μ^A_Σ being the components of the vielbein. Decomposing the indices A and Σ in (I.3.192) one finds:

$$\varepsilon^{ab} = \omega^{ab}_\mu \varepsilon^\mu + h^{ab} \qquad \text{(I.3.193a)}$$

$$\varepsilon^a = V^a_\mu \varepsilon^\mu \qquad \text{(I.3.193b)}$$

where we have set $\omega^{ab}_{\rho\sigma}\,\eta^{\rho\sigma} \equiv h^{ab}$, $\eta^{\rho\sigma}$ being the Lorentz parameters of a generic infinitesimal Lorentz transformation h^{ab} on the fiber, and $V^a_{(\rho\sigma)} \equiv 0$ by a coordinate choice.

Let us now substitute (I.3.193b) into the r.h.s. of (I.3.190b); recalling that $J_{ab}\rfloor R^a = 0$ because of SO(1,3) factorization one finds:

$$\ell_\varepsilon V^a = \mathcal{D}(V^a_\mu \varepsilon^\mu) + \varepsilon^\mu \partial_\mu \rfloor R^a_{\rho\sigma} dx^\rho \wedge dx^\sigma =$$

$$= \mathcal{D} V^a_\mu \varepsilon^\mu + V^a_\mu d\varepsilon^\mu + 2\varepsilon^\mu R^a_{\mu\sigma} dx^\sigma$$

$$(\ell_\varepsilon V^a)_\rho = (\mathcal{D}_\rho V^a_\mu - \mathcal{D}_\mu V^a_\rho)\varepsilon^\mu + \mathcal{D}_\mu V^a_\rho \varepsilon^\mu + V^a_\mu \partial_\rho \varepsilon^\mu +$$

$$+ 2\varepsilon^\mu R^a_{\mu\rho} = \mathcal{D}_\mu V^a_\rho \varepsilon^\mu + V^a_\mu \partial_\rho \varepsilon^\mu =$$

$$= (\partial_\mu V^a_\rho)\varepsilon^\mu + V^a_\mu \partial_\rho \varepsilon^\mu + \omega^{ab}_\mu V_{b|\rho} \varepsilon^\mu \quad . \qquad \text{(I.3.194)}$$

We see that the final result differs from the genuine general coordinate transformation (I.1.220) of the coordinate vector V^a_μ by the term $\varepsilon^\mu\, \omega^{ab}_\mu\, V_{b\rho}$, which can be interpreted as a field-dependent Lorentz transformation of parameter $\varepsilon^{ab} - h^{ab}$ since from Eq. (I.3.193a):

$$\varepsilon^\mu \omega^{ab}_\mu V_{b\rho} = (\varepsilon^{ab} - h^{ab}) V_{b\rho} \quad . \qquad \text{(I.3.195)}$$

In other words a diffeomorphism on the soft group manifold gives rise, on a Lorentz vector, to a diffeomorphism on the base space plus a field dependent Lorentz transformation.

CHAPTER I.4

POINCARE GRAVITY

I.4.1 - Poincaré Gravity

In this chapter we utilize the vielbein V^a and the spin connection ω^{ab} to describe the Einstein theory of gravitation.

On one hand this formalism reveals that gravity is a gauge theory, precisely the gauge theory of the Poincaré group ISO(1,3) (ISO(1,D-1) in a D-dimensional space-time); on the other hand, however, the action from which we deduce the gravitational field equations is essentially different from the Yang-Mills action utilized in ordinary gauge theories.

To understand this difference and to clarify the formal properties of "gravity" is essential for the formulation of its supersymmetric extension, namely "supergravity".

We begin by writing the Einstein-Cartan action:

$$A = \int_{M_4} R^{ab}(\omega) \wedge V^c \wedge V^d \varepsilon_{abcd} \quad . \qquad \text{(I.4.1)}$$

The notations are those utilized in Chapter I.2 for the study of an n-dimensional Riemannian manifold M_n. In our case $n = 4$. In particular according to (I.2.35)

$$R^{ab} = d\omega^{ab} - \omega^{a}{}_{c} \wedge \omega^{cb} \qquad (I.4.2)$$

is the curvature 2-form and V^a is the vielbein.

We take units such that the gravitational coupling constant k is equal to 1. Let us show the equivalence of Eq. (I.4.1) with the action of gravity written in tensor formalism. Expanding R^{ab} on the complete 2-form basis $V^i \wedge V^j$ (see (I.2.34c) we get:

$$R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} = R^{ab}{}_{ij} V^i \wedge V^j \wedge V^c \wedge V^d \varepsilon_{abcd} =$$

$$= R^{ab}{}_{ij} V^i{}_{\mu} V^j{}_{\nu} V^c{}_{\rho} V^d{}_{\sigma} \varepsilon_{abcd}\, dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} =$$

$$= R^{ab}{}_{ij} V^i{}_{\mu} V^j{}_{\nu} V^c{}_{\rho} V^d{}_{\sigma} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd}\, d^4x =$$

$$= R^{ab}{}_{ij} \varepsilon^{ijcd} \varepsilon_{abcd} \det V\, d^4x = -4 R^{ij}{}_{ij} \det V\, d^4x \quad . \qquad (I.4.3)$$

Now

$$R^{ij}{}_{ij} \equiv R^{\mu\nu}{}_{\mu\nu} = R \qquad (I.4.4)$$

is the scalar curvature and $\det V \equiv \sqrt{-g}$ is the square root of the metric determinant $(g = \det g_{\mu\nu})$. Hence we get:

$$\int_{M_4} R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} = -4 \int_{M_4} R \sqrt{-g}\, d^4x \quad . \qquad (I.4.5)$$

Let us examine the group-theoretical significance of Eq. (I.4.1).

There are two gauge fields, the spin connection ω^{ab} and the vielbein V^a:

$$\omega^{ab} = \omega^{ab}_{\mu}\, dx^{\mu} \qquad (I.4.6)$$

$$V^a = V^a_{\mu}\, dx^{\mu} \quad . \qquad (I.4.7)$$

Working in <u>first order formalism</u> both gauge fields are treated as independent. The equation $R^a = 0$ which allows to express ω^{ab} in terms of V^a is not taken as an "a priori constraint", rather, as we are going to see, it follows as a variational equation from (I.4.1). This means that "off the mass-shell" the connection ω^{ab} is not necessarily Riemannian.

The key observation is that $\{V^a, \omega^{ab}\}$, considered as a single entity, constitute a multiplet in the adjoint representation of the Poincaré group. That is we can write:

$$\mu^A(x) T_A \equiv \omega^{ab}(x) J_{ab} + V^a(x) P_a \qquad (I.4.8)$$

where

$$\mu^A(x) = \mu^A_{\mu}(x)\, dx \qquad (I.4.9)$$

is the gauge field of the Poincaré group, J_{ab} and P_a being the generators of the Lorentz transformations and of the four dimensional translations, respectively. Hence gravity, as we claimed, is the "gauge theory" of the Poincaré group.

