Writing arbitrary coefficients for all the terms which are permissible we work out the field equations and we impose that the rheonomic conditions should come out as solution of the outer projections. This fixes all the coefficients and the action (II.6.96) is uniquely singled out.

All the mechanisms, concepts and techniques discussed in this chapter will be essential for the development of supergravity theory in Part Three.

CHAPTER II.7

Γ-MATRIX ALGEBRA AND SPINORS IN $4 \leq D \leq 11$

II.7.1 - The construction of Γ-matrices

In order to describe spinor fields and hence supersymmetric theories one needs the Dirac gamma matrices. These form the Clifford algebra

$$\{\Gamma_a, \Gamma_b\} = 2\,\eta_{ab} \qquad \text{(II.7.1)}$$

where η_{ab} is the invariant metric of the D-dimensional Lorentz group SO(1,D-1):

$$\eta_{ab} = \mathrm{diag}(+, -, -, \ldots, -) \qquad \text{(II.7.2)}$$

To study the general properties of the Clifford algebra (II.7.1) one can use group-theoretical techniques: we prefer a very pedestrian approach based on the direct construction of the gamma matrices.

We begin by fixing our conventions. The matrix $\Gamma_0=\Gamma^0$ corresponding to the plus sign in the signature (II.7.2) is hermitean:

$$\Gamma_0^{\dagger} = \Gamma_0 \tag{II.7.3}$$

the matrices $\Gamma_i=-\Gamma^i$ $(i=1,2,\dots,D-1)$ corresponding to the minus signs in the signature (II.7.3) are antihermitean:

$$\Gamma_i^{\dagger} = - \Gamma \tag{II.7.4}$$

We subdivide the range of dimensions in the even and odd sector

<u>D = 2ν= even</u>

In this case the representation of the Clifford algebra has dimension

$$\dim \Gamma_a = 2^{\frac{D}{2}} = 2^{\nu} \tag{II.7.5}$$

In other words the gamma $\{\Gamma_a\}$ are $2^\nu \times 2^\nu$ matrices.

The proof is easily obtained by iteration. Suppose that we have the gamma matrices γ_a corresponding to the case $\nu' = \nu - 1$

$$\{\gamma_{a'}, \gamma_{b'}\} = 2\,\eta_{a'b'}, \quad (a' = 0, 1, \dots, D-3) \tag{II.7.6}$$

and that they are $2^{\nu'}$ dimensional. We write down the following $2^{\nu'+1}$-dimensional matrices

$$\Gamma_{a'} = \left(\begin{array}{c|c} 0 & \gamma_{a'} \\ \hline \gamma_{a'} & 0 \end{array}\right) \quad ; \quad \Gamma_{D-2} = \left(\begin{array}{c|c} i & 0 \\ \hline 0 & -i \end{array}\right)$$

$$\Gamma_{D-1} = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right) \tag{II.7.7}$$

and we verify that they satisfy the Clifford algebra (II.7.1); furthermore they have the correct hermiticity properties:

$$\Gamma_{D-2}^{\dagger} = - \Gamma_{D-2} \quad ; \quad \Gamma_{D-1}^{\dagger} = - \Gamma_{D-1} \tag{II.7.8}$$

The matrices (II.7.7) can be interpreted as the following tensor product of the γ_a -matrices with the Pauli sigma-matrices:

$$\Gamma_{a'} = \gamma_{a'} \otimes \sigma_1 \quad ; \quad \Gamma_{D-2} = \mathbb{1} \otimes i\sigma_3 \quad ; \quad \Gamma_{D-1} = \mathbb{1} \otimes i\sigma_2 \tag{II.7.9}$$

To complete the proof of our statement we just have to show that for $\nu=2$, corresponding to $D=4$, we have a 4-dimensional representation of the gamma matrices. This is a well-known result; for example one can use the representation:

$$\gamma_0 = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right) \quad ; \quad \gamma_{1,2,3} = \left(\begin{array}{c|c} 0 & \sigma_{1,2,3} \\ \hline -\sigma_{1,2,3} & 0 \end{array}\right) \tag{II.7.10}$$

