2 \bar{e} \bar{\psi} \gamma^{ab} \varepsilon + 2 \bar{\theta}^{ab}{}_c \varepsilon V^c \qquad \text{(III.3.207a)}$$

$$\delta V^a = i \bar{\varepsilon} \gamma^a \psi \qquad \text{(III.3.207b)}$$

$$\delta\psi = \mathcal{D}\varepsilon - i \bar{e} \gamma_a \varepsilon V^a . \qquad \text{(III.3.207c)}$$

It is easy to verify, in the same way as we did for the $\overline{\text{Osp}(4/1)}$ case, that these transformations are an invariance of the action (III.3.196) (but not a symmetry).

In second order formalism, instead, the $\delta\omega^{ab}$ variation is given by the chain rule and it is different from (III.3.207a), the difference being proportional to the l.h.s. of the gravitino space-time equation, in complete analogy to what happens in the $\overline{\text{Osp}(4/1)}$ case (see (III.3.129)). We leave these verifications to the reader.

III.3.12 - Building rules for supergravity theories using rheonomy and Bianchi identities

In previous sections we saw that the rheonomic constraints, so far regarded as an yield of the variational equations, enforce, once inserted into the Bianchi identities, the propagation equations of the physical fields.

This suggests another line of approach for the construction of a supergravity theory.

One can forget about the Lagrangian and assume the rheonomy principle as the fundamental starting point of the classical supergravity theory.

The space-time equations of the fields will then be obtained as a result of the integrability of the rheonomic parametrizations through the use of Bianchi identities.

To be explicit, one assumes the rheonomic principle in the usual form, that is:

a) the outer components of the curvature 2-forms in superspace are to be expressed in terms of the inner components, that is of the field-strengths of the physical fields, according to equations (III.3.165). If one further assumes the rules:

b) Lorentz gauge invariance of the R^A rheonomic parametrization.

c) Homogeneous scaling of all the terms involved in order not to violate the fundamental properties of Eqs. (III.3.134-135).

Then the structure of the superspace curvature is completely determined, except possibly for a few constant coefficients. The Bianchi identities fix the coefficients and moreover give also the space-time differential integrability constraints on the inner curvatures, to be identified with the space-time equations of motion of the physical theory. The rheonomic parametrization itself fixes, on the other hand, the on-shell supersymmetry transformations of the theory, as it has been remarked in the comment following (III.3.107). In this way the implementation of the rheonomy principle a), plus Lorentz gauge invariance b) and the right scaling behaviour c) into the Bianchi identities gives an equivalent description of the on-shell theory as it would be derived from the action principle.

To see how this works in a particular simple example let us rederive once again $\overline{Osp(4/1)}$ supergravity using a), b) and c).

Let us start with the structure equations and the Bianchi identities of the $\overline{Osp(4/1)}$ theory, namely Eqs. (III.3.5) and (III.3.9).

We write down the most general parametrization of the curvature in superspace which is rheonomic, Lorentz covariant and scale invariant. For simplicity we also add the kinematical constraint:

$$R^a = 0 . \tag{III.3.208}$$

As we have already pointed out $R^a{}_{mn} = 0$ is a kinematical constraint which allows the transition from first to second order, by expressing the non propagating field ω^{ab}_μ in terms of the physical field V^a_μ and ψ_μ. Moreover the absence of outer components in R^a, $\varepsilon\rfloor R^a = 0$, implies that the supersymmetry transformation law of the vielbein field, given by the general formula (III.3.21b), reduces to the standard form:

$$\delta_\varepsilon V^a = (\nabla\varepsilon)^a = i\,\bar\varepsilon\,\gamma^a\psi \tag{III.3.209}$$

(that is, identical with a Q-gauge transformation).