The field strength associated to μ^A is defined as the Poincaré Lie algebra-valued curvature 2-form

$$R^A = d\mu^A + \frac{1}{2} C^A_{BC} \mu^B \wedge \mu^C \quad . \qquad (I.4.10)$$

Splitting the index $A \equiv (ab, a)$, we get:

$$R^{ab} = d\omega^{ab} - \omega^{a}{}_{c} \wedge \omega^{cb} \qquad (I.4.11a)$$

$$R^a = dV^a - \omega^a{}_b \wedge V^b \equiv \mathscr{D} V^a \qquad \text{(I.4.11b)}$$

which coincide with Eqs. (I.3.174-175). The associated Bianchi identities are given by Eqs. (I.2.62) or (I.3.176) which we rewrite here for completeness:

$$\mathscr{D} R^{ab} = 0 \qquad \text{(I.4.12a)}$$

$$\mathscr{D} R^a + R^{ab} \wedge V_b = 0 \quad . \qquad \text{(I.4.12b)}$$

Therefore the Lorentz-algebra valued curvature is the field strength of the spin connection while the vector valued curvature (or torsion) is the field strength of the vielbein field.

Let us emphasize that, although $\mu^A \equiv (\omega^{ab}, V^a)$ is a Yang-Mills potential and $R^A \equiv (R^{ab}, R^a)$ the corresponding field strength the action (I.4.1) is not of the Yang-Mills type; a Yang-Mills action for μ^A would have the following form:

$$\int_{M_4} R^A \wedge {}^*R_A \equiv \int_{M_4} R^A_{\mu\nu} R_{A|\rho\sigma} \varepsilon^{\rho\sigma\alpha\beta} dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta =$$

$$= -4 \int_{M_4} R^A_{\mu\nu} R_{A|\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g}\, d^4x \quad . \qquad \text{(I.4.13)}$$

The main differences between an action of the form (I.4.13) and the Einstein-Cartan action (I.4.1) are the following:

a) A Yang-Mills action is invariant under the whole gauge group G of which the μ^A.s are the Lie algebra valued potentials.

The action (I.4.1) instead is not invariant under the whole gauge group ISO(1,3), but only under the Lorentz subgroup SO(1,3).

The invariance under Lorentz gauge transformations is manifest since R^{ab}, V^a and ε^{abcd} are good Lorentz tensors.

To show the non invariance of (I.4.1) under a gauge translation we recall that under any Poincaré gauge transformation we have (see Eq. (I.3.141))

$$\delta_{\text{gauge}}\, \mu^A = \nabla \varepsilon^A \qquad \text{(I.4.14)}$$

where ∇ is the Poincaré covariant derivative, and ε^A is the gauge parameter: $\varepsilon^A \equiv (\varepsilon^{ab}, \varepsilon^a)$.

The Lorentz content of (I.4.14) is easily obtained by setting

$$\underline{t} \rfloor R^{ab} = \underline{t} \rfloor R^a = 0 \qquad \text{(I.4.15)}$$

in Eqs. (I.3.183); we obtain

$$\delta^{\text{gauge}}\, \omega^{ab} = (\nabla\varepsilon)^{ab} = \mathscr{D}\varepsilon^{ab} \qquad \text{(I.4.16a)}$$

$$\delta^{\text{gauge}}\, V^a = (\nabla\varepsilon)^a = \mathscr{D}\varepsilon^a + \varepsilon^{ab} V_b \qquad \text{(I.4.16b)}$$

and setting $\varepsilon^{ab} = 0$ we get the infinitesimal action of the gauge translation on the fields

$$\delta\omega^{ab} = 0 \qquad \text{(I.4.17a)}$$

$$\delta V^a = \mathscr{D}\varepsilon^a \quad . \qquad \text{(I.4.17b)}$$

Since (I.4.17a) implies $\delta R^{ab} = 0$, the variation of the action under (I.4.17) is:

$$\delta A = 2 \int_{M_4} R^{ab} \wedge \mathscr{D}\varepsilon^c \wedge V^d \varepsilon_{abcd} = 2 \int R^{ab} \wedge R^d \varepsilon^c \varepsilon_{abcd} \neq 0 \quad .$$

where we have used (I.4.12a) and (I.4.11b). Notice that we cannot use the constraint $R^a = 0$ since it is not invariant under the gauge translation:

$$\delta R^a = \delta \mathscr{D} V^a = \mathscr{D} \delta V^a = \mathscr{D}\mathscr{D} \varepsilon^a = - R^{ab} \varepsilon_b \neq 0 \quad .$$

The non invariance of the Cartan-Einstein action under a gauge translation seems at first sight strange since one usually thinks of a translation as a coordinate transformation. This however is not right since the generator of a coordinate transformations is not a gauge translation, rather a Lie derivative. Indeed, the Lie-derivative along the tangent vector

$$\varepsilon = \varepsilon^\mu \partial_\mu \equiv \varepsilon^a \tilde{P}_a + \omega^{pq}_\mu J_{pq} \varepsilon^\mu \tag{I.4.18}$$

where

$$\varepsilon^a = V^a_\mu \varepsilon^\mu \tag{I.4.19a}$$

and

$$\tilde{P}_a = V^\mu_a (\partial_\mu - \omega^{pq}_\mu J_{pq}) \tag{I.4.19b}$$

yields the transformation laws (I.3.190):

$$\ell_\varepsilon \omega^{ab} = \underline{\varepsilon} \rfloor R^{ab} \tag{I.4.20a}$$

$$\ell_\varepsilon V^a = \mathscr{D} \varepsilon^a + \underline{\varepsilon} \rfloor R^a \quad . \tag{I.4.20b}$$

The Einstein-Cartan action is obviously invariant under general coordinate transformations generated by Lie derivatives of the type (I.4.20). Indeed since the integrand of (I.4.1) is written using only exterior

products of forms and exterior derivatives d thereof, invariance under diffeomorphisms is guaranteed by the general law of transformation of forms under diffeomorphisms (see Eqs. (I.1.170-171)).

Furthermore invariance under diffeomorphisms can be directly checked using the explicit form of the Lie derivative (I.1.236). We obtain

$$\delta^{\mathrm{Diff}} \mathscr{A} = \int_{M_4} \ell_\varepsilon \mathscr{L} = \int_{M_4} (d\, \underline{\varepsilon} \rfloor + \underline{\varepsilon} \rfloor d) \mathscr{L} \quad . \tag{I.4.21}$$

Now the second term is zero since the 5-form $d\mathscr{L}$ vanishes identically on the 4-dimensional space-time M_4: hence $\delta A = 0$ since the first term is a total derivative.