In D=2ν one can construct the matrix

$$\Gamma_{D+1} = \alpha_D\, \Gamma_0 \Gamma_1 \Gamma_2, \dots, \Gamma_{D-1} \quad ; \quad |\alpha_D|^2 = 1 \tag{II.7.11}$$

where α_D is a normalization factor to be fixed in such a way that

$$\Gamma^2_{D+1} = \mathbb{1} \qquad \text{(II.7.12)}$$

By direct evaluation one can verify that

$$\{\Gamma_a, \Gamma_{D+1}\} = 0 \qquad \text{(II.7.13)}$$

namely Γ_{D+1} is the generalization of the γ_5-matrix of 4-dimensions.

The normalization α_D is easily derived. We have

$$\Gamma_0\Gamma_1,\ldots,\Gamma_{D-1} = (-)^{\frac{1}{2}D(D-1)}\Gamma_{D-1}\Gamma_{D-2},\ldots,\Gamma_1\Gamma_0 \qquad \text{(II.7.14)}$$

and therefore, imposing Eq. (II.7.12) we find:

$$\alpha_D^2(-)^{\frac{1}{2}D(D-1)}(-)^{D-1} = 1 \qquad \text{(II.7.15)}$$

This implies

$$\alpha_D = 1 \quad \text{if} \quad \nu = 2\mu+1 = \text{odd}$$

$$\alpha_D = i \quad \text{if} \quad \nu = 2\mu = \text{even} \qquad \text{(II.7.16)}$$

With the same token we can show that Γ_{D+1} is hermitean. Indeed

$$\Gamma^\dagger_{D+1} = \alpha^*_D\,(-)^{\frac{1}{2}D(D-1)}(-)^{D-1}\Gamma_0\Gamma_1,\ldots,\Gamma_{D-1} = \Gamma_{D+1} \qquad \text{(II.7.17)}$$

$D = 2\nu+1$ = odd

In this case the Clifford algebra (II.7.1) is represented by $2^\nu \times 2^\nu$ matrices. It suffices to take the gamma matrices Γ_a, corresponding to the even case $D' = D-1$ and add to them the matrix $i\Gamma_{D'+1} = \Gamma_{D-1}$ which is antihermitean and anticommutes with all the other ones.

II.7.2 - The charge conjugation matrix

Since Γ_a and their transposed Γ_a^T satisfy the same Clifford algebra it follows that there must be a similarity transformation connecting these two representations of the same algebra. Such statement relies on Schur's lemma and it is proved in the following way. Introducing the notation

$$\Gamma_{a_1,\ldots,a_n} = \Gamma_{[a_1}\Gamma_{a_2},\ldots,\Gamma_{a_n]} = \frac{1}{n!}\sum_P (-)^{\delta_P}\Gamma_{a_{P(1)}},\ldots,\Gamma_{a_{P(n)}} \qquad \text{(II.7.18)}$$

where Σ_P denotes the sum over permutations, we can easily convince ourselves that the union $\mathbb{1}$, Γ_{a_1}, $\Gamma_{a_1a_2}$,..., $\Gamma_{a_1..a_D}$ constitutes a finite group of $2^{[D/2]}$-dimensional matrices.

Furthermore the groups generated by Γ_a, $-\Gamma_a$ or Γ_a^T are the same. Hence by Schur's lemma two irreducible representations of the same group, with the same dimension and defined over the same vector space, must be equivalent, that is there must be a similarity transformation which connects the two. The matrix realizing such a similarity is called the charge conjugation matrix. Instructed by this discussion we define the charge conjugation matrix by the following equations

$$C_{(-)} \Gamma_a C^{-1}_{(-)} = -\Gamma_a^T \qquad \text{(II.7.19a)}$$

$$C_{(+)} \Gamma_a C^{-1}_{(+)} = \Gamma_a^T \qquad \text{(II.7.19b)}$$

$C_{(-)}$ connects the representation generated by Γ_a to that generated by $-\Gamma_a^T$, while $C_{(+)}$ relates the Γ_a and Γ_a^T representations. In even dimensions both $C_{(-)}$ and $C_{(+)}$ exist, while in odd dimensions only one is possible. Indeed in odd dimensions Γ_{D-1} is proportional to $\Gamma_0\Gamma_1 \ldots \Gamma_{D-2}$ so that the $C_{(-)}$ and $C_{(+)}$ of D-1 dimensions yield the same result for Γ_{D-1}, namely either $+\Gamma_{D-1}$ or $-\Gamma_{D-1}$. This decides which C exists in a given odd dimension.

