Gelfand spectra of C*-algebras in topos theory

Masters thesis, by Martijn Caspers

Date: August 2008

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"Proof is the idol before whom the pure mathematician tortures himself."

Sir Arthur Eddington (1882-1944)

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Preface

The aim of this thesis is to compute Gelfand spectra of internal C*-algebras in a topos. The text gives a brief introduction to topos theory, starting from the basic definitions and ending with the internal Mitchell-Bénabou language of a topos. Once we developped enough topos theory we turn our attentention to the generalized Gelfand spectrum in topos theory. We show how to compute these spectra and work out some examples.

This thesis consists of two parts. The first four chapters contain a brief introduction into topos theory, the underlying topos logic and the Mitchell-Bénabou language. This theory was developped in the second half of the 20th century and a vast amount of literature is available, see for example the reference list. A special note should be devoted to the book by Moerdijk and MacLane 'Sheaves in Geometry and Logic: A First Introduction to Topos Theory', which is a very nice caleidoscope of subjects in topos theory. The first chapter of this thesis contains precisely all definitions between a category and a topos. It introduces the notation we use and gives the important examples for this thesis. A reader who is familiar with for example [10] can skip this chapter, except for its last proof which becomes important. The second chapter contains all proofs of theorems widely used in topos theory. This chapter has been added for completeness and we will refer to these pages several times. In chapters three and four we introduce the underlying logic of a topos and the Mitchell-Bénabou language respectively. These chapters show the power of topos theory and the Mitchell-Bénabou language shows that a topos is a really nice generalization of set theory. In chapter 4 some examples are given to emphasize this fact. The Mitchell-Bénabou language is important for the definitions in the last chapters. However, this language has a wide amount of properties which would take a whole book to describe (see [1] for example). Here we restrict ourselves to the central example of monicity and prove an important fact which is used extensively in the last chapters: equality in the Mitchell-Bénabou language is equivalent to equality of morphisms (proposition 4.3.6).

Whereas most theory in the first part of this thesis can be found in literature, the second part is largely new. We turn our attention to an article written by Chris Heunen, Klaas Landsman and Bas Spitters [7]. This article introduces a very specific topos in which the authors define the notion of the localic spectrum of a C*-algebra, which is a generalization of the spectrum of a commutative C*-algebra as used in the Gelfand-Naimark theorem. In terms of the Mitchell-Bénabou language they apply generalizations of set theoretic constructions to their topos. In chapter 5 we introduce these constructions and make the definitions as described in [7] more precise and exlicit. We show that all constructions are actually well-defined and doing so, we prove some important theorems that allow us to compute the spectrum. In chapter 6 it is time to use the tools developed in chapter 5 to compute spectra of C*-algebras. At the start of writing this thesis, our main goal was to compute the localic spectrum of the 2×2-matrices over the complex numbers. This result is presented in chapter 6. Furthermore, we show what the spectrum of a finite-dimensional, commutative C*-algebra looks like and give the tools to compute the spectrum of any finite-dimensional (non-commutative) C*-algebra as long as we can compute its commutative subalgebras. The case of the 2×2 matrices then drops out as a special case. Finally, we dare to taste a bit of the general unital, commutative case: the continuous complex valued functions on a compact Hausdorff space. Here the spectrum becomes quite complicated and a nice representation has not yet been found.

Concerning the prerequisites, we assume the reader has the level of an M. Sc. student. There is no need to know anything about category theory, to which the first chapter gives an introduction. However, this introduction is very brief and it would be more interesting to read [10] to see that category theory can be applied to a wide variety of subjects in mathematics and really unifies these subjects, so that it can be used in 'algebraic geometry', 'algebraic topology', et cetera. We do assume that the reader knows the very basics of lattice theory and logic. Furthermore, we assume the reader is familiar with C*-algebras and the Gelfand theorem.

Chapter 1

Categories and topoi

"Man muss immer generalisieren" - Carl Jacobi

This chapter introduces briefly the main concepts of category theory, starting from the definition of a category and ending with the definition of a topos (plural: topoi or toposes), a special kind of category. A reader who is familiar with category theory can skip the first paragraph, although one might find it instructive to see what notation is used in the rest of the thesis. The main definitions will be clarified by some examples, which are worked out to some extent. Most of this material can be found in [10]. For more examples and applications of topoi, see for instance [5] and [11].

1.1 Categories

In this chapter we define categories, functors, natural transformations and some definitions which are closely related to them. The most important examples for this thesis will be covered.

1.1.1 Categories and examples thereof

Definition 1.1.1. *A* ***category*** $\mathcal{C}$ *consists of the following data:*

- *A class of objects, denoted as* $Obj(\mathcal{C})$*.*
- *A class of morphisms, denoted as* $Mor(\mathcal{C})$*.*
- *A map called* $source : Mor(\mathcal{C}) \to Obj(\mathcal{C})$*.*
- *A map called* $target : Mor(\mathcal{C}) \to Obj(\mathcal{C})$*.*
- *A map called* $identity : Obj(\mathcal{C}) \to Mor(\mathcal{C})$*. The image of an object A under the identity map is usually denoted as Id_A and is called the identity on A.*
- *A composition map called* $\circ : P \to Mor(\mathcal{C})$*, defined on*

$$P = \{(f, g) \in Mor(\mathcal{C}) \times Mor(\mathcal{C}) | target(g) = source(f)\}.$$

P is called the set of composable pairs. We will write $f \circ g$ or simply fg instead of $\circ(f, g)$.

These data satisfy the following rules:

1. *For $A \in Obj(\mathcal{C})$, $source(Id_A) = target(Id_A) = A$.*
2. *If f is a morphism with $source(f) = A$ and $target(f) = B$, then $f \circ Id_A = f = Id_B \circ f$.*
3. *For (f, g) and (g, h) composable pairs, $(f \circ g) \circ h = f \circ (g \circ h)$.*

Note that we assumed the objects and morphisms to form a class, which is not necessarily a set. This is because the categories we wish to examine have a large number of objects, i.e. the objects do not form a set anymore. If both the objects and morphisms form a set, the category is called *small*. All other categories are called *large* necessarily. Going into great detail about the difference between small and large categories is beyond the scope of this thesis, but one should at least notice this choice in the definition.

The morphisms are sometimes depicted as arrows. In some literature they are even called 'arrows'. The source map and target map indicate from which object an arrow 'departs' and at which object the arrow 'arrives'. In this thesis we use the notation $f : A \to B$ to say 'f is a morphism such that $source(f) = A$ and $target(f) = B$'. We now state the notion of an isomorphism.

Definition 1.1.2. *In a category $\mathcal{C}$ a morphism $f : A \to B$ is called an* ***isomorphism*** *if there exists a morphism $g : B \to A$ such that $Id_A = gf$ and $Id_B = fg$. If $f : A \to B$ is an isomorphism, then A and B are called isomorphic. Notation $A \simeq B$.*

The most instructive way of grasping the concept of a category is by seeing some examples:

1. The empty category $\mathcal{C}_\mathbf{0}$, consisting of no objects and no morphisms.
2. The category $\mathcal{C}_\mathbf{1}$, consisting of 1 object and 1 morphism, which is necessarily the identity on that object.
3. The category $\mathcal{C}_\mathbf{2}$,

$$\bullet \longrightarrow \star,$$

   consisting of two objects (the bullet and the star) and three morphisms (two identities and the arrow in the above diagram). Although this is not a very complicated category, we will refer to this example several times.
4. The category **Set** of all small sets. For this category one takes a universe U of sets and defines the small sets as the sets in that universe. This construction is needed in order to make sure the objects form a class. The morphisms are functions between sets. The other maps are evident.
5. The category **Grp** of (small) groups with the group homomorphisms as morphisms.
6. Let X be a topological space. Let P be the set of points of X and for $p, q \in P$, let $\Pi(p, q)$ be the set of homotopy classes of paths from p to q. Then this forms a category with the elements of P as objects and the union $\cup_{(p,q)}\Pi(p, q)$ as the class of morphisms. Composition is the composition of paths.
7. The category **cCStar** of commutative, unital C^*-algebra's with the (unital) C^*-morphisms as morphisms.

8. A category $\mathcal{P}$ is called a *preorder* if for every two objects $A, B \in Obj(\mathcal{P})$ there is at most one morphism from A to B. Conversely, if P is some set with a partial order, P can be regarded as a category if we take the elements of P as objects and define a morphism to be a pair $(p, q) \in P \times P$ such that $p \leq q$. Define $source((p, q)) = p$, $target((p, q)) = q$ and $(q, r) \circ (p, q) = (p, r)$. Note that the transitivity of the partial order shows that this composition is a valid operation. With this composition, (p, p) is the identity on p. Conversely, a preorder $\mathcal{P}$ regarded as a category gives rise to a partial order by putting for any two objects $p, q \in Obj(P)$, $p \leq q$ if and only if there is a morphism from p to q. The elements of the partial order are the objects of $\mathcal{P}$ modulo isomorphism. In that case $\mathcal{P}$ is a preorder and there exists a morphism between objects p and q, we denote it by $\subseteq_{p,q}: p \to q$ or simply $\subseteq: p \to q$.

9. If $\mathcal{C}$ is a category, we can define the opposite category $\mathcal{C}^{op}$ by:

   - $Obj(\mathcal{C}^{op}) := Obj(\mathcal{C})$.
   - $Mor(\mathcal{C}^{op}) := Mor(\mathcal{C})$.
   - $source^{op} := target$. So the source map in the opposite category is the target map of the original category.
   - $target^{op} := source$.
   - $Id_A^{op} := Id_A$.
   - If g and f are composable morphisms in $\mathcal{C}$, i.e. $target(f) = source(g)$, then the composition in the opposite category is defined by:

   $$f^{op} \circ^{op} g^{op} := (g \circ f)^{op}.$$

   Remark that by switching the source and target maps in the opposite category this composition is well-defined.

Examples 3, 4 and 8 are very important for this thesis. The other examples will be useful too.

An important concept will be the one of 'Hom-sets'.

Definition 1.1.3. *Let $\mathcal{C}$ be a category. Let $A, B \in Obj(C)$. We define the* ***Hom-set*** *of A and B as:*

$$\boldsymbol{Hom}_{\mathcal{C}}(A, B) = \{f \mid f \in Mor(\mathcal{C}), \quad source(f) = A, \quad target(f) = B\}.$$

Do not be misled by this terminology. A Hom-set does not necessarily have to be a set (but it certainly is a class).

1.1.2 Special objects and morphisms

We define two special objects and two special types of morphisms.

Definition 1.1.4. *An* ***initial object 0*** *in a category $\mathcal{C}$ is an object such that for each object C in $\mathcal{C}$ there is a unique morphism from* ***0*** *to C. This morphism will be denoted by* $!$.

Definition 1.1.5. *A* ***terminal object 1*** *in a category $\mathcal{C}$ is an object such that for each object C in $\mathcal{C}$ there is a unique morphism from C to* ***1***. *This morphism will be denoted by* $!$ *as well. This might lead to some confusion, but we will always indicate which morphism is meant.*

For example, in **Set** the initial object is the empty set, since there is only one function from $\emptyset$ to any other set. The terminal object is any one-point set, since any function to the one-point set must be constant. Remark that if we have two initial objects $\mathbf{0}$ and $\mathbf{0}'$, then they must be isomorphic. By definition there are unique morphisms $! : \mathbf{0} \to \mathbf{0}'$ and $!' : \mathbf{0}' \to \mathbf{0}$. The composite $!' \circ !$ is the unique morphism from $\mathbf{0}$ tot itself, which must be the identity. For the same reason $! \circ !' = Id_{\mathbf{0}'}$. Hence $\mathbf{0} \simeq \mathbf{0}'$. In the same way one finds that $\mathbf{1}$ is unique up to isomorphism.

Definition 1.1.6. *A morphism m is called **monic** or 'a monic' if $mf = mg$ implies $f = g$, i.e. it is left-cancellable. In diagrams we will denote monics by $\hookrightarrow$.*

Definition 1.1.7. *A morphism e is called **epi** or 'an epi' if $fe = ge$ implies $f = g$, i.e. it is right-cancellable. In diagrams we will denote epis by $\twoheadrightarrow$.*

It may be checked that in **Set** monic is equivalent to injective and epi is equivalent to surjective. However, this is not always true. For example, one can take a category which does not have a notion of injectivity, like example 6. Another example is the embedding of $\mathbb{Q}$ in $\mathbb{R}$ as topological spaces with the euclidian topology. The embedding is not surjective, but it is epi.

1.1.3 Functors

Next we define 'morphisms between categories', the so-called functors.

Definition 1.1.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **functor** F from $\mathcal{C}$ to $\mathcal{D}$ consists of:*

- *A function $F^0 : Obj(\mathcal{C}) \to Obj(\mathcal{D})$.*
- *A function $F^1 : Mor(\mathcal{C}) \to Mor(\mathcal{D})$.*

These functions satisfy the following properties. For any $f, g \in Mor(\mathcal{C})$ such that f and g are composable, the following relations holds:

- $F^0(source(f)) = source(F^1(f))$.
- $F^0(target(f)) = target(F^1(f))$.
- $F^1(f \circ g) = F^1(f) \circ F^1(g)$.
- $F^1(Id_A) = Id_{F^0(A)}$.

We usually write $F : \mathcal{C} \to \mathcal{D}$, saying F is a functor from $\mathcal{C}$ to $\mathcal{D}$. Remark that this notation is consistent with the idea of viewing F as a morphism between categories. For convenience of notation we write $F(A) := F^0(A)$ and $F(f) := F^0(f)$ if A is an object and f is a morphism of $\mathcal{C}$.

There exists a notion of a *contravariant functor*. The definition is the same as definition 1.1.8 except for the relations. They become:

- $F^0(source(f)) = target(F^1(f))$.
- $F^0(target(f)) = source(F^1(f))$.
- $F^1(f \circ g) = F^1(g) \circ F^1(f)$.

- $F^1(Id_A) = Id_{F^0(A)}$. Remark that this relation is the same as for a functor.

The intuitive idea is that a contravariant functor reverses the direction of the morphisms, whence a contravariant functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ can be regarded as a functor from the opposite category $\mathcal{C}^{op}$ to $\mathcal{D}$. The functors of definition 1.1.8 are often called *covariant functors*.

Again we give some examples.

1. Let $U : \mathbf{Grp} \to \mathbf{Set}$ be the *forgetful functor*, forgetting the group structure of **Grp**. This is a functor mapping the objects of **Grp** to **Set**.

2. If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors, the composite $G \circ F$ is a functor. Here the composite maps are defined as $(G \circ F)^0 = G^0 \circ F^0$ and $(G \circ F)^1 = G^1 \circ F^1$.

3. Let $\mathcal{C}$ be a category with small Hom-sets, i.e. the Hom-sets are sets. Let A be an object. Then there is a functor $\mathbf{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set}$, defined on objects as:

$$\mathbf{Hom}_{\mathcal{C}}(A, -) : B \mapsto \mathbf{Hom}_{\mathcal{C}}(A, B),$$

   and on the morphisms as:

$$\mathbf{Hom}_{\mathcal{C}}(A, -) : f \mapsto f_*,$$

   where $f_*(g) = f \circ g$, i.e. composing with f on the left. This functor is called the *covariant Yoneda functor.*

4. Let $\mathcal{C}$ be a category with small Hom-sets. Let A be an object. Then there is a functor $\mathbf{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^{op} \to \mathbf{Set}$, defined on objects as:

$$\mathbf{Hom}_{\mathcal{C}}(-, A) : B \mapsto \mathbf{Hom}(B, A),$$

   and on the morphisms as:

$$\mathbf{Hom}_{\mathcal{C}}(A, -) : f \mapsto f^*,$$

   where $f^*(g) = g \circ f$ is composing with f on the right. This functor is called the *contravariant Yoneda functor.*

5. Let $\mathcal{J}$ and $\mathcal{C}$ be categories. Let C be an object of $\mathcal{C}$. Define the *constant functor* $\Delta_C : \mathcal{J} \to \mathcal{C}$ on objects as $\Delta_C^0 : Obj(\mathcal{J}) \to Obj(\mathcal{C}) : J \mapsto C$ and on morphisms as $\Delta_C^1 : Mor(\mathcal{J}) \to Mor(\mathcal{C}) : f \mapsto Id_C$.

The functors defined in 3, 4 and 5 will become important later on.

1.1.4 Natural transformations

Next we define the morphisms between functors, the so-called natural transformations.

Definition 1.1.9. *Let F and G be functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A* ***natural transformation*** *τ from F to G is a function*

$$\tau : Obj(\mathcal{C}) \to Mor(\mathcal{D}),$$

such that for any object A in $\mathcal{C}$

$$\tau(A) \in \boldsymbol{Hom}_{\mathcal{D}}(F(A), G(A)),$$

and such that the following diagram commutes for every $f \in \boldsymbol{Hom}_{\mathcal{C}}(A, B) \subseteq Mor(\mathcal{C})$:

$$\begin{tikzcd} F(A) \arrow[r, "\tau(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\tau(B)"'] & G(B). \end{tikzcd}$$

Furthermore, τ is a natural isomorphism if there is a natural transformation σ from G to F such that for any object $A \in Obj(\mathcal{C})$, one has $\tau(A) \circ \sigma(A) = Id_{G(A)}$ and $\sigma(A) \circ \tau(A) = Id_{F(A)}$. We write $F \simeq G$. If we view the functors between $\mathcal{C}$ and $\mathcal{D}$ as objects and the natural transformations as morphisms between these objects, this notation coincides with definition 1.1.2.

We write $\tau : F \to G$ when τ is a natural transformation from the functor F to the functor G. If $\sigma : G \to H$ is another natural transformation, then there is a way of composing σ and τ. This is done point-wise, i.e. for every object A, $(\sigma \circ \tau)(A) = \sigma(A) \circ \tau(A)$. In some literature this is called the *horizontal composition*.

The next definition is the central object of this thesis. It is the notion of a *functor category*. The idea is to regard functors as objects and natural transformation as the morphisms between functors.

Definition 1.1.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We define the category $\mathcal{D}^{\mathcal{C}}$ to be the category whose objects are the functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are the natural transformations between the functors. For an object functor F, the identity Id_F is the natural transformation assigning to every object $A \in Obj(\mathcal{C})$ the identity $Id_{F(A)}$. The composition of natural transformation is defined as point-wise composition, i.e. for two (composable) natural transformations τ and σ, one has $(\tau \circ \sigma)(A) = \tau(A) \circ \sigma(A)$. This category is referred to as a* ***functor category****.*

If F is an object of $\mathcal{D}^{\mathcal{C}}$ and if $f : A \to B$ is a morphism in $\mathcal{C}$, then we write

$$f^F_{A,B} := F(f).$$

When we write $f^F_{A,B}$ without specifying f,A and B, f is implicitly defined to be a morphism from A to B in $\mathcal{C}$. We call the morphisms $f^F_{A,B}$ the intrinsic maps (of F).

One easily checks that this indeed defines a category. It is instructive to think about this definition a bit more. As an example we study the category $\mathbf{Set}^{\mathcal{C}_\mathbf{2}}$. An object of this category is a functor $F : \mathcal{C}_\mathbf{2} \to \mathbf{Set}$. This functor can be regarded as two sets, namely $F^0(\bullet)$ and $F^0(\star)$, together with a morphism between those sets, namely $F^1(\rightarrow)$. So the objects of $\mathbf{Set}^{\mathcal{C}_2}$ are the diagrams of the form $\{\} \rightarrow \{\}$, where $\{\}$ are sets. In the literature, diagrams in a category $\mathcal{C}$ are defined as functors from a category $\mathcal{J}$ (which 'has the form of the diagram') to the category $\mathcal{C}$. Intuitively the definition of a diagram is quite clear. A morphism in $\mathbf{Set}^{\mathcal{C}_2}$ between objects F and G is a natural transformation $\tau : F \rightarrow G$, i.e. a pair of morphisms such that the following diagram commutes:

$$\begin{tikzcd}
F^0(\bullet) \arrow[r, "F^1(\rightarrow)"] \arrow[d, "\tau(\bullet)"'] & F^0(\star) \arrow[d, "\tau(\star)"] \\
G^0(\bullet) \arrow[r, "G^1(\rightarrow)"] & G^0(\star).
\end{tikzcd}$$

So the morphisms in $\mathbf{Set}^{\mathcal{C}_2}$ are pairs of arrows forming commutative diagrams.

To complete this paragraph we give two more examples of categories. These categories turn out to be topoi and are widely used in the literature.

- As a special case of definition 1.1.10 we take $\mathbf{Set}^{\mathcal{P}}$, where $\mathcal{P}$ is a preorder. This category will become important.
- Let $\mathcal{C}$ be a category. Define the *category of presheaves on* $\mathcal{C}$ to be $\mathbf{Set}^{\mathcal{C}^{op}}$. This is the category which has as objects the contravariant functors $\mathcal{C} \rightarrow \mathbf{Set}$. If F is an object of $\mathbf{Set}^{\mathcal{C}^{op}}$ and $f : A \rightarrow B$ is a morphism in $\mathcal{C}$, then the intrinsic morphism $F(f)$ is denoted as $f^F_{B,A}$.
- Let X be a topological space and let $\mathcal{O}(X)$ be its set of opens. Regard $\mathcal{O}(X)$ as a preorder and hence as a category using the inclusion to define a partial order. Recall that if $V, U \in \mathcal{O}(X)$ and V is a subset of U, we denote the corresponding morphism by $\subseteq : V \rightarrow U$. An object F in $\mathbf{Set}^{\mathcal{O}(X)^{op}}$ is called a *sheaf* is it satisfies the following properties:
  - If U is an open set in X, if $\{V_i\}$ is an open covering of U and if $p, q \in F(U)$ are elements such that for each i, $\subseteq^F_{U,V_i}(p) = \subseteq^F_{U,V_i}(q)$, then $p = q$.
  - If U is an open set in X, if $\{V_i\}$ is an open covering of U and if we have elements $p_i \in F(V_i)$ such that for every i, j $\subseteq^F_{V_i,V_i \cap V_j}(p_i) = \subseteq^F_{V_j,V_i \cap V_j}(p_j)$, then there is an element $p \in F(U)$ such that for every i, $\subseteq^F_{U,V_i}(p) = p_i$.

  The morphisms between sheaves are just the morphisms of presheaves.

1.2 Topoi

A topos is a special kind of category. One might think of it as a generalization of the concept of a set. The most important properties of a set can be translated into a categorical formulation. If one takes these formulations as axioms, you could wonder what categories satisfy these axioms. These set-like categories are called topoi. Formally topoi are categories in which objects have subobjects (analogous to subsets), power objects (analogous to the power set of a set), and finite (co)limits (which allows us to do set arithmetics).

1.2.1 Limits

Here we introduce the notion of a categorical limit. The name 'limit' may let you think of an analytic limit in terms of ϵ and δ. However, a categorical limit is totally different. Limits are objects that are unique for some universal property. We state the precise definition now.

Definition 1.2.1. *Let $\mathcal{J}$ and $\mathcal{C}$ be categories and let $F : \mathcal{J} \to \mathcal{C}$ be a functor. We define the **limit** of F to be an object C in $\mathcal{C}$ together with a natural transformation τ from Δ_C (the constant functor) to F, such that for every other object D in $\mathcal{C}$ and natural transformation σ from Δ_D to F there is a unique morphism $f : D \to C$ such that for every object J in $\mathcal{J}$, $\sigma(J) = \tau(J) \circ f$.*

Remark that if both (C, τ) and (C', τ') are limits, there exist unique morphisms $f : C \to C'$ and $g : C' \to C$ such that $\tau(J) = \tau'(J) \circ f$ and $\tau'(J) = \tau(J) \circ g$. Now $\tau(J) = \tau(J) \circ g \circ f$, so $g \circ f$ is the unique morphism h such that $\tau(J) = \tau(J) \circ h$. Obviously this h is the identity: $g \circ f = Id_C$. In the same way $f \circ g = Id_{C'}$. Hence limits are unique up to isomorphism, provided that they exist.

Sometimes we will omit the natural transformation τ, just saying C is the limit of F. We will denote this as $\lim F = C$.

The definition of a limit is a very abstract one. However, often limits can be depicted as a diagram. Remark that the functor $F : \mathcal{J} \to \mathcal{C}$ can be represented as a diagram 'of the form $\mathcal{J}$' with on each vertex an object of $\mathcal{C}$ and between them morphisms of $\mathcal{C}$. Then the limit is an object of $\mathcal{C}$ together with 'commutative morphisms from the object to the diagram' (the *limiting cone*). One should think of limits of functors as 'limits of diagrams'. This is best depicted in some important examples below. These examples will be used extensively in this thesis.

1. Let $\mathcal{J}$ be the category consisting of 2 objects and only the two identity morphisms:

$$^{J_1}\bullet \qquad \bullet^{J_2}.$$

The limit of a functor $F : \mathcal{J} \to \mathcal{C}$ is called the *product* of $F(J_1)$ and $F(J_2)$, denoted as $F(J_1) \times F(J_2)$. For example, take $\mathcal{C}$ =**Set**. Let A_1 and A_2 be two sets. Let F be the functor $F : J_i \to A_i$. Then the following diagram shows the limiting product $F(J_1) \times F(J_2) = A_1 \times A_2$ is just the usual cartesian product of sets along with its projections as the natural transformation.

$$\begin{tikzcd} & A_1 \\ B \arrow[ur] \arrow[r, dotted, "\exists!"] \arrow[dr] & A_1 \times A_2 \arrow[u, "\tau(J_1) := p_1"'] \arrow[d, "\tau(J_2) := p_2"] \\ & A_2. \end{tikzcd}$$

In the future the 'projections' corresponding to the natural transformation of a product will always be denoted by p or p_i, where i is an index.

2. Let $\mathcal{J}$ be the category:

$$^{J_1}\bullet \rightrightarrows \bullet^{J_2}.$$

The limit of a functor $F : \mathcal{J} \to \mathcal{C}$ is called an *equalizer*. Again take for example $\mathcal{J} = \mathbf{Set}$. Let A_1 and A_2 be sets and f, g both maps from A_1 to A_2 as indicated in this diagram:

$$A_1 \underset{g}{\overset{f}{\rightrightarrows}} A_2.$$

Remark that this diagram represents a functor $F : \mathcal{J} \to \mathcal{C}$. Its limit is the set $\{a \in A_1 \mid f(a) = g(a)\}$, as the following diagram obviously shows.

$$\begin{tikzcd}
\{a \in A_1 \mid f(a) = g(a)\} \arrow[r, "inclusion"] & A_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A_2 \\
C \arrow[u, dotted, "\exists!"] \arrow[ur] & &
\end{tikzcd}.$$

3. Let $\mathcal{J}$ be the category:

$$\begin{tikzcd}
 & \bullet^{J_1} \arrow[d] \\
{}_{J_3}\bullet \arrow[r] & \bullet_{J_2}
\end{tikzcd}.$$

   The limit of a functor $F : \mathcal{J} \to \mathcal{C}$ is called a *pullback*. In **Set** the limit of the diagram

$$\begin{tikzcd}
 & A_1 \arrow[d, "f"] \\
A_2 \arrow[r, "g"'] & A_3
\end{tikzcd}.$$

   is the set

$$\{(a, b) \in A_1 \times A_3 \mid f(a) = g(b)\}$$

   with the projections on A_1 and A_3 as morphisms.

The next definition is the dual of the previous one. It is exactly the same except that the limiting cone must be a class of morphisms from the diagram to the object (instead of in the other direction).