Sometimes the gauge translations generated by $\varepsilon^a P_a = \varepsilon^a V^\mu_a \partial_\mu$ (P_a being left-invariant) are confused with the general coordinate transformations generated by the (non left-invariant) tangent vector $\varepsilon^a \tilde{P}_a$, the relation between the two generators being given in (I.3.191). As we have just seen, however, the associated transformations (I.4.20) and (I.4.17) are quite distinct; actually the former leaves the action (I.4.1) invariant, while the latter does not.

What really people do when speaking of "equivalence between the two kinds of transformations" is to observe that the gauge transformation (I.4.17) can be traded with the diffeomorphism (I.4.20) if one keeps $R^a = 0$ (second order formalism) and amend the transformation law of ω^{ab}, which is a dependent field, in such a way that it coincides with (I.4.20a).

Since the transformation law of ω^{ab} is uninteresting in second order formalism one finds that on the vielbein field V^a_μ the two transformations are the same.

It is evident however from our discussion, that what one is really performing is in any case a general coordinate transformation, since the $\delta\omega^{ab}$ as calculated from (I.4.20b) at $R^a = 0$ exactly reproduces (I.4.20a).

As a final remark we notice that the algebra of the diffeomorphisms being given by (I.1.239), it closes with structure functions rather than with structure constants as it would be the case for the group of translations. Indeed using (I.1.239) and (I.3.132) and setting $\varepsilon_1 = \varepsilon_1^a \tilde{P}_a$, $\varepsilon_2 = \varepsilon_2^b \tilde{P}_b$ one has:

$$[\ell_{\varepsilon_1}, \ell_{\varepsilon_2}] = \ell_{[\varepsilon_1,\varepsilon_2]} \equiv \ell_{\varepsilon_3} \qquad \text{(I.4.22a)}$$

where ε_3 is given by:

$$\varepsilon_3 = [\varepsilon_1^a \tilde{P}_a, \varepsilon_2^b \tilde{P}_b] = \varepsilon_1^a \varepsilon_2^b (C^A_{ab} - 2R^A_{ab})\tilde{T}_A =$$

$$= -2\, \varepsilon_1^a \varepsilon_2^b (R^c_{ab} \tilde{P}_c + R^{cd}_{ab} J_{cd}) \qquad \text{(I.4.22b)}$$

and we have used the fact that the structure constants of two translations are zero for ISO(1,3).

b) A second difference we want to discuss between the Einstein-Cartan and the Yang-Mills action is the following: the action (I.4.1) is linear in the curvature forms R^A, while the Yang-Mills action (I.4.13) is quadratic. A quadratic action is necessary in ordinary Yang-Mills theories to produce second order propagation equations for the potential A_μ. How is it, then, that the Cartan-Einstein action, which is linear, gives second order propagation equations for the graviton?

The answer is that we are using first order formalism. As anticipated, the variation of (I.4.1) in $\delta\omega^{ab}$ yields the torsion equation

$$R^a = 0$$

which can be algebraically solved for the spin connection ω^{ab} in terms of the vielbein first order derivatives. Substituting this result into the other field equation, obtained by varying (I.4.1) with respect to δV^a, we get a second order differential equation for the vielbein V^a.

Let us study how this works in more detail. Varying (I.4.1) with respect to the vielbein field we get:

$$2R^{ab} \wedge V^c \varepsilon_{abcd} = 0 \quad . \qquad \text{(I.4.23)}$$

In order to retrieve from (I.4.23) the corresponding equation for the components $R^{ab}{}_{mn}$ we proceed as follows: we expand the 2-form R^{ab} along a complete basis of vielbeins as in (I.2.34c) and we obtain:

$$2R^{ab}{}_{mn} V^m \wedge V^n \wedge V^c \varepsilon_{abcd} = 0 \quad . \qquad \text{(I.4.24a)}$$

Setting

$$V^m \wedge V^n \wedge V^c = \varepsilon^{mnc\ell}\, \Omega_\ell \qquad \text{(I.4.24b)}$$

where Ω_ℓ is a non zero 3-form, one deduces:

$$R^{ab}{}_{mn} \varepsilon^{mnc\ell} \varepsilon_{abcd} = -3!\, \delta^{mn\ell}_{abd} R^{ab}{}_{mn} = 0$$

that is

$$R^{a\ell}{}_{b\ell} - \frac{1}{2} \delta^a_b R = 0 \qquad \text{(I.4.25)}$$

which is the usual Einstein field equation of pure gravity (the only difference being that we are using intrinsic components of the curvature instead of the world-components). Equation (I.4.25) is a 1st-order equation for the field ω^{ab}_μ.

Another equation is obtained if we vary the independent field ω^{ab}; this variation is easily derived using the following formula:

$$\delta R^A = \nabla(\delta\mu^A) \qquad \text{(I.4.26)}$$

which is an immediate consequence of the definitions (I.3.122) and (I.3.126). In our case (I.4.26) becomes:

$$\delta R^{ab} = \nabla\delta\omega^{ab} = \mathscr{D}\,\delta\omega^{ab} \qquad \text{(I.4.27a)}$$

where the last equality follows from the definition of the Poincaré covariant derivatives of an adjoint multiplet (see Eq. (I.4.16a)).

Therefore the $\delta\omega^{ab}$ variation of (I.4.1) yields

$$\int \delta R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} = \int \mathscr{D}(\delta\omega^{ab}) \wedge V^c \wedge V^d \varepsilon_{abcd} =$$

$$= 2\int \delta\omega^{ab} \wedge \mathscr{D}V^c \wedge V^d \varepsilon_{abcd} \quad . \qquad \text{(I.4.27b)}$$

Notice that there is no minus sign in the partial integration since we are partially integrating a 1-form. It follows that:

$$R^c \wedge V^d \varepsilon_{abcd} = 0 \qquad \text{(I.4.28)}$$

where we have used the definition (I.4.11b).

It is easy to verify that (I.4.28) implies

$$R^c = 0 \quad . \qquad \text{(I.4.29)}$$

Indeed, let us expand the torsion R^c along the vielbein basis

$$R^c = R^c_{mn} V^m \wedge V^n \qquad \text{(I.4.30)}$$

(I.4.28) becomes:

$$R^c_{mn} V^m \wedge V^n \wedge V^d \varepsilon_{abcd} = 0 \quad . \qquad \text{(I.4.31)}$$

Setting as before:

$$V^m \wedge V^n \wedge V^d = \varepsilon^{mnd\ell}\,\Omega_\ell \qquad \text{(I.4.32)}$$

we get:

$$R^c{}_{mn}\,\varepsilon^{mnd\ell}\,\varepsilon_{abcd} = -R^c{}_{mn}\,3!\,\delta^{mn\ell}_{abc} = 0 \qquad \text{(I.4.33)}$$

that is

$$R^\ell{}_{ab} + 2\,R^m{}_{m[a}\,\delta^\ell_{b]} = 0 \quad . \qquad \text{(I.4.34)}$$

Contracting ℓ with a we obtain:

$$R^\ell{}_{\ell b} = 0 \quad . \qquad \text{(I.4.35)}$$

Hence we find:

$$R^\ell{}_{ab} = 0 \quad . \qquad \text{(I.4.36)}$$

Therefore formula (I.4.29) holds.