Another important property of the charge conjugation matrix follows by iterating (II.7.19). Using Schur's lemma one finds that $C_{(+)} = \alpha C_{(-)}^T$ (idem for $C_{(-)}$) so that iterating again $\alpha^2=1$. In other words $C_{(+)}$ and $C_{(-)}$ are either symmetric or antisymmetric.

It is very important to decide which is the case in every dimension.

We do not dwell on the derivation which can be based either on general arguments or on an explicit construction in a gamma matrix basis. We simply collect the results in Table II.7.I.

TABLE II.7.I

CHARGE CONJUGATION MATRICES IN $4 \leq D \leq 11$

D	$C^*_{(+)} = C_{(+)}$ (real)	$C^*_{(-)} = C_{(-)}$ (real)
4	$C_{(+)}^T = -C_{(+)}$; $C_{(+)}^2 = -1$	$C_{(-)}^T = -C_{(-)}$; $C_{(-)}^2 = -1$
5	$C_{(+)}^T = -C_{(+)}$; $C_{(+)}^2 = -1$	
6	$C_{(+)}^T = -C_{(+)}$; $C_{(+)}^2 = -1$	$C_{(-)}^T = C_{(-)}$; $C_{(-)}^2 = 1$
7		$C_{(-)}^T = C_{(-)}$; $C_{(-)}^2 = 1$
8	$C_{(+)}^T = C_{(+)}$; $C_{(+)}^2 = 1$	$C_{(-)}^T = C_{(-)}$; $C_{(-)}^2 = 1$
9	$C_{(+)}^T = C_{(+)}$; $C_{(+)}^2 = 1$	
10	$C_{(+)}^T = C_{(+)}$; $C_{(+)}^2 = 1$	$C_{(-)}^T = C_{(-)}$; $C_{(-)}^2 = -1$
11		$C_{(-)}^T = C_{(-)}$; $C_{(-)}^2 = -1$

II.7.3 - Majorana, Weyl and Majorana-Weyl spinors

The Dirac conjugate of a spinor ψ is defined by

$$\bar{\psi} = \psi^{\dagger}\Gamma_0 \qquad (II.7.20)$$

and the charge conjugate of ψ is given by:

$$\psi^c = C\,\bar{\psi}^T \qquad (II.7.21)$$

where C is the charge conjugation matrix. When we have the option in (II.7.21) we can use either $C_{(+)}$ or $C_{(-)}$. By definition a Majorana spinor λ satisfies the following condition

$$\lambda = \lambda^c = C\,\bar{\lambda}^T = C\,\Gamma_0^T\,\lambda^* \qquad (II.7.22)$$

which means that λ is its own conjugate. Eq. (II.7.22) is not always self consistent. Indeed by iterating it a second time

$$\lambda = C\,\bar{\lambda}^{cT} = C\,\Gamma_0^T\,C^*\,\Gamma_0^{\dagger}\,\lambda = C\,\Gamma_0^T\,C^*\,\Gamma_0\,\lambda \qquad (II.7.23)$$

we get the consistency condition

$$C\,\Gamma_0^T\,C = \Gamma_0 \qquad (II.7.24)$$

in which we have used the reality of $C(C^* = C)$.

It can be shown that there are two possible solutions to equation (II.7.24): either $C_{(-)}$ is antisymmetric, or $C_{(+)}$ is symmetric. Hence looking at Table II.7.I we conclude that Majorana spinors exist only in

$$D = 4,\ 8,\ 9,\ 10,\ 11 \qquad (II.7.25)$$

In D = 4,10,11 they are defined using $C_{(-)}$, which is antisymmetric while in D = 8,9 they are defined using $C_{(+)}$ which is symmetric.