Definition 1.2.2. *Let $\mathcal{J}$ and $\mathcal{C}$ be categories and let $F : \mathcal{J} \to \mathcal{C}$ be a functor. We define the* ***colimit*** *of F to be an object C in $\mathcal{C}$ and a natural transformation τ from F to Δ_C such that for every other object D of $\mathcal{C}$ and natural transformation σ from F to Δ_D there is a unique morphism $f : C \to D$ such that for every object J in $\mathcal{J}$, $\sigma(J) = f \circ \tau(J)$.*

In the same spirit as limits, colimits are unique up to isomorphism. We list some important examples of colimits.

1. Let $\mathcal{J}$ be the category:

$$\overset{J_1}{\bullet} \qquad \overset{J_2}{\bullet}.$$

The colimit of a functor $F : \mathcal{J} \to \mathcal{C}$ is called the *coproduct* of $F(J_1)$ and $F(J_2)$, denoted as $F(J_1) + F(J_2)$. For example, take $\mathcal{C}$ =**Set**. Let A_1 and A_2 be two sets. Let F be the functor $F : J_i \to A_i$. Then the following diagram shows the limiting product $F(J_1) + F(J_2) = A_1 \coprod A_2$ is just the disjoint union of sets along with its inclusions as the natural transformation.

$$\begin{tikzcd}
A_1 \arrow[d, "\tau(J_1):=i_{A_1}"'] \arrow[dr] & \\
A_1 \coprod A_2 \arrow[r, dotted, "\exists!"'] & B \\
A_2 \arrow[u, "\tau(J_2):=i_{A_2}"] \arrow[ur] &
\end{tikzcd}.$$

2. Let $\mathcal{J}$ be the category:

$$\overset{J_1}{\bullet} \rightrightarrows \overset{J_2}{\bullet}.$$

The colimit of a functor $F : \mathcal{J} \to \mathcal{C}$ is called a *coequalizer*. Again take for example $\mathcal{C} =$ **Set**. Let A_1 and A_2 be sets and f, g both maps from A_1 to A_2 as indicated in this diagram:

$$\begin{tikzcd} A_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A_2 \end{tikzcd}.$$

Remark that this diagram represents a functor F. Its colimit is the set

$$\{b \mid b \in A_2\} / \sim,$$

where $\sim$ is the equivalence relation generated by the relation $b\mathrm{R}b'$ if and only if there exists an $a \in A_1$ such that $b = f(a)$ and $b' = g(a)$. The following diagram then obviously shows that this set is the colimit of the diagram.

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A_2 \arrow[r] \arrow[dr] & \{[b] \mid b \in B\} / \sim \arrow[d, dotted] \\
& & C.
\end{tikzcd}$$

A limit or colimit is said to be *finite* if the categorie $\mathcal{J}$ has a finite number of objects and morphisms. We say that for a given category $\mathcal{C}$ all (co)limits exist if for every category $\mathcal{J}$ and for every functor $F : \mathcal{J} \to \mathcal{C}$ the (co)limit exists. In chapter 2 we will prove a theorem that says all (finite) limits exist if all (finite) products and all equalizers exist. Dually, all (finite) colimits exist if all (finite) coproducts and all coequalizers exist.

1.2.2 Subobjects

Now we generalize the notion of a subset, which will be called a subobject. In **Set** every subset A of a given set B can be described by its characteristic function (from B to $\{0,1\}$), i.e. the function that is defined to be 1 at elements of A and 0 elsewhere. Conversely, every characteristic function describes a subset of B. This correspondence can be generalized by the following definition.

Definition 1.2.3. *A category $\mathcal{C}$ with a terminal object is said to have a* ***subobject classifier*** *if there exists a diagram*

$$\boldsymbol{1} \overset{t}{\hookrightarrow} \Omega.$$

with the following property. For any diagram

$$A \overset{m}{\hookrightarrow} B.$$

there exist a unique morphism χ such that the following square is a pullback, i.e. A is the limit of the bottom right part of the diagram.

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, hook, "m"'] & \boldsymbol{1} \arrow[d, hook, "t"] \\ B \arrow[r, dotted, "\exists!\chi"'] & \Omega. \end{tikzcd}$$

Remark that in this definition m and t are monic.

Sometimes we briefly call Ω the subobject classifier, assuming that it is clear which morphism t is meant. For the moment we call a diagram $A \hookrightarrow B$ a *subobject* of B which is usually abbreviated by saying A is a subobject of B. The exact definition of a subobject will be postponed to chapter 3. For now this definition will suffice. The map χ is called the characteristic morphism of the subobject. It turns out there is a 'bijection up to isomorphism' between subobjects of B and maps from B to Ω (i.e. the characteristic morphisms). For convenience of notation we write $true_B \equiv t\circ!$, where ! is the unique map from B to **1**.

The details of subobjects and their classifiers will be postponed to chapter 3. Here we will only prove the uniqueness of a subobject classifier and give some examples.

Proposition 1.2.4. *If a category $\mathcal{C}$ has a subobject classifier $t : \boldsymbol{1} \hookrightarrow \Omega$, then it is unique up to isomorphism. That is, if $t' : \boldsymbol{1} \hookrightarrow \Omega'$ is another subobject classifier, then there is an isomorphism f such that the following diagram commutes.*

$$\begin{tikzcd} \boldsymbol{1} \arrow[r, "Id_{\boldsymbol{1}}"] \arrow[d, "t"'] & \boldsymbol{1} \arrow[d, "t'"] \\ \Omega \arrow[r, "f"'] & \Omega'. \end{tikzcd}$$

Proof:

$$\begin{tikzcd}
\mathbf{1} \arrow[d, "t"] & \mathbf{1} \arrow[l, "Id_{\mathbf{1}}"'] \arrow[d, "t'"] & \mathbf{1} \arrow[l, "Id_{\mathbf{1}}"'] \arrow[d, "t"] \\
\Omega & \Omega' \arrow[l, dotted, "g"] & \Omega. \arrow[l, dotted, "f"]
\end{tikzcd}$$

The definition of the subobject classifier gives unique morphisms f and g that turn the above squares into pullbacks. The pullback lemma (see chapter 2) proves that the outer rectangle is a pullback. Now $g \circ f$ has the unique property of making the outer diagram a pullback. It is clear Id_Ω has this property. We conclude $g \circ f = Id_\Omega$. In the same way $f \circ g = Id_{\Omega'}$, which proves the proposition. $\square$

In **Set** the terminal object is any one-point set. We claim the subobject classifier is defined by the diagram:

$$\begin{tikzcd}
\{1\} \arrow[d, hook, "i"] \\
\{0,1\}\,.
\end{tikzcd}$$

Here i is the inclusion. Remark that in a subobject diagram $m : A \hookrightarrow B$ the set A can be regarded as a subset of B (since m is monic A is isomorphic to its image under m). So let A be a subset of B, then we claim the characteristic morphism χ_A is the unique morphism turning the following square into a pullback.

$$\begin{tikzcd}
A \arrow[r, "!"] \arrow[d, hook, "m"'] & \{1\} \arrow[d, hook, "i"] \\
B \arrow[r, dotted, "\chi_A"'] & \{0,1\}\,.
\end{tikzcd}$$

Indeed the diagram commutes. Furthermore, suppose we have a diagram:

$$\begin{tikzcd}
C \arrow[dr, dotted] \arrow[drr, "h"] \arrow[ddr, "k"'] & & \\
& A \arrow[r, "!"] \arrow[d, hook, "m"] & \{1\} \arrow[d, hook, "i"] \\
& B \arrow[r, "\chi_A"'] & \{0,1\}\,,
\end{tikzcd}$$

then we can show that the dotted morphism is just k, which maps to a smaller target set. To see this, remark that for any $c \in C$, one has $\chi_A k(c) = ih(c) = 1$. So $k(c) \in A$, so we can restrict the target of k to A.

The map χ_A is unique. Suppose $b \in B, b \in A$, then $\chi(b) = 1$, since the diagram must commute. Suppose $b \in B, b \notin A$, then $\chi(b) = 0$, since A must be the pullback of the bottom right limiting diagram.

Another example is given by the category $\mathbf{Set}^{\mathcal{C}_2}$ of functors from $\mathcal{C}_2$ to $\mathbf{Set}$. The objects are just diagrams in $\mathbf{Set}$ of the form $A \to B$ and the morphisms are the natural transformations. It is easy to see that a natural transformation $\tau : F \to G$ is monic if and only if both 'components' are injective, i.e.

$$\begin{tikzcd}
F(\bullet) \arrow[r, hook, "\tau(\bullet)"] \arrow[d] & G(\bullet) \arrow[d] \\
F(\star) \arrow[r, hook, "\tau(\star)"'] & G(\star).
\end{tikzcd}$$

both $\tau(\bullet)$ and $\tau(\star)$ are injective. We claim the right-hand side of the following cubic diagram is the subobject classifier.

$$\begin{tikzcd}
 & T \arrow[rr, "!"] \arrow[dd, hook, "\tau_2"] & & \{1\} \arrow[dd, hook, "i_1"] \\
S \arrow[ur] \arrow[rr, "!"] \arrow[dd, hook, "\tau_1"'] & & \{1\} \arrow[ur, "!"] \arrow[dd, hook, "i_2"'] & \\
 & B \arrow[rr, dotted, "\chi_2"] & & \{0,1\} \\
A \arrow[ur, "h"] \arrow[rr, dotted, "\chi_1"'] & & \{0,1,2\} \arrow[ur, "f"] & .
\end{tikzcd}$$

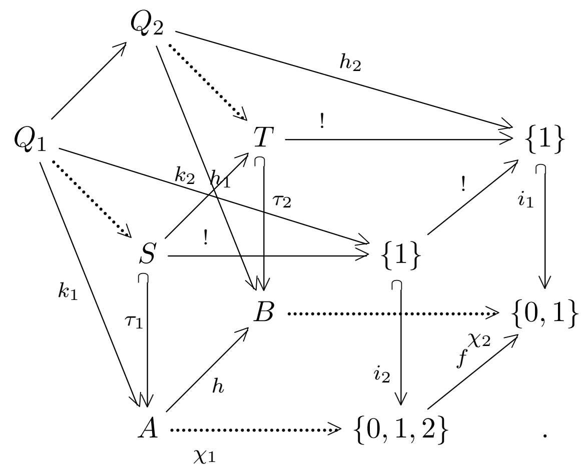
The morphisms i_1 and i_2 are inclusions. The map f is defined as:

$$\begin{array}{rccc} f: & 0 & \mapsto & 0 \\ & 1 & \mapsto & 1 \\ & 2 & \mapsto & 1 \end{array}.$$

In order to prove the claim, we may take S and T to be subsets of A and B respectively, like in the previous example. Let χ_2 be the characteristic function of T and let χ_1 be defined as follows:

$$\begin{array}{rcccl} \chi_1: & a & \mapsto & 0 & \text{if } a \notin S \text{ and } h(a) \notin T. \\ & a & \mapsto & 1 & \text{if } a \in S. \\ & a & \mapsto & 2 & \text{if } a \notin S \text{ and } h(a) \in T. \end{array}$$

This obviously makes the cube commutative. To check if this cube is a pullback we look at the following diagram.



.

For $n \in \{1,2\}$, for all $q_n \in Q_n$ notice that $\chi_n k_n(q_n) = i_n h_n(q_n) = 1$. Hence $k_n(q_n)$ is in S or T, depending on wether n is 1 or 2, respectively. We conclude that k_n factors (obviously uniquely) through τ_n in such a way that the diagram commutes. The map χ_n is unique in making the cube a pullback. For all $a \in A$, $\chi_1(a) = 1$ if and only if $a \in S$ by demanding the square becomes a pullback. Also for all $b \in B$, $\chi_1(b) = 1$ if and only if $b \in T$. Hence the map χ_2 is determined. The commutativity of the bottom square now determines the map χ_1 and we conclude that these χ's are unique.

The elements of A that are mapped to 2 by χ_1 could be interpreted as "those that eventually will be part of the subobject". The condition that the bottom square has to commute forces the front bottom right object ($\Omega(\bullet)$) to be a larger set than $\{0,1\}$. The question 'is a certain element part of the subobject?' cannot simply be answered by *yes* or *no*. The answer has become more subtle, varying between *no*, *yes* and *not yet: after applying h it will be.*

The last example we examine, is a generalization of the previous ones and will be important for the most important topos in this thesis. Let $\mathcal{C}$ be an arbitrary category. We can prove that $\mathbf{Set}^{\mathcal{C}}$ has a subobject classifier. To see this, we introduce the notion of a *cosieve*.

Definition 1.2.5. *Let $\mathcal{C}$ be a category and let C be an object in $\mathcal{C}$. A **cosieve** on C is a subset S of*

$$\{f \in Mor(\mathcal{C}) \mid source(f) = C\},$$

which has the following property: if $f \in S$, then $hf \in S$ for all h which have the same source as the target of f.

Now let

$$\Omega(C) = \{S | S \text{ is a cosieve on } C\}.$$

For a morphism $f : C \to D$ in $\mathcal{C}$, recall that we defined $f^{\Omega}_{C,D} := \Omega(f)$ and define

$$f^{\Omega}_{C,D} : \Omega(C) \to \Omega(D) : S \mapsto \{h \in Mor(\mathcal{C}) \mid source(f) = D, hf \in S\}$$

and note that if S is a cosieve on C, then $f(S)$ is indeed a cosieve on D. The cosieve on an object C containing all morphisms with source C is called the *full cosieve* (on C). Remark that a cosieve on C is full if and only if it contains the identity on C.

The terminal object in $\mathbf{Set}^{\mathcal{C}}$ is obviously the functor $\Delta_{\{1\}}$ sending each object C in $\mathcal{C}$ to the one-point set $\{1\}$. Define a morphism (natural transformation) $t : \mathbf{1} \to \Omega$ by

$$t(C) : \mathbf{1}(C) \to \Omega(C) : 1 \mapsto \text{the full cosieve on } C.$$

We can prove that the diagram

$$\mathbf{1} \xrightarrow{t} \Omega.$$

is the subobject classifier. Doing this is just a generalization of the proof that $\mathbf{Set}^{\mathcal{C}_2}$ has a subobject classifier. Again, a morphism in $\mathbf{Set}^{\mathcal{C}}$ (i.e. a natural transformation) is monic if and only if all components are injective. Again if F is an object in $\mathbf{Set}^{\mathcal{C}}$ and G is a subobject of F, we may take $G(C)$ to be a subset of $F(C)$ for all objects C. The morphisms $\chi(C)$ that form the characteristic natural transformation between F and Ω are defined as follows:

$$\chi(C) : F(C) \to \Omega(C) : a_c \mapsto \left\{ f : C \to D | D \in Obj(\mathcal{C}),\ f^F_{C,D}(a_c) \in G(D) \right\}.$$

As in the example of $\mathbf{Set}^{\mathcal{C}_2}$, this combines to a commutative diagram of natural transformations. The diagram forms a pullback and χ is proved to be unique in having this property. The conclusion is that $\mathbf{Set}^{\mathcal{C}}$ has a subobject classifier. As a special case $\mathbf{Set}^{\mathcal{C}_2}$ has a subobject classifier and we compute:

1. $\Omega(\bullet) = \{S \mid S \text{ is a cosieve on } \bullet\}$ consists of the cosieves

$$\begin{aligned} 0 &:= \emptyset; \\ 1 &:= \{\bullet \to \star\}; \\ 2 &:= \{Id_\bullet, \bullet \to \star\}. \end{aligned}$$

2. $\Omega(\star) = \{S \mid S \text{ is a cosieve on } \star\}$ consists of the cosieves

$$\begin{aligned} 0 &:= \emptyset; \\ 1 &:= \{Id_\star\}. \end{aligned}$$

3. The map $f^\Omega_{\bullet,\star}$ indeed is defined as:

$$\begin{aligned} f^\Omega_{\bullet,\star} :\ 0 &\mapsto 0; \\ 1 &\mapsto 1; \\ 2 &\mapsto 1. \end{aligned}$$

As another example we prove that **Grp** does not have a subobject classifier. Consider the pullback:

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, hook] & \mathbf{1} = \{0\} \arrow[d, hook, "t"] \\ B \arrow[r, "\chi"'] & \Omega. \end{tikzcd}$$

Suppose it does have a subobject classifier. Then in order to make the above diagram a pullback A must be isomorphic to the kernel of χ. However, if the image of A under the left monic is not a normal subgroup of B, this certainly is not the case. Even if we consider the category of abelian groups **Ab**, in which every subgroup is normal, there is no subobject classifier. Because then the image of χ is isomorphic to the group B modulo A. Since every group is the quotient of itself by the trivial group, every group must be a subgroup of Ω, which is impossible.

1.2.3 Powerobjects

Now we like to generalize the notion of the powerset of a set. This is done using the definition of adjoint functors and the notion of cartesian closed categories.

Definition 1.2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that F and G are* ***adjoint functors*** *if there exists a natural isomorphism*

$$\tau : \boldsymbol{Hom}_{\mathcal{D}}(F(-), -) \to \boldsymbol{Hom}_{\mathcal{C}}(-, G(-)).$$

Remark that $\boldsymbol{Hom}_{\mathcal{D}}(F(-), -)$ and $\boldsymbol{Hom}_{\mathcal{C}}(-, G(-))$ can be regarded as a functor from $\mathcal{C}^{op} \times \mathcal{D}$ to ***Set****. In this case F is called a* ***left adjoint****, and G is called a* ***right adjoint****.*

If a functor F has a right adjoint G, then it is unique up to isomorphism. That is, if G' is a right adjoint for F too, then there is a natural isomorphism from G to G'. Conversely, if G has a left adjoint, it is unique up to isomorphism, too. A proof can be found in [10].

Adjoint functors are very important in category theory. The most useful result is theorem 2.4.1, showing that if a functor has a left adjoint, then the functor preserves all limits. Similarly, if a functor has a right adjoint, then it preserves all colimits. The exact meaning of this formulation is stated in theorem 2.4.1.

Definition 1.2.7. *A category $\mathcal{C}$ is said to be* ***cartesian closed*** *if it has finite products and if for any given object B the functor $A \mapsto A \times B$ (with the obvious map on morphisms) has a right adjoint, called $-^B$. That is, there is a natural isomorphism*

$$\boldsymbol{Hom}_{\mathcal{C}}(A \times B, C) \simeq \boldsymbol{Hom}_{\mathcal{C}}(A, C^B). \tag{1.1}$$

If $h : C \to C'$, we denote the morphism after applying $-^B$ by $h^B : C^B \to C'^B$.

Note that we require naturality in A and C, not necessarily in B.

The category **Set** is cartesian closed and the above notation reduces to the familiar meaning on sets. I.e. for a map $f : A \times B \to C$:

$$\begin{array}{rlcl} \tau : & \mathbf{Hom}_{\mathbf{Set}}(A \times B, C) & \to & \mathbf{Hom}_{\mathbf{Set}}(A, C^B) \\ & ((a, b) \mapsto f(a, b)) & \mapsto & (a \mapsto (b \mapsto f(a, b))) \end{array}$$

is a natural isomorphism.

Next, we prove some properties using of the Yoneda lemma, which is stated in chapter 2 (lemma 2.1.1).

Proposition 1.2.8. *The following properties hold in any cartesian closed category $\mathcal{C}$:*

1. *If both C^B and $(C^B)'$ have the property of (1.1), then $C^B \simeq (C^B)'$;*
2. *$C^{\mathbf{1}} \simeq C$;*

3. $C^B \times D^B \simeq (C \times D)^B$.

Proof: For (1) we have

$$\begin{aligned}\mathbf{Hom}_{\mathcal{C}}(A, C^B) &\simeq \mathbf{Hom}_{\mathcal{C}}(A \times B, C) \\ &\simeq \mathbf{Hom}_{\mathcal{C}}(A, (C^B)').\end{aligned}$$

Now, property (1) follows from the Yoneda lemma 2.1.1. The other properties follow from this lemma by:

$$\begin{aligned}\mathbf{Hom}_{\mathcal{C}}(A, C^{\mathbf{1}}) &\simeq \mathbf{Hom}_{\mathcal{C}}(A \times \mathbf{1}, C) \\ &\simeq \mathbf{Hom}_{\mathcal{C}}(A, C)\end{aligned} \; ;$$

$$\begin{aligned}\mathbf{Hom}_{\mathcal{C}}(A, C^B \times D^B) &\simeq \mathbf{Hom}_{\mathcal{C}}(A, C^B) \times \mathbf{Hom}_{\mathcal{C}}(A, D^B) \\ &\simeq \mathbf{Hom}_{\mathcal{C}}(A \times B, C) \times \mathbf{Hom}_{\mathcal{C}}(A \times B, D) \\ &\simeq \mathbf{Hom}_{\mathcal{C}}(A \times B, C \times D) \\ &\simeq \mathbf{Hom}_{\mathcal{C}}(A, (C \times D)^B).\end{aligned}$$

□

So where is the power object? If a category is both cartesian closed and has a subobject classifier there is a correspondence between subobjects of an object A and morphisms $\chi : A \to \Omega$. Since the category is cartesian closed such a morphism corresponds to a morphism $\hat{\chi} : \mathbf{1} \to \Omega^A$. Now in **Set** these morphisms correspond to points in Ω^A since $\mathbf{1} = \{1\}$. Remark that $\Omega^A = \{0, 1\}^A$ is simply the powerset of A. Hence we have found our generalization of the powerset of A: it is Ω^A.

1.2.4 Topoi

We conclude this paragraph by stating the definition of a topos. There are many ways to define a topos, that of course are all equivalent. Here we state a very categorical definition, which is based on the previous paragraphs and is most suitable for this thesis.

Definition 1.2.9. *A category $\mathcal{T}$ is a* ***topos*** *if:*

1. *it has all finite limits and colimits;*
2. *it is cartesian closed;*
3. *it has a subobject classifier.*

We have already seen that **Set** is a topos. Also, $\mathbf{Set}^{\mathcal{C}}$ turns out to be a topos. For $\mathbf{Set}^{\mathcal{C}}$ we have proved that it has a subobject classifier. It has all finite limits, since limits can be taken point-wise (see theorem 2.4.5). Here we will prove it is cartesian closed too. The reason we do not postpone this proof to the second chapter is that we need the details of the proof for further use. All proofs in the second chapter will not be of any further interest, although their results will be. The proof below is very technical in nature and seems not very elegant at all. It shows how abstract category theory can be. However, after chapter 2 we skip the technical details more and more to avoid long pages of computations.

Proposition 1.2.10. *The category $\mathbf{Set}^{\mathcal{C}}$ is cartesian closed.*

Proof: First remark that theorem 2.4.5 below tells us that this category has finite products. Hence it remains to show that if F, G and H are functors from $\mathcal{C}$ to **Set** we must define a functor H^G such that there is a natural isomorphism

$$\tau : \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(F \times G, H) \simeq \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(F, H^G).$$

In what follows, A and B are objects of $\mathcal{C}$. One might expect $H^G = \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(G(-), H(-))$. However, there is no way we can regard $\mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(G(-), H(-))$ as a functor from $\mathcal{C}$ to $\mathbf{Set}$ (although it can be regarded as a functor from $\mathcal{C}^{op} \times \mathcal{C}$ to $\mathbf{Set}$). We show that the correct definition is

$$H^G(A) = \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(\mathbf{Hom}_{\mathcal{C}}(A, -) \times G(-), H(-)),$$

the set of natural transformations between the functors $\mathbf{Hom}(A, -) \times G$ and H. Note that the outer $\mathbf{Hom}$ is taken in $\mathbf{Set}^{\mathcal{C}}$ while the inner $\mathbf{Hom}$ is taken in $\mathcal{C}$. Since H^G must be a functor, we must specify the function on the morphisms $f : A \to B$ of $\mathcal{C}$.

$$\begin{array}{rrcl} H^G(f): & \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(\mathbf{Hom}_{\mathcal{C}}(A, -) \times G(-), H(-)) & \to & \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(\mathbf{Hom}_{\mathcal{C}}(B, -) \times G(-), H(-)); \\ & \sigma & \mapsto & (f^* \times Id_G)^* \sigma. \end{array}$$

Remark that for composable morphisms f and g in $\mathcal{C}$,

$$\begin{array}{rcl} H^G(g) \circ H^G(f) & = & (g^* \times Id_G)^* \circ (f^* \times Id_G)^* \\ & = & ((f^* \times Id) \circ (g^* \times Id_G))^* \\ & = & ((f^* \circ g^*) \times Id_G)^* \\ & = & (((g \circ f)^*) \times Id_G)^* \\ & = & H^G(g \circ f); \\ & & \\ H^G(Id_A) & = & (Id_A^* \times Id_G)^* \\ & = & Id_{H^G(A)}. \end{array}$$

Hence H^G is a functor, since other properties are trivial. We conclude H^G is an object in $\mathbf{Set}^{\mathcal{C}}$.

Next we define the natural isomorphism τ.

$$\begin{array}{rrcl} \tau: & \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(F \times G, H) & \to & \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(F, H^G); \\ & \sigma & \mapsto & \hat{\sigma}; \\ & & & \\ \hat{\sigma}(A): & F(A) & \to & H^G(A) = \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(\mathbf{Hom}_{\mathcal{C}}(A, -) \times G, H); \\ & a & \mapsto & \hat{\sigma}(A)(a); \\ & & & \\ ((\hat{\sigma}(A))(a))(B): & \mathbf{Hom}_{\mathcal{C}}(A, B) \times G(B) & \to & H(B); \\ & (g, x) & \mapsto & \sigma(B)(g^F_{A,B}(a), x). \end{array}$$

First we check that $\hat{\sigma}$ is indeed a natural transformation. This is done by checking wether the following diagram commutes for every $f : A \to B$ in $\mathcal{C}$.

$$\begin{tikzcd} F(A) \arrow[r, "\hat{\sigma}(A)"] \arrow[d, "f^F_{A,B}"'] & \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(\mathbf{Hom}_{\mathcal{C}}(A, -) \times G, H) \arrow[d, "(f^* \times Id_G)^*"] \\ F(B) \arrow[r, "\widehat{\sigma(B)}"'] & \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(\mathbf{Hom}_{\mathcal{C}}(B, -) \times G, H) \end{tikzcd} . \qquad (1.2)$$

This is done by the following computation.

$$\begin{array}{rcl}
(((f^* \times Id_G)^*((\hat{\sigma}(A))(a)))(C))(g,x) & = & ((((\hat{\sigma}(A))(a))(f^* \times Id_G))(C))(g,x) \\
& = & (((\hat{\sigma}(A))(a))(C)(f^* \times Id_G)(C))(g,x) \\
& = & ((\hat{\sigma}(A))(a))(C)(gf,x) \\
& = & \sigma(C)((gf)^F_{A,C},x) \\
& = & \sigma(C)(g^F_{B,C}f^F_{A,B}a,x) \\
& = & (((\hat{\sigma}(B))(f^F_{A,B}(a)))(C))(g,x).
\end{array}$$

We must check if the morphism τ is natural in F and H. So let $\eta : F' \to F$ and $\mu : H \to H'$ be morphisms in $\mathbf{Set}^{\mathcal{C}}$. Then we must check if the following diagram commutes:

$$\begin{tikzcd}
\mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(F \times G, H) \arrow[r, "\tau"] \arrow[d, "(\eta \times Id_G)^*\mu_*"'] & \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(F, H^G) \arrow[d, "\nu^*\mu_{**}"] \\
\mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(F' \times G, H') \arrow[r, "\tau"'] & \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(F', H'^G).
\end{tikzcd}$$

This is indeed the case, as the following computation shows:

$$\begin{array}{rcl}
(((\tau(\mu\sigma(\nu \times Id_G))(A))(a))(B))(g,x) & = & ((\mu\sigma(\nu \times Id_G))(B))(g^{F'}_{A,B}a,x) \\
& = & \mu(B)(\sigma(B)(\nu(B)g^{F'}_{A,B}a,x)) \\
& = & \mu(B)(\sigma(B)(g^F_{A,B}\nu(A)a,x)) \\
& = & \mu(B)(((\hat{\sigma}(A))(\nu(A)(a))(B))(g,x)) \\
& = & ((((\mu_*\hat{\sigma}\nu)(A))(a))(B))(g,x).
\end{array}$$

The next step is to define the inverse of τ. This is done by

$$\begin{array}{lrcl}
\tau^{-1}: & \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(F, H^G) & \to & \mathbf{Hom}_{\mathbf{Set}^{\mathcal{C}}}(F \times G, H); \\
& \sigma & \mapsto & \check{\sigma}; \\
& & & \\
\check{\sigma}(A): & F(A) \times G(A) & \to & H(A); \\
& (a,b) & \mapsto & \sigma(A)(a)(A)(Id_A, b).
\end{array}$$

It follows from a similar computation that τ^{-1} indeed defines a natural transformation and that $\check{\sigma}$ is a natural transformation. The last thing to check is whether τ^{-1} is indeed the inverse of τ:

$$\begin{array}{rcl}
(\check{\hat{\sigma}}(A))(a,b) & = & (((\hat{\sigma}(A))(a))(A))(Id_A, b); \\
& = & (\sigma(A))(a,b), \\
& & \\
(((\hat{\check{\sigma}}(A))(a))(B))(g,b) & = & (\check{\sigma}(B))g^F_{A,B}(a),x) = \\
& = & (((\sigma(B))(g^F_{A,B}(a)))(B))(Id_B, b) = \\
& = & (((\sigma(A))(a))(B))(g,b),
\end{array}$$

where the last equality follows from diagram 1.2. ☐

In particular it now follows that $\mathbf{Set}^{\mathcal{C}}$ is a topos for any category $\mathcal{C}$.