Therefore it suffices to consider the general parametrization of R^{ab} and ρ. Let us use the general superspace ansatz given by Eqs. (III.3.60):

$$R^{ab} = R^{ab}{}_{cd}\, V^c \wedge V^d + \bar\theta^{ab}{}_c\psi \wedge V^c + \bar\psi \wedge K^{ab}\psi \tag{III.3.210a}$$

$$\rho = \rho_{ab}\, V^a \wedge V^b + H_c\psi \wedge V^c + \Omega_{\alpha\beta}\,\psi^\alpha \wedge \psi^\beta . \tag{III.3.210b}$$

The rheonomy principle applied to (III.3.210) implies that $\theta^{ab|c}$, K^{ab}, H_c, $\Omega_{\alpha\beta}$ should be constructed in terms of the inner components

$$R^{ab}{}_{mn};\ \rho_{mn};\ R^a{}_{mn} \equiv 0\ , \qquad \text{(III.3.211)}$$

in a Lorentz covariant way. Moreover the scale invariance of (III.3.210) under (III.3.145) implies the following requirements:

$$H_c \to w^{-1}\ H_c \qquad \text{(III.3.212a)}$$

$$\Omega_{\alpha\beta} \to w^{-1/2}\ \Omega_{\alpha\beta} \qquad \text{(III.3.212b)}$$

$$K_{ab} \to w^{-1}\ K_{ab} \qquad \text{(III.3.212c)}$$

$$\theta^{ab}{}_c \to w^{-3/2}\ \theta^{ab}{}_c\ . \qquad \text{(III.3.212d)}$$

From the scale properties of R^{ab}, ρ (III.3.145a,c) we also find

$$R^{ab}{}_{cd} \to w^{-2}\ R^{ab}{}_{cd} \qquad \text{(III.3.213a)}$$

$$R^a{}_{cd} \to w^{-1}\ R^a{}_{cd} \qquad \text{(III.3.213b)}$$

$$\rho_{cd} \to w^{-3/2}\ \rho_{cd} \qquad \text{(III.3.213c)}$$

since V^a scales as w^1.

It is now easy to see that the γ-matrix-valued Lorentz tensors H_c and K_{ab} could satisfy (III.3.212a,c) only if they were constructed in terms of $R^a{}_{mn}$, which however has been set equal to zero. We have therefore:

$$H_c = K_{ab} = 0\ . \qquad \text{(III.3.214)}$$

The same conclusion also holds for the spinor valued matrix $\Omega_{\alpha\beta}$ which should be expressed in terms of ρ_{ab}: again we cannot match the scale behaviour. Hence

$$\Omega_{\alpha\beta} = 0 \qquad \text{(III.3.215)}$$

$\theta^{ab|c}$, however, can be expressed in terms of the ρ_{mn} components because their scale powers coincide. By taking the ψVV projection of the torsion-Bianchi (III.3.9b) with $R^a = 0$ one obtains:

$$\bar{\theta}^{ab}{}_c \psi \wedge V_b \wedge V^c + i\ \bar{\psi} \wedge \gamma^a\ \rho_{bc}\ V^b \wedge V^c = 0 \qquad \text{(III.3.216)}$$

or

$$\frac{1}{2}\ (\bar{\theta}_{ab|c} - \bar{\theta}_{ac|b})\psi = +\ i\ \bar{\rho}_{bc}\ \gamma_a \psi\ . \qquad \text{(III.3.217)}$$

This equation has the same Lorentz content as (I.2.40) and can be solved in the same way; one finds:

$$\bar{\theta}_{ab|c} = 2\ i\ \bar{\rho}_{c[a}\ \gamma_{b]} - i\ \bar{\rho}_{ab}\ \gamma_c\ . \qquad \text{(III.3.218)}$$

Thus we have fixed the superspace curvatures as follows:

$$R^{ab} = R^{ab}{}_{cd}\ V^c \wedge V^d + (2\ i\ \bar{\rho}_{c[a}\ \gamma_{b]} - i\ \bar{\rho}_{ab}\gamma_c)\psi \wedge V^c \qquad \text{(III.3.219a)}$$

$$R^a = 0 \qquad \text{(III.3.219b)}$$

$$\rho = \rho_{ab}\ V^a \wedge V^b\ . \qquad \text{(III.3.219c)}$$

At this point we call the reader's attention on the fact that the parametrization (III.3.219a) is different from that found by solving the equations of motion in superspace (Eqs. (III.3.72)). Indeed the $\psi \wedge V$ component of R^{ab} is different in (III.3.219a) from that given in (III.3.74a). This fact might seem to contradict our statement that the Lagrangian approach gives the same result as the rheonomy principle inserted in the Bianchi identities.