From $R^a = 0$ (and $\omega^{ab} = -\omega^{ba}$) we deduce that the Riemannian manifold M^4 is endowed with a Riemannian spin connection. ω^{ab} is given in terms of the vielbein through formula (I.2.44) and (I.2.45). Inserting (I.2.44) into Eq. (I.4.25) which is 1st-order in the ω^{ab}_μ we get a second-order equation for the vielbein field (since (I.2.44) is first order in $\partial_\nu V^a_\mu$).

The conclusion is that starting from the Cartan-Einstein action (I.4.1), which is linear in the curvature, the propagation of the vielbein field V^a_μ is obtained via the torsion mechanism $R^a = 0$, which allows the elimination of the spin connection in terms of V^a_μ. Therefore only the degrees of freedom of V^a_μ are physical since they correspond to a propagating field.

I.4.2 - Extension to the soft group manifold

The Cartan-Einstein Lagrangian has still another striking difference as compared with the Yang-Mills one (I.4.13). It is built using only forms, wedge products and exterior derivative with exclusion of the Hodge duality operator * (see Eqs. (I.1.191-192)):

$$R^A \to {}^*R^A \Leftrightarrow R^A{}_{mn} \to \frac{1}{2}\varepsilon_{mn}{}^{pq} R^A{}_{pq} \qquad (I.4.37)$$

appearing instead in (I.4.13). As a consequence, the equations of motion, stating that certain 3-forms are zero, can be naturally extended to a larger manifold by an inclusion mapping:

$$M_4 \to \tilde{G} \supset M_4 \quad . \qquad (I.4.38)$$

In presence of the Hodge duality operator this would be forbidden since in the definition (I.1.191) of the operator * the dimension of the manifold enters in an essential way.

In our case the forms $\{\omega^{ab}, V^a\}$, being Yang-Mills potentials subject to the gauge transformations (I.4.16), are already defined on a larger manifold $\tilde{G} \supset M_4$ which is the principal fiber bundle

$$\tilde{G} = \tilde{G}[\tilde{G}/H, H] \qquad (I.4.39)$$

where $\tilde{G}$ is defined by the structure Eqs. (I.3.174-175) and

$$\tilde{G}/H \simeq M_4 \qquad (I.4.40a)$$

$$H \equiv SO(1,3) \quad . \qquad (I.4.40b)$$

As discussed in the previous chapter, this means that the inclusion mapping extending the fields from M_4 to $\tilde{G}$ is given by the Lorentz transformations (I.2.48) and (I.2.51-52) and that the curvatures

(I.4.11) are horizontal. It is therefore possible to extend the field equations (I.4.23) and (I.4.28) to $\tilde{G} \supset M_4$.

We will now show that it is not necessary to start with the fiber bundle structure (I.4.39) in constructing the action (I.4.1).

Indeed the fiber bundle structure can be obtained as a result of the (suitable extended) variational principle, if we start with a field μ^A defined on the soft Poincaré group manifold $\widetilde{ISO(1,3)}$. In other words, we will show that SO(1,3) horizontality of the curvatures can be obtained as a variational equation from the same Cartan-Einstein action.

According to the discussion of the previous chapter we start with a set of fields μ^A which are Poincaré Lie algebra valued soft 1-forms spanning a basis of the cotangent plane to the 10-dimensional soft Poincaré manifold $\widetilde{ISO(1,3)}$.

The group curvatures are given by

$$R^A = d\mu^A + \frac{1}{2} C^A{}_{BC}\mu^B \wedge \mu^C \qquad (I.4.41)$$

or, in terms of SO(1,3) representations by (I.3.174-175). A priori these forms are not horizontal. The Lagrangian for the fields $\mu^A \equiv \omega^{ab}$, V^a is formally taken to be the same as before

$$\mathscr{L} = R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} \qquad (I.4.42)$$

but, being a 4-form, it must be integrated on a 4-dimensional submanifold of $\tilde{G}$.

Therefore we write

$$\mathscr{A} = \int_{M_4 \subset \tilde{G}} R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} \qquad (I.4.43)$$

where M^4 is any 4-dimensional submanifold of $\tilde{G}$.

In principle when we vary (I.4.43) we should consider not only arbitrary variations of the fields ω^{ab}, V^a but also arbitrary variations of M_4. Observe, however, that a variation of M_4 can always be compensated by a diffeomorphism on the fields ω^{ab}, V^a under which the Lagrangian is invariant. It suffices, therefore, to restrict our attention to the field variations.

The equations of motion

$$R^{ab} \wedge V^c \varepsilon_{abcd} = 0 \qquad (I.4.44)$$

$$R^a \wedge V^b \varepsilon_{abcd} = 0 \qquad (I.4.45)$$

are identical in form to the previous ones, (I.4.23) and (I.4.28), except for the fact that they hold now on the whole 10-dimensional manifold $\tilde{G}$.

To examine their content we must expand the curvatures R^A on a cotangent basis of $\tilde{G}$ which is given by the set of 2-forms:

$$\mu^A \wedge \mu^B \equiv \{V^a \wedge V^b;\ \omega^{ab} \wedge V^c;\ \omega^{ab} \wedge \omega^{cd}\} \quad . \qquad (I.4.46)$$

Hence the expansion of the curvature in this local frame is

$$R^A = R^A_{BC}\mu^B \wedge \mu^C = R^A_{ab}V^a \wedge V^b + 2R^A_{a,bc}V^a \wedge \omega^{bc} + $$
$$+ R^A_{ab,cd}\omega^{ab} \wedge \omega^{cd} \quad . \qquad (I.4.47)$$

It is now easy to verify that on the larger manifold $\tilde{G}$ the equations of motion (I.4.44-45) imply SO(1,3) horizontality of R^A, besides the usual implications for $R^a{}_{mn}$ and $R^{ab}{}_{cd}$ obtained on M_4.