1.2.5 Rational numbers objects

In this paragraph we will try find a generalization of the rational numbers $\mathbb{Q}$ in **Set** to a topos $\mathcal{T}$. This can not be done for every topos, but in the case of **Set**, $\mathbf{Set}^{\mathcal{C}}$ and $\mathbf{Set}^{\mathcal{C}^{op}}$, with $\mathcal{C}$ a category, it can. To do so one requires the topos $\mathcal{T}$ to have a *natural numbers object*, i.e. an object $\mathbf{N}$ together with morphisms 0 and s such that for every diagram

$$\begin{tikzcd} \mathbf{1} \arrow[r, "0"] \arrow[d, equal] & \mathbf{N} \arrow[r, "s"] \arrow[d, dotted, "i"] & \mathbf{N} \arrow[d, dotted, "i"] \\ \mathbf{1} \arrow[r, "0'"] & X \arrow[r, "s'"] & Y, \end{tikzcd}$$

there exists a unique morphism i making the diagram commutative. This object plays a similar role as the natural numbers in **Set**. Not every topos has a natural numbers object, but $\mathbf{Set}^{\mathcal{C}}$, $\mathbf{Set}^{\mathcal{C}^{op}}$ and (obviously) **Set** do have one. In $\mathbf{Set}^{\mathcal{C}}$ and $\mathbf{Set}^{\mathcal{C}^{op}}$ the natural numbers object is the constant functor $\Delta_{\mathbb{N}}$. Notice that there is a morphism $+ : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$, which in the case of $\mathbf{Set}^{\mathcal{C}}$ is defined by letting $+(C) : \mathbf{N}(C) \times \mathbf{N}(C) \to \mathbf{N}(C)$ be addition in **Set**. We may use this to define the *integral numbers object* $\mathbf{Z}$ as the coequalizer of the diagram:

$$\begin{tikzcd} E \arrow[r, shift left, "(p_1 f, p_2 g)"] \arrow[r, shift right, "(p_2 g, p_1 f)"'] & \mathbf{N} \times \mathbf{N} \arrow[r] & \mathbf{Z}. \end{tikzcd}$$

Here E is the upper left object in the following pullback:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{N} \times \mathbf{N} \arrow[d, "+"] \\ \mathbf{N} \times \mathbf{N} \arrow[r, "+"'] & \mathbf{N}. \end{tikzcd}$$

E is the object such that $E(C)$ consists of 4-tuples (m, n, m', n') such that $m + n = m' + n'$. In the case of $\mathbf{Set}^{\mathcal{C}}$ and $\mathbf{Set}^{\mathcal{C}^{op}}$ a short calculation gives $\mathbf{Z} = \Delta_{\mathbb{Z}}$. Note that this object comes with a morphism $m : \mathbf{Z} \times \mathbf{N} \to \mathbf{Z}$ that is locally the multiplication operator, i.e. $m(C) : \mathbf{Z}(C) \times \mathbf{N}(C) \to \mathbf{Z}(C)$ is multiplication in **Set**. In a similar way one defines the *rational numbers object* $\mathbf{Q}$, which in case of $\mathbf{Set}^{\mathcal{C}}$ and $\mathbf{Set}^{\mathcal{C}^{op}}$ becomes

$$\mathbf{Q} = \Delta_{\mathbb{Q}}.$$

In section 4.5 we will argue why we would call this a generalization of the set-theoretic rational numbers $\mathbb{Q}$. In chapter 5 we will need the rational numbers object in order to define a generalized norm in topos theory.

Chapter 2

Topos theorems

"I mean the word proof not in the sense of the lawyers, who set two half proofs equal to a whole one, but in the sense of a mathematician, where half proof = 0, and it is demanded for proof that every doubt becomes impossible." - Karl Friedrich Gauss

Throughout this chapter $\mathcal{T}$ will be a topos and any morphism or object will be taken within this topos unless specified otherwise. The aim of this chapter is to prove some often used theorems in topos theory. Some of these theorems may hold in a category which has only some of the properties of a topos, but for convenience we prove the statements within a topos $\mathcal{T}$. Most of this material can be found in [10] and [11], however sometimes without proof.

2.1 The Yoneda lemma

If A and B are objects of a category $\mathcal{C}$ one might wonder what the natural transformations between the Yoneda functors $\mathbf{Hom}_{\mathcal{C}}(A,-)$ and $\mathbf{Hom}_{\mathcal{C}}(B,-)$ look like. The Yoneda lemma and its immediate corollary give the answer to this question.

Lemma 2.1.1. *(**Yoneda lemma**) If $F:\mathcal{C}\to$**Set** is a functor and A is an object of the category $\mathcal{C}$, then there is a bijection of sets*

$$k:\boldsymbol{Hom}_{\boldsymbol{Set}^{\mathcal{C}}}(\boldsymbol{Hom}_{\mathcal{C}}(A,-),F)\to F(A),$$

which sends each natural transformation $\tau:\boldsymbol{Hom}_{\mathcal{C}}(A,-)\to F$ to $(\tau(A))(Id_A)$.

Proof: Let $f:A\to B$ be a morphism in $\mathcal{C}$. Then the following diagram shows that $k(\tau)$ determines the natural transformation τ, i.e. if you know what $(\tau(A))(Id_A)$ is, τ is completely determined. Hence k is injective. It is surjective too, since for every $x\in F(A)$ one can define τ by putting $(\tau(A))(Id_A)=x$.

$$\begin{tikzcd}
\mathbf{Hom}_{\mathcal{C}}(A,A) \arrow[r, "\tau(A)"] \arrow[d, "f_*"'] & F(A) \arrow[d, "f^F_{A,B}"] \\
\mathbf{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\tau(B)"'] & F(B).
\end{tikzcd}$$

□

Corollary 2.1.2. *Let A, B be objects of a category $\mathcal{C}$, then every natural transformation from $\boldsymbol{Hom}_{\mathcal{C}}(A,-)$ to $\boldsymbol{Hom}_{\mathcal{C}}(B,-)$ is of the form f^* for a map $f \in \boldsymbol{Hom}_{\mathcal{C}}(B,A)$.*

Proof: This is the special case of lemma 2.1.1 with $F = \mathbf{Hom}_{\mathcal{C}}(B,-)$. □

2.2 Pullbacks

Here we will examine two properties of pullbacks. Together with products and equalizers the pullback is the most important type of limit in topos theory.

Theorem 2.2.1. *Let the diagram below be a pullback square:*

$$\begin{tikzcd} A \arrow[r, "n"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, "m"'] & D. \end{tikzcd}$$

If m is monic, then n is monic.

Proof: Suppose we have morphisms $h, k : E \to A$, such that $nh = nk$. Then $m(fk) = g(nk) = g(nh) = m(fh)$, so $fk = fh$. By the universal property of a pullback there is a unique morphism from $l : E \to A$ such that $nh = nl = nk$ and $fh = fl = fk$. The uniqueness of the pullback proves $h = k$. □

The next lemma will be used extensively in the next chapters. In the literature one refers to this lemma as the 'pullback lemma' or simply 'PBL'.

Lemma 2.2.2. ***(Pullback lemma, PBL)***

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F. \end{tikzcd}$$

1. *If in the above diagram both squares are pullbacks, then the outer rectangle is a pullback.*
2. *If the outer rectangle is a pullback and the right square is a pullback, then the left square is a pullback.*

Proof:

1.

$$\begin{tikzcd} G \arrow[dr, dotted] \arrow[drr, dotted] \arrow[drrr] \arrow[ddr] & & & \\ & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ & D \arrow[r] & E \arrow[r] & F. \end{tikzcd}$$

Suppose we have morphisms from G to C and D as in the diagram. The pullback property of the right and left square gives the unique dotted morphisms to B and A respectively.

2.

$$
\begin{tikzcd}
G \arrow[dr, dotted] \arrow[drr] \arrow[drrr, dashed] \arrow[ddr] & & & \\
& A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
& D \arrow[r] & E \arrow[r] & F.
\end{tikzcd}
$$

Suppose we have morphisms from G to B and D as indicated in the diagram. This gives a morphism from G to C such that the diagram commutes. Since the outer rectangle is a pullback, there is a unique morphism from G to A such that the diagram commutes. The morphism $G \to D$ equals the factorization $G \to A \to D$. Remark that the morphism $G \to A \to B$ is a morphism such that $G \to A \to B \to C$ equals $G \to C$ and $G \to A \to B \to E$ equals $G \to D \to E$. Since the right square is a pullback, such a morphism is unique and hence it equals the morphism $G \to B$. ☐

2.3 Epi-monic factorization

In **Set** every morphism can be written as a composition of an epi with a monic. The epi is the map to the direct image of the morphism and the monic is the inclusion in the target set of the morphism. This can be generalized to a topos. This is the first result that shows topoi look like sets.

Definition 2.3.1. *In a topos $\mathcal{T}$, let $f : A \to B$ be a morphism. A monic m is called the* ***image*** *of f if f factors through m and whenever f factors through a monic m', m factors through m'. We denote the source of m by* ***Im****(f), thus $f : A \to \boldsymbol{Im}(f) \xrightarrow{m} B$.*

This definition leads to an important theorem about (unique) epi-monic factorization.

Theorem 2.3.2. *In a topos $\mathcal{T}$, let $f : A \to B$ be a morphism. Then $f = me$ where m is an image of f (thus monic) and e is epi.*

Proof: the proof makes use of two facts:

1. For every morphism $f : A \to B$ there exists a so-called *cokernel pair* of f. This is a pair of morphisms s, t from B to an object C which has the universal property $sf = tf$, i.e. if there is another pair $u, v : B \to C'$ such that $uf = vf$, then there is a unique morphism w such that the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[dr, shift left, "u"] \arrow[dr, shift right, "v"'] & C \arrow[d, dotted, "w"] \\
& & C'.
\end{tikzcd}
$$

The proof of this fact is by taking s, t to be the morphisms obtained from the colimit of

$$\begin{tikzcd} & B \\ A \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & B. \end{tikzcd}$$

This colimit is called a *pushout*.

2. Every monic $m : C \hookrightarrow D$ is the equalizer of two morphisms. It follows from the definition of a subobject classifier that m equalizes the characteristic morphism of m and the map $true_D$.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, equal] & & A \arrow[d, equal] & \\
A \arrow[r, dotted, "e"'] \arrow[d, equal] & M \arrow[r, dotted, "m"'] & B \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, equal] & C \arrow[d, "w"] \\
A \arrow[r, "g"'] & N \arrow[r, "h"'] & B \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & D.
\end{tikzcd}$$

Take $s, t : B \to C$ the cokernel pair of f and let m be the equalizer of s, t. Since f equalizes s, t, it factors through m, say $f = me$. Now take another factorization $f = hg$. By (2) h is an equalizer, say of u, v. Since h equalizes u, v, so does f. By the uniqueness property of a cokernel pair there is a morphism w making the diagram above commutative. Now note that $um = wsm = wtm = vm$, hence m equalizes u and v, hence m factors through h. This proves that m is the image of f.
Recall that an equalizer is always monic. It remains to prove that e is epi. Suppose m is an isomorphism. Then $s = t$, since m is the equalizer of s and t. Now since s, s is the cokernel pair of f, e must be epi.
Now suppose that we have $f = me$ without m being an isomorphism. Take an epi-monic factorization of e, $e = m'e'$.

$$A \xrightarrow{e'} M' \xrightarrow{m'} M \xrightarrow{m} B.$$

We find f factors through mm' and so does the image m, say $m = mm'q$ for some q. By m being monic this implies $1 = m'q$. This gives $m'qm' = m'$ and since m' is monic $1 = qm'$. So m' is an isomorphism and e' is epi. Hence $e = m'e'$ is epi. □

In the previous theorem we proved the existence of an epi-monic factorization. The factorization turns out to be unique up to isomorphism as the following theorem and its corollary shows.

Theorem 2.3.3. *If $f = me$ and $f' = m'e'$ with m, m' monic and e, e' epi, then for morphisms r, t in the diagram below, there is a unique map s such that the diagram commutes.*

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[dd, "r"'] & C \arrow[rr, "m"] \arrow[dd, dotted, "s"] & & B \arrow[dd, "t"] \\
& & P \arrow[ur] \arrow[dl] & \\
A' \arrow[r, "e'"'] & C' \arrow[rr, "m'"'] & & B'.
\end{tikzcd}$$

Proof: First suppose m is the image of f. Let P be the pullback of t along m', like in the diagram. Because m' is monic $P \to B$ is monic (theorem 2.2.1). Since $tf = m'e'r$, f factors through $P \to B$ by the pullback property. Since m was assumed to be the image of f and $P \to B$ is monic, m factors through $P \to B$. Hence there is a map s such that the right square commutes. Since m' is monic, this s is unique. Since $m'se = tme = m'e'r$ it follows from the fact that m' is monic that $se = e'r$ and so the left square commutes.
Now suppose $f = \tilde{m}\tilde{e}$ is an arbitrary epi-monic factorization of f. Then look at:

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, equal] & C \arrow[r, "m"] \arrow[d, "s"'] & B \arrow[d, equal] \\
A' \arrow[r, "\tilde{e}"] & C' \arrow[r, "\tilde{m}"] & B'.
\end{tikzcd}$$

s is monic, since $\tilde{m}s = m$. s is epi since $se = \tilde{e}$. Now as in fact (1) of theorem 2.3.2, s is the equalizer of some cokernel pair. Because it is epi, the cokernel pair must be a pair of equal morphisms. But if s is the equalizer of two identical morphisms, s must be an isomorphism. Hence $\tilde{m}$ is the image of f. □

Corollary 2.3.4. *An epi-monic factorization of a morphism f is unique up to isomorphism.*

Proof: This follows straight from theorem 2.3.3 taking $f' = f$, $r = Id_A$ and $t = Id_B$. □

2.4 Limits

In chapter 1 we introduced limits and stated some of their properties. We promised to give the proofs in this chapter. First we take care of the preservation of limits under adjoint functors.

Theorem 2.4.1. *Let $F : \mathcal{J} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. Let $H : \mathcal{D} \to \mathcal{C}$ be a left adjoint for G. Suppose* $\lim F = C$. *Then* $\lim GF = GC$.

Proof: Call the natural isomorphism obtained from the adjunction

$$\varphi : \mathbf{Hom}_{\mathcal{D}}(D, GC) \tilde{\to} \mathbf{Hom}_{\mathcal{C}}(HD, C).$$

Let D be an object in $\mathcal{D}$ and τ a natural transformation from Δ_D to GF. Compare the following diagrams in the category $\mathcal{D}$ and $\mathcal{C}$ respectively:

$$
\begin{tikzcd}
 & & GFj_\alpha \arrow[dd, "GFf_{\alpha\beta}"] \\
D \arrow[urr, "\tau_\alpha"] \arrow[drr, "\tau_\beta"'] \arrow[r, dotted] & GC \arrow[ur, "G\sigma_\alpha"] \arrow[dr, "G\sigma_\beta"'] & \\
 & & GFj_\beta.
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & Fj_\alpha \arrow[dd, "Ff_{\alpha\beta}"] \\
HD \arrow[urr, "\varphi\tau_\alpha"] \arrow[drr, "\varphi\tau_\alpha"'] \arrow[r, dotted, "\exists!k"] & C \arrow[ur, "\sigma_\alpha"] \arrow[dr, "\sigma_\beta"'] & \\
 & & Fj_\beta.
\end{tikzcd}
$$

By adjunction the first diagram commutes if and only if the second diagram commutes. In the second diagram there exist a unique morphism k such that the diagram commutes. By adjunction this gives the unique (!) (dotted) morphism $\varphi^{-1}(k)$ making the first diagram commutative. □

Of course, there is a dual version of the previous theorem in terms of colimits and right adjoints.

Theorem 2.4.2. *Let $F : \mathcal{J} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. Let $H : \mathcal{D} \to \mathcal{C}$ be a right adjoint for G. Suppose colim $F = C$. Then colim $GF = GC$.*

Proof: The proof is obtained in exactly the same way as theorem 2.4.1 by comparing the following diagrams.

$$
\begin{tikzcd}
GFj_\alpha \arrow[dd, "GFf_{\alpha\beta}"'] \arrow[dr, "G\sigma_\alpha"] \arrow[drr, "\tau_\alpha"] & & \\
 & GC \arrow[r, dotted] & D \\
GFj_\beta \arrow[ur, "G\sigma_\beta"] \arrow[urr, "\tau_\beta"'] & & .
\end{tikzcd}
\qquad
\begin{tikzcd}
Fj_\alpha \arrow[dd, "Ff_{\alpha\beta}"'] \arrow[dr, "\sigma_\alpha"] \arrow[drr, "\varphi\tau_\alpha"] & & \\
 & C \arrow[r, dotted, "\exists!k"] & HD \\
Fj_\beta \arrow[ur, "\sigma_\beta"] \arrow[urr, "\varphi\tau_\beta"'] & & .
\end{tikzcd}
$$

□

One of the properties of a category in order to be a topos is having all finite limits and colomits. In general it is very hard to check if a given category has all finite (co)limits. The following theorem will make life a lot easier; one only has to check wether a category has (co)equalizers and finite (co)products.

Theorem 2.4.3. *Let $\mathcal{J}$ and $\mathcal{C}$ be categories. If $\mathcal{C}$ has equalizers and if $\mathcal{C}$ has all products of length card(Obj($\mathcal{J}$)) and card(Mor($\mathcal{J}$)) then the limit of every functor $F : \mathcal{J} \to \mathcal{C}$ exists.*

Proof: In the following diagram A, B and C are objects in $\mathcal{J}$ and the product $\prod_A$ is taken over all objects of $\mathcal{J}$. f and g are morphisms of $\mathcal{J}$. $\prod_f$ is taken over all morphisms of $\mathcal{J}$. p denotes the projection on the object in its index.

$$
\begin{tikzcd}
F(target(g)) \arrow[r, phantom, "="] & F(target(g)) & F(B) \\
\prod_f F(target(f)) \arrow[u, "p_{target(g)}"] \arrow[d, "p_{target(g)}"'] & \prod_A F(A) \arrow[l, shift left, "h"'] \arrow[l, shift right, "k"] \arrow[u, "p_{target(g)}"'] \arrow[ur, "p_B"] \arrow[d, "p_{source(g)}"] & C \arrow[l, "e"'] \arrow[u] \\
F(target(g)) & F(source(g)) \arrow[l, "F(g)"] & .
\end{tikzcd}
$$

By definition of the left product in the diagram, there is a morphism h such that the upper square commutes for every g and there is a morphism k such that the lower square commutes for every g. Let e be the equalizer of h and k. Write $\tau(B) := p_B e$. Then for a morphism $g : B \to B'$ of $\mathcal{J}$:

$$\begin{array}{rcl} F(g) \circ \tau(B) & = & F(g)p_B e \\ & = & p_{B'} ke \\ & = & p_{B'} he \\ & = & p_{B'} e \\ & = & \tau(B'). \end{array}$$

So the morphisms $\tau(B)$ combine into a natural transformation from Δ_C to F.

Now suppose we have an object D together with a natural transformation σ from Δ_D to F. Then the morphisms $\sigma(A)$ combine to a morphsim $m : D \to \prod_A F(A)$. The property that σ is a natural transformation from Δ_D to F is equivalent to the equation $hm = km$. Hence $m = en$ for a morphism $n : D \to C$ by the unique property of an equalizer. Then σ factors through τ, i.e. $\sigma(B) = \tau(B) \circ n$. □

So we see that if $\mathcal{T}$ has all finite products and equalizers, then every finite limit exist since for these limits the cardinality of $\mathrm{Obj}(\mathcal{J})$ and $\mathrm{Mor}(\mathcal{J})$ is finite. The same argument can be used for finite colimits as the next theorem shows.

Theorem 2.4.4. *Let $\mathcal{J}$ and $\mathcal{C}$ be categories. If $\mathcal{C}$ has equalizers and if $\mathcal{C}$ has all products of length card(Obj($\mathcal{J}$)) and card(Mor($\mathcal{J}$)), then the limit of every functor $F : \mathcal{J} \to \mathcal{C}$ exist.*

Proof: The theorem is proved by the following diagram:

$$\begin{tikzcd}
F(source(g)) \arrow[d, "i_{source(g)}"'] \arrow[r, phantom, "="] & F(source(g)) \arrow[d, "i_{source(g)}"] & F(B) \arrow[dl, "i_B"] \arrow[d] \\
\coprod_f F(source(f)) \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & \coprod_A F(A) \arrow[r, "e"] & C \\
F(source(g)) \arrow[u, "i_{source(g)}"] \arrow[r, "F(g)"'] & F(target(g)) \arrow[u, "i_{target(g)}"'] & .
\end{tikzcd}$$

□

The next theorem is might look complicated at first sight. It is not! The theorem is a simple result of a diagram chase, which is worked out for a simple but non-trivial case.

Theorem 2.4.5. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{J}$ be categories (not necessarily topoi). Let $F : \mathcal{J} \to \mathcal{C}^{\mathcal{D}}$ and let*

$$\begin{array}{rrcl} G_D : & \mathcal{C}^{\mathcal{D}} & \to & \mathcal{C}; \\ & F & \mapsto & F(D); \\ & \tau & \mapsto & \tau(D). \end{array}$$

Note that this defines a functor. Then if ($lim\ G_D F$) exists in $\mathcal{C}$ for all objects D of $\mathcal{D}$, then $lim\ F$ exists and one has

$$(lim\ F)(D) = (lim\ G_D F). \tag{2.1}$$

Let $\sigma : \Delta_{limF} \to F$ be the natural transformation implied by the limit of F and let $\tau_D : \Delta_{limG_DF} \to G_DF$ be the natural transformation implied by the limit of G_DF. Then we have

$$\sigma(D) = \tau_D.$$

Furthermore, the intrinsic morphisms of $(lim\ F)$ are determined by this equality.

Proof: The proof is a simple diagram chase. To keep things clear, we prove the theorem for $\mathcal{J} = \mathcal{C}_\mathbf{2}$. In this case the limit of the functor F can be represented as the following diagram:

$$\begin{tikzcd}
K \arrow[r] \arrow[dr] & H \arrow[d, "f"] \\
& G.
\end{tikzcd}$$

Where $H := F(\bullet)$, $G := F(\star)$, $f := F(\to)$ and $K = \lim F$. Let $g : D \to D'$ be a morphism of $\mathcal{D}$. Then the uniqueness of the point-wise limit of $G_{D'}F = K(D')$ shows that there is a unique dotted morphism making the diagram below commutative:

$$\begin{tikzcd}
& K(D') \arrow[rr, "\tau_{D'}(\bullet)"] \arrow[ddrr, "\tau_{D'}(\star)"] & & H(D') \arrow[dd, "f(D')"] \\
K(D) \arrow[ur, dotted, "g^K_{D,D'}"] \arrow[rr, "\tau_D(\bullet)"] \arrow[ddrr, "\tau_D(\star)"'] & & H(D) \arrow[ur, "g^H_{D,D'}"'] \arrow[dd, "f(D)"] & \\
& & & G(D') \\
& & G(D) \arrow[ur, "g^G_{D,D'}"'] & .
\end{tikzcd}$$

By a simple diagram chase, we have $(Id_D)^K_{D,D} = Id_{K(D)}$ and if $h : D' \to D''$ is a morphism in $\mathcal{D}$, then $h^K_{D',D''} g^K_{D,D'} = (hg)^K_{D,D''}$. This defines the intrinsic maps of K and determines K as a functor, hence an object of $\mathcal{C}^\mathcal{D}$. The above diagram shows that the morphisms τ_D indeed combine into a natural transformation $\sigma : \Delta_{\lim F} \to F$.

Suppose L is another object that has a natural transformation $\rho : \Delta_L \to F$. We have proved the theorem if ρ factors through σ.

$$\begin{tikzcd}
& L(D') \arrow[r, "\beta(D')"] & K(D') \arrow[rr, "\tau_{D'}(\bullet)"] \arrow[ddrr, "\tau_{D'}(\star)"'] & & H(D') \arrow[dd, "f(D')"] \\
L(D) \arrow[ur, dotted, "g^L_{D,D'}"] \arrow[r, "\beta(D)"] & K(D) \arrow[ur, "g^K_{D,D'}"] \arrow[rr, "\tau_D(\bullet)"] \arrow[ddrr, "\tau_D(\star)"'] & & H(D) \arrow[ur, "g^H_{D,D'}"'] \arrow[dd, "f(D)"] & \\
& & & & G(D') \\
& & & G(D) \arrow[ur, "g^G_{D,D'}"'] &
\end{tikzcd}.$$

Since we took the limit point-wise, there are unique (!) morphisms $\beta(D) : L(D) \to K(D)$ and $\beta(D') : L(D') \to K(D')$ such that the above diagram commutes except maybe for the part where the dotted morphism is included. Note that

$$\alpha(\bullet) := g^H_{D,D'}\tau_D(\bullet)\beta(D)$$

and

$$\alpha(\star) := g^G_{D,D'}\tau_D(\star)\beta(D)$$

combine into a natural transformation $\alpha : G_D\Delta_L \to G_{D'}F$. Hence α must factor uniquely through $\tau_{D'}$. From the diagram above it follows that α factors both through $g^K_{D,D'}\beta(D)$ and $\beta(D')g^L_{D,D'}$. Hence these morphisms must be equal and the part where the dotted morphism is included commutes. Hence ρ factors uniquely through σ. □

2.5 Special objects

This section shows some properties of the initial and terminal object of a topos. We first show they always exist.

Proposition 2.5.1. *Every topos $\mathcal{T}$ has an initial object **0** and a terminal object **1**.*

Proof: Let $F : \mathcal{C}_\mathbf{0} \to \mathcal{T}$ be a functor. Notice that there is only one such functor. The inital object is the colimit of F and the terminal object is the limit of F, because in this case the definition of the (co)limit is equivalent to the definition of an initial object or a final object. □

For the next proof it is instructive to introduce the slice category. In the next chapters we will study diagrams of the form $B \to A$ for a fixed object A. These diagrams can be seen as objects of what is called the *slice category*.