However we observe that the two parametrizations (III.3.74a) and (III.3.219a) differ only by the l.h.s. of the gravitino space time equation. Indeed the two expressions (III.3.72) and (III.3.218) just determine two different supersymmetry transformation laws for $\delta\omega^{ab}$ (III.3.74a) determines through (III.3.107a) the rule which, as we have seen, is the on-shell first-order supersymmetry transformation leaving the action invariant; (III.3.219a), instead, gives rise through (III.3.107a) to a $\delta\omega^{ab}$ supersymmetry variation which coincides with (III.3.127) and therefore is the on-shell supersymmetry transformation leaving the action invariant in 2nd order formalism. The two laws have been shown to differ by the l.h.s. of gravitino equation in (III.3.129). They are therefore both viable as supersymmetry transformations of the equations of motion. Hence the two different parametrizations (III.3.74a) and (III.3.219a) differ only by the l.h.s. of a space-time equation of motion and are therefore on-shell equivalent.

Actually the fact that the Bianchi identities imply the equations of motion was deduced in Sect. III.6 using both the rheonomic constraints given by the variational principle and the Bianchi identities. If we use only the Bianchi identities then the derivation is somewhat different; in particular the Einstein equation, which was deduced in Sect. III.6 from $\mathcal{D}R^{ab}=0$ cannot be obtained in the same way since, using the new parametrization (III.3.219a) we obtain in the $\psi\psi V$ sector only the cyclic identity on the Riemann tensor.

As we presently show the shortest way to reobtain the bosonic equation for the Riemann tensor is to recall that the gravitino and Einstein equation must transform into each other under a supersymmetry transformation (or a Lie derivative ℓ_ε in the $\tilde{D}_\alpha$ direction). Equivalently we may differentiate the gravitino equation and take its content along the ψ-1 form. Let us see how this works.

First we deduce the gravitino equation from the Bianchi identities and the parametrizations (III.3.219). In Sect. III.6 it was deduced from the ψVV sector of the torsion-Bianchi. In absence of the rheonomic constraints given by the variational equations, this does not work in the present case; indeed the use of (III.3.219) in the same sector, ψVV, of the Torsion Bianchi would just give an identity. Instead we take the $\psi\psi V$ sector of the gravitino-Bianchi (III.3.9c); using Eqs. (III.3.219) one finds

$$\rho_{ab}\,\bar{\psi}\wedge\gamma^a\psi\wedge V^b+\frac{1}{4}\gamma^{ab}\psi\wedge\bar{\psi}(\gamma_c\,\rho_{ab}-2\,\gamma_a\,\rho_{bc})\wedge V^c=0\,. \qquad \text{(III.3.220)}$$

Using the Fierz decomposition:

$$\psi\wedge\bar{\psi}=\frac{1}{4}\gamma^a\psi\wedge\gamma_a\psi-\frac{1}{8}\gamma^{ab}\psi\wedge\gamma_{ab}\psi \qquad \text{(III.3.221)}$$

after some γ-matrix algebra one gets two equations for the coefficients of $\psi\wedge\gamma_a\psi\wedge V^b$ and $\psi\wedge\gamma_{pq}\psi\wedge V^c$ respectively:

$$\frac{1}{8}\rho_{ab}+\frac{i}{16}\gamma_5\,\varepsilon_{ab}{}^{cd}\,\rho_{cd}+\frac{1}{16}\delta_{ab}\,\gamma^{pq}\,\rho_{pq}-\frac{1}{4}\gamma_{[a}\,\gamma^p\,\rho_{b]q}=0 \qquad \text{(III.3.222)}$$

$$-\gamma_{pq}\,\gamma^a\,\rho_{ac}-4\,\gamma^b{}_{[q}\,\gamma^c\,\rho_{p]b}=0\,. \qquad \text{(III.3.223)}$$