Using (I.4.47) with $(A) \equiv (ab)$ we have:

$$R^{ab}{}_{pq}V^p \wedge V^q \wedge V^c \varepsilon_{abcd} + 2R^{ab}{}_{p,\ell m}V^p \wedge \omega^{\ell m} \wedge V^c \varepsilon_{abcd} +$$
$$+ R^{ab}{}_{\ell m,rs}\omega^{\ell m} \wedge \omega^{rs} \wedge V^c \varepsilon_{abcd} = 0 \quad . \qquad (I.4.48)$$

Since VVV, VωV and ωωω are independent 3-forms each term of (I.4.48) must be separately zero:

$$R^{ab}{}_{pq}V^p \wedge V^q \wedge V^c \varepsilon_{abcd} = 0 \qquad (I.4.49a)$$

$$R^{ab}{}_{p,\ell m}V^p \wedge \omega^{\ell m} \wedge V^c \varepsilon_{abcd} = 0 \qquad (I.4.49b)$$

$$R^{ab}{}_{\ell m,rs}\omega^{\ell m} \wedge \omega^{rs} \wedge V^c \varepsilon_{abcd} = 0 \qquad (I.4.49c)$$

Equation (I.4.49a) is formally the same as Eq. (I.4.22) and we deduce again (I.4.25). Moreover from (I.4.49b) and (I.4.49c) it easily follows that:

$$R^{ab}{}_{p,\ell m} = 0 \qquad (I.4.50a)$$

$$R^{ab}{}_{\ell m,rs} = 0 \qquad (I.4.50b)$$

In an analogous way the torsion equation gives:

$$R^a{}_{bc} = 0 \qquad (I.4.51a)$$

$$R^a{}_{p,\ell m} = 0 \qquad (I.4.51b)$$

$$R^a{}_{\ell m,rs} = 0 \quad . \qquad (I.4.51c)$$

Eqs. (I.4.50a,b) and (I.4.51b-c) are encompassed by the single equation:

$$R^A(J_{\ell m}, \tilde{T}_B) \equiv R^A_{\ell m,B} = 0 \qquad (I.4.52)$$

which is equivalent to:

$$J_{\ell m} \rfloor R^A = 0 \quad . \tag{I.4.53}$$

Since the SO(1,3) horizontality condition is satisfied, we may restrict the equations (I.4.44) and (I.4.45) to the base space $M_4 \equiv \tilde{G}/SO(1,3)$ and they coincide with (I.4.23) and (I.4.28).

Eq. (I.4.53) enforcing the SO(1,3) fiber bundle structure of the theory is due to its Lorentz gauge invariance, that is to the absence of the bare field ω^{ab} in the Lagrangian. Hence the Lorentz gauge invariance of the extended action (I.4.43) is responsible for the factorization of the Lorentz coordinates: effectively the theory lives only on the base space $M_4 = \widetilde{ISO(1,3)}/SO(1,3)$. This is so because the fields depend on the "Lorentz coordinates" only via the finite Lorentz transformations (I.3.148). Since the Lagrangian is invariant (by construction!) under such transformations, the dependence on the Lorentz coordinates disappears.

In supergravity theories we will always confine ourselves, for obvious physical reasons, to Lorentz invariant Lagrangians, so that, starting from soft super-group manifolds, factorization of the Lorentz coordinates will always be guaranteed and the fields will effectively depend only on the base space coordinates.

The use of the entire (soft)group manifold $\tilde{G}$ instead of $\tilde{G}/SO(1,3)$ has therefore a rather formal value. Furthermore we have pursued a pedagogical goal since, in the future, we will compare the "almost factorization" of the supersymmetry parameters of supergravity theories, due to rheonomy, with the complete factorization of the Lorentz parameters due to Lorentz invariance.

It is in this spirit that in the next section we will insist on giving the building principles for a geometrical theory on a soft group-manifold $\tilde{G}$, rather than on a coset manifold $\tilde{G}/H$. The problems connected with the extension from $\tilde{G}/H$ to $\tilde{G}$, which is in itself trivial in the present case without supersymmetry, are however similar to the problems connected with the extension from space to superspace which is non trivial and crucial for the geometric formulation of supergravities.

I.4.3 - Building rules for the gravity Lagrangians

Let us summarize our discussion. We started with the potential μ^A and its corresponding curvature (I.4.11) defined on the whole soft group manifold $\tilde{G} \equiv ISO(1,3)$. Variation of the action (I.4.43) gave the 3-form equations of motion (I.4.44) and (I.4.45).

These imply the vanishing of the curvature R^A along the Lorentz directions (I.4.53) and the consequent factorization of the Lorentz parameters through gauge transformations.

Projection of the equations of motion along the directions of the base space

$$\tilde{G}/H = \widetilde{ISO(1,3)}/SO(1,3) \equiv M_4 \tag{I.4.54}$$

identified with the physical space-time, gave the equations of motion on M_4 (I.4.23) and (I.4.28) for the factorized curvatures and potentials.

As we have seen, they are the usual Einstein equations of gravity in first order formalism.

One may wonder how one could have invented the Lagrangian (I.4.43) possessing all the aforementioned good properties without previous knowledge of gravitational theory.

It is worthwhile to note that (I.4.1), or its extended form (I.4.43), can be uniquely determined using a small set of building rules which appear to be very different from the usual ones used in the derivation of the Einstein action in the theory of gravitation. The formal nature of these principles will prove useful in finding generalizations of gravity Lagrangians to supergravity Lagrangians, one of the main goals of this book.

Before giving and discussing the aforementioned building principle for the construction of the action let us discuss the general philosophy behind them.

We observe the following: if we want to identify the space-time components of the 1-forms $\{\omega^{ab}, V^a\}$ with the physical fields V^a_μ and

ω_μ^{ab}, without destroying their geometrical meaning, we should construct the action in a way consistent with the equations (I.4.8,9), (I.4.11,12) defining (ω^{ab}, V^a) and their curvatures (R^{ab}, R^a).

Now Eqs. (I.4.11-12) have a number of properties and a symmetries that we want to be conserved by the action describing the physical theory. They are:

i) Coordinate invariance: this is an obvious consequence of the fact that (I.4.11-12) are equations among forms, where only the coordinate invariant operations of exterior product and derivative are used; in other words the equations defining the curvatures and their Bianchi identities have an intrinsic geometrical meaning.

ii) SO(1,3) gauge invariance: in fact all the equations (I.4.11-12) are covariantly defined in terms of good Lorentz tensors. We notice that, in contrast, (I.4.11-12) are not invariant under (I.4.17), the gauge translation.

iii) $R^A = 0$ is a solution of (I.4.11-12): indeed in this case (I.4.11) reduce to the Maurer-Cartan equations for the ISO(1,3) left-invariant 1-forms V^a, ω^{ab} and (I.4.12) to the Jacobi identities for the structure constants.

iv) Rigid scale invariance: (I.4.11-12) are invariant under the rigid transformation

$$\omega^{ab} \to \omega^{ab} \quad ; \quad V^a \to eV^a \tag{I.4.55a}$$

$$R^{ab} \to R^{ab} \quad ; \quad R^a \to eR^a \tag{I.4.55b}$$

where e is a constant non zero parameter.