Definition 2.5.2. *Let $\mathcal{T}$ be a topos and let A be an object of $\mathcal{T}$. The **slice category** $\mathcal{T}/A$ is defined as the following category:*

- *The objects are diagrams in $\mathcal{T}$ of the form $f : B \to A$.*

- *A morphism from an object $f : B \to A$ to an object $g : C \to A$ is a morphism $h : B \to C$ such that the following diagram commutes:*

$$\begin{tikzcd}
B \arrow[rr, "h"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\
& A &
\end{tikzcd}.$$

The next theorem is stated without proof. The reason is that the proof is very technical, not very instructive to read and it would take about three pages to work out. The result is simple, but very important.

Theorem 2.5.3. *Let $\mathcal{T}$ be a topos and let A be an object of $\mathcal{T}$. Then the slice category $\mathcal{T}/A$ is a topos.*

Proof: See for instance [11]. □

A morphism $f : A \to B$ induces a functor $f^{-1} : \mathcal{T}/B \to \mathcal{T}/A$ by pulling back, as the following definition shows.

Definition 2.5.4. *Let $f : A \to B$ be a morphism between objects A and B in a topos $\mathcal{T}$. This gives a functor $f^{-1} : \mathcal{T}/B \to \mathcal{T}/A$, by pulling back a morphism $S \to B$ along f.*

$$\begin{tikzcd}
A \times_B S \arrow[r] \arrow[d] & S \arrow[d] \\
A \arrow[r, "f"'] & B.
\end{tikzcd}$$

$$\begin{array}{rccl}
f^{-1} : & \mathcal{T}/B & \to & \mathcal{T}/A; \\
& (S \to B) & \mapsto & (A \times_B S \to B)\,.
\end{array}$$

For a morphism $k : T \to S$ in $\mathcal{T}/A$ one defines $f^{-1}(k)$ as the unique dotted morphism making the following diagram commutative:

$$\begin{tikzcd}
& A \times_B T \arrow[r] \arrow[dl, dotted, "f^{-1}(k)"'] \arrow[ddl] & T \arrow[dl, "k"] \arrow[ddl] \\
A \times_B S \arrow[r] \arrow[d] & S \arrow[d] & \\
A \arrow[r, "f"'] & B &
\end{tikzcd}.$$

We will refer to f^{-1} as the ***pullback functor****.*

By diagram chasing this definition is well defined. A very important result is that the functor f^{-1} has both a left and right adjoint. The proof of this is postponed to the next chapter, though we will use the result here to prove the following proposition.

Proposition 2.5.5. *If $f : A \to \mathbf{0}$ is a morphism to the initial object in a topos $\mathcal{T}$, then f is an isomorphism. Hence A is isomorphic to the initial object. We say the initial object of a topos is strict.*

Proof: In the slice category $\mathcal{T}/\mathbf{0}$ the object $Id_{\mathbf{0}} : \mathbf{0} \to \mathbf{0}$ is both initial and terminal. Since $k^{-1} : \mathcal{T}/\mathbf{0} \to \mathcal{T}/A$ has a left adjoint, $k^{-1}(Id_{\mathbf{0}} : \mathbf{0} \to \mathbf{0})$ must be the the terminal object in $\mathcal{T}/A$, which is $Id_A : A \to A$. Since k^{-1} has a right adjoint $k^{-1}(Id_{\mathbf{0}} : \mathbf{0} \to \mathbf{0})$ must be the initial object in $\mathcal{T}/A$, which is $! : \mathbf{0} \to A$. So we see that both the diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "Id_A"'] & \mathbf{0} \arrow[d, "!=Id_{\mathbf{0}}"] \\ A \arrow[r, "f"'] & \mathbf{0}. \end{tikzcd} \qquad \begin{tikzcd} \mathbf{0} \arrow[r, "f"] \arrow[d, "!"'] & \mathbf{0} \arrow[d, "!=Id_{\mathbf{0}}"] \\ A \arrow[r, "f"'] & \mathbf{0}. \end{tikzcd}$$

are pullbacks. Hence A must be isomorphic to $\mathbf{0}$ (in $\mathcal{T}$). Since A is initial, this isomorphism must be f. $\square$

Chapter 3

The algebra of subobjects

"Logic is invincible because in order to combat logic it is necessary to use logic." - Pierre Boutroux

In this chapter $\mathcal{T}$ is a topos and all objects and morphisms are taken within this topos unless specified otherwise. We will explore the structure of the subobjects of a given object in a topos. We will begin with defining a subobject. Remark that the definition we gave in chapter 1 only sufficed for the previous chapters. In this chapter we make it more precise and work towards a lattice structure on the subobjects of a given object. Most of this work is based on [5], although some of the proofs are neater and a few corollaries have been added.

3.1 Subobjects

Definition 3.1.1. *A* ***presubobject*** $f : B \hookrightarrow A$ *in a topos* $\mathcal{T}$ *is an object* B *in* $\mathcal{T}$ *together with a monic* $f : B \hookrightarrow A$.

Mostly we omit the morphism f if it is clear which morphism is meant. We will just write $B \hookrightarrow A$ saying B is a presubobject of A. When it is clear of which object B is a presubobject, we simply speak about the presubobject B.

The following definition of a subobject should not surprise the reader at all. In the previous chapters we had a notion of uniqueness up to isomorphism for many of the definitions. For example **0**, **1**, limits, colimits, Ω, ... are all unique up to isomorphism. The structure we would like to study in this chapter is the one of the presubobjects modulo isomorphism.

Definition 3.1.2. *We define an equivalence relation on the presubobjects of an object* A *by* $(B \hookrightarrow A) \simeq (C \hookrightarrow A)$ *if and only if there exists an isomorphism* h *such that the following diagram commutes:*

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[dr, hook] & & C \arrow[dl, hook'] \\ & A & \end{tikzcd}.$$

It is easily verified that this relation is symmetric and satisfies reflexivity and transitivity.

Definition 3.1.3. *Define* $Sub(A)$ *to be the class of presubobjects of* A *modulo* $\simeq$*-equivalence. The elements of* $Sub(A)$ *are called* ***subobjects****.*

The first proof of this chapter is not hard at all. We have already mentioned it: there is a correspondence between subobjects and characteristic morphisms. Although the proof is easy, its consequences are important as we will see later. From this point we alter our notation of a characteristic morphism. If $B \hookrightarrow A$ is a subobject we write χ_B for the characteristic morphism of B.

Theorem 3.1.4. *In a topos $\mathcal{T}$, there is a bijective correspondence between $Sub(A)$ and characteristic morphisms $A \to \Omega$.*

Proof: Since $\mathcal{T}$ is a topos every subobject $B \in Sub(A)$ admits a unique $\chi_B : A \to \Omega$ such that:

$$\begin{tikzcd} B \arrow[r, "!"] \arrow[d, hook, "f"] & \mathbf{1} \arrow[d, hook, "t"] \\ A \arrow[r, "\chi_B"'] & \Omega, \end{tikzcd}$$

is a pullback square. This assignment is:

- Well-defined. If $B, C \in Sub(A)$ and $B \simeq C$, then the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\sim"] \arrow[d, hook] & B \arrow[r] \arrow[d, hook, "f"] & \mathbf{1} \arrow[d, hook, "t"] \\ A \arrow[r, "Id_A"'] & A \arrow[r, "\chi_B"'] & \Omega. \end{tikzcd}$$

  Both the left and right square are pullbacks. By the PBL the whole rectangle is a pullback. So χ_B must be the unique morphism turning

$$\begin{tikzcd} C \arrow[r] \arrow[d, hook] & \mathbf{1} \arrow[d, hook, "t"] \\ A \arrow[r, "\chi_B"'] & \Omega, \end{tikzcd}$$

  into a pullback.

- Injective. Suppose both $B, C \in Sub(A)$ have characteristic morphism χ. Then both B and C are the pullbacks of

$$\begin{tikzcd} & \mathbf{1} \arrow[d, hook, "t"] \\ A \arrow[r, "\chi"'] & \Omega. \end{tikzcd}$$

  Limits are unique up to isomorphism, so we may conclude $B \simeq C$.

- Surjective. For a given characteristic morphism $\chi : A \to \Omega$, its pullback along $t : \mathbf{1} \to \Omega$ will yield a subobject B with χ as its characteristic morphism.

□

From now on we will always consider subobjects instead of presubobjects. We will denote the subobjects like presubobjects. This can be done without confusion of notation, since every claim made below will be true for presubobjects as well as for subobjects.

3.2 Lattices and subobjects

We will prove that $Sub(A)$ has more structure than just the structure of a set. $Sub(A)$ will appear to be a Heyting algebra. This algebra is defined as a lattice together with some extra structure.

Definition 3.2.1. *A **lattice** is a poset L together with binary operations $\cup$ and $\cap$ and nullary operations $\bot$ and $\top$, such that for every $x, y, z \in L$:*

$$\begin{array}{rcl} x \cap y \geq z & \text{iff} & x \geq z \text{ and } y \geq z; \\ x \cup y \leq z & \text{iff} & x \leq z \text{ and } y \leq z; \\ x \cap \bot = \bot & & ; \\ x \cup \top = \top. & & \end{array}$$

A morphism of lattices is a map that preserves $\cap$, $\cup$, $\bot$ and $\top$. Such a map is called a lattice homomorphism.

The symbols $\cap$ and $\cup$ are sometimes called the *inf* and *sup*, respectively. The symbols $\bot$ and $\top$ are called the bottom and top element, respectively. Some authors refer to the defintion above as a lattice with top and bottom element and define a lattice without having these elements. Since all the lattices in this thesis do have a top and bottom element we prefer this definition.

Definition 3.2.2. *A **Heyting algebra** is a lattice L with a binary operation $\Rightarrow$, such that for every $x, y, z \in L$:*

$$x \leq (y \Rightarrow z) \quad \text{iff} \quad x \cap y \leq z.$$

A morphism of Heyting algebras is a map that preserves $\cap$, $\cup$, $\bot$ and $\top$ as well as $\Rightarrow$.

Heyting algebras where introduced by Arend Heyting and play a role in intuitionistic logic. In the next chapter we will introduce (an interpretation of) a logical language which will turn out to be intuitionistic, i.e. the law of the excluded middle is not necessarily true. For chapters 5 and 6 we will need the definition of a locale, which we state now.

Definition 3.2.3. *A **locale** L is a Heyting algebra that is complete, i.e. for every subset U of L (possibly infinite) there exists an element $z \in L$ such that x is the least upper bound of U:*

$$\forall x(x \in U \Rightarrow x \leq z) \text{ and } (\forall x(x \in U \Rightarrow x \leq y)) \Rightarrow (z \leq y).$$

Notation: $z = \bigvee U$ (z is the supremem of U). A morphism f from a locale L to a locale M is a morphism of Heyting algebras $\bar{f}$ from M to L that preserves the (infinite) supremum. Note that a morphism from L to M is a function in the reversed direction preserving the structure of a locale.

A locale L is called ***compact*** *if for any subset U of L such that $\top = \bigvee U$, there is a finite subset $V \subset U$ such that $\top = \bigvee V$.*

For a locale L and $x, y \in L$, we say x is ***rather below*** *y if there exists an element $z \in L$ such that:*

$$x \wedge z = 0 \text{ and } z \vee y = 1.$$

Moreover, we say x is ***completely below*** *y if there is a family $z_q \in L$ indexed by rationals $0 \leq q \leq 1$, for which*

$$z_0 = x, z_p \text{ is rather below } z_q \text{ if } p < q, \text{ and } z_1 = y.$$

Now, a locale L is called ***completely regular*** *if for every $x \in L$*

$$x = \bigvee \{y \in L \mid y \text{ is completely below } x\}.$$

If X is a topological space, the set of opens $\mathcal{O}(X)$ is a locale. If $f : X \to Y$ is a map of topological spaces, it leads to a map of locales $g : \mathcal{O}(X) \to \mathcal{O}(Y)$ by the inverse image function: $\bar{g} = f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$. Furthermore, if X is compact Hausdorff, the locale is a compact completely regular locale. The correspondence between topological spaces and their underlying topology will be usefull for chapter 5. In this chapter our attention is focused on the algebra structure of $Sub(A)$. It turns out to be an Heyting algebra as the following series of definitions an propositions shows.

Proposition 3.2.4. *$Sub(A)$ has the structure of a partially ordered set by writing $B \subseteq C$ if and only if there is a morphism h such that the following diagram commutes:*

$$\begin{tikzcd}
B \arrow[rr, "h"] \arrow[dr, hook] & & C \arrow[dl, hook'] \\
& A &
\end{tikzcd}.$$

Proof:

- Reflexivity is obvious, by taking $h = Id$.
- Transitivity follows by:

$$\begin{tikzcd}
B \arrow[r, "h"] \arrow[dr, hook] & C \arrow[r, "k"] \arrow[d, hook] & D \arrow[dl, hook'] \\
& A &
\end{tikzcd}.$$

  Since the outer triangles commute, the large triangle commutes, which amounts to $B \subseteq D$.

- For anti-symmetry. Suppose $B \subseteq C$ and $C \subseteq B$, thus:

$$\begin{tikzcd}
B \arrow[r, "h"] \arrow[dr, hook, "f"'] & C \arrow[r, "k"] \arrow[d, hook] & B \arrow[dl, hook', "f"] \\
& A &
\end{tikzcd}.$$

  We have $f \circ Id_B = f \circ k \circ h$. Since f is monic, we conclude $Id_B = k \circ h$. In the same way $Id_C = h \circ k$. Hence $k = h^{-1}$, so $B \simeq C$.

□

Proposition 3.2.5. *$Sub(A)$ has a bottom element $\perp$, namely the initial object and its unique morphism $! : \mathbf{0} \hookrightarrow A$. $Sub(A)$ has a top element $\top$, namely the subobject $Id_A : A \hookrightarrow A$.*

Proof: Since $\mathbf{0}$ is the initial object, it admits a unique morphism $\mathbf{0} \hookrightarrow A$. This morphism is automatically monic: since in a topos $\mathbf{0}$ is strict (see proposition 2.5.5), any map $B \to \mathbf{0}$, must be the unique (identity)map $\mathbf{0} \to \mathbf{0}$. Now since $\mathbf{0}$ is the initial object, the following diagram commutes for any subobject $f : B \hookrightarrow A$:

$$\begin{tikzcd} 0 \arrow[rr, "!"] \arrow[dr, hook, "!"'] & & B \arrow[dl, hook', "f"] \\ & A & \end{tikzcd}.$$

It follows from the commutativity of

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, hook, "f"'] & & A \arrow[dl, hook', "Id_A"] \\ & A & \end{tikzcd},$$

that for every subobject $f : B \hookrightarrow A$, $B \subseteq A$. Hence $Id_A : A \hookrightarrow A$ is the top element of the poset $Sub(A)$. □

In order to turn $Sub(A)$ into a Heyting algebra, we wish to define $\cap$, $\cup$ and $\Rightarrow$. This can be done as follows.

Definition 3.2.6. *Let $f : B \hookrightarrow A$ and $g : C \hookrightarrow A$ subobjects of A. Define the subobject $B \cap C \hookrightarrow A$ of A by pulling back f allong g. That is:*

$$\begin{tikzcd} B \cap C \arrow[r, hook, "k"] \arrow[d, hook, "h"] & B \arrow[d, hook, "f"] \\ C \arrow[r, hook, "g"'] & A. \end{tikzcd}$$

Proposition 3.2.7. *Definition 3.2.6 really defines the $\cap$ in the poset $Sub(A)$. I.e. if $D \subseteq B$ and $D \subseteq C$, then $D \subseteq B \cap C$.*

Proof: Since f and g are monic, so are h and k (proposition 2.2.1), hence $f \circ k$ is. We conclude $f \circ k : B \cap C \hookrightarrow A$ is a subobject of A.
Now suppose we have a subobject $m : D \hookrightarrow A$ such that $D \subseteq B$ and $D \subseteq C$. Then:

$$\begin{tikzcd}
D \arrow[drr, "i"] \arrow[dr, dotted] \arrow[ddr, "j"'] & & \\
& B\cap C \arrow[r, hook, "k"] \arrow[d, hook, "h"] & B \arrow[d, hook, "f"] \\
& C \arrow[r, hook, "g"] & A.
\end{tikzcd}$$

By assumption $f \circ i = m = g \circ j$, so by definition of a pullback the dotted morphism exists, which means $D \subseteq B \cap C$. ☐

The definition of $\cap$ already gives some interesting results. For example,

$$(B \cap C) \cap D = B \cap (C \cap D)$$

is true in an arbitrary partial order with a meet [15]. In our topos the $\cap$ is interpreted as a pullback and hence this equation is an equality of objects which tells us pulling back $B \hookrightarrow A$ along $C \hookrightarrow A$ and then pulling back the result along $D \hookrightarrow A$ is the same (up to isomorphism) as pulling back $C \hookrightarrow A$ along $D \hookrightarrow A$ and then pulling back the result along $B \hookrightarrow A$.

Next we define $\cup$.

Definition 3.2.8. *Let $f : B \hookrightarrow A$ and $g : C \hookrightarrow A$ be subobjects of A. Define the subobject $m : B \cup C \hookrightarrow A$ of A by taking the image of the coproduct of $f : B \hookrightarrow A$ and $g : C \hookrightarrow A$. That is:*

$$\begin{tikzcd}
B+C \arrow[dr, "e"] & & B \arrow[ll, "k"'] \arrow[dd, hook, "f"] \\
& B\cup C \arrow[dr, hook, "m"] & \\
C \arrow[uu, "h"] \arrow[rr, hook, "g"'] & & A.
\end{tikzcd}$$

Proposition 3.2.9. *Definition 3.2.8 really defines the $\cup$ in the poset $Sub(A)$. I.e. if $B \subseteq D$ and $C \subseteq D$, then $B \cup C \subseteq D$.*

Proof: Suppose we have a subobject $j : D \hookrightarrow A$ such that $B \subseteq D$ and $C \subseteq D$. The universal property of the coproduct amounts to the following commutative diagram:

$$\begin{tikzcd}
B+C \arrow[dr, "i"] & & B \arrow[ll, "k"'] \arrow[dd, hook, "f"] \\
& D \arrow[dr, hook, "j"] & \\
C \arrow[uu, "h"] \arrow[rr, hook, "g"'] & & A.
\end{tikzcd}$$

Note that both $m \circ e$ in definition 3.2.8 and $j \circ i$ in the above diagram are the unique morphism such that both f and g factor through $B + C$. Hence they are equal: $m \circ e = j \circ i$. Now split i into its epi-monic factorization:

$$\begin{tikzcd}
 & N \arrow[r, hook] & D \arrow[dr, hook, "j"] & \\
B + C \arrow[ur, two heads] \arrow[dr, two heads, "e"'] & & & A \\
 & B \cup C \arrow[uu, dotted] \arrow[urr, hook, "m"'] & &
\end{tikzcd} .$$

According to theorem 2.3.3 there is an isomorphism (the dotted arrow) $M \to N$ such that the above diagram commutes. It now follows that $B \cup C \subseteq D$. □

We will denote the morphism m in the above definition as $f \cup g$. Finally, we define the implication $\Rightarrow$.

Definition 3.2.10. *Let $f : B \hookrightarrow A$ and $g : C \hookrightarrow A$ be subobjects of A. Define the subobject $e : (B \Rightarrow C) \hookrightarrow A$ as the equalizer of the characteristic morphisms χ_B and $\chi_{B\cap C}$.*

$$\begin{tikzcd}
(B \Rightarrow C) \arrow[r, hook, "e"] & A \arrow[r, shift left, "\chi_B"] \arrow[r, shift right, "\chi_{B\cap C}"'] & \Omega.
\end{tikzcd}$$

Proposition 3.2.11. *Definition 3.2.10 really defines the $\Rightarrow$ in the poset $Sub(A)$. I.e. for subobjects $B, C, D \in Sub(A)$: $D \subseteq (B \Rightarrow C)$ if and only if $D \cap B \subseteq C$.*

Proof: First note that e in definition 3.2.10 really is monic since it is an equalizer.
Write $m : D \hookrightarrow A$ for the subobject D. By definition of the partial order and the equalizer one has $D \subseteq (B \Rightarrow C)$ if and only if m factors through e if and only if m equalizes χ_B and $\chi_{B\cap C}$. We find:

$$\begin{array}{rll}
D \subseteq (B \Rightarrow C) & \text{iff} & \chi_B \circ m = \chi_{B\cap C} \circ m; \\
 & \text{iff} & D \cap B = D \cap B \cap C; \\
 & \text{iff} & D \cap B \subseteq C.
\end{array}$$

In the second iff-statement we used proposition 3.1.4 as well as the fact that in

$$\begin{tikzcd}
D \cap B \arrow[r] \arrow[d, hook] & B \arrow[r] \arrow[d, hook] & 1 \arrow[d, hook, "true"] \\
D \arrow[r, "m"'] & A \arrow[r, "\chi_B"'] & \Omega,
\end{tikzcd}$$

both the left and right square are pullbacks. By the PBL the outer rectangle is a pullback, hence $\chi_B \circ m$ is the characteristic morphism of $D \cap B$. Replacing B by $B \cap C$, one readily finds that $\chi_{B\cap C} \circ m$ is the characteristic morphism of $D \cap B \cap C$. □

Finally we summarize the result in one theorem.

Theorem 3.2.12. *With $\cap$, $\cup$ and $\Rightarrow$ defined as above, $Sub(A)$ is a Heyting algebra.*

You could wonder if $Sub(A)$ has the structure of a boolean algebra, i.e. if there is some operation $\neg$ on subobjects such that $B \cap \neg B = \bot$ and $B \cup \neg B = \top$. This is not necessarily the case. For example, one might look at the topos $\mathbf{Set}^{\mathcal{C}_2}$. Let F be the functor, depicted as a diagram:

$$\{0,1\} \xrightarrow{!} \{1\}\,.$$

Let G be the subfunctor

$$\{0\} \xrightarrow{!} \{1\}\,.$$

By applying theorem 2.4.5 to the definition of $\cap$ and $\cup$ and noticing that the epi-monic factorization can be taken pointwise, the $\cup$ and $\cap$ are the naive point-wise meet and intersection. Then $\neg G$ must be equal to

$$\{1\} \xrightarrow{?} \emptyset,$$

which is not a proper function anymore.

Since we are doing category theory, the obvious next step is to define morphisms between algebras $Sub(A)$ and $Sub(B)$. We recall definition 2.5.4.

Definition 3.2.13. *Let $f : A \to B$ be a morphism between objects A and B in a topos $\mathcal{T}$. This gives a functor $f^{-1} : \mathcal{T}/B \to \mathcal{T}/A$, by pulling back a morphism $S \to B$ along f.*

$$\begin{tikzcd}
A \times_B S \arrow[r] \arrow[d] & S \arrow[d] \\
A \arrow[r, "f"'] & B.
\end{tikzcd}$$

$$\begin{array}{rlcl} f^{-1}: & \mathcal{T}/B & \to & \mathcal{T}/A \\ & (S \to B) & \mapsto & (A \times_B S \to B) \end{array}$$

For a morphism $k : T \to S$ in $\mathcal{T}/A$ one defines $f^{-1}(k)$ by the unique dotted morphism making the following diagram commutative.

$$\begin{tikzcd}
 & A \times_B T \arrow[rr] \arrow[dl, dotted, "f^{-1}(k)"'] \arrow[ddl] & & T \arrow[dl, "k"] \arrow[ddl] \\
A \times_B S \arrow[rr] \arrow[d] & & S \arrow[d] & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}\quad .$$

Definition 3.2.14. *Since the pullback of a monic is monic (proposition 2.2.1), f^{-1} in definition 2.5.4 descents to a morphism from $Sub(B)$ to $Sub(C)$. Since limits are unique up to isomorphism the pullback functor is well-defined on the equivalence classes defined in definition 3.1.2.*

The most important property of the pullback functor is that it preserves the structure of the Heyting algebra.

Proposition 3.2.15. *The map f^{-1} of definition 3.2.14 is a morphism of Heyting algebras.*

Proof: One can give a proof by diagram chasing, using the universal property of pullbacks, the fact that pulling back an epi (monic) gives again an epi (monic), theorem 2.3.3 about epi-monic factorization and the PBL. □

The proof of the last theorem is not very elegant. In the next chapter, we will give a short proof of proposition 3.2.15 using theorem 3.1.4. Actually, proposition 3.2.15 will turn out to be a special case of a more general situation. Namely, we will observe that f^{-1} has a left and right adjoint. Then f^{-1} preserves limits as well as colimits, and hence it perserves $\cap$ and $\cup$ since they are the product, respectively coproduct of the lattice $Sub(B)$ as a preorder.

3.3 The internal Heyting algebra

Theorem 3.1.4 gives a correspondence between the Heyting algebra $Sub(A)$ of an object A in a topos $\mathcal{T}$ and the characteristic morphisms $A \to \Omega$. We may use this correspondence to turn the set $\mathbf{Hom}_{\mathcal{T}}(A, \Omega)$ into a Heyting algebra. For $B, C \in Sub(A)$, put:

$$\begin{array}{rclcl}
\chi_B \cap \chi_C & = & \chi_{B\cap C} & & ; \\
\chi_B \cup \chi_C & = & \chi_{B\cup C} & & ; \\
\chi_B \Rightarrow \chi_C & = & \chi_{B\Rightarrow C} & & ; \\
\bot & = & \neg & = & \chi_\bot ; \\
\top & = & true_A & = & \chi_\top .
\end{array}$$

The last two items define the initial and terminal object in $\mathbf{Hom}_{\mathcal{T}}(A, \Omega)$. It follows directly from the definitions that these are indeed the initial and terminal objects.
The poset structure on $\mathbf{Hom}_{\mathcal{T}}(A, \Omega)$ is given by the obvious relation:

$$\begin{array}{rll}
\chi_B \subseteq \chi_C & \text{iff} & B \subseteq C; \\
& \text{iff} & \chi_B \cap \chi_C = \chi_B; \\
& \text{iff} & B \cap C = B.
\end{array}$$

The aim of this paragraph is to give a more direct description of the operators $\cap, \cup$ and $\Rightarrow$ on $\mathbf{Hom}_{\mathcal{T}}(A, \Omega)$.

Proposition 3.3.1. *Let $\wedge : \Omega \times \Omega \to \Omega$ be the characteristic morphism of $(t, t) : \mathbf{1} \hookrightarrow \Omega \times \Omega$. Then for $B, C \in Sub(A)$, $\wedge \circ (\chi_B, \chi_C) = \chi_B \cap \chi_C$.*

Proof: We are finished if the outer rectangle in the following diagram is a pullback.

$$\begin{tikzcd}
B \cap C \arrow[r, "!"] \arrow[d, hook] & 1 \arrow[r, "\sim"] \arrow[d, hook, "(t,t)"] & 1 \arrow[d, hook, "t"] \\
A \arrow[r, "(\chi_B, \chi_C)"'] & \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega
\end{tikzcd}$$

The right square is a pullback by definition of $\wedge$. If we can prove the left square is a pullback, the PBL shows the outer rectangle is a pullback. Suppose we have a commutative diagram:

$$\begin{tikzcd}
D \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& B \cap C \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, "{(t,t)}"] \\
& A \arrow[r, "{(\chi_B,\chi_C)}"'] & \Omega \times \Omega.
\end{tikzcd} \tag{3.1}$$

We need to prove that there is a unique morphism from D to $B \cap C$, indicated with the dotted arrow in the diagram above. From this diagram, we obtain:

$$\begin{tikzcd}
D \arrow[drr, dashed] \arrow[drrr] \arrow[ddrr] \arrow[ddr, dashed] \arrow[dddr] & & & \\
& B \cap C \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 1 \arrow[d, "t"] \\
& C \arrow[r] \arrow[d] & A \arrow[r, "\chi B"'] \arrow[d, "\chi C"] & \Omega \\
& 1 \arrow[r, "t"'] & \Omega & .
\end{tikzcd} \tag{3.2}$$

The bottom and the right squares are pullbacks. Hence there are unique morphisms from D to B and from D to C making the diagram commute. The upper left square is a pullback, hence there is a unique morphism from D to $B \cap C$ making the diagram commute. Since diagram 3.1 is equivalent to diagram 3.2, this is the unique dotted morphism making diagram 3.1 commute.
If you do not accept this last argument, do a diagram chase on diagram 3.1. $\square$

Proposition 3.3.2. *Let* $\vee : \Omega \times \Omega \to \Omega$ *be the characteristic morphism of* $(true_\Omega, 1_\Omega) \cup (1_\Omega, true_\Omega) : \Omega \cup \Omega \hookrightarrow \Omega \times \Omega$. *Then for* $B, C \in Sub(A)$, $\vee \circ (\chi_B, \chi_C) = \chi_B \cup \chi_C$.