Using Eqs. (III.3.81), (III.3.100a-b) one easily recognizes that the l.h.s. of Eq. (III.3.222,223) are proportional to the l.h.s. of the gravitino field equation E_p; therefore we find:

$$E_p\overset{\text{def.}}{=}\gamma_5\,\gamma_r\,\varepsilon_p{}^{rst}\,\rho_{st}=0\,. \qquad \text{(III.3.224)}$$

To retrieve the Einstein equation we could make a Lie derivative of Eq. (III.3.224); it is equivalent and more convenient, however, to rewrite Eq. (III.3.224) as a 3-form in superspace, that is to use Eq. (III.3.52c) (at $R^a=0$) and to differentiate it; we obtain:

$$\mathcal{D}(\gamma_5\,\gamma_m\rho\wedge V^m)=\gamma_5\,\gamma_m\mathcal{D}\rho\wedge V^m+\frac{i}{2}\gamma_5\,\gamma_m\rho\wedge\bar{\psi}\wedge\gamma^m\psi=0\,. \qquad \text{(III.3.225)}$$

Using the gravitino-Bianchi, Eq. (III.3.225) becomes:

$$\gamma_5\,\gamma_m(-\frac{1}{4}R^{ab}\,\gamma_{ab}\wedge\psi\wedge V^m+\frac{i}{2}\rho\wedge\bar{\psi}\wedge\gamma^m\psi)=0\,. \qquad \text{(III.3.226)}$$

In the $VVV\psi$-sector we find

$$\gamma_m \gamma_{ab}\psi \wedge R^{ab}{}_{cd} V^c \wedge V^d \wedge V^m = 0 \qquad \text{(III.3.227)}$$

that is

$$i \gamma_5 \gamma^t \psi \, \varepsilon_{mabt} R^{ab}{}_{cd} V^c \wedge V^d \wedge V^m +$$
$$+ 2 \gamma_b \psi \wedge R^{mb}{}_{cd} V^c \wedge V^d \wedge V^m = 0 \, . \qquad \text{(III.3.228)}$$

The second term is zero, due to the cyclic identity, which is valid at $R^a = 0$; the first term, using $V^c \wedge V^d \wedge V^m \equiv \varepsilon^{cdmf} \Omega_f$ gives the Einstein equation

$$R^{ma}{}_{mb} - \frac{1}{2} \delta^a{}_b R^{mn}{}_{mn} = 0 \qquad \text{(III.3.229)}$$

as promised.

Therefore we can conclude that the rheonomy principle a) implemented in the Bianchi identities (plus b) and c)) has the same on-shell content as Eqs. (III.3.52) derived from the extended action obtained via the A, B, C, D, E rules of Sect. III.2.9.

This has been explicitly verified in the case of N=1, D=4 supergravity, but it is true in general as one can easily guess and as will be explicitly shown in more complicated theories.

Thus we have two alternative ways for constructing a classical supergravity theory: either by using the extended action principle and using the building rules A-E of Sect. III.3.9; or by using the Bianchi identities and the principles a), b), c) of this section. In both cases the use of the rheonomy principle is essential.

Actually what proves to be the most convenient practice in constructing more general theories than the one examined so far is the combined use of the Lagrangian approach and of the Bianchi identities. The doubling of information gained in this way eliminates a lot of labour which would be required relying on either of the two approaches alone. In the following we shall see examples of these parallel constructions.

In Table III.3.I we give a resumé of D=4, N=1 anti de Sitter (Osp(4/1)) supergravity. The case of Poincaré $(\overline{\text{Osp}(4/1)})$ supergravity is obtained by simply setting in all the formulae $\bar{e} = 0$.

TABLE III.3.I

Summary of D=4 N=1 De Sitter and Poincaré supergravity

De Sitter supergravity: $\bar{e} \neq 0$. Poincaré supergravity $\bar{e} = 0$.