Majorana spinors do not exist in:

$$D = 5,\ 6,\ 7 \qquad (II.7.26)$$

Weyl spinors, on the contrary exist in every even dimension; by definition they are eigenstates of the Γ_{D+1}-matrix, corresponding to the +1 or -1 eigenvalue:

$$\Gamma_{D+1}\,\psi_{\binom{L}{R}} = \pm\,\psi_{\binom{L}{R}} \qquad (II.7.27)$$

As a matter of convention the spinors belonging to the positive eigenvalue are named "left-handed", while those belonging to the negative one are called "right-handed".

In some special dimensions we can define Majorana-Weyl spinors which are both eigenstates of Γ_{D+1} and satisfy Eq. (II.7.22). In order for this to be possible we must have

$$C\,\Gamma_0^T\,\Gamma_{D+1}^*\,\psi^* = \Gamma_{D+1}\,\psi \qquad (II.7.28)$$

Using (II.7.24) equation (II.7.28) becomes

$$C\,\Gamma_0^T\,\Gamma_{D+1}^*\,\Gamma_0^T\,C^{-1} = \Gamma_{D+1} \qquad (II.7.29)$$

Since Γ_{D+1} is hermitean this relation can also be written as

$$C^{-1}\,\Gamma_{D+1}\,C = \Gamma_0^T\,\Gamma_{D+1}^T\,\Gamma_0^T = -\,\Gamma_{D+1}^T \tag{II.7.30}$$

Recalling that Γ_{D+1} is defined by equation (II.7.11), we can verify in which dimensions this relation holds.

If $C = C_{(+)}$ we have

$$C^{-1}\,\Gamma_{D+1}\,C = \alpha_D\,\Gamma_0^T\,\Gamma_1^T,\ldots,\Gamma_{D-1}^T = (-)^{\frac{1}{2}D(D-1)}\,\Gamma_{D+1}^T \tag{II.7.31}$$

while if $C = C_{(-)}$ we find

$$C^{-1}\,\Gamma_{D+1}\,C = (-)^{\frac{1}{2}D(D-1)+D}\,\Gamma_{D+1}^T \tag{II.7.32}$$

Hence we get

$$\begin{aligned} d = 4 &\qquad C^{-1}\,\Gamma_5\,C = \Gamma_5^T \\ d = 8 &\qquad C^{-1}\,\Gamma_9\,C = \Gamma_9^T \\ d = 10 &\qquad C^{-1}\,\Gamma_{11}\,C = -\,\Gamma_{11}^T \end{aligned} \tag{II.7.33}$$

and we see that in the range $4 \le D \le 11$ the only dimension for which Majorana-Weyl spinors can be defined is D=10.

The results are summarized in Table II.7.II.

TABLE II.7.II

SPINORS IN $4 \le D \le 11$

D	Dirac	Majorana	Weyl	Majorana-Weyl
4	Yes	Yes	Yes	No
5	Yes	No	No	No
6	Yes	No	Yes	No
7	Yes	No	No	No
8	Yes	Yes	Yes	No
9	Yes	Yes	No	No
10	Yes	Yes	Yes	Yes
11	Yes	Yes	No	No

II.7.4 - Useful formulae in Γ-matrix algebra

In every dimension it is important to know which $\Gamma_{a_1 \dots a_n}$ matrices are symmetric and which are antisymmetric. By this we mean the following

$$(C\,\Gamma_{a_1 \dots a_n})^T = C\,\Gamma_{a_1 \dots a_n} \Rightarrow \text{symmetric}$$

$$(C\,\Gamma_{a_1 \dots a_n}) = -\,C\,\Gamma_{a_1 \dots a_n} \Rightarrow \text{antisymmetric} \qquad (II.7.34)$$

C being either $C_{(+)}$ or $C_{(-)}$. In odd-dimensions there is no ambiguity; in those even dimensions where Majorana spinors exist we choose C to coincide with the charge conjugation matrix entering the Majorana condition. Finally in D=6 where no criterion is available we select $C = C_{(-)}$.