We split the proof in three steps:

1. We first prove the following square to be a pullback:

$$\begin{tikzcd}
B \arrow[r, hook, "f"] \arrow[d, "\chi_C \circ f"] & A \arrow[d, "{(\chi_B,\chi_C)}"] \\
\Omega \arrow[r, "{(true_\Omega,1_\Omega)}"'] & \Omega \times \Omega.
\end{tikzcd}$$

Suppose we have an object D such that the following diagram commutes:

$$\begin{tikzcd}
D \arrow[dr, dotted, "l"] \arrow[drr, "k"] \arrow[ddr, "m"'] & & & \\
& B \arrow[r, hook, "f"'] \arrow[d, "\chi_C \circ f"] & A \arrow[dd, "\chi_B"'] \arrow[dr, "\chi_C"] & \\
& \Omega \arrow[rr, "1_\Omega"] \arrow[d, "!"] \arrow[dr, "true_\Omega"] & & \Omega \\
& 1 \arrow[r, "t"] & \Omega & .
\end{tikzcd}$$

The rectangle '$B - A - \Omega - 1$' is a pullback, hence there exists an l as indicated, such that

$$\begin{aligned} f \circ l &= k; \\ ! \circ \chi_C \circ f \circ l &= ! \circ m. \end{aligned}$$

From the diagram we obtain $m = 1_\Omega \circ m = \chi_C \circ k$. Hence

$$\chi_C \circ f \circ l = \chi_C \circ k = m.$$

So the whole diagram commutes. Furthermore, since f is monic it follows that l is a unique such arrow, for if $l' \circ f = k = l \circ f$, then $l' = l$. Looking closely, one sees that both the above diagrams represent the same situation and the morphism l is the unique morphism turning the first diagram into a pullback.

2. In the same spirit, the following square is a pullback:

$$\begin{tikzcd}
C \arrow[r, hook, "g"] \arrow[d, "\chi_B \circ g"] & A \arrow[d, "(\chi_B{,}\chi_C)"] \\
\Omega \arrow[r, "(1_\Omega{,}true_\Omega)"'] & \Omega \times \Omega.
\end{tikzcd}$$

3. Now take a look at the diagram:

$$\begin{tikzcd}
B + C \arrow[r, "i_1 \circ \chi_B \circ f + i_2 \circ \chi_C \circ g"] \arrow[d, two heads] & \Omega + \Omega \arrow[d, two heads] & \\
B \cup C \arrow[r, dotted] \arrow[d, hook, "f \cup g"'] & \Omega \cup \Omega \arrow[r, "!"] \arrow[d, hook, "(true_\Omega{,}1_\Omega) \cup (1_\Omega{,}true_\Omega)"] & 1 \arrow[d, "t"] \\
A \arrow[r, "(\chi_B{,}\chi_C)"'] & \Omega \times \Omega \arrow[r, "\vee"'] & \Omega.
\end{tikzcd}$$

In the left outer rectangle, we took the epi-monic factorization of $f+g$ and $(true_\Omega, 1_\Omega)+(1_\Omega, true_\Omega)$. Since the pullback functor $(\chi_B, \chi_C)^{-1}$ has a left adjoint (as we will prove in theorem 3.4.1), it preserves colimits. Combining this with step 1 and step 2, the left (outer) rectangle is a pullback. Theorem 2.3.3 gives a unique map from $B \cup C$ to $\Omega \cup \Omega$. This turns the bottom left square into a pullback. To see this, pull back (χ_B, χ_C) along $(true_\Omega, 1_\Omega) \cup (1_\Omega, true_\Omega)$ and pull the result back along $\Omega+\Omega \to \Omega\cup\Omega$. The PBL tells us this double pullback is a pullback itself, so that the outer left square is a pullback too. Since the pullback of a monic is monic (theorem 2.2.1), or alternatively, since the pullback functor has a left adjoint (as we will prove later) and since the pullback of an epi is epi since the pullback functor has a right adjoint (as we will prove later), it follows that the two morphisms obtained from the double pullback must form the epi-monic factorization of $f+g$. This factorization is unique, hence the bottom square must be the first pullback.
The (bottom) right square is a pullback by definition. The PBL gives that the bottom (outer) rectangle is a pullback. Hence $\vee \circ (\chi_B, \chi_C)$ is the characteristic morphism of $B \cup C$, which is $\chi_B \cup \chi_C$ by definition. □

Proposition 3.3.3. *Let* $\rightarrow : \Omega \times \Omega \to \Omega$ *be the characteristic morphism of the equalizer* e *of*

$$\begin{tikzcd} \leq \arrow[r, "e"] & \Omega \times \Omega \arrow[r, shift left, "\wedge"] \arrow[r, shift right, "p_1"'] & \Omega, \end{tikzcd}$$

where p_1 *is the projection on the first coordinate. Then for* $B, C \in Sub(A)$, $\rightarrow \circ(\chi_B, \chi_C) = (\chi_B \Rightarrow \chi_C)$.

Proof: Our strategy will be the same as in propopositions 3.3.1 and 3.3.2. Take the diagram

$$\begin{tikzcd} D \arrow[dr, dotted, "l"] \arrow[drr, "k"] \arrow[ddr, "m"'] & & & \\ & B \Rightarrow C \arrow[r, dotted, "h"'] \arrow[d, "f"] & \leq \arrow[r, "!"] \arrow[d, hook, "e"] & 1 \arrow[d, "t"] \\ & A \arrow[r, "(\chi_B, \chi_C)"'] & \Omega \times \Omega \arrow[r, "\rightarrow"'] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "\wedge"] & \Omega \\ & & \Omega. & \end{tikzcd},$$

where f denotes the equalizer of both compositions. Note that $p_1 \circ (\chi_B, \chi_C) = \chi_B$ and $\wedge \circ (\chi_B, \chi_C) = \chi_B \cap \chi_C$.
The right square is a pullback (by definition). If we can prove that the left square is a pullback, the PBL tells us that the whole rectangle is a pullback, which proves the proposition. First note that since

$$p_1 \circ (\chi_B, \chi_C) \circ f = \wedge \circ (\chi_B, \chi_C) \circ f,$$

$(\chi_B, \chi_C) \circ f$ equalizes p_1 and $\wedge$. Hence there is a unique morphism $h : (B \Rightarrow C) \to \leq$, that makes the left square commute. This square actually turns out to be a pullback. Suppose we have an object D and morphisms k, m as indicated in the diagram. Now,

$$p_1 \circ (\chi_B, \chi_C) \circ m = p_1 \circ e \circ k = \wedge \circ e \circ k = \wedge \circ (\chi_B, \chi_C) \circ m.$$

So m equalizes χ_B and $\chi_B \cap \chi_C$, hence it factors uniquely through f by l. To see that for this unique l one has $h \circ l = k$, compute

$$e \circ k = (\chi_B, \chi_C) \circ m = (\chi_B, \chi_C) \circ f \circ l = e \circ h \circ l.$$

Since e is monic, we conclude $h \circ l = k$. □

We are now in a position to prove proposition 3.2.15. A morphism $f : A \to B$ gives a morphism $f^{-1} : Sub(B) \to Sub(A)$ as defined in the previous chapter. The corresponding map on the characteristic morphisms is

$$f^* : \mathbf{Hom}_{\mathcal{T}}(A, \Omega) \to \mathbf{Hom}_{\mathcal{T}}(B, \Omega) : \chi_T \mapsto \chi_T \circ f.$$

Since the left and right squares in the following diagram are pullbacks, the PBL turns the outer rectangle into a pullback. Hence $f^*\chi_T$ is indeed the characteristic morphism of $f^{-1}(T)$.

$$\begin{tikzcd}
f^{-1}(T) \arrow[r] \arrow[d, hook] & T \arrow[r, "!"] \arrow[d, hook] & 1 \arrow[d, hook] \\
A \arrow[r, "f"'] & B \arrow[r, "\chi_T"'] & \Omega.
\end{tikzcd}$$

Proof (of proposition 3.2.15): For $S, T \in Sub(B)$ one has

$$\begin{aligned}
f^*(\chi_{T \cup S}) &= \wedge \circ (\chi_T, \chi_S) \circ f \\
&= \wedge \circ (\chi_T \circ f, \chi_S \circ f) \\
&= \chi_{f^{-1}(T)} \cap \chi_{f^{-1}(S)}.
\end{aligned}$$

With theorem 3.1.4 we conclude

$$f^{-1}(T \cap S) = f^{-1}(T) \cap f^{-1}(S).$$

Replacing $\wedge$ and $\cap$ by $\vee$ and $\cup$ or $\to$ and $\Rightarrow$ in the above equations, one finds

$$f^{-1}(T \cup S) = f^{-1}(T) \cup f^{-1}(S)$$
$$f^{-1}(T \Rightarrow S) = f^{-1}(T) \Rightarrow f^{-1}(S).$$

□

Propositions 3.3.1, 3.3.2 and 3.3.3 and the definition of f^* gave a direct description of the Heyting algebra structure of $\mathbf{Hom}_{\mathcal{T}}(A, \Omega)$. Now if we apply the functor $-^A$ obtained from the cartesian closedness of our topos, we find morphisms:

$$\begin{array}{lrcl}
\wedge^A : & \Omega^A \times \Omega^A & \to & \Omega^A; \\
\vee^A : & \Omega^A \times \Omega^A & \to & \Omega^A; \\
\to^A : & \Omega^A \times \Omega^A & \to & \Omega^A; \\
\perp : & \mathbf{1} & \to & \Omega^A; \\
\top : & \mathbf{1} & \to & \Omega^A.
\end{array}$$

Note that we used the isomorphism $(\Omega \times \Omega)^A \simeq \Omega^A \times \Omega^A$ here. These morphisms have the property that resemble of a Heyting algebra in **Set**. For example, $\wedge^A$ is associative, since the following diagram commutes:

$$\begin{tikzcd}
\Omega^A \times \Omega^A \times \Omega^A \arrow[r, "Id_{\Omega^A} \times \wedge^A"] \arrow[d, "\wedge^A \times Id_{\Omega^A}"'] & \Omega^A \times \Omega^A \arrow[d, "\wedge^A"] \\
\Omega^A \times \Omega^A \arrow[r, "\wedge^A"'] & \Omega^A.
\end{tikzcd}$$

In the same way one finds that $\wedge^A$ is commutative, and that it satisfies the idempotence law ($\phi \wedge^A \phi = \phi$) and the unit law ($\mathbf{1} \wedge^A \phi = \phi$). The dual relations for $\vee^A$ are true too, i.e. $\vee^A$ satisfies associativity, commutativity, idempotence and the unit law ($\mathbf{0} \vee^A \phi = \phi$). $\wedge^A$ and $\vee^A$ satisfy the absorbtion laws. Furthermore, if E is the equalizer of

$$E \xrightarrow{e} \Omega^A \times \Omega^A \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} \Omega^A,$$

then both the following squares are pullbacks:

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & E \arrow[d, "e"] \\
\Omega^A \times \Omega^A \times \Omega^A \arrow[r, "\wedge^A \times Id_{\Omega^A}"'] & \Omega^A \times \Omega^A.
\end{tikzcd}
\quad
\begin{tikzcd}
Q \arrow[r] \arrow[d] & E \arrow[d, "e"] \\
\Omega^A \times \Omega^A \times \Omega^A \arrow[r, "Id_{\Omega^A} \times \Rightarrow^A"'] & \Omega^A \times \Omega^A.
\end{tikzcd}$$

Note that both diagrams contain the same object Q. An object L with morphisms $\wedge$, $\vee$, $\Rightarrow$, $\top$ and $\bot$ that satisfies all the above properties is called an *internal Heyting algebra* in the topos $\mathcal{T}$. This is a generalization of the concept of a Heyting algebra in **Set**. Thus we have proved Ω^A is an internal Heyting algebra.

3.4 ∃ and ∀

In this section we will prove that the pullback functor has both a left and a right adjoint. These are called ∃ and ∀, respectively and are generalizations of the symbols ∃ and ∀ in **Set**. In the next chapter we will see that most of the set theoretic properties of ∃ and ∀ generalize to topoi. Defining the topos-theoretic operations ∃ and ∀ is not as simple as in the case of **Set**, however.

Definition 3.4.1. *Let $f : A \to B$ be a morphism in a topos. The induced map $f^{-1} : Sub(B) \to Sub(A)$ has a left adjoint as a morphism of preorders. The left adjoint is given by:*

$$\begin{tikzcd}
S \arrow[r, two heads, "e"] \arrow[d, hook, "i"'] & \exists_f S \arrow[d, hook, "m"] \\
A \arrow[r, "f"'] & B,
\end{tikzcd}$$

where $S \twoheadrightarrow^e \exists_f(S) \hookrightarrow^m B$ is the epi-monic factorization of the composite $S \to^i A \to^f B$.

We prove that this definition is correct.

Proposition 3.4.2. *The function $\exists_f$ in definition 3.4.1 is the left adjoint of f^{-1} as morphism of preorders.*

Proof: First observe that the isomorphism

$$\mathbf{Hom}_{Sub(B)}(\exists_f S, T) \simeq \mathbf{Hom}_{Sub(A)}(S, f^{-1}T)$$

is equivalent to the statement: a morphism u exists such that the diagram below commutes if and only if a morphism v exists such that the diagram below commutes.

$$\begin{tikzcd}
 & f^{-1}T \arrow[rr, "k"] \arrow[ddl, "e"] & & T \arrow[ddl, "s"] \\
S \arrow[ur, dotted, "u"] \arrow[rr, two heads, "r"] \arrow[d, hook, "i"'] & & \exists_f S \arrow[ur, dotted, "v"] \arrow[d, hook, "m"] & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}.$$

Now if u exists, then $mr = fi = fru = sku$. Now according to theorem 2.3.3, sku has a unique epi-monic factorization. Hence there is a unique isomorphism between $\exists_f S$ and $\mathbf{Im}(sku) \simeq \mathbf{Im}(ku)$ (s was already monic), which gives the unique (!) morphism v.
If v exists, $s \circ v \circ e = m \circ e = f \circ i$. By the universal property of a pullback there is a unique morphism u such that the diagram commutes. $\square$

Some properties one might expect to be true for the left adjoint $\exists_f$ are infect false. For instance, one might ask if $\exists_f$ preserves $\cap$, $\cup$ and $\Rightarrow$. This is in general only true for some of these logical connectives. To see that some examples of properties do not hold, take the topos **Set**. Let $X = \{0,1\}$ and $Y = \{0,1\}$. Let $p : X \times Y \to X$ be the projection on the first coördinate. Then:

- The following diagram is in general not a pullback:

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, hook, "i"'] & \exists_p A \arrow[d, hook, "m"] \\
X \times Y \arrow[r, "p"'] & X.
\end{tikzcd}$$

  To see this, choose $A = \{(0,0)\}$. Then $\exists_p A = \mathbf{Im}(p \circ i) = e\left(\{(0,0)\}\right) = \{0\}$. But then the pullback of m along p is

$$\begin{tikzcd}
\{(0,0),(0,1)\} \arrow[r, two heads, "e"] \arrow[d, hook, "i"'] & \{0\} \arrow[d, hook, "m"] \\
\{0,1\} \times \{0,1\} \arrow[r, "p"'] & \{0,1\}.
\end{tikzcd}$$

Hence the first diagram can not be pullback diagram.

- $\exists_p$ does not preserve $\cap$. Take $A = \{(0,1)\} \subseteq X \times Y$ and $B = \{(0,0)\} \subseteq X \times Y$. Then $\exists_p(A \cap B) = \exists_p(\emptyset) = \emptyset$. But $\exists_p(A) \cap \exists_p(B) = \{0\} \cap \{0\} = \{0\}$.
- $\exists_p$ does not preserve $\Rightarrow$. In **Set** one can easily show that for subobjects A and B of $X \times Y$ that $A \Rightarrow B = B^c \cup A$. Take $A = \emptyset$ and $B = \{(1,0)\}$. Then $\exists_p(A \Rightarrow B) = \{0,1\} \neq \{0\} = \exists_p(A) \Rightarrow \exists_p(B)$.
- $\exists_p$ does preserve $\cup$. Let A be an object in a topos. Then we can regard $Sub(A)$ as a preorder, hence a category. $\cup$ is the coproduct of this category. Since $\exists_p$ has a right adjoint (namely p^{-1}) it must preserve colimits, hence in particular the coproduct (theorem 2.4.2).

We will prove that f^{-1} has a right adjoint too. This is not as easy as proving the existence of its left adjoint. We will split the proof in two steps, making use of the slice category (see chapter 2). The proof can be found in [11]. This proof uses the theorem stated in chapter 2, proving that the slice category $\mathcal{T}/A$ is a topos. The most important proposition is the following one.

Proposition 3.4.3. *Let $\mathcal{T}$ be a topos and let A and B be objects of $\mathcal{T}$. Let f be a morphism from A to B. The functor $f^{-1} : \mathcal{T}/B \to \mathcal{T}/A$ has a right adjoint, called π_f.*

Proof: The proof is obtained in two steps:

1. First assume B is the terminal object $\mathbf{1}$. We find that

$$\begin{tikzcd} S \times A \arrow[r, "p_1"] \arrow[d, "p_2"] & S \arrow[d] \\ A \arrow[r, "f"'] & \mathbf{1}, \end{tikzcd}$$

is a pull back square, since it is easily verified that every pullback over $\mathbf{1}$ is just the product. We conclude that $f^{-1}(S) = S \times A$.

Now let $h : T \to A$ be an object over A. A morphism from $S \times A$ to T over A is a morhpism $t : S \times A \to T$ such that

$$h \circ t = p_2.$$

We use the adjunction of cartesian closedness to apply the functor $-^A$ on both sides of the equality. The morphisms t correspond to morphisms $t' : S \to T^A$ such that $h^A \circ t'$ (the LHS of the equality) equals the composite $S \to^! \mathbf{1} \to^j A^A$ (the RHS of the equality). Here j is the morphism corresponding by cartesian closedness to the identity on A. This can be seen from the following diagrams:

$$\begin{tikzcd} \mathbf{Hom}_{\mathcal{T}}(S \times A, T) \arrow[r, "\sim", "\varphi"'] \arrow[d, "h_*"'] & \mathbf{Hom}_{\mathcal{T}}(S, T^A) \arrow[d, "(h^A)_*"] \\ \mathbf{Hom}_{\mathcal{T}}(S \times A, A) \arrow[r, "\varphi"'] & \mathbf{Hom}_{\mathcal{T}}(S, A^A), \end{tikzcd}$$

$$\varphi(h \circ t) = \varphi(h_* \circ t) = (h^A)_* \circ \varphi(t) = h^A \circ t';$$

$$\begin{tikzcd}
\mathbf{Hom}_{\mathcal{T}}(A, A) \arrow[r, "\sim", "\phi"'] \arrow[d, "p_2^*"'] & \mathbf{Hom}_{\mathcal{T}}(1, A^A) \arrow[d, "!^*"] \\
\mathbf{Hom}_{\mathcal{T}}(S \times A, A) \arrow[r, "\phi"'] & \mathbf{Hom}_{\mathcal{T}}(S, A^A),
\end{tikzcd}$$

$$\phi(p_2) = \phi(p_2^* \circ Id_A) =!^* \circ \phi(Id_A) = j \circ !.$$

Now consider the pullback square:

$$\begin{tikzcd}
S \arrow[dr, dotted, "t''"] \arrow[drr, bend left, "t'"] \arrow[ddr, bend right, "!"'] & & \\
& \mathbf{1} \times_{A^A} T^A \arrow[r] \arrow[d] & T^A \arrow[d, "h^A"] \\
& \mathbf{1} \arrow[r, "j"'] & A^A.
\end{tikzcd}$$

By the universal property of pullbacks, every t' corresponds to a morphism $t'' : S \to 1 \times_{A^A} T^A$. We conclude that every morphism $t : S \times A \to T$ over A, corresponds bijectively to a unique morphism $t'' : S \to \mathbf{1} \times_{A^A} T^A$ (over $\mathbf{1}$ if you like). Hence this defines the desired right adjoint.

2. Now suppose B is not necessarily equal to $\mathbf{1}$. The morphism $f : A \to B$ can be regarded as an object of the slice category $\mathcal{T}/B$. An object over $f : A \to B$ in the double slice category $(\mathcal{T}/B)/(f : A \to B)$ is a commutative square

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "h"] & B \arrow[d, "Id_B"] \\
A \arrow[r, "f"'] & B.
\end{tikzcd}$$

Now for each h such that this diagram commutes, the arrow g is determined. Conversely, every such g defines an h. This correspondence defines a functor $F : (\mathcal{T}/B)/(f : A \to B) \to \mathcal{T}/A$ as well as a functor $G : \mathcal{T}/A \to (\mathcal{T}/B)/(f : A \to B)$, which are inverses of each other (i.e. the maps on the objects are mutual inverses and the maps on the morphisms are mutual inverses). Where we take the obvious map on morphisms.
Now we can regard f^{-1} as a functor

$$f^{-1} : (\mathcal{T}/B)/\mathbf{1} \simeq \mathcal{T}/B \to \mathcal{T}/A \simeq (\mathcal{T}/B)/(f : A \to B).$$

Hence, since $\mathcal{T}/B$ and $\mathcal{T}/A$ are toposes, we are in the situation of step 1 and so f^{-1} has a right adjoint. This is the required π_f. □

Corollary 3.4.4. *Let $f : A \to B$ be a map between objects in a topos. The induced map $f^{-1} : Sub(B) \to Sub(A)$ has a right adjoint as map of partially ordered sets, called $\forall_f$.*

Proof: In order to prove this theorem, it is enough to show that if $m : C \hookrightarrow A$ is monic, then $\pi_f m$ is monic. To see this, notice that since π_f is a right adjoint, it must preserve the terminal object of $\mathcal{T}/A$, which is $Id_A : A \hookrightarrow A$.
Suppose m is a monic in $\mathcal{T}/A$. Then

$$\begin{array}{rcll} (\pi_f m)h & = & (\pi_f m)k & \text{iff} \\ m(f^{-1}(h)) & = & m(f^{-1}(k)) & \text{iff} \\ h & = & k. & \end{array}$$

Hence π_f preserves monics. So if m is a monic, it is preserved under the following operations:

$$\begin{tikzcd} C \arrow[d, "m"'] \\ A \end{tikzcd} \Rightarrow \begin{tikzcd} C \arrow[r, "m"] \arrow[dr, "m"'] & A \arrow[d, "Id_A"] \\ & A \end{tikzcd} \Rightarrow \begin{tikzcd} \pi_f C \arrow[r, "\pi_f m"] \arrow[dr, "\pi_f m"'] & B \arrow[d, "Id_B"] \\ & B \end{tikzcd} \Rightarrow \begin{tikzcd} \pi_f C \arrow[d, "\pi_f m"'] \\ A \end{tikzcd} .$$

Hence π_f restrict to the 'monic objects'. This restriction is by definition $\forall_f$. □

We examine a few properties of this adjoint functor. Consider once more the topos **Set**. Let $X = \{0, 1\}$ and $Y = \{0, 1\}$. Let $p : X \times Y \to X$ be the projection on the first coördinate. Then:

- There is not necessarily a pullback diagram of the form

$$\begin{tikzcd} A \arrow[r, dotted] \arrow[d, hook] & \forall_p A \arrow[d, hook] \\ X \times Y \arrow[r, "p"'] & X. \end{tikzcd}$$

  Indeed the construction of $\forall_p$ does not give the dotted morphism. And actually such a morphism does not have to exist. For an example, take $A = \{(0, 0)\}$. Then $\forall_p(A) = \emptyset$ (this follows from direct calculation, or in a more elegant way, by making use of the correspondence $\mathbf{Hom}_{Sub(X\times Y)}(p^{-1}-, -) \simeq \mathbf{Hom}_{Sub(X)}(-, \forall_f -)$). Then there is no morphism making the following diagram a pullback:

$$\begin{tikzcd} \{(0,0)\} \arrow[r, dotted] \arrow[d, hook] & \emptyset \arrow[d, hook] \\ X \times Y \arrow[r, "p"'] & X. \end{tikzcd}$$

- Unlike $\exists_f$, $\forall_f$ does not preserve $\cup$. Take $A = \{(0, 1)\} \subseteq X \times Y$ and $B = \{(0, 0)\} \subseteq X \times Y$. Then $\forall_p(A \cup B) = \forall_p(\{(0, 0), (0, 1)\}) = \{0\}$ (again this follows from $\mathbf{Hom}_{Sub(X\times Y)}(p^{-1}-, -) \simeq \mathbf{Hom}_{Sub(X)}(-, \forall_f -)$). But $\forall_p(A) \cup \forall_p(B) = \emptyset \cup \emptyset = \emptyset$.

- $\forall_f$ does not preserve $\Rightarrow$. In **Set**, $A \Rightarrow B = B^c \cup A$. Take $A = \emptyset$ and $B\{(1,0)\}$. Then $\forall_p(A \Rightarrow B) = \{0\} \neq \{0,1\} = \forall_p(A) \Rightarrow \forall_p(B)$.
- Unlike $\exists_f$, $\forall_f$ does preserve $\cap$. Let A be an object in a topos. Then we can regard $Sub(A)$ as a preorder, hence a category, whose product is $\cap$. Since $\forall_f$ has a left adjoint (namely p^{-1}), it must preserve limits, hence in particular, the product.

For objects A and B in a topos and a morphism $f : A \to B$, we would like to know how the characteristic morphisms of $S \in Sub(A)$ and $\exists_f S \in Sub(B)$ as well as $\forall_f S \in Sub(B)$ depend on each other.

Definition 3.4.5. *Let A and B be objects in a topos and let $f : A \to B$ be a morphism from A to B. For $S \in Sub(A)$, define:*

$$\exists_f \chi_S := \chi_{\exists_f S}$$
$$\forall_f \chi_S := \chi_{\forall_f S}$$

Now we have shown the algebra of subobjects as a logical structure, namely the structure of a Heyting algebra. Furthermore we have a pullback functor f^{-1} and a notion of $\exists_f$ and $\forall_f$. We will use these definitions to introduce an internal logical language in a topos.

Chapter 4

The Mitchell-Bénabou language

"Such is the advantage of a well constructed language that its simplified notation often becomes the source of profound theories." - Pierre-Simon de Laplace

We have defined a logical structure using $\cap$, $\cup$, $\Rightarrow$ on objects of the form $Sub(A)$ (externally) or Ω^A (internally), where A is an object in some topos $\mathcal{T}$, as well as quantifiers $\forall_f$ and $\exists_f$ relating $Sub(A)$ and $Sub(B)$, where both A and B are objects of a topos $\mathcal{T}$ and $f : A \to B$ is a morphism. Now we would like to build logical formulae from it. This can be done by using *terms*: expressions of variables, logical connectives and quantifiers that are defined inductively. In any topos $\mathcal{T}$, these formal expressions have an interpretation, which is a morphism in $\mathcal{T}$, see definition 4.1.1. The target of the interpretation of a term is called the *type* of the term. If the interpretation of a term is a morphism which target is Ω, the term is called a *formula*. In the previous chapters we used the characters A and B to specify objects. In this chapter we use X, Y, X_1, X_2, et cetera to denote objects. This notation will remind one more of variables. The basic definitions and theorems in this chapter can be found in [11] and [1].