Accordingly we shall require that the action constructed in terms of ω^{ab}, V^a and R^{ab}, R^a will respect all the symmetries and properties expressed by i)-iv). This leads us to formulate the following building rules:

i) The Lagrangian must be geometrical: by that we mean that it must be a 4-form constructed using the potential 1-form μ^A on $\tilde{G}$ and the diffeomorphic invariant operations among them, the wedge product "$\wedge$" and the exterior differential "d".

Actually the requirement that the only physical fields of the theory should be given by the Lie algebra valued 1-forms μ^A turns out to be too restrictive for more general theories. In particular, when coupling matter multiplets to gravity or supergravity, or considering extended supergravities, one must also allow new fields which are 0-forms, i.e. functions on $\tilde{G}$. For the moment we restrict ourselves to these "strong geometricity" allowing; the presence of 0-forms will be discussed in the next chapters.

Notice that we have excluded the duality operator $\mu \to {}^*\mu$ since it depends on the dimensions of the embedding space. As our Lagrangian is a 4-form it must be integrated on a 4-dimensional surface embedded in the ten-dimensional manifold $\tilde{G}$. The duality mapping would bring potentials and their curvatures out of the 4-dimensional integration domain.

ii) The Lagrangian must be invariant under the subgroup $H \equiv SO(1,3)$ of G. To this we also add the obvious physical requirement that it must be a scalar density of definite parity.

iii) The Lagrangian must be such that the eqs. of motion should admit as a particular solution the zero-curvature solution:

$$R^A = 0 \qquad A = \{ab;\ a\} \tag{I.4.56}$$

so that the corresponding potential μ^A are given by the left invariant 1-forms σ^A.

The solution (I.4.56) will be referred to as the "vacuum" of the theory. In our case $G \equiv ISO(1,3)$ and on G we have:

$$\sigma^A \equiv \begin{cases} \omega^{ab} = (\Lambda^{-1}(\eta)d\Lambda)^{ab} & \text{(I.4.57a)} \\ V^a = (\Lambda^{-1}(\eta))^a_b\ \delta^b_\mu\ dx^\mu & \text{(I.4.57b)} \end{cases}$$

or in a particular cross section $(\Lambda = 1)\ G/H \equiv R^4$

$$\sigma^A \equiv \begin{cases} \omega^{ab} = 0 & \text{(I.4.58a)} \\ V^a = \delta^a_\mu\, dx^\mu & \text{(I.4.58b)} \end{cases}$$

which correspond to the vielbein and spin connection of flat Minkowski space.

iv) Finally we impose that the Lagrangian should scale homogeneously with respect to the transformation (I.4.55): if it were not so the equations of motion derived from it would give relations among the curvatures R^{ab}, R^a depending on the parameter e; this would be inconsistent with the Bianchi identities (I.4.12) which scale homogeneously in e.

Let us see how one can retrieve the action (I.4.43) from these principles.

Condition i) implies that the Lagrangian is a 4-form expressible as a polynomial (in the exterior calculus sense) in the μ^A.s and the curvature R^A. Indeed the exterior differential $d\mu^A$ can be written in terms of the curvature R^A: moreover the exterior differential dR^A is linear in R^A due to the Bianchi identities. Therefore the most general Lagrangian is given by:

$$\mathcal{L} = \Lambda^{(4)} + R^A \wedge \nu^{(2)}_A + \frac{1}{2} R^A \wedge R^B\, \nu^{(0)}_{AB} + \text{total differential} \qquad \text{(I.4.59)}$$

since the Lagrangian is defined modulo a total divergence.

Here $\Lambda^{(4)}$, $\nu^{(2)}_A$, $\nu^{(0)}_{AB}$ are polynomials of degree four, two and zero, respectively, in the μ^A's and their coefficients are constant tensors.

$$\Lambda^{(4)} = C_{ABCD}\mu^A \wedge \mu^B \wedge \mu^C \wedge \mu^D \qquad \text{(I.4.60a)}$$

$$\nu_A = C_{APQ}\mu^P \wedge \mu^Q \qquad \text{(I.4.60b)}$$

$$\nu_{AB} = C_{AB} \qquad \text{(I.4.60c)}$$

$\Lambda^{(4)}$ is a scalar, $\nu^{(2)}_A$ is in the coadjoint representation and $\nu^{(0)}_{AB}$ in the coadjoint $\otimes$ coadjoint representation.

Moreover requirement ii) implies that the constant tensors C_{ABCD}, C_{APQ}, C_{AB} be Lorentz (SO(1,3)) invariant tensors.

Now we show that the quadratic terms can always be dropped since they are equivalent to a total differential.

Indeed the only constant tensors C_{AB} which are invariant under SO(1,3) are the following

$$C_{AB} \equiv \begin{cases} C_{(ab),(cd)} \equiv \varepsilon_{abcd} \\ C'_{ab,cd} \equiv \delta^{ab}_{cd} \\ C_{a,b} \equiv \delta^a_b \end{cases} \qquad \text{(I.4.61)}$$

therefore we can write:

$$R^A \wedge R^B V_{AB} = c_1 R^{ab} \wedge R^{cd}\, \varepsilon_{abcd} + c_2 R^{ab} \wedge R_{ab} + c_3 R^a \wedge R_a \qquad \text{(I.4.62)}$$

where c_1, c_2, c_3 are constants.

The first two terms are closed forms. Indeed

$$d(R^{ab} \wedge R^{cd}\, \varepsilon_{abcd}) = \mathcal{D}(R^{ab} \wedge R^{cd}\, \varepsilon_{abcd}) = 0 \qquad \text{(I.4.63)}$$

where $\mathcal{D}$ is the Lorentz covariant derivative.

In this proof we have used the fact the $R^{ab} \wedge R^{cd} \varepsilon_{abcd}$ is Lorentz invariant and the Bianchi identity (I.4.12a).

In the same way we find:

$$d(R^{ab} \wedge R_{ab}) = 0 \quad . \tag{I.4.64}$$

From these results we conclude that $R^{ab} \wedge R^{cd} \varepsilon_{abcd}$ and $R^{ab} \wedge R_{ab}$ are locally exact. Explicitly we can write:

$$R^{ab} \wedge R^{cd} \varepsilon_{abcd} = d(\varepsilon_{abcd}\omega^{ab} \wedge R^{cd} - \varepsilon_{abcd}\omega^{a\ell} \wedge \omega_\ell{}^b \wedge \omega^{cd}) \tag{I.4.65}$$

$$R^{ab} \wedge R_{ab} = d(\omega^{ab} \wedge R_{ab} - \frac{1}{3}\omega^\ell{}_a \wedge \omega_\ell{}^m \wedge \omega_m{}^a) \quad . \tag{I.4.66}$$

Since our manifold M_4 is without boundary the integral is either zero or a topological number; indeed

$$-\frac{1}{8\pi^2}\int_{M_4} R^{ab} \wedge R_{ab} \equiv p_1 \tag{I.4.67}$$

$$\frac{1}{32\pi^2}\int_{M_4} \varepsilon_{abcd}R^{ab} \wedge R^{cd} = E \tag{I.4.68}$$

where the integer p_1 is the first Pontriagyn number and the integer E is the Euler characteristic of the manifold M_4.