With these conventions and using the general inversion formula

$$\Gamma_{m_1 \dots m_n} = (-)^{n(n-1)/2}\,\Gamma_{m_n m_{n-1} \dots m_1} \qquad (II.7.35)$$

we obtain the results of Table II.7.III

TABLE II.7.III

SYMMETRIC AND ANTISYMMETRIC Γ-MATRICES

D	Symmetric	Antisymmetric
4	γ_a, γ_{ab}	$\mathbb{1}$, γ_5, $\gamma_5\,\gamma_a$
5	$\Gamma_{a_1 a_2}$	$\mathbb{1}$, Γ_a
6	$\mathbb{1}$, Γ_7, $\Gamma_7\Gamma_{a_1 a_2}$, $\Gamma_{a_1 a_2 a_3}$	Γ_a, $\Gamma_7\Gamma_a$, $\Gamma_{a_1 a_2}$
7	$\mathbb{1}$, $\Gamma_{a_1 a_2 a_3}$	Γ_a, $\Gamma_{a_1 a_2}$
8	$\mathbb{1}$, Γ_a, $\Gamma_9\Gamma_{a_1 .. a_3}$, $\Gamma_{a_1 .. a_4}$	$\Gamma_9\Gamma_a$, $\Gamma_{a_1 a_2}$, $\Gamma_9\Gamma_{a_1 a_2}$, $\Gamma_{a_1 a_2 a_3}$
9	$\mathbb{1}$, Γ_a, $\Gamma_{a_1 .. a_4}$, $\Gamma_{a_1 .. a_5}$	$\Gamma_{a_1 a_2}$, $\Gamma_{a_1 .. a_3}$
10	Γ_a, $\Gamma_{a_1 a_2}$, $\Gamma_{11}\Gamma_a$, $\Gamma_{11}\Gamma_{a_1 .. a_4}$ $\Gamma_{a_1 .. a_5}$	$\mathbb{1}$, $\Gamma_{a_1 .. a_3}$, $\Gamma_{a_1 .. a_4}$, $\Gamma_{11}\Gamma_{a_1 a_2}$ $\Gamma_{11}\Gamma_{a_1 .. a_3}$
11	Γ_a, $\Gamma_{a_1 a_2}$, $\Gamma_{a_1 .. a_5}$	$\mathbb{1}$, $\Gamma_{a_1 .. a_3}$, $\Gamma_{a_1 .. a_4}$

Note that in $D = 2\nu + 1$ = odd we consider Γ-matrices with, at most, $[D/2]$ indices. Those with more indices are redundant since they are proportional to the ones considered. Indeed we can utilize the duality relation:

$$\Gamma_{a_1 \ldots a_n} = \text{const}\ \varepsilon_{a_1 \ldots a_n b_1 \ldots b_{D-n}} \Gamma^{b_1 \ldots b_{D-n}} \qquad \text{(II.7.36)}$$

Finally we write some formulae of invaluable help in practical calculations:

$$\Gamma^{a_1 \ldots a_n c_1 \ldots c_q} \Gamma_{c_1 \ldots c_q b_1 \ldots b_n} =$$

$$= \sum_{k=1}^{\inf(n,m)} C_k(q,n,m)\, \delta^{[a_1 \ldots a_k}_{[b_1 \ldots b_k} \Gamma^{a_{k+1} \ldots a_n]}{}_{b_{k+1} \ldots b_m]} \qquad \text{(II.7.37a)}$$

$$C_k(q,n,m) =$$

$$= (-)^{q\frac{1}{2}(q-1)} (-)^{\frac{k}{2}(k-(-1)^{n-1})} \binom{n}{k}\binom{m}{k} q!\,k! \binom{D-n-m+k}{q} \qquad \text{(II.7.37b)}$$

$$\Gamma^{a_1 \ldots a_n b} = \Gamma^{a_1 \ldots a_n} \Gamma^b - n\, \Gamma^{[a_1 \ldots a_{n-1}} \delta^{a_n]}_b \qquad \text{(II.7.37c)}$$

$$\Gamma^{b a_1 \ldots a_n} = \Gamma^b \Gamma^{a_1 \ldots a_n} - n\, \delta^{[a_1}_b \Gamma^{a_2 \ldots a_n]} \qquad \text{(II.7.37d)}$$