4.1 The language

In this chapter we introduce the language formulated by W. Mitchell and J. Bénabou. Recall that our perspective so far has been that a topos is a generalization of the category of sets. In this chapter we see how well the three defining properties (i.e. limits, subobjects and cartesian closedness) where chosen. Indeed, it turns out that a whole logical language can be built within a topos.

Definition 4.1.1. *We define a* ***term*** *in a topos $\mathcal{T}$ to be an expression that can be obtained by the following inductive steps:*

1. *For an object X, the character x is a term of type X. Its interpretation is the identity Id_X.*
2. *Let $f : X \to Y$ be a morphism and let τ be a term of type X. Then the formal expression $f\tau$ is a term of type Y. Its interpretation is the morphism $f \circ \tau$.*
3. *Let σ be a term of type X with interpretation $\sigma : U \to X$ and let τ be a term of type Y with interpretation $\tau : V \to Y$. Then (σ, τ) is a term of type $X \times Y$. Its interpretation is the morphism $(\sigma \circ p_1, \tau \circ p_2) : U \times V \to X \times Y$.*
4. *Let σ and τ be terms of type X, interpreted by $\sigma : U \to X$ and $\tau : V \to X$ respectively. Then $\sigma = \tau$ is a term of type Ω, i.e. a formula. Its interpretation is the morphism*

$$\sigma = \tau : U \times V \xrightarrow{(\sigma \circ p_1, \tau \circ p_2)} X \times X \xrightarrow{\delta_X} \Omega.$$

Here δ_X is the characteristic morphism of the diagonal morphism $\Delta_X : X \to X \times X$. Note that the equality sign '=' is not an equality of morphisms. For example, σ and τ may have different sources.

5. *Let σ be a term of type X with interpretation $\sigma : U \to X$ and let τ be a term of type Ω^X with interpretation $\tau : V \to \Omega^X$. This yields a term $\sigma \in \tau$ of type Ω, interpreted by*

$$\sigma \in \tau : U \times V \longrightarrow X \times \Omega^X \xrightarrow{e} .\Omega.$$

Here e is the morphism in $\boldsymbol{Hom}_{\mathcal{T}}(X \times \Omega^X, \Omega)$ corresponding to $Id_{\Omega^X} \in \boldsymbol{Hom}_{\mathcal{T}}(\Omega^X, \Omega^X)$ by cartesian closedness.

6. *Let θ be a term of type Y^X, interpreted by $\theta : V \to Y^X$ and let σ be a term of type X, interpreted by $\sigma : U \to X$. This gives a term $\theta(\sigma)$ of type Y, interpreted by*

$$\theta(\sigma) : V \times U \longrightarrow Y^X \times X \xrightarrow{e} Y.$$

7. *Let x be a variable of type X and σ be a term of type Z with interpretation $\sigma : X \times U \to Z$. This yields a term $\lambda x \sigma$ of type Z^X whose interpretation is the transpose of σ.*

$$\lambda x \sigma : U \longrightarrow Z^X.$$

8. *Let ϕ and ψ be terms of type Ω with interpretation $\phi : U \to \Omega$ and $\psi : V \to \Omega$ respectively. Then the following expressions and their interpretations define terms of type Ω.*

$$\phi \wedge \psi : U \times V \xrightarrow{(\phi p_1, \psi p_2)} \Omega \times \Omega \xrightarrow{\wedge} \Omega;$$

$$\phi \vee \psi : U \times V \xrightarrow{(\phi p_1, \psi p_2)} \Omega \times \Omega \xrightarrow{\vee} \Omega;$$

$$\phi \Rightarrow \psi : U \times V \xrightarrow{(\phi p_1, \psi p_2)} \Omega \times \Omega \xrightarrow{\rightarrow} \Omega.$$

9. *Let ϕ be a term of type Ω with interpretation $\phi : U \times X \to \Omega$. Then the following expressions and their interpretation define terms of type Ω:*

$$\exists x \phi : U \xrightarrow{\exists_p \phi} \Omega;$$
$$\forall x \phi : U \xrightarrow{\forall_p \phi} \Omega.$$

Here $p : U \times X \to U$ is the projection on U.

It is important to remark is that the definition of a term is a *formal* expression defined by the inductive steps in above definition. These formal expressions are just strings of logical signs, variables and functions having a priori nothing to do with morphisms within a topos. Regarding them as morphisms in a topos can be done though (by above definition), and is called the interpretation of the logical language.

In the next part of the thesis expressions that are obtained using inductive steps (5), (6) and (7) will not occur. These steps are quoted for the completeness of the definition. The Mithcell-Bénabou language has a wide amount of properties, many of which can be found in [1] and [11]. Some proofs in the literature make use of all of these steps, but the details of this language distract too much from the scope of this thesis. Nevertheless, we will give a few examples which demonstrate the power of the Mitchell-Bénabou language. Our main example will be to prove that for a morphism f in a topos the formula $\forall x \forall x' fx = fx' \Rightarrow x = x'$ 'is true' if and only if f is monic. After briefly playing with this language I will refer to literature.

If $\phi : X_1 \times \ldots \times X_n \to \Omega$ is a formula, then we write $\phi(x_1, \ldots, x_n)$ to explicitly indicate its source.

Definition 4.1.2. *A **formula** is a term of type Ω.*

We now state when a formula is true.

Definition 4.1.3. *A formula $\phi(x) : X \to \Omega$ is called **true**, written $\models \phi(x)$, if it factors through $t : 1 \to \Omega$.*

$$\begin{tikzcd} & \mathbf{1} \arrow[d, "t"] \\ X \arrow[ur, "!"] \arrow[r, "\phi(x)"'] & \Omega. \end{tikzcd}$$

This definition can be extended in an obvious way to formulae of multiple variables.

For our convenience we introduce a different, but equivalent notion of 'true'.

Corollary 4.1.4. *We say a formula $\phi(x) : X \to \Omega$ is true if and only if the following square is a pullback:*

$$\begin{tikzcd} X \arrow[r, "!"] \arrow[d, "Id_X"] & 1 \arrow[d, "t"] \\ X \arrow[r, "\phi(x)"'] & \Omega. \end{tikzcd}$$

Proof: The proof is straightforward. □

We will not distinguish between these definitions.

If $\phi(x) : X \to \Omega$ is a formula we will write $\{x \mid \phi(x)\}$ for the corresponding subobject of X.

4.2 Some examples

At this point the language built, in the previous chapters is very abstract. Before building any further machinery, it is instructive to see some examples of the use of this language. This chapter uses a lot of

diagram chasing to tackle its problems. This will be useful to understand the definitions of the language. Furthermore, it will give a feeling for the theory built up in the next chapter.
Actually, the logical language will produce quite useful results. Many of the logical formulae which are true intuitively *are* indeed true in the topos logic. This shows the power of the language.

Example 4.2.1. *Let $\phi(x) : X \to \Omega$ and $\psi(x) : X \to \Omega$ be formulae. Then we can show that $\{x|\phi(x)\} \subseteq \{x|\psi(x)\}$ if and only if $\phi(x) \to \psi(x)$ is true.*

Remark that:

$$\begin{array}{ll} \{x|\phi(x)\} \subseteq \{x|\psi(x)\} & \text{iff} \\ \{x|\phi(x)\} \cap \{x|\psi(x)\} = \{x|\phi(x)\} & \text{iff} \\ \phi(x) \cap \psi(x) = \phi(x) & \text{(equality as morphisms).} \end{array}$$

Hence we must show that $\phi(x) \cap \psi(x) = \phi(x)$ if and only if $\models (\phi(x) \to \psi(x))$. Now look at the diagram:

$$\begin{tikzcd} \{x|\phi(x) \to \psi(x)\} \arrow[r, "h"'] \arrow[d, "f"] & \leq \arrow[r, "!"] \arrow[d, hook, "e"] & \mathbf{1} \arrow[d, "t"] \\ X \arrow[r, "(\phi(x),\psi(x))"'] & \Omega \times \Omega \arrow[r, "\to"'] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "\wedge"] & \Omega \\ & \Omega. & \end{tikzcd}$$

and remark that f was defined as the equalizer of $\phi(x)$ and $\phi(x) \cap \psi(x)$. Now if $\phi(x) \to \psi(x)$ is true, we find f is the identity. Hence $\phi(x) \cap \psi(x) = \phi(x)$. If $\phi(x) \cap \psi(x) = \phi(x)$, the equalizer of $\phi(x)$ and $\phi(x) \cap \psi(x)$ apparently is $Id_X : X \to X$. Hence we conclude that $\phi(x) \to \psi(x)$ is true.

Example 4.2.2. *In this example both x and x' will be variables of type X. Let $f : X \to Y$ be a morphism. The following is statement holds: $\forall x \forall x' fx = fx' \Rightarrow x = x'$ is true if and only if f is monic.*

Remark that 'monic' has the intuitive meaning of 'injective'. But since in a topos we are dealing with morphisms rather than functions, there is no proper meaning of 'injective'. In the topos **Set** the morphisms are functions and one can easily prove monic is equivalent to injective. The formula $\forall x \forall x' fx = fx' \Rightarrow x = x'$ is intuitively the definition of injectivity. However, in the Mitchell-Bénabou language this expression has a far more complicated meaning. Therefore it may be surprising that this formula is true if and only if f is monic. We will prove this using example 4.2.1.

First suppose the formula is true. Consider:

$$\begin{tikzcd} \left\{(x,x')|fx = fx' \Rightarrow x = x'\right\} \arrow[r] & X \times X \arrow[d, "p_1"] \\ \left\{(x,x')|\forall x' fx = fx' \Rightarrow x = x'\right\} \arrow[r] & X \arrow[d, "!"] \\ \left\{(x,x')|\forall x \forall x' fx = fx' \Rightarrow x = x'\right\} \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, "true_{\mathbf{1}} = (\forall x \forall x' fx = fx' \Rightarrow x = x')"] \\ \mathbf{1} \arrow[r, "t"'] & \Omega. \end{tikzcd}$$

The bottom square is a pullback. Hence we conclude $\{(x,x')|\forall x \forall x' fx = fx' \Rightarrow x = x'\} = \mathbf{1}$. By adjunction we know:

$$
\begin{array}{lr}
\mathbf{Hom}_{Sub(\mathbf{1})}(\mathbf{1},\mathbf{1}) & = \\
\mathbf{Hom}_{Sub(\mathbf{1})}(\forall x \left\{(x,x')|\forall x' fx = fx' \Rightarrow x = x'\right\}, \forall x \left\{(x,x')|\forall x' fx = fx' \Rightarrow x = x'\right\}) & \simeq \\
\mathbf{Hom}_{Sub(X)}(!^{-1}\forall x \left\{(x,x')|\forall x' fx = fx' \Rightarrow x = x'\right\}, \left\{(x,x')|\forall x' fx = fx' \Rightarrow x = x'\right\}) & = \\
\mathbf{Hom}_{Sub(X)}(p_1^{-1}\mathbf{1}, \{(x,x')|\forall x' fx = fx' \Rightarrow x = x'\}) & = \\
\mathbf{Hom}_{Sub(X)}(X, \{(x,x')|\forall x' fx = fx' \Rightarrow x = x'\}) &
\end{array}
$$

Since $\mathbf{Hom}_{Sub(\mathbf{1})}(\mathbf{1},\mathbf{1})$ contains exactly one morphism, so does $\mathbf{Hom}_{Sub(X)}(X, \{(x,x')|\forall x' fx = fx' \Rightarrow x = x'\})$. Since X is the terminal object in $Sub(X)$, we must have

$$\left\{(x,x')|\forall x' fx = fx' \Rightarrow x = x'\right\} = X$$

.
In the same way we find:

$$\left\{(x,x')|fx = fx' \Rightarrow x = x'\right\} = X \times X.$$

Hence we conclude that the formula $fx = fx' \Rightarrow x = x'$ is true. This means (by example 4.2.2) that $\{(x,x')|fx = fx'\} \subseteq \{(x,x')|x = x'\}$. Now consider:

$$
\begin{tikzcd}
A \arrow[d, dotted] \arrow[dr, dashed, "{(g,h)}"] & X \arrow[r, "!"] \arrow[d, "\Delta_X"] & 1 \arrow[d] \\
\left\{(x,x')|fx = fx'\right\} \arrow[r] \arrow[ur] \arrow[d] & X \times X \arrow[r, "\delta_X"'] \arrow[d, "{(fp_1,fp_2)}"] & \Omega \\
Y \arrow[r, "\Delta_Y"] \arrow[d, "!"] & Y \times Y \arrow[d, "\delta_Y"] & \\
\mathbf{1} \arrow[r] & \Omega & .
\end{tikzcd}
$$

The two lower squares are pullbacks. We would like to prove f is monic, so suppose $fg = fh$. Then $(fp_1, fp_2) \circ (g,h)$ factors through Δ_Y. Since the upper square is a pullback, (g,h) factors through $\{(x,x')|fx = fx'\}$. With the inclusion $\{(x,x')|fx = fx'\} \subseteq \{(x,x')|x = x'\}$ we find (g,h) factors through Δ_X. Hence $g = h$ and f is monic.

The converse is also true. Suppose f is monic. Since $\forall x$ and $\forall x'$ are right adjoints, they preserve the terminal object. Hence it is enough to check if $fx = fx' \Rightarrow x = x'$ is true. Consider:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta_X"] & Y \arrow[r, "!"] \arrow[d, "\Delta_Y"] & 1 \arrow[d, "t"] \\
X \times X \arrow[r, "{(f,f)}"'] & Y \times Y \arrow[r, "\delta_Y"'] & \Omega.
\end{tikzcd}
$$

The right square is a pullback by definition. By diagram chasing the left square is a pullback. The PBL makes the whole rectangle is a pullback, so by the universal property of the subobject classifier one has $\delta_Y \circ (f, f) = \delta_X$. Switching from characteristic morphisms to subobjects this equation becomes $\{(x, x')|fx = fx'\} = \{(x, x')|x = x'\}$. So example 4.2.1 makes $fx = fx' \Rightarrow x = x'$ true. Since $\forall x$ and $\forall x'$ are right adjoints, they preserve the terminal object and hence $\forall x \forall x' fx = fx' \Rightarrow x = x'$ is true.

Of course, the equivalent statement is true for epis. Clearly we could view 'epi' as a generalization of 'surjective' in the topos **Set**.

Example 4.2.3. *Let f be a morphism between objects X and Y. Let x be a variable of type X and y a variable of type Y. Then $\forall y \exists x fx = y$ is true if and only if f is epi.*

Like in example 4.2.2, it suffices to prove that $\exists x fx = y$ is true if and only if f is epi. This can be expressed by the following diagram:

$$\begin{tikzcd}
& X \arrow[r] \arrow[dl] \arrow[d, "{(Id_X,f)}"] & Y \arrow[r, "!"] \arrow[d, "\Delta_Y"'] & 1 \arrow[d, "t"] \\
Y & X \times Y \arrow[l, "p_2"] \arrow[r, "{(fp_1,Id_Y p_2)}"'] & Y \times Y \arrow[r, "\delta_Y"'] & \Omega.
\end{tikzcd}$$

Now clearly, f is epi if and only if the left diagonal morphism is epi if and only if $\exists x fx = y$ is true. $\square$

The next example is concerned with unique factorization. Suppose we have functions $m : X \to Z$ and $f : Y \to Z$ in **Set** such that m is injective. Suppose furthermore $\forall y \exists x\, mx = fy$. Then it is clear that f factors uniquely through m. This can be generalized to an arbitrary topos.

Example 4.2.4. *Let $m : X \to Z$ be a monic morphism and $f : Y \to Z$ a morphism. Suppose*

$$\models \forall y \exists x\, mx = fy.$$

Then there is a unique morphism g such that $mg = f$.

First, we make the meaning of the formula $\models \forall y \exists x\, mx = fy$ explicit. Like in example 4.2.2, it follows that $\models \exists x\, mx = fy$.

$$\begin{tikzcd}
\{y|\exists x\, mx = fy\} \arrow[d, hook, "Id"'] & \{(x,y)|mx = fy\} \arrow[l, two heads] \arrow[r, "me"] \arrow[dl, two heads] \arrow[d, "e"] & Z \arrow[r] \arrow[d, "\Delta"] & 1 \arrow[d, "t"] \\
Y & X \times Y \arrow[l, "p_2"] \arrow[r, "{(mp_1,fp_2)}"'] & Z \times Z \arrow[r, "\delta"'] & \Omega.
\end{tikzcd}$$

The middle square is a pullback (by definition) and by a simple diagram chase it turns out that $\{(x, y)|mx = fy\}$ is such that the following diagram is an equalizer:

$$\begin{tikzcd}
\{(x,y)|mx = fy\} \arrow[r, "e"] & X \times Y \arrow[r, shift left, "mp_1"] \arrow[r, shift right, "fp_2"'] & Z.
\end{tikzcd}$$

Now look at

$$\begin{tikzcd}
 & E \arrow[dl] \arrow[d, "e"] \arrow[drr] & & \\
Y \arrow[drr, "f"'] & X \times Y \arrow[l, "p_2"] \arrow[r, "p_1"'] & X \arrow[r, "!_X"'] \arrow[d, "m"] & \mathbf{1} \arrow[d, "t"] \\
 & & Z \arrow[r, "\chi_m"'] & \Omega.
\end{tikzcd}$$

Remark that $\chi_m f p_2 e = \chi_m m p_1 e = t!_X p_1 e = t!_Y p_2 e$, where $!_Y$ is the unique morphism from Y to $\mathbf{1}$. Since $p_2 e$ is epi, $\chi_m f = t!_Y$. Hence there is a unique morphism $g : Y \to X$ such that the diagram above commutes (g is not indicated in the diagram). This morphism makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, hook, "m"] & Z \\
Y \arrow[u, "g"] \arrow[ur, "f"'] & .
\end{tikzcd}$$

Since m is monic, this g is unique.

Finally, we show that limits in a topos are generalizations of the limits in **Set**. That is, in **Set** one can write the limit of a diagram as a subobject of the product of the objects in that diagram. The subobject is implied by the morphisms in the diagram and can be expressed by a characteristic morphism. Since subobjects and characteristic morphisms are generalized from a set to a topos it turns out that limits can be defined in exact the same way. We give two examples, pullbacks and equalizers.

Example 4.2.5. *The following diagram is a pullback:*

$$\begin{tikzcd}
\{(x,y)|fx = fy\} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z.
\end{tikzcd}$$

Note that this is exactly the pullback of **Set**. That is, pulling back f along g yields the set of pairs $(x, y) \in X \times Y$ such that $fx = gy$. In previous chapter we gave the expression $fx = gy$ an interpretation in an arbitrary topos, which is:

$$\begin{tikzcd}
 & \{(x,y)|fx = fy\} \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "\Delta_Z"] & 1 \arrow[d] \\
U \arrow[r, dotted, "(h,k)"'] & X \times Y \arrow[r, "(fp_1, gp_2)"'] & Z \times Z \arrow[r, "\delta_Z"'] & \Omega.
\end{tikzcd}$$

Suppose there is an object U and a morhpism (h, k) such that $fh = gk$. Then $(fp, gq) \circ (h, k)$ factors through Δ_Z. The left square is a pullback, hence there is a unique morphism from U to $\{(x, y)|fx = fy\}$ such that the diagram commutes. This proves that $\{(x, y)|fx = fy\}$ is the pullback of f along g.

Example 4.2.6. *$\{x|fx = gx\}$ is the equalizer of the diagram:*

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y.$$

The proof is similar as in example 4.2.5.

4.3 Some further examples

The aim of this section is to establish some techniques to handle the formulae that are introduced in the beginning of this chapter. In the previous section many examples were solved by diagram chasing. In this section we introduce theorems which enable you to solve problems using logical rules. This will lead to much shorter and more elegant solutions to many problems. Also, it will give a good logical tool to prove deeper results of the Mitchell-Bénabou language. The aim of this chapter is not to give a general overview of true formulae. We restrict ourselves to some elegant examples.

We start with a proposition which has already been proved in example 4.2.1. For convenience we quote it again.

Proposition 4.3.1. *Let $\phi(x) : X \to \Omega$ and $\psi(x) : X \to \Omega$ be formulae. Then $\{x|\phi(x)\} \subseteq \{x|\psi(x)\}$ if and only if $\models \phi(x) \Rightarrow \psi(x)$.*

Proof: see example 4.2.1. □

Let's start with some simple formulae.

Theorem 4.3.2. *Let $\phi(x)$ and $\psi(x)$ be formulae. Then the following formulae are true:*

1. $\models \phi(x) \wedge \psi(x) \Rightarrow \phi(x)$.
2. $\models \phi(x) \wedge \psi(x) \Rightarrow \psi(x)$.
3. $\models \phi(x) \Rightarrow \phi(x) \vee \psi(x)$.
4. $\models \psi(x) \Rightarrow \phi(x) \vee \psi(x)$.

Proof: According to proposition 4.3.1, proving that one of the formulae above is true amounts to proving an inclusion:

1. $$\begin{array}{ll} \{x|\phi(x) \wedge \psi(x)\} \subseteq \{x|\phi(x)\} & \text{iff} \\ \{x|\phi(x)\} \cap \{x|\psi(x)\} \subseteq \{x|\phi(x)\} & . \end{array}$$

2. $$\begin{array}{ll} \{x|\phi(x) \wedge \psi(x)\} \subseteq \{x|\psi(x)\} & \text{iff} \\ \{x|\phi(x)\} \cap \{x|\psi(x)\} \subseteq \{x|\psi(x)\} & . \end{array}$$

3. $$\begin{array}{ll} \{x|\phi(x)\} \subseteq \{x|\phi(x) \vee \psi(x)\} & \text{iff} \\ \{x|\phi(x)\} \subseteq \{x|\phi(x)\} \cup \{x|\psi(x)\} & . \end{array}$$

4.

$$\begin{array}{ll}\{x|\psi(x)\} \subseteq \{x|\phi(x) \vee \psi(x)\} & \text{iff} \\ \{x|\psi(x)\} \subseteq \{x|\phi(x)\} \cup \{x|\psi(x)\} & .\end{array}$$

Each time, the last inclusion is obviously true. □

If $\phi(x_1, \ldots, x_n)$ is a formula with variables $x_1, \ldots, x_n$, there is a way of 'extending its source', i.e. turning it into a formula of more variables: $\phi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)$, $m > n$. This is done by the following diagram:

$$\phi(x_1, \ldots, x_m) := X_1 \times \ldots \times X_m \xrightarrow{p} X_1 \times X_n \xrightarrow{\phi(x_1,\ldots,x_n)} \Omega,$$

where p is the obvious projection.

Lemma 4.3.3. *Let $\phi(x_1, \ldots, x_n)$ be a formula with variables $x_1, \ldots, x_n$. Let $\phi(x_1, \ldots, x_m)$, $m > n$ be its extension. Then:*

1. *If $\models \phi(x_1, \ldots, x_n)$, then $\models \phi(x_1, \ldots, x_m)$.*
2. *If the projection $X_1 \times \ldots \times X_n \to X_1 \times \ldots \times X_m$ is epi and $\models \phi(x_1, \ldots, x_m)$, then $\models \phi(x_1, \ldots, x_n)$.*

Proof:

1. The statement is obvious from the following diagram:

$$\begin{tikzcd} & & \mathbf{1} \arrow[d] \\ X_1 \times \ldots \times X_m \arrow[r, "p"] & X_1 \times \ldots \times X_n \arrow[ur] \arrow[r, "\phi(x_1,\ldots,x_n)"] & \Omega. \end{tikzcd}$$

   If $\phi(x_1, \ldots, x_n)$ factors through $\mathbf{1}$, so does $\phi(x_1, \ldots, x_n) \circ p$.

2. We have:

$$\begin{tikzcd} & & \mathbf{1} \arrow[d, "t"] \\ X_1 \times \ldots \times X_m \arrow[urr, "!_{X_1 \times \ldots \times X_m}"] \arrow[r, "p"] & X_1 \times \ldots \times X_n \arrow[ur] \arrow[r, "!_{X_1 \times \ldots \times X_n}", "\phi(x_1,\ldots,x_n)"'] & \Omega. \end{tikzcd}$$

   Now,

$$\phi(x_1, \ldots, x_n) \circ p = t \circ !_{X_1 \times \ldots \times X_m} = t \circ !_{X_1 \times \ldots \times X_n} \circ p.$$

   If p is epi, we find:

$$\phi(x_1, \ldots, x_n) = t \circ !_{X_1 \times \ldots \times X_n}.$$

□

The following theorem is a very powerful one. It proves that the Modus Ponens is often true in topos logic. For many topoi the conditions for the theorem are easily checked. For example the topoi **Set** and $\mathbf{Set}^{\mathcal{C}}$ satisfy these conditions. We refer to topoi in which the Modus Ponens is true as topoi with Modus Ponens.

Theorem 4.3.4. ***(Modus Ponens).*** *Suppose ϕ and ψ are formulae. Let $X_1, \ldots, X_n$ be the variables of ψ and let $X_1, \ldots, X_m$ be the variables of ϕ and ψ. Suppose the projection $X_1 \times \ldots \times X_m \to X_1 \times \ldots \times X_n$ is epi. If $\models \phi$ and $\models \phi \Rightarrow \psi$ then $\models \psi$.*

Proof: First, extend both ϕ and ψ to the variables $X_1, \ldots, X_m$. Call the extensions $\bar{\phi}$ and $\bar{\psi}$. Remark that lemma 4.3.3 gives $\models \bar{\phi}$ and $\models \bar{\phi} \Rightarrow \bar{\psi}$, which is equivalent to $\big\{(x_1, \ldots, x_m)|\bar{\phi}(x_1, \ldots, x_m)\big\} = \mathbf{1}$ and $\big\{(x_1, \ldots, x_m)|\bar{\phi}(x_1, \ldots, x_m)\big\} \subseteq \big\{(x_1, \ldots, x_m)|\bar{\psi}(x_1, \ldots, x_m)\big\}$. From this, we conclude

$$\big\{(x_1, \ldots, x_m)|\bar{\psi}(x_1, \ldots, x_m)\big\} = \mathbf{1}.$$

Hence $\models \bar{\psi}$. Now, lemma 4.3.3 gives $\models \psi$. □

Suppose we want to prove a statement of the form

'If $\models \phi$, then $\models \psi$'.

In a topos with Modus Ponens, it then is enough to check whether the formula $\phi \Rightarrow \psi$ is true. In general this statement is easier to prove, since it amounts to proving an inclusion of subobjects. Thanks to the Modus Ponens the following theorem becomes useful.