A) The Osp(4/1) curvatures

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} + 4 \bar{e}^2 V^a \wedge V^b + \bar{e} \bar{\psi} \wedge \gamma^{ab} \psi$$

$$R^a = \mathscr{D} V^a - \frac{i}{2} \bar{\psi} \wedge \gamma^a \psi$$

$$\rho = \mathscr{D} \psi - i \bar{e} \gamma_a \psi \wedge V^a \, .$$

B) The Osp(4/1) Bianchi identities

$$\mathscr{D} R^{ab} - 8 \bar{e}^2 R^{[a} \wedge V^{b]} + 2 \bar{e} \bar{\psi} \wedge \gamma^{ab} \rho = 0$$

$$\mathscr{D} R^a + R^{ab} \wedge V_b - i \bar{\psi} \wedge \gamma^a \rho = 0$$

$$\mathscr{D} \rho - i \bar{e} \gamma_a \psi \wedge R^a - \frac{1}{4} \gamma_{ab} \psi \wedge R^{ab} = 0 \, .$$

C) The Action

$$\mathscr{A} = \int_{M^4 \subset M^{4/4}} \mathscr{L}$$

where:

$$M^{4/4} = \widetilde{\text{Osp}(4/1)}/SO(1,3)$$

$$\mathscr{L} = R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} + 4 \bar{\psi} \wedge \gamma_5 \gamma_a \rho \wedge V^a -$$
$$- 4 \bar{e}^2 \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d -$$
$$- 2 \bar{e} \varepsilon_{abcd} \bar{\psi} \wedge \gamma^{ab} \psi \wedge V^c \wedge V^d \, .$$

....

TABLE III.3.I continued

D) The Field-Equations in Superspace

$$2\,\varepsilon_{abcd}\, R^{ab} \wedge V^c + 4\,\bar\psi \wedge \gamma_5\, \gamma_d \rho = 0$$

$$2\,\varepsilon_{abcd}\, R^c \wedge V^d = 0$$

$$8\,\gamma_5\,\gamma_a \rho \wedge V^a - 4\,\gamma_5\,\gamma_a \psi \wedge R^a = 0\,.$$

E) The Rheonomic Parametrization of the Curvatures

$$R^a = 0$$

$$R^{ab} = R^{ab}{}_{mn}\, V^m \wedge V^n + \bar\theta^{ab}{}_c \psi \wedge V^c$$

$$\rho = \rho_{mn}\, V^m \wedge V^n$$

where: determination of θ^{ab}_c from the field equations gives

$$\bar\theta^{ab}_c \equiv \overset{(I)}{\bar\theta^{ab}_c} = -\,\varepsilon^{abrs}\,\bar\rho_{rs}\,\gamma_5\,\gamma_c - \delta^{[a}_c\,\varepsilon^{b]mst}\,\bar\rho_{st}\,\gamma_5\,\gamma_m$$

determination of θ^{ab}_c from Bianchi identities gives

$$\bar\theta^{ab}_c \equiv \overset{(II)}{\bar\theta^{ab}_c} = 2\,i\,\bar\rho_{c.}{}^{[a}\,\gamma^{b]} - i\,\bar\rho^{ab}\,\gamma_c\,.$$

F) Inner Field Equations

$$R^{ab}{}_{bm} - \frac{1}{2}\,\delta^a{}_b\, R^{mn}{}_{mn} = 0$$

$$R^a{}_{mn} = 0$$

$$\varepsilon^{rsmn}\,\gamma_5\,\gamma_s\,\rho_{mn} = 0\,.$$

....

TABLE III.3.I continued

G) Supersymmetry Transformation Laws

$$\delta_\varepsilon\, \omega^{ab} = -\,2\,\bar\varepsilon\,\bar\psi\,\gamma^{ab}\varepsilon + 2\,\bar\theta^{ab}_c\,\varepsilon\, V^c$$

$$\delta_\varepsilon\, V^a = i\,\bar\varepsilon\,\gamma^a \psi$$

$$\delta_\varepsilon\, \psi = \mathcal{D}\varepsilon - i\,\bar\varepsilon\,\gamma_a\,\varepsilon\, V^a\,.$$

They are

i) Symmetry (= closed algebra) of the inner equations F)

ii) Invariance (on-shell closed, off-shell open algebra) of the action C) in first order formalism if $\bar\theta^{ab}_c = \overset{(I)}{\bar\theta^{ab}_c}$, in second-order formalism if $\bar\theta^{ab}_c \equiv \overset{(II)}{\bar\theta^{ab}_c}$.