In any case the two terms (I.4.65) and (I.4.66) give no contribution to the variation of the action and we can drop them. (Let us stress, however, that this conclusion holds because c_1 and c_2 are constant numbers; if we allow c_1 and c_2 to be functions on $\tilde{G}$ which is the case when we couple gravity to matter fields, then the contribution of these two terms to the variation of the action is not zero since in the partial integration the derivative hits c_1 and c_2).

We may arrive at the same conclusion in a quicker way by using the requirement iv), namely the homogeneous scaling of all the terms of the Lagrangian under (I.4.55). Since as we shall see in a moment the linear terms of the Lagrangian contain of course the Einstein term $R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd}$ which scales as $[e^2]$ under (I.4.55), the same must be true for all the other terms; we see that the first two terms in (I.4.62) have the wrong scale $[e^0] \equiv 1$ so that they must not be included in the Lagrangian in order to have a consistent theory. (Again the argument fails if those terms appear multiplied by functions on $\tilde{G}$ which scale as $[e^2]$ due to the presence of some dimensional constant).

The last term in (I.4.62)

$$R^a \wedge R_a \tag{I.4.69}$$

(which scales as $[e^2]$) can be reduced to a linear term in the curvature R^A through partial integration. Indeed we have:

$$R^a \wedge R_a \equiv \mathscr{D}V^a \wedge \mathscr{D}V_a = \mathscr{D}(V^a \wedge \mathscr{D}V_a) + V^a \wedge \mathscr{D}R_a =$$
$$= d(V^a \wedge \mathscr{D}V_a) + V^a \wedge (-R_{ab} \wedge V^b) \tag{I.4.70}$$

where we have used the Bianchi identity (I.4.12b). Therefore

$$R^a \wedge R_a = - R^{ab} \wedge V_a \wedge V_b + \text{total divergence} \tag{I.4.71}$$

so that (I.4.60) just redefines the coefficient of $R^{ab} \wedge V_a \wedge V_b$ already present in the general term $R^A \wedge \nu_A$ of Eq. (I.4.59).

Therefore the most general Lagrangian can be rewritten as follows

$$\mathscr{L} = \Lambda^{(4)} + R^A \wedge \nu_A \quad . \tag{I.4.72}$$

Now we observe that requirement ii) allows the appearance of ω^{ab}, the bare gauge field of SO(1,3), only through the SO(1,3)-covariant curvature R^{ab}; therefore (I.4.72) becomes

$$\mathscr{L} = \alpha\, \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d + \beta\, \varepsilon_{abcd} R^{ab} \wedge V^c \wedge V^d + \gamma\, R^{ab} \wedge V_a \wedge V_b \tag{I.4.73}$$

and the constant G-tensor C_{ABCD} and C_{APQ} have been identified with the Lorentz invariant tensors as follows:

$$C_{ABCD} \rightarrow C_{abcd} \equiv \alpha\, \varepsilon_{abcd} \tag{I.4.74a}$$

$$C_{APQ} \rightarrow \begin{cases} C_{(ab)cd} \equiv \beta\, \varepsilon_{abcd} \\ C'_{(ab)cd} \equiv \gamma\, \delta^{ab}_{cd} \end{cases} \tag{I.4.74b}$$

with α, β, γ constant numbers.

Moreover requirement iii) implies $\alpha = 0$; indeed if we vary the action with respect to the vielbein field V^d, we find:

$$2\gamma R^{ad} \wedge V_a + 4\alpha\, \varepsilon_{abcd} V^a \wedge V^b \wedge V^c + 2\beta\, \varepsilon_{abcd} R^{ab} \wedge V^c = 0 \quad . \tag{I.4.75}$$

Requiring the "vacuum" (or flat group-manifold)

$$R^{ab} = R^a = 0 \tag{I.4.76}$$

to be a solution of (I.4.75) implies:

$$4\alpha\, \varepsilon_{abcd} V^a \wedge V^b \wedge V^c = 0 \tag{I.4.77}$$

and this can be true only if $\alpha = 0$ since $V^a \wedge V^b \wedge V^c$ is an independent 3-form on $\tilde{G}$.

Finally since ii) requires a definite parity for $\mathscr{L}$ we must discard either

$$\beta\, R^{ab} \wedge V^c \wedge V^d\, \varepsilon_{abcd} \tag{I.4.78}$$

or

$$\gamma\, R^{ab} \wedge V_a \wedge V_b \quad . \tag{I.4.79}$$

The equations of motion resulting from the second choice are

$$R^{ab} \wedge V_a = 0 \tag{I.4.80}$$

$$R^a \wedge V^b - R^b \wedge V^a = 0 \tag{I.4.81}$$

which are identically satisfied by the choice

$$R^a = 0 \tag{I.4.82}$$

since Bianchi identity (I.4.12b) implies (I.4.80) when (I.4.82) holds. The curvature R^{ab} remains therefore completely free. We conclude that (I.4.79) is not a physical Lagrangian.

We are thus left with the Einstein-Cartan action

$$\mathscr{A} = \int_{M_4 \subset \tilde{G}} R^{ab} \wedge V^c \wedge V^d\, \varepsilon_{abcd} \tag{I.4.83}$$

extended to the soft group manifold $\tilde{G}$, or, if horizontality has been assumed, to its restriction to $M_4 \equiv \tilde{G}/SO(1,3)$.

I.4.4 - Gravity in de Sitter and anti de Sitter space

In the previous section we have discussed the formal properties of the Cartan-Einstein formulation of pure gravity. This study will prove to be extremely useful when we try to construct more sophisticated theories generalizing gravity: that is supergravity, matter coupling in gravity and supergravity, higher dimensional theories.

In this section we present the very simple extension of the Cartan-Einstein Lagrangian to the case where the potentials μ^A are defined on a de Sitter or anti de Sitter soft group manifold $\tilde{G}$. The two cases are respectively:

$$\tilde{G}_{d.S.} = \widetilde{SO(1,4)} \qquad (I.4.84a)$$

or

$$\tilde{G}_{A.d.S.} = \widetilde{SO(2,3)} \quad . \qquad (I.4.84b)$$

In the following we restrict ourselves to the anti-de Sitter case; the modification needed for the de Sitter case were discussed in Section I.3.7.