Theorem 4.3.5. *Let $\phi(x_1, \ldots, x_n)$ be a formula with variables $X_1, \ldots, X_n$ and let $\phi(x, x_1, \ldots, x_n)$ be its extension to variables $X, X_1, \ldots, X_n$. Let $p : X \times X_1 \times \ldots \times X_n \to X_1 \times X_n$ be the obvious projection. Let $f : Y \to X$ be a morphism. Then:*

1. $\models \phi(x_1, \ldots, x_n) \Rightarrow (\forall x \phi(x, x_1, \ldots, x_n))$.
2. $\models (\forall x \phi(x, x_1, \ldots, x_n))(x, x_1, \ldots, x_n) \Rightarrow \phi(x, x_1, \ldots, x_n)$,
   where the part to the left of $\Rightarrow$ should be regarded as the extension of the formula $\forall x \phi(x, x_1, \ldots, x_n)$ to the variables $x, x_1, \ldots, x_n$.
3. *If $\models \phi(x, x_1, \ldots, x_n)$, then $\models \phi(fy, x_1, \ldots, x_n)$.*

Proof:

1.
$$\begin{array}{ll} \models \phi(x_1, \ldots, x_n) \Rightarrow (\forall x \phi(x, x_1, \ldots, x_n)) & \text{iff} \\ \{x_1 \times \ldots \times x_n \mid \phi(x_1, \ldots, x_n)\} \subseteq \{x_1 \times \ldots \times x_n \mid \forall x \phi(x, x_1, \ldots, x_n)\} & \text{iff} \\ p^{-1}\{x_1 \times \ldots \times x_n \mid \phi(x_1, \ldots, x_n)\} \subseteq \{x_1 \times \ldots \times x_n \mid \forall x \phi(x, x_1, \ldots, x_n)\} & \end{array}$$

   The last inclusion holds, since it actually is an equality.

2.
$$\begin{array}{ll} \models (\forall x \phi(x, x_1, \ldots, x_n))(x, x_1, \ldots, x_n) \Rightarrow \phi(x, x_1, \ldots, x_n) & \text{iff} \\ \{x \times x_1 \times \ldots \times x_n \mid (\forall x \phi(x, x_1, \ldots, x_n))(x, x_1, \ldots, x_n)\} \subseteq \{x \times x_1 \times \ldots \times x_n \mid \phi(x, x_1, \ldots, x_n)\} & \text{iff} \\ \{x \times x_1 \times \ldots \times x_n \mid (\forall x \phi(x, x_1, \ldots, x_n))(x, x_1, \ldots, x_n)\} \subseteq \{x \times x_1 \times \ldots \times x_n \mid \phi(x, x_1, \ldots, x_n)\} & \end{array}$$

3. The proof is exactly the same as in lemma 4.3.3 (2), with p replaced by $f \times Id_{X_1 \times \ldots \times X_n}$.

□

The following proposition will be of great use too:

Proposition 4.3.6. *Let $f : X \to Y$ and $g : X \to Y$ be morphisms. Then, in a topos with Modus Ponens $f = g$ (equality as morphisms!) if and only if:*

$$\models \forall x fx = gx$$

Proof: This is almost immediate from the universal property of a pullback:

$$\begin{tikzcd} & Y \arrow[r, "O_Y"] \arrow[d, "\Delta_Y"] & \mathbf{1} \arrow[d, "t"] \\ X \arrow[ur, dotted] \arrow[urr, dotted] \arrow[r, "(f,g)"'] & Y \times Y \arrow[r, "\delta_Y"'] & \Omega. \end{tikzcd}$$

The left dotted morphism exists such that the diagram commutes if and only if the right dotted morphism exists such that the diagram commutes. The existence of the left dotted morphism is equivalent to $f = g$ (equality as morphisms). The existence of the right dotted morphism is equivalent to $\models fx = gx$ (as a formula). Theorem 4.3.5 now gives $f = g$ if and only if $\models \forall x fx = gx$. □

Now let us examine example 4.2.2 again.

Example 4.3.7. *In a category with Modus Ponens, $\models \forall x \forall x' fx = fx' \Rightarrow x = x'$ if and only if f is monic.*

First the only if part. Suppose $fh = fg$, then $\models fhx = fgx$ according to theorem 4.3.6. From $\models \forall x \forall x' fx = fx' \Rightarrow x = x'$ we obtain $\models fx = fx' \Rightarrow x = x'$ according to theorem 4.3.5 (2). So $\models fhy = fgy \Rightarrow hy = gy$ by applying theorem 4.3.5 (3) twice. Modus Ponens gives $\models hx = gx$ and hence $h = g$.

The converse is probably best done in previous section. □

4.4 Partial truth

In this section we briefly describe the notion of partial truth, since in the next chapter we will refer to some articles which make use of this construction. Since we will not use the details of these articles, we just give the definition. Some important theorems about partial truth can be found in [11]. In particular the Kripke-Joyal semantics form an important tool for problems involving partial truth.

Definition 4.4.1. *Let $\phi(x_1, \ldots, x_n) : X_1, \ldots, X_n \to \Omega$ be a formula in a topos. Let U be an object of the topos and $\alpha : U \to X_1 \times \ldots \times X_n$ a morphism. We write:*

$$U \models \phi(\alpha)$$

if there exists a morphism $\tilde{\alpha}$ such that the following diagram commutes.

$$\begin{tikzcd} & \{(x_1, \ldots, x_n) | \phi(x_1, \ldots, x_n)\} \arrow[r] \arrow[d, hook] & 1 \arrow[d, hook] \\ U \arrow[ur, dotted, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & X \arrow[r, "\phi(x)"'] & \Omega. \end{tikzcd}$$

Sometimes we abbreviate this as $U \models \phi$.

The following theorem shows the link to the previous section.

Theorem 4.4.2. *Let $\phi(x_1, \ldots, x_n) : X_1, \ldots, X_n \to \Omega$ be a formula in a topos. Then $\models \phi$ if and only if $U \models \phi(\alpha)$ for all objects U and morphisms α.*

Proof: Suppose that $\models \phi$. Then the pullback property of the (right) square in the diagram below tells us there is a (dotted) morphism $\tilde{\alpha}$ for every α.

$$\begin{tikzcd}
 & \{(x_1, \ldots, x_n) | \phi(x_1, \ldots, x_n)\} \arrow[r] \arrow[d, hook] & 1 \arrow[d, hook] \\
U \arrow[ur, dotted, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & X \arrow[ur] \arrow[r, "\phi(x)"'] & \Omega.
\end{tikzcd}$$

Conversely, suppose $U \models \phi(\alpha)$ for every U and α then certainly $X_1 \times \ldots \times X_n \models \phi(Id_{X_1 \times \ldots \times X_n})$. Hence ϕ factors through $\mathbf{1}$. □

A formula ϕ might not be true for all U, but if one puts a U with a morphism in front of it, it becomes 'true on U', i.e. $U \models \phi$. For deeper results about partial truth, see for example [5] or [11].

4.5 Internal categories

The Mitchell-Bénabou language forms an important tool for generalizing objects that are defined as a set with a certain structure. For example, a group is a set togheter with an addition + and a special element 0, satisfying the properties:

$$\begin{array}{llcll}
\models \forall a \forall b \forall c & (a+b)+c & = & a+(b+c) & ; \\
\models \forall a & (a+0) & = & 0 & ; \\
\models \forall a \exists b & (a+b) & = & 0 & .
\end{array}$$

Since these properties, expressed in the Mitchell-Bénabou-language, have a (generalized) interpretation in any topos, there is no need to require that a group needs to be a *set* with some additional structure. It is enough to require that a group is an object G within a topos $\mathcal{T}$, together with morphisms:

$$\begin{array}{rlcl}
0: & \mathbf{1} & \to & G; \\
+: & B \times B & \to & B,
\end{array}$$

such that these morphisms satisfy the above properties written in the Mitchell-Bénabou language. Such an object G is called an *internal group*. In an analogous way, one can define *internal fiels*, *internal lattices*, et cetera. In chapter 1, we introduced the rational numbers object $\mathbf{Q}$ and like one would expect, one can proof that this is an internal field. More general, there is a notion of an *internal category* whose the definition can be found in [8]. This definition is quite abstract and we rather give specific definitions of internal lattices, internal locales, et cetera.

Chapter 5

The construction of the spectrum

"The mathematician is fascinated with the marvelous beauty of the forms he constructs, and in their beauty he finds everlasting truth." - G.B. Shaw

In this chapter we turn our attention to a generalization of the Gelfand theorem to topos theory. The original theorem proved by I.M. Gelfand and M. Naimark shows that there is a duality between unital commutative C*-algebras and compact Hausdorff spaces (in **Set**). If we consider the locale formed by the underlying topology of a compact Hausdorff space instead of the space itself, this duality becomes one between unital commutative C*-algebras and compact completely regular locales.

$$\mathbf{cCStar} \leftrightarrow \mathbf{KRegLoc}$$

By the work of Mulvey and Banaschewski this duality may be generalized to any topos $\mathcal{T}$, see [12], [13] and [14]. This is done by introducing a generalization of a C*-algebra to the topos $\mathcal{T}$, i.e. an object of $\mathcal{T}$ which has all the properties of a C*-algebra (we state the precise definition later in this chapter), called an *internal C*-algebra* of $\mathcal{T}$. Also the definition of a locale may be generalized to an object called an *internal locale* of $\mathcal{T}$. Mulvey and Banaschewski showed that there is a duality between the internal C*-algebras and the internal compact completely regular locales of any topos $\mathcal{T}$.

In this thesis we study an article by Heunen, Landsman and Spitters [7]. The authors start with a unital C*-algebra A in **Set** and construct a particular topos $\mathcal{T}(A)$ from this C*-algebra. Furthermore, they define an internal C*-algebra of $\mathcal{T}(A)$, called $\bar{A}$. In the next two chapters we try to compute the compact completely regular locale corresponding to $\bar{A}$ by the morphism **cCStar** $\rightarrow$ **KRegLoc**. This locale is called the spectrum of A and is denoted as $\Sigma(A)$, or briefly Σ, since A is fixed from the start of the construction. In this chapter we write out the details of the construction of the spectrum of the C*-algebra A and show that every step is well-defined. The construction consists of the following steps:

1. In section 5.1, we define the topos $\mathcal{T}(A)$ and the internal C*-algebra $\bar{A}$.

2. In section 5.2 we introduce some theory about lattices in topoi and construct an internal lattice in a topos, called $L_{\widetilde{\bar{A}}_{sa}}$. We first consider a generalization of a free distributive lattice to topos theory (paragraph 5.2.1) and then turn our attention to the more general case of generalized distributive lattices freely generated by an object *subject to relations* in a topos. Paragraph 5.2.3 should be considered an intermezzo: this paragraph only considers lattices in **Set** and is heavily based on [2] and [3]. This paragraph is included to show that the constructions that follow are well defined. Furthermore, it gives a nice representation of the lattices that we are interested in, in **Set**.

3. Finally, in section 5.3 we introduce a morphism $\mathcal{A} : \Omega^{L_{\widetilde{A}_{sa}}} \to \Omega^{L_{\widetilde{A}_{sa}}}$ and define the spectrum $\Sigma(A)$ as a certain subobject of $\Omega^{L_{\widetilde{A}_{sa}}}$.

During the construction we will introduce some theorems that allow us to compute the spectrum. The Mitchell-Bénabou language will be of great use. However, we sometimes avoid some technicalities of this language by making use of elementary category theory or lattice theory.

The actual computations are done in chapter 6, where our main examples will be the spectra of the C*-algebras $\mathbb{C}^2$ and $M(2,\mathbb{C})$, the 2×2-matrices. We will spend some words on $\mathbb{C}$, $\mathbb{C}^n$ and $C(X)$, with X a compact Hausdorff space, too.

5.1 The topos $\mathcal{T}(A)$

The construction of the spectrum starts with a unital C*-algebra A, which is defined as a *set* with the usual structure of a Banach Space over $\mathbb{C}$ with unit and involution called * satisfying $\|a^*a\| = \|a\|^2$. The reason we emphasize that this construction has a *set* as input is that we are going to construct (i.e. define) a C*-algebra that is not a set anymore, but a generalization of a set: an object in a certain topos. In what follows A will always be our unital C*-algebra as a set, unless specified otherwise.

Now we introduce $\mathcal{C}(A)$ as the following preorder:

$$\mathcal{C}(A) \quad = \quad \{B \mid B \text{ is a unital commutative C*-subalgebra of } A\}. \tag{5.1}$$

Here the unit in B is supposed to be inherited from A. The partial order of $\mathcal{C}(A)$ is given by the inclusion. This partial order has a meet, given by the intersection of unital, commutative C*-subalgebras. It does not have a join, since the join of two commutative C*-subalgebras does not have to be commutative. An important remark is that $\mathcal{C}(A)$ has a bottom element, which is the unique commutative, unital C*-subalgebra of dimension 1: $\mathbb{C} \cdot 1$. The topos we are interested in will be the following one.

Definition 5.1.1. *Let $\mathcal{T}(A)$ denote the topos $\mathbf{Set}^{\mathcal{C}(A)}$, where $\mathcal{C}(A)$ is the categorical preorder (5.1).*

In the previous chapters we proved that $\mathcal{T}(A)$ is a topos. The partial order and the bottom element of $\mathcal{C}(A)$ will turn $\mathcal{T}(A)$ into a topos with nice properties, as we will see. The next step is to introduce a special object in this topos, which is called the *tautological functor*.

Definition 5.1.2. *Let $\bar{A} : \mathcal{C}(A) \to$ **Set** denote the functor that sends a unital, commutative C*-subalgebra $B \subseteq A$ to itself. I.e. it is defined on objects as:*

$$\bar{A} : B \mapsto B.$$

$\bar{A}$ is defined on morphisms (in the preorder $\mathcal{C}(A)$) as the inclusion of C-subalgebras.*

What is so interesting about this tautological object? First, it turns out to be an internal C*-algebra in the topos $\mathcal{T}(A)$. We introduce the internal C*-algebra in an analogous way to the internal categories of section 4.5. The only problem is the analytic part of the definition of a C*-algebra: the norm and its completeness. We will not cover this in full detail, but only elaborate a bit.

A norm of a C*-algebra A in **Set** can be regarded as a subset N of $A \times \mathbb{Q}^+$, where one requires $(a, q) \in N$ if and only if $\|a\| < q$. Since this concept is set theoretic in nature, it may be generalized to a topos that has a rational numbers object $\mathbf{Q}$. Recall that we defined the rational numbers object in chapter 1. Then define

$$\mathbf{Q}^+ = \{q \mid \exists r\,(q = r^2)\} \subseteq \mathbf{Q}.$$

Now a norm on A is a subobject of $A \times \mathbf{Q}^+$ satisfying certain axioms. These axioms can be found in [14] as well as the axioms for an internal C*-algebra to be complete. Now, we state the definition of an internal C*-algebra.

Definition 5.1.3. *An **internal C*-algebra** in a topos $\mathcal{T}$ is an object B of the topos, together with morphisms*

$$\begin{array}{rlcll} 0: & \mathbf{1} & \to & B & ; \\ \lambda: & \mathbb{C} \times B & \to & B & \text{(scalar multiplication);} \\ \cdot: & B \times B & \to & B & \text{(multiplication);} \\ +: & B \times B & \to & B & ; \\ *: & B & \to & B & , \end{array}$$

such that:

$$\begin{array}{lrcll} \models \forall a \forall b \forall c & (a+b)+c & = & a+(b+c) & ; \\ \models \forall a \forall b & a+b & = & b+a & ; \\ \models \forall a & 0+a & = & a & ; \\ \models \forall a \exists b & a+b & = & 0 & ; \\ \models \forall a \forall \mu \forall \nu & \lambda(\mu, \lambda(\nu, a)) & = & \lambda(\mu\nu, a) & ; \\ \models \forall a \forall b \forall \mu & \lambda(\mu, a+b) & = & \lambda(\mu, a) + \lambda(\mu, b) & ; \\ \models \forall a \forall b \forall c & (a \cdot b) \cdot c & = & a \cdot (b \cdot c) & ; \\ \models \forall a \forall b \forall c & a \cdot (b+c) & = & (a \cdot b) + (a \cdot c) & ; \\ \models \forall a & a^{**} & = & a & ; \\ \models \forall a \forall b & (a+b)^* & = & a^* + b^* & ; \\ \models \forall a \forall b & (ab)^* & = & b^* a^* & ; \\ \models \forall a \forall \mu & \lambda(\mu, a)^* & = & \lambda(\bar{\mu}, a^*) & . \end{array}$$

Moreover, B has a norm $N \subseteq B \times \mathbf{Q}^+$, in which it is complete.

Furthermore, we call the C-algebra B commutative if*

$$\models \forall a \forall b\; a \cdot b = b \cdot a.$$

We say the C-algebra B is unital if there is a morphism $1 : \mathbf{1} \to B$ such that*

$$\models \forall a\, ((1 \cdot a = a) \wedge (a \cdot 1 = a)).$$

It is very easy to verify that $\bar{A}$ as in definition 5.1.2 satisfies the axioms of definition 5.1.3 for the (pointwise) addition, multiplication and scalar multiplication. For the completeness part we refer to [7], where it is proved that $\bar{A}$ is complete using the Kripke-Joyal semantics. It is easy to see that $\bar{A}$ is commutative and has a unit (since we have chosen all algebras of $\mathcal{C}(A)$ to be commutative and unital). We state the result as a theorem.

Theorem 5.1.4. *$\bar{A}$ is an internal commutative unital C*-algebra of $\mathcal{T}(A)$.*

Proof: See [7]. □

5.2 Free lattices

The next step of constructing the spectrum is to calculate the free distributive lattice (with top and bottom) generated by $\bar{A}$ subject to certain relations which we will introduce later. First, we study the free distributive lattice generated by $\bar{A}$ without subjecting it to any relation since this object turns out to be quite interesting. Recall that a lattice is always assumed to have a top and a bottom element.

5.2.1 Construction of free distributive lattices

For a free distributive lattice in a topos we need the notion of an *internal distrivutive lattice*. The definition is analogous to the one of an internal Heyting algebra and the one of an internal C*-algebra. An internal distrivutive lattice in a topos $\mathcal{T}$ is an object L of the topos together with morphisms

$$\begin{array}{rrcl} \vee: & L \times L & \to & L; \\ \wedge: & L \times L & \to & L; \\ \top: & \mathbf{1} & \to & L; \\ \bot: & \mathbf{1} & \to & L, \end{array}$$

satisfying the obvious expressions in the Mitchell-Bénabou language for associativity, commutativity, the unit law, idempotence and distributivity for both $\vee$ and $\wedge$. Furthermore, it satisfies the absorption laws. In **Set** these relations reduce to the familiar definition of a distributive lattice. Now, we formally state the definition of a free distributive lattice in a topos.

Definition 5.2.1. *Let $\mathcal{T}$ be a topos, and let S be an object in that topos. An internal distributive lattice L_S is called freely generated by S if there is a morphism $i : S \to L_S$ such that for every internal distributive lattice M and morphism $f : S \to M$ there is a unique morphism $g : L_S \to M$ such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "i"'] & M \\ L_S \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

and such that g commutes with the internal lattice structure of L_S and M, i.e. the following diagrams commute:

$$\begin{tikzcd} L_S \times L_S \arrow[r, "\vee"] \arrow[d, "g\times g"'] & L_S \arrow[d, "g"] \\ M \times M \arrow[r, "\vee"'] & M \end{tikzcd} \qquad \begin{tikzcd} L_S \times L_S \arrow[r, "\wedge"] \arrow[d, "g\times g"'] & L_S \arrow[d, "g"] \\ M \times M \arrow[r, "\wedge"'] & M \end{tikzcd}$$

$$\begin{tikzcd} L_S \arrow[r, "g"] & M \\ \mathbf{1} \arrow[u, "\bot"] \arrow[ur, "\bot"'] & \end{tikzcd} \qquad \begin{tikzcd} L_S \arrow[r, "g"] & M \\ \mathbf{1} \arrow[u, "\top"] \arrow[ur, "\top"'] & \end{tikzcd}.$$

The choice of the notation i for the map between the object and the free distributive lattice it generates may remind the reader of an inclusion. In many cases it will be an 'inclusion', i.e. the morphism is monic. However, after we subject the lattice to relations, i will not be monic anymore. Nevertheless, we will still speak about the inclusion in the free distributive lattice, since this is a convenient way of indicating that a variable has to be regarded as an element of the free distributive lattice.

As an example we calculate the free distributive lattice generated by a set S in the topos **Set**. We claim the lattice obtained by the following procedure is the distributive lattice freely generated by S. Construct a set L inductively:

1. $S \subseteq L$. $\bot, \top \in L$ (as formal symbols).
2. If ϕ and ψ are elements of L, so are the formal expressions $\phi \vee \psi$ and $\phi \wedge \psi$.

Now divide out the following equational laws in L.

$$\begin{array}{ll}
\textit{commutativity}: & \phi \wedge \psi = \psi \wedge \phi \\
\textit{associativity}: & \phi \wedge (\psi \wedge \rho) = (\phi \wedge \psi) \wedge \rho \\
\textit{unit law}: & \phi \wedge \top = \phi \\
\textit{idempotence}: & \phi \wedge \phi = \phi \\
 & \\
\textit{commutativity}: & \phi \vee \psi = \psi \vee \phi \\
\textit{associativity}: & \phi \vee (\psi \vee \rho) = (\phi \vee \psi) \vee \rho \\
\textit{unit law}: & \phi \vee \bot = \phi \\
\textit{idempotence}: & \phi \vee \phi = \phi \\
 & \\
\textit{absorbtion}: & \phi \vee (\phi \wedge \psi) = \phi \\
 & \phi \wedge (\phi \vee \psi) = \phi \\
 & \\
\textit{distributivity}: & \phi \vee (\psi \wedge \rho) = (\phi \vee \psi) \wedge (\phi \vee \rho)
\end{array}$$

Define $L_S = L/\sim_l$, where the $\sim_l$ stands for the equivalence relation generated by above relations. For convenience of notation we do not add extra notation to elements of L_S to indicate that its elements are actually equivalence classes. The internal lattice structure is given by:

$$\begin{array}{lll}
\textit{bottom element} & \bot & \\
\textit{top element} & \top & \\
\textit{meet} & \wedge : L_S \times L_S \to L_S : (\phi, \psi) \mapsto \phi \wedge \psi & \text{where the last wedge is defined as above} \\
\textit{join} & \vee : L_S \times L_S \to L_S : (\phi, \psi) \mapsto \phi \vee \psi & \text{where the last vee is defined as above.}
\end{array}$$

It is important to remark that one has to prove that this defines a distributive lattice. This is done in [15]. Now if we have a diagram in **Set** like:

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "i"'] & M \\
L_S \arrow[ur, dotted, "g"'] &
\end{tikzcd},$$

where i is the obvious inclusion, then we can define g inductively by:

1. Define
$$\begin{array}{lll} g: & s \mapsto f(s) & s \in S; \\ g: & \top \mapsto \top & ; \\ g: & \bot \mapsto \bot. & \end{array}$$

2. If $\phi(s_1, \ldots, s_n, \bot, \top) \in L_S$ is an expression in which the elements $s_1, \ldots, s_n \in S$ and possibly $\bot$ or $\top$ appear, define
$$g : \phi(s_1, \ldots, s_n) \mapsto \phi(g(s_1), \ldots, g(s_n)).$$

The hard part is proving that the function g is well defined. This relates to the problem of determining which of the formal expressions that are defined above, are equivalent. This problem is known as the 'word problem' (for free lattices) and has been solved. Indeed one can prove that g is well defined. It is obvious that $f = gi$. By definition g is indeed a morphism of lattices (i.e. it preserves meets, joins, $\top$ and $\bot$). Furthermore, g is unique in having this property, since there is no free choice for the initial step 1 and the inductive step 2 of the definition of g in order to let g respect the lattice structure of L_S and M. We restate the result:

Theorem 5.2.2. *In **Set** the distributive lattice freely generated by a set S is defined as the object L_S above.*

The theorem may not be surprising, but it is of great use to compute free lattices generated by an object in $\mathbf{Set}^{\mathcal{C}}$. The following theorem shows that a free lattice in this category can easily be computed, namely point-wise (sometimes called *locally*).

Theorem 5.2.3. *Let $\mathcal{C}$ be a category. In the topos $\mathbf{Set}^{\mathcal{C}}$ the free distributive lattice generated by an object S can be computed point-wise. That is, for any object $C \in \mathcal{C}$, $L_S(C) = L_{S(C)}$ and the internal meets and joins are defined by the local meets and joins.*

Proof:

1. First we must define the intrinsic maps $f^{L_S}_{C,D}$. Recall that this notation is defined as $f^{L_S}_{C,D} = L_S(f)$, where f is a morphism in $\mathcal{C}$ from C to D. Since $L_S(D) = L_{S(D)}$ is a distributive lattice in **Set** by definition of $L_{S(D)}$ and $L_S(C) = L_{S(C)}$ is the free distributive lattice generated by $S(C)$, there is a unique map $f^{L_S}_{C,D}$ as in the diagram below such that $f^{L_S}_{C,D}$ preserves the meet and join in **Set**:

$$\begin{tikzcd} S(C) \arrow[r, "f^S_{C,D}"] \arrow[d, "i_{S(C)}"'] & S(D) \arrow[d, "i_{S(D)}"] \\ L_{S(C)} \arrow[r, dotted] & L_{S(D)}. \end{tikzcd}$$

Remark that by uniqueness of this map and some diagram chasing the following equations hold.

$$\begin{array}{rcl} Id^{L_S}_{C,C} & = & Id_{L_S(C)}; \\ g^{L_S}_{D,E} \circ f^{L_S}_{C,D} & = & (g \circ f)^{L_S}_{C,E}. \end{array}$$

Hence the assignment $(f : C \to D) \mapsto f^{L_S}_{C,D}$ is functorial.

2. The meet and join in L_S commute with the intrinsic maps $f_{C,D}^{L_S}$. Since this meet and join are defined locally in L_S this claim becomes a tautology; we actually defined the maps $f_{C,D}^{L_S}$ in such a way they commute with the meet and join.

3. Now suppose M is an internal lattice in $\mathbf{Set}^{\mathcal{C}}$.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "i"'] & M \\ L_S \arrow[ur, dotted, "g"'] & \end{tikzcd}.$$

We have to prove there exists a unique g such that this diagram commutes. The *uniqueness* is the easy part. For if there exists such a morphism g, the diagram must at least commute locally, i.e.

$$\begin{tikzcd} S(C) \arrow[r, "f(C)"] \arrow[d, "i"'] & M(C) \\ L_S(C) \arrow[ur, dotted, "g(C)"'] & \end{tikzcd},$$

commutes. Note that $M(C)$ is a distributive lattice in **Set**, since the local meets and joins inherited from M equip $M(C)$ with a distributive lattice structure. For every C there exists at most one such $g(C)$ since $M(C)$ is a distributive lattice in **Set** and $L_S(C)$ is the distributive lattice freely generated by $S(C)$ in **Set**.

For the *existence* of g, note that we can define a global map by combining the unique local maps which arise from the local diagrams in **Set**. If this is a well-defined map it automatically commutes with the meet and join in L_A and M, since it does commute locally. In order to let this map g be well-defined, we need to check if it commutes with the intrinsic maps of L_S and M, i.e. for any morphism $k : C \to D$ in $\mathcal{C}$ the following equation should hold.

$$g(D)k_{C,D}^{L_S} = k_{C,D}^{M} g(C) \tag{5.2}$$

Both the left hand side and right hand side are maps from $L_S(C)$ to $M(D)$. The left hand side is the unique map arising from

$$\begin{tikzcd} L_S(C) \arrow[rrr, dotted] & & & M(D) \\ S(C) \arrow[u] \arrow[r] & L_S(C) \arrow[r, "k_{C,D}^{L_S}"'] & L_S(D) \arrow[r, "g(D)"'] & M(D), \end{tikzcd} \tag{5.3}$$

and the right hand side is the unique map arising from

$$\begin{tikzcd} L_S(C) \arrow[rrr, dotted] & & & M(D) \\ S(C) \arrow[u] \arrow[r] & L_S(C) \arrow[r, "g(C)"'] & M(C) \arrow[r, "k_{C,D}^{M}"'] & M(D). \end{tikzcd} \tag{5.4}$$

For $x \in S(C)$, one has

$$\begin{aligned} g(D)k^{L_S}_{C,D}i(C)(x) &= g(D)i(D)k^S_{C,D}(x) \\ &= f(D)k^S_{C,D}(x) \\ &= k^M_{C,D}f(C)(x) \\ &= k^M_{C,D}g(C)i(C)(x). \end{aligned}$$

So the composition of the horizontal arrows in (5.3) and (5.4) are equal. The uniqueness of the dotted arrow then results in (5.2).