Furthermore if $\Theta_{a_1 \ldots a_m}$ is an irreducible $(3/2)^m (1/2)^{[D/2]-m}$ spinor, namely an antisymmetric spinor tensor satisfying the Γ-trace condition

$$\Gamma^{a_1} \theta_{a_1 \ldots a_m} = 0 \qquad \text{(II.7.38)}$$

then we have:

$$\Gamma^{a_1 \ldots a_n c_1 \ldots c_q} \theta^{c_1 \ldots c_q b_1 \ldots b_m} =$$

$$= (-)^{q\frac{1}{2}(q+1)} \frac{n!}{(n-q)!} \Gamma^{[a_1 \ldots a_{n-q}} \theta^{a_{n-q+1} \ldots a_n] b_1 \ldots b_m} \qquad \text{(II.7.39a)}$$

$$\bar\theta^{b_1 \ldots b_m c_1 \ldots c_q} \Gamma^{c_1 \ldots c_q a_1 \ldots a_n} =$$

$$= (-)^{q\frac{1}{2}(q+1)} \frac{n!}{(n-q)!} \bar\theta^{b_1 \ldots b_m [a_1 \ldots a_q} \Gamma^{a_{q+1} \ldots a_n]} \qquad \text{(II.7.39b)}$$

$$\Gamma^{c_1 \ldots c_q} \Gamma_{[c_1 \ldots c_q a_1 \ldots a_n} \theta_{b_1 \ldots b_m]} =$$

$$= (-)^{q\frac{(q-1)}{2}} \frac{(q+n)!\,(n+m)!\,(D-n-2m)!}{n!\,(n+m+q)!\,(D-n-q-2m)!} \Gamma_{[a_1 \ldots a_n} \theta_{b_1 \ldots b_m]} \qquad \text{(II.7.39c)}$$

$$\bar\theta_{[b_1 \ldots b_m} \Gamma_{a_1 \ldots a_n c_1 \ldots c_q]} \Gamma^{c_1 \ldots c_q} =$$

$$= (-)^{q\frac{(q-1)}{2}} \frac{(q+n)!\,(n+m)!\,(D-n-2m)!}{n!\,(n+m+q)!\,(D-n-2m-q)!} \cdot \bar\theta_{[b_1 \ldots b_m} \Gamma_{a_1 \ldots a_n]} \qquad \text{(II.7.39d)}$$

Moreover in D=4 we write the explicit form of the duality relation on γ_{ab}

$$\varepsilon_{abcd}\gamma_{cd} = 2\,i\,\gamma_5\,\gamma_{ab} \qquad (II.7.40)$$

and we conclude the chapter with another useful formula valid in every dimension:

$$\Gamma^{c_1..c_q}\,\Gamma_{a_1..a_n}\,\Gamma_{c_1..c_q} = I_q^{(n)}\,\Gamma_{a_1...a_n} \qquad (II.7.41)$$

In (II.7.41) the coefficient $I_n^{(q)}$ is determined by the recurrence relation:

$$I_q^{(n)} = I_1^{(n)} I_{(q-1)}^{(n)} - (q-1)(D-q+2)I_{(q-2)}^{(n)} \qquad (II.7.42a)$$

$$I_0^{(n)} = 1 \qquad (II.7.42b)$$

$$I_1^{(n)} = D - 2n \qquad (II.7.42c)$$

CHAPTER II.8

FIERZ IDENTITIES AND GROUP THEORY

II.8.1 - Introduction

This chapter is very technical but nonetheless very important for all what follows. It deals with a very specific problem which arises in the development of both globally and locally supersymmetric field theories.

As we saw in Chapter II.6, in order to construct the action of a supersymmetric field-theory model we have, in general, to solve exterior form equations on superspace which arise either as Bianchi identities or as field equations associated to a Lagrangian which is itself an exterior form.

A complete cotangent frame on superspace is provided by the vielbein V^a and the gravitino 1-form ψ^A which is a spin 1/2 representation of the Lorentz group SO(1,D-1) and has, moreover, an index A enumerating the supersymmetries $(A=1,2,\ldots,N)$.

Henceforth an arbitrary p-form $\omega^{(p)}$ on superspace can be expanded as follows