As we are going to see the new Lagrangian corresponds in tensor calculus formalism to ordinary gravity plus a cosmological term. To construct the action of anti de Sitter gravity we apply the building rules discussed in the previous section.

We start from the soft 1-form μ^A of the SO(2,3) Lie algebra:

$$\mu = \mu^A T_A = \mu^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}} \qquad \hat{a}, \hat{b} = (0,1,\ldots,4) \quad . \qquad (I.4.85)$$

We use the formalism developed in Section I.3.8 with $D=4$; here we just rewrite the anti de Sitter curvatures and Bianchi identities:

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} + 4\bar{e}^2 V^a \wedge V^b \equiv \mathcal{R}^{ab} + 4\bar{e}^2 V^a \wedge V^b \qquad (I.4.86a)$$

$$R^a = dV^a - \omega^a{}_b \wedge V^b \equiv \mathcal{D} V^a \qquad (I.4.86b)$$

$$2\mathcal{D} R^{ab} + 8\bar{e}^2 V^{[a} \wedge R^{b]} = 0 \qquad (I.4.87a)$$

$$\mathcal{D} R^a + R^{ab} \wedge V_b = 0 \quad . \qquad (I.4.87b)$$

From Eqs. (I.4.86a) we see that, for $\bar{e} \neq 0$, the (anti) de Sitter curvature R^{ab} differs from the Lorentz curvature

$$\mathcal{R}^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} \qquad (I.4.88)$$

by the term $4\bar{e}^2 V^a \wedge V^b$.

In particular zero anti-de Sitter curvature corresponds to a constant Riemann tensor: indeed we have

$$R^{ab} = 0 \Rightarrow \mathcal{R}^{ab} = - 4\bar{e}^2 V^a \wedge V^b \qquad (I.4.89)$$

which implies:

$$\mathcal{R}^{ab}{}_{cd} = - 4\bar{e}^2 \delta^{ab}_{cd} \quad . \qquad (I.4.90)$$

Hence the anti de Sitter "vacuum" $(R^a = R^{ab} = 0)$ is a 4-dimensional manifold characterized by a constant negative curvature $(- 4\bar{e}^2)$.

Since we are going to require Lorentz invariance we assume that the anti de Sitter curvature is already horizontal; hence all the fields live on the base space $M_4 \equiv \widetilde{SO}(2,3)/SO(1,3)$. The extension to the soft group manifold can be done exactly in the same way as in the Poincaré case.

The Lagrangian in the anti de Sitter case can be written down following closely the procedure and the notations of the Poincaré case. Indeed if one decomposes the adjoint SO(2,3) indices of the general Lagrangian (I.4.59) with respect to SO(1,3) and uses the Lorentz invariant tensors δ^a_b and ε_{abcd}, as required by Lorentz gauge invariance, then one gets exactly the same terms as in (I.4.62) and (I.4.73) the only difference being that the curvature R^{ab} is given by (I.4.86a) instead of (I.4.11a). Using

$$R^{ab} = \mathcal{R}^{ab} + 4\bar{e}^2 V^a \wedge V^b \qquad \text{(I.4.91)}$$

where $\mathcal{R}^{ab}$ is given by (I.4.88), we see that the argument which permits to drop the quadratic terms is still valid. Indeed we have:

$$R^{ab} \wedge R^{cd}\,\varepsilon_{abcd} = \mathcal{R}^{ab} \wedge \mathcal{R}^{cd}\,\varepsilon_{abcd} + 8\bar{e}^2 V^a \wedge V^b \wedge \mathcal{R}^{cd}\,\varepsilon_{abcd} + 16\bar{e}^4 V^a \wedge V^b \wedge V^c \wedge V^d\,\varepsilon_{abcd} \qquad \text{(I.4.92)}$$

and

$$R^{ab} \wedge R_{ab} = \mathcal{R}^{ab} \wedge \mathcal{R}_{ab} + 8\bar{e}^2 V^a \wedge V^b \wedge \mathcal{R}_{ab} \quad . \qquad \text{(I.4.93)}$$

Hence the new quadratic terms differ from those occurring in the Poincaré case by terms of lower order in the curvatures and these just redefine the constant coefficients of the linear terms. The third quadratic term $R^a \wedge R_a$, (Eq. (I.4.70)), is eliminated exactly as before. Therefore we are left with the Lagrangian linear in the curvatures given in Eq. (I.4.73). By the same argument which leads from (I.4.73) to (I.4.75) we get

$$\mathscr{L}_{(AdS)} = \alpha\,\varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d + \beta\, R^{ab} \wedge V^c \wedge V^d\,\varepsilon_{abcd} \qquad \text{(I.4.94)}$$

where R^{ab} is given by Eq. (I.4.86a) and where we have already discarded the parity non conserving term $R^{ab} \wedge V_a \wedge V_b$ (see the discussion following (I.4.79)).

A non trivial difference with respect to the Poincaré case comes into play when we require the existence of the "vacuum" solution; indeed, recalling (I.4.86a) the variation of the vielbein field gives:

$$(4\alpha + 8\bar{e}^2\beta)\varepsilon_{abcd} V^a \wedge V^b \wedge V^c + 2\beta\,\varepsilon_{abcd} R^{ab} \wedge V^c = 0 \quad . \qquad \text{(I.4.95)}$$

In order for the vacuum $R^{ab} = R^a = 0$ to be a solution we must set:

$$\alpha = -2\bar{e}^2\beta \quad . \qquad \text{(I.4.96)}$$

Choosing $\beta = 1$ the Lagrangian (I.4.94) becomes

$$\mathscr{L}_{(AdS)} = R^{ab} \wedge V^c \wedge V^d\,\varepsilon_{abcd} - 2\bar{e}^2 V^a \wedge V^b \wedge V^c \wedge V^d\,\varepsilon_{abcd} \qquad \text{(I.4.97)}$$

or in terms of the Lorentz curvature $\mathcal{R}^{ab}$

$$\mathscr{L}_{(AdS)} = \mathcal{R}^{ab} \wedge V^c \wedge V^d\,\varepsilon_{abcd} + 2\bar{e}^2 V^a \wedge V^b \wedge V^c \wedge V^d\,\varepsilon_{abcd} \quad . \qquad \text{(I.4.98)}$$

In the tensor formalism (I.4.98) reads:

$$\mathscr{L}_{(AdS)} = -4(\mathcal{R}^{ab}{}_{ab} + 12\bar{e}^2)\sqrt{-g}\,d^4x \quad . \qquad \text{(I.4.99)}$$

This is the Einstein Lagrangian with the addition of the cosmological term $12\bar{e}^2$.

Finally we observe that in the contraction limit $\bar{e}^2 \to 0$ from (I.4.98) we recover the Cartan-Einstein action (I.4.1) (or its extended form (I.4.43)).