□

5.2.2 Relations and the lattice $L^{\sim}_{\bar{A}_{sa}}$

For the construction of the spectrum we first have to take the self-adjoint part of our object $\bar{A}$, which is defined as the following subobject of $\bar{A}$:

$$\bar{A}_{sa} = \{a | a^* = a\}.$$

It follows immediately from the fact that the adjoint * is defined locally, that $\bar{A}$ is the local self-adjoint part of every C*-subalgebra B of A. That is, $\bar{A}_{sa}$ is given on objects as:

$$B \mapsto B_{sa}.$$

In this paragraph we are going to define the distributive lattice freely generated by the object $\bar{A}_{sa}$ subject to relations. This turns out to be an object, called $L^{\sim}_{\bar{A}_{sa}}$, together with a morphism, called $i^{\sim} : \bar{A}_{sa} \to L^{\sim}_{\bar{A}_{sa}}$. Suppose a is a variable of type $\bar{A}$ (see definition 4.1.1 for its interpretation), then we write D_a for the term $i^{\sim}a$. Now the relations we need in order to define the spectrum are the following terms of the Mitchell-Bénabou-language.

$$\begin{array}{llcll} \models & D_1 & = & \top & (5) \\ \models \forall a & D_a \wedge D_{-a} & = & \bot & (6) \\ \models \forall a & D_{-a^2} & = & \bot & (7) \\ \models \forall a \forall b & D_{a+b} & \leq & D_a \vee D_b & (8) \\ \models \forall a \forall b & D_{ab} & \leq & (D_a \wedge D_b) \vee (D_{-a} \wedge D_{-b}). & (9) \end{array}$$

By definition of $\leq$, the last two relations are equivalent to:

$$\begin{array}{llcll} \models \forall a \forall b & D_{a+b} \vee (D_a \vee D_b) & = & D_a \vee D_b & (8) \\ \models \forall a \forall b & D_{ab} \vee ((D_a \wedge D_b) \vee (D_{-a} \wedge D_{-b})) & = & (D_a \wedge D_b) \vee (D_{-a} \wedge D_{-b}) & (9). \end{array}$$

How is this lattice defined? The reader may, meanwhile have acquired some intuition about this.

Definition 5.2.4. *The free distribuitive lattice generated by $\bar{A}_{sa}$ subject to relations (5) - (9) is an internal distributive lattice $L^{\sim}_{\bar{A}_{sa}}$ that satisfies (5) - (9)together with a morphism:*

$$\bar{A}_{sa} \xrightarrow{i^{\sim}} L^{\sim}_{\bar{A}_{sa}}.$$

Furthermore, if there is another internal distributive lattice M satisfying (5) - (9) together with a map $f : \bar{A}_{sa} \to M$, then f factors uniquely through $i^{\sim}$ by a morphism of lattices.

Remarks and notation: Of course this definition can be generalized to distributive lattices generated by an arbitrary object subject to an arbitrary finite number of relations in an arbitrary topos. For any such object we adopt the **notation** D_a to indicate the term $i^{\sim}a$, specifying which relations, which generating object and which topos define the morphism $i^{\sim}$. Especially, we are going to use both the interpretation in **Set** and its generalization to $\mathcal{T}(A)$ a lot.

Here we study the relations (5) - (9). We first prove that in **Set** the distributive lattice freely generated by a set S subject to (5) - (9) always exists.

Theorem 5.2.5. *In **Set** the lattice freely generated by a set S subject to relations (5) - (9) exists.*

Proof: First, construct L_S, the free lattice generated by S. Then find the smallest equivalence relation $\sim$ on L_S such that $L_S/\sim$ is a lattice when it inherits the lattice structure of L_S and such that $L_S/\sim$ satisfies the relations (5) - (9). Suppose there is another lattice M subject to the given relations. Then, by definition of the free lattice, there exists a morphism $L_S \to M$ which respects the lattice structure of L_S and M. By definition of the equivalence relation, if two elements of L_S are equivalent by $\sim$, then their images under $L_S \to M$ are equal. Hence there is a unique lattice homomorphism $(L_S/\sim) \to M$ making the diagram below commutative:

$$
\begin{tikzcd}
S \arrow[r] \arrow[d] & M \\
L_S \arrow[d] \arrow[ur, dotted] & \\
L_S^{\sim} \arrow[uur, dotted] & .
\end{tikzcd}
$$

□

Now note that by theorem 4.3.6 relations (5) - (9) can be expressed as equalities of morphisms in any topos. For example, relation (7) is equivalent to the following diagram being commutative:

$$
\begin{tikzcd}
\bar{A}_{sa} \arrow[r, "-\cdot^2"] \arrow[d, "!"'] & \bar{A}_{sa} \arrow[d, "i"] \\
\mathbf{1} \arrow[r, "\perp"'] & L_{\bar{A}_{sa}}.
\end{tikzcd}
$$

All the other relations can be expressed in this way, too. The next theorem shows that the free distributive lattice generated by $\bar{A}_{sa}$ subject to relations (5) - (9) can be computed point-wise. The proof is similar to theorem 5.2.3, but involves some extra diagram chasing. For our convenience we will only subject the free distributive lattice in the proof below to relation (7). There is no special reason why this relations is chosen. The proof for subjecting the lattice to all the relations is analogous to the one below. The reader can easily work out the details on a huge amount of scratch paper.

Theorem 5.2.6. *Let S be an object of **Set**$^{\mathcal{C}}$ and let $\alpha : S \to S$ be a morphism. Then $L_S^{\sim}$, the distributive lattice generated freely by S subject to relation*

$$D_{\alpha(s)} = \perp, \tag{5.10}$$

can be computed locally. That is, for any object $C \in \mathcal{C}$, $L^{\sim}_{S}(C) = L^{\sim}_{S(C)}$ and the internal meets and joins are defined by the local meets and joins.

Proof: The proof is similar to theorem 5.2.3. Notice that α will play the role of the function $s \mapsto -s^2$. Furthermore, theorem 5.2.5 proves that the local distributive lattices generated by the local sets subject to relation (5.10) exist. We need three steps:

1. First we must define the internal maps $f^{L^{\sim}_{S}}_{C,D}$

$$\begin{tikzcd}
 & S(C) \arrow[rrr, "f^S_{C,D}"] \arrow[d, "i(C)"'] & & & S(D) \arrow[d, "i(D)"] & \\
 & L^{\sim}_{S}(C) \arrow[rrr, dotted] & & & L^{\sim}_{S}(D) & \\
 & & \mathbf{1}(C) \arrow[ul, "\perp(C)"'] \arrow[rrr, "!"] & & & \mathbf{1}(D) \arrow[ul, "\perp(D)"'] \\
S(C) \arrow[uur, "i^{\sim}(C)"] \arrow[rrr, "f^S_{C,D}"] & & & S(D) \arrow[uur, "i^{\sim}(D)"] & & \\
 & S(C) \arrow[ul, "(\alpha)(C)"] \arrow[uur] \arrow[rrr, "f^S_{C,D}"'] & & & S(D) \arrow[ul, "(\alpha)(D)"] \arrow[uur, "!"'] &
\end{tikzcd} .$$

We know that the non-dotted arrows form a commutative diagram. Note that this commutativity is exactly the condition stating that there exists a unique morphism $f^{L^{\sim}_{S}}_{C,D}$ as indicated in the diagram. By uniqueness and an extensive diagram chase we find:

$$\begin{aligned} Id^{L^{\sim}_{S}}_{C,C} &= Id_{L^{\sim}_{S}(C)}; \\ g^{L^{\sim}_{S}}_{D,E} \circ f^{L^{\sim}_{S}}_{C,D} &= (g \circ f)^{L^{\sim}_{S}}_{C,E}. \end{aligned}$$

Hence the assignment $(f : C \to D) \mapsto f^{L^{\sim}_{S}}_{C,D}$ is functorial.

2. $f^{L^{\sim}_{S}}_{C,D}$ is a lattice homomorphism by construction.

3. Now suppose M is an internal lattice in $\mathbf{Set}^{\mathcal{C}}$ and $f : A \to M$ is a morphism such that the right diagram below commutes.

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "i^{\sim}"'] & M \\
L^{\sim}_{S} \arrow[ur, dotted, "g"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
S \arrow[r, "\alpha"] \arrow[d, "!"'] & S \arrow[d, "f"] \\
\mathbf{1} \arrow[r, "\perp"'] & M
\end{tikzcd} .$$

We have to prove that there exists a unique g such that the left diagram above commutes. First we prove *uniqueness*. If there exists such a morphism g, the diagrams must at least commute locally, i.e.

$$
\begin{tikzcd}
S(C) \arrow[r, "f(C)"] \arrow[d, "i^{\sim}(C)"'] & M(C) \\
L_S(C) \arrow[ur, dotted, "g(C)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
S(C) \arrow[r, "\alpha(C)"] \arrow[d, "!"'] & S(C) \arrow[d, "f(C)"] \\
\mathbf{1}(C) \arrow[r, "\perp(C)"'] & M(C)
\end{tikzcd},
$$

commute. For every C there exists at most one such $g(C)$, since $M(C)$ is a lattice subject to relation (5.10) in **Set** and $L^{\sim}_S(C)$ is the lattice freely generated by $S(C)$ subjected to relation (5.10) in **Set**.
For the *existence* of g, note that we can define a global map by combining the local maps obtained by the local diagrams in **Set**. We only need to prove that this map commutes with the intrinsic maps $f^{L^{\sim}_S}_{C,D}$, i.e. for any morphism $k : C \to D$ in $\mathcal{C}$ the following equation should hold:

$$
g(D)k^{L^{\sim}_S}_{C,D} = k^M_{C,D}\,g(C). \tag{5.11}
$$

Both the left-hand side and the right-hand side are maps from $L^{\sim}_S(C)$ to $M(D)$. The left-hand side is the unique map arising from

$$
\begin{tikzcd}
L^{\sim}_S(C) \arrow[rrrd, dotted] & & & \\
S(C) \arrow[u, "i^{\sim}"] \arrow[r, "i^{\sim}"'] & L^{\sim}_S(C) \arrow[r, "k^{L^{\sim}_S}_{C,D}"'] & L^{\sim}_S(D) \arrow[r, "g(D)"'] & M(D)
\end{tikzcd}
\quad
\begin{tikzcd}
S(C) \arrow[r, "\alpha(C)"] \arrow[d, "!(C)"'] & S(C) \arrow[d, "g(D)k^{L^{\sim}_S}_{C,D}i"] \\
\mathbf{1}(C) \arrow[r, "\perp"'] & M(D)
\end{tikzcd}, \tag{5.12}
$$

and the right-hand side is the unique map arising from

$$
\begin{tikzcd}
L^{\sim}_S(C) \arrow[rrrd, dotted] & & & \\
A(C) \arrow[u, "i^{\sim}"] \arrow[r, "i^{\sim}"'] & L^{\sim}_S(C) \arrow[r, "g(C)"'] & M(C) \arrow[r, "k^M_{C,D}"'] & M(D)
\end{tikzcd}
\quad
\begin{tikzcd}
A(C) \arrow[r, "\alpha(C)"] \arrow[d, "!(C)"'] & A(C) \arrow[d, "k^M_{C,D}g(C)i"] \\
1(C) \arrow[r, "\perp"'] & M(D)
\end{tikzcd}, \tag{5.13}
$$

For $x \in S(C)$, one has

$$
\begin{aligned}
g(D)k^{L^{\sim}_S}_{C,D}i^{\sim}(C)(x) &= g(D)i^{\sim}(D)k^S_{C,D}(x) \\
&= f(D)k^S_{C,D}(x) \\
&= k^M_{C,D}f(C)(x) \\
&= k^M_{C,D}g(C)i^{\sim}(C)(x).
\end{aligned}
$$

Hence the compositions of the horizontal arrows in the left diagrams of (5.12) and (5.13) are equal. The uniqueness of the dotted arrow then results in (5.11).

□

We derive a short corollary from the theorem above.

Corollary 5.2.7. *For any C*-algebra A, the free distributive lattice generated by $\bar{A}$ subject to relations (5) - (9) exists. This lattice is denoted as $L^{\sim}_{\bar{A}_{sa}}$.*

5.2.3 Intermezzo: representing the local lattice

The two theorems in this paragraph will be quite important for for what follows. We first introduce a few lemmas that give us tools to compute the distributive lattice freely generated by a commutative unital C*-algebra subject to (5) - (9) in **Set**. We will use the Gelfand theorem to infer that if A is a commutative unital C*-algebra, then it is of the form $C(X)$, where X is a compact Hausdorff space. In this situation, A_{sa} is the space of real valued continuous functions on X. The theorems below are based on [2] and [3].

In this paragraph all statements are taken within **Set**. Hence, in this paragraph, we write D_a for $i^{\sim}a : A_{sa} \to L^{\sim}_{A_{sa}}$, the inclusion of the variable a of type A_{sa} into the distributive lattice freely generated by A_{sa}, subject to relations (5) - (9).

Lemma 5.2.8. *Let $A = C(X)$ be a commutative unital C*-algebra. Suppose $a, b \in A_{sa}$ and $a \leq b$ (i.e. $\forall x \in X\ a(x) \leq b(x)$). Then for $D_a, D_b \in L^{\sim}_{A_{sa}}$, $D_a \leq D_b$.*

Proof:

$$\begin{array}{rcll} D_a & = & D_{(a-b)+b} & \\ & \leq & D_{a-b} \vee D_b & \text{(by relation (8))} \\ & = & \perp \vee D_b & \text{(by relation (7))} \\ & = & D_b & \end{array}$$

□

For $a, b \in C(X)_{sa}$ define $(a \vee b) \in C(X)_{sa}$ and $(a \wedge b) \in C(X)_{sa}$ to be the point-wise maximum and minimum of a and b, respectively. Let $a^+ = a \vee 0$ and $a^- = a \wedge 0$. Note that $a = a^+ + a^-$.

Lemma 5.2.9. *Let $A = C(X)$ be a commutative unital C*-algebra. For $a \in A_{sa}$, $D_a = D_{a^+}$ and $D_{-a} = D_{-a^-}$.*

Proof: Lemma 5.2.8 gives $D_a \leq D_{a^+}$. Conversely:

$$\begin{array}{rcll} D_{a^+} & = & D_{a \vee 0} & \\ & \leq & D_a \vee D_0 & \text{(by relation (8))} \\ & = & D_a & \text{(by relation (7)).} \end{array}$$

We obtain $D_a = D_{a^+}$, from which $D_{-a^-} = D_{(-a)^+} = D_{-a}$. □

Lemma 5.2.10. *Let $A = C(X)$ be a commutative unital C*-algebra. For $a \in A_{sa}$ and $n \in \mathbb{N}^*$, one has $D_a = D_{na}$.*

Proof:

$$\begin{array}{rcll} \top & = & D_1 & \text{(by relation (5))} \\ & = & D_{n\frac{1}{n}} & \\ & \leq & D_{\frac{1}{n}} & \text{(by applying relation (8) } n \text{ times)} \end{array}$$

Now relation (9) gives $D_{na} \leq D_a$ and $D_a = D_{\frac{1}{n}na} \leq D_{na}$, from which the lemma follows. □

Theorem 5.2.11. *Let $A = C(X)$ be a commutative unital C*-algebra. Then the map of sets $i : A_{sa} \to L^{\sim}_{A_{sa}}$ is surjective, i.e. every element of the lattice $L^{\sim}_{A_{sa}}$ has a representative of the form D_a for some $a \in A_{sa}$.*

First notice that the proof of theorem 5.2.7 tells us $L^{\sim}_{A_{sa}}$ is equal to the free lattice generated by A_{sa} modulo some equivalence relation. We have proved the theorem if we can show that:

1. $\perp$ is equal to D_a for some $a \in A_{sa}$. This is true, since $\perp = D_{-1}$ by relation (7).
2. $\top$ is equal to D_a for some $a \in A_{sa}$. This is true, since $\top = D_1$ by relation (5).
3. $D_a \vee D_b$ is equal to D_c for some $c \in A_{sa}$. We prove $D_c = D_{a\vee b}$. The property $D_a \vee D_b \leq D_{a\vee b}$ follows straight from lemma 5.2.8. For the other inequality, we note

$$\begin{array}{rcll} D_{a\vee b} & = & D_{(a-b)^+ + b} & \\ & \leq & D_{(a-b)^+} \vee D_b & \text{(by relation (8))} \\ & \leq & D_{(a-b)^+} \vee (D_b \vee D_a); & \\ & & & \\ D_{a\vee b} & \leq & D_{(b-a)^+} \vee (D_b \vee D_a). & \end{array}$$

Now, the following inequality follows:

$$\begin{array}{rcl} D_{a\vee b} & \leq & (D_{(b-a)^+} \vee (D_b \vee D_a)) \wedge (D_{(a-b)^+} \vee (D_b \vee D_a)) \\ & = & (D_{(b-a)^+} \wedge D_{(a-b)^+}) \vee (D_b \vee D_a) \vee ((D_b \vee D_a) \wedge D_{(a-b)^+}) \vee ((D_b \vee D_a) \wedge D_{(b-a)^+}) \\ & = & (D_{b-a} \wedge D_{a-b}) \vee (D_b \vee D_a) \\ & = & D_b \vee D_a. \end{array}$$

The first equality follows from distributivity, the second one follows from lemma 5.2.9 and the last one follows from relation (6).

4. $D_a \wedge D_b$ is equivalent to D_c for some $c \in A_{sa}$. We prove that $D_c = D_{a\wedge b}$. $D_a \wedge D_b \geq D_{a\wedge b}$ follows straight from lemma 5.2.8. For the other inequality:

$$\begin{array}{rcll} D_a & = & D_{(a\wedge b)+(a-b)^+} & \\ & \leq & D_{a\wedge b} \vee D_{(a-b)^+} & \text{, by relation (8)} \\ & & & \\ D_b & \leq & D_{a\wedge b} \vee D_{-(a-b)^-}. & \end{array}$$

So:

$$\begin{array}{rcl} D_a \wedge D_b & \leq & (D_{a\wedge b} \vee D_{(a-b)^+}) \wedge (D_{a\wedge b} \vee D_{-(a-b)^-}) \\ & = & D_{a\wedge b} \vee (D_{(a-b)^+} \wedge D_{-(a-b)^-}) \vee (D_{a\wedge b} \wedge D_{(a-b)^+}) \vee (D_{a\wedge b} \wedge D_{-(a-b)^-}) \\ & = & D_{a\wedge b} \vee (D_{(a-b)} \wedge D_{-(a-b)}) \\ & = & D_{a\wedge b} \vee \perp \\ & = & D_{a\wedge b}. \end{array}$$

So all elements in $L_{A_{sa}}$ that are obtained by the inductive steps in the proof of theorem 5.2.2 are equivalent to D_a for some $a \in A_{sa}$. This proves the theorem. $\square$

The theorem above tells us that every element of $L^{\sim}_{A_{sa}}$ is of the form D_a for some $a \in A_{sa}$. So if we want to compute $L^{\sim}_{A_{sa}}$, we can start with the set A_{sa} and divide out relations (5) - (9) in A_{sa} instead of dividing out these relations in $L_{A_{sa}}$. This makes the lattice far easier to compute. Futhermore, this justifies to denote all the elements of the lattice $L^{\sim}_{A_{sa}}$ as D_a for a fixed element $a \in A_{sa}$.

Notation: recapulating the discussion above, the notation D_a can have three meanings:

1. We write D_a for the term $i^{\sim}a$ if $i^{\sim} : \bar{A}_{sa} \to L^{\sim}_{\bar{A}_{sa}}$ is defined by the morphism of definition 5.2.6 in the topos $\mathcal{T}(A)$. When this is meant, we will emphasize that this term is interpreted in $\mathcal{T}(A)$.

2. We write D_a for the term $i^{\sim}a$ if $i^{\sim} : A_{sa} \to L^{\sim}_{A_{sa}}$ is defined by the morphism of definition 5.2.6 in the topos **Set**. We will emphasize that this term has to be interpreted in **Set**.

3. We write D_a for the equivalence class of $L^{\sim}_{A_{sa}}$ that contains a. In this case, we will always specify the value of a and replace 'a' in the notation by its value. So, for example, we will always write D_1, D_{-1}, et cetera.

This notation is not as ambiguous as it seems. The interpretations of the terms of the Mitchell-Bénabou language in **Set** are exactly the conventional set-theoretic meanings of the expressions in terms of variables, logical connectives and quantifiers. Since in **Set** it is our practise to denote D_a for both the meanings as stated in 2 and 3, this notation will not lead to confusion. Also, note that meaning number 1 is a generalization of meaning number 2. Adopting the notation from meaning number 2 for meaning number 1 will not lead to confusion as long as we specify in which topos we are working.

Returning to the computations in **Set** we prove one last theorem, which tells exactly which equivalence relation we must divide out in order to compute the distributive lattice generated by A_{sa} subject to (5) - (9) in **Set**.

Theorem 5.2.12. *Let $A = C(X)$ be a commutative unital C*-algebra. Let L be the set A_{sa} modulo the following equivalence relation:*

$$a \sim_+ b \text{ if and only if } \exists n \in \mathbb{N}^* : a^+ \leq nb^+ \text{ and } \exists m \in \mathbb{N}^* : b^+ \leq ma^+.$$

Suppose L is equipped with the structure of the partial order $\preceq_+$ defined by:

$$a \preceq_+ b \text{ if and only if } \exists n \in \mathbb{N}^* : a^+ \leq nb^+.$$

Then L is a lattice isomorphic to $L^{\sim}_{A_{sa}}$.

Proof: Theorem 5.2.11 and the discussion after its proof show that the free lattice can be computed by dividing out the equivalence relation generated by (5) - (9) in A_{sa} with its partial order on functions. Now clearly at least the equivalence relation above must at least be divided out, since if

$$\exists n \in \mathbb{N}^* : a^+ \leq nb^+ \text{ and } \exists m \in \mathbb{N}^* : b^+ \leq ma^+$$

then

$$D_a = D_b,$$

by lemma 5.2.9 and lemma 5.2.10. The lattice $L = A_{sa}/\sim_+$ satisfies (5) - (9). Hence $\sim_+$ is the smallest equivalence relation $\sim$ such that $A_{sa}/\sim$ is a lattice satisfying (5) - (9), and hence L must be the free lattice. One easily verifies that $\preceq_+$ is a well-defined partial order and if $a \leq b$, then $a \preceq_+ b$. Hence $\preceq_+$ is the partial order on L that is inherited from the partial order on functions on A_{sa}. $\square$

The theorem above gives a very nice representation of the free lattice generated by A_{sa}. Now that we now what this lattice looks like, we proceed to the next step, the construction of the spectrum.

5.3 The spectrum

Thusfar, we started with a commutative unital C*-algebra A in **Set**. We defined the topos $\mathcal{T}(A)$ and the internal C*-algebra $\bar{A}$. By means of section 5.2 we defined the distributive lattice freely generated by $\bar{A}_{sa}$ subject to relations (5) - (9) in $\mathcal{T}(A)$. In this section we introduce a morphism $\mathcal{A} : \Omega^{L_{\bar{A}_{sa}}^{\sim}} \to \Omega^{L_{\bar{A}_{sa}}^{\sim}}$ of $\mathcal{T}(A)$ and define the spectrum Σ as a subobject of $\Omega^{L_{\bar{A}_{sa}}^{\sim}}$ by means of this morphism $\mathcal{A}$.
In this section V, V_0 or any object that includes the character 'V' in its notation is supposed to be a *set*. We use the character U to indicate a subobject of $L_{\bar{A}_{sa}}^{\sim}$ or more generally something that 'looks like' or 'corresponds to' a subfunctor of $L_{\bar{A}_{sa}}^{\sim}$. The symbols C and D will always be objects in $\mathcal{C}(A)$, i.e. commutative unital C*-subalgebras of A. Recall that $L_{\bar{A}_{sa}}^{\sim}$ was an object of $\mathcal{T}(A)$. Hence $L_{\bar{A}_{sa}}^{\sim}(C)$ for an object $C \in \mathcal{C}(A)$ is a set.

5.3.1 The map $\mathcal{A}_0$

In the following construction we define a map $\mathcal{A}_0 : Sub(L_{\bar{A}_{sa}}^{\sim}) \to Sub(L_{\bar{A}_{sa}}^{\sim})$ of subobjects of $L_{\bar{A}_{sa}}^{\sim}$ in the topos $\mathcal{T}(A)$. This is done by defining a map $\mathcal{A}_0(C) : \mathcal{P}(L_{\bar{A}_{sa}}^{\sim}(C)) \to \mathcal{P}(L_{\bar{A}_{sa}}^{\sim}(C))$ of subsets of $L_{\bar{A}_{sa}}^{\sim}(C)$ for all objects $C \in \mathcal{C}(A)$ and combining them together into a global map (here $\mathcal{P}$ denotes the power set). The maps $\mathcal{A}_0(C)$ are called the (local) *completions*.

To define the completions we first need to know something about joins of finite subsets of a lattice. Suppose L is an internal lattice in **Set** and suppose V_0 is a finite subset of L. We define $\bigvee V_0$ to be the supremum of the set V_0. Thus $\bigvee V_0$ is the element $z \in L$ such that

$$\models (\forall x\, x \in V_0 \Rightarrow x \leq z) \wedge (\forall x (\forall y\, y \in V_0 \Rightarrow y \leq x) \Rightarrow z \leq x).$$

Here the variables x and y are of type L. The expression above has an interpretation in the Mitchell-Bénabou language of the topos **Set**, which coincides with its familiar meaning in set theory. Note that $\bigvee V_0$ always exists: it is the join of all the elements of V_0, which is finite by definition. Note that by this definition $\bigvee \emptyset$ is defined to be the bottom element $\perp$ of L.

We start with a short lemma in **Set**.

Lemma 5.3.1. *Suppose C is a unital commutative C*-algebra in **Set**. Let $D_a, D_b \in L_{C_{sa}}^{\sim}$ such that $D_a = D_b$. Then for every $q \in \mathbb{Q}$ there exists an $r \in \mathbb{Q}$ such that*

$$D_{a-q} \leq D_{b-r}.$$

Proof: Choose an $n \in \mathbb{N}^*$ such that $a \leq nb$, which can be done by theorem 5.2.12. Then

$$\begin{aligned} D_{a-q} &\leq D_{nb-q} \\ &= D_{n(b-\frac{q}{n})} \\ &= D_{b-\frac{q}{n}}. \end{aligned}$$

Hence $r = \frac{q}{n}$ works. $\square$

Now if C is an object of $\mathcal{C}(A)$ and U is a subfunctor of $L_{\widetilde{A}_{sa}}$, we define locally what is called an ideal generated by $U(C)$ with respect to the covering $\lhd$ by:

$$\mathcal{A}_0(C)U(C) = \left\{ D_a \in L_{\widetilde{A}_{sa}}(C) \mid D_a \lhd U(C) \right\},$$

where $D_a \lhd U(C)$ is defined as:

$$\text{For every } q \in \mathbb{Q}^+ \text{ there exists a finite subset } V_0 \text{ of } U(C) \text{ such that } D_{a-q} \leq \bigvee V_0. \tag{5.14}$$

We say $U(C)$ *covers* D_a. It follows from lemma 5.3.1 that if $D_a = D_b$ and $D_a \lhd U(C)$, then $D_b \lhd U(C)$. Hence $\mathcal{A}_0(C)$ is well-defined. Furthermore, the completions $\mathcal{A}_0(C)U(C)$ glue together to a functor called $\mathcal{A}_0 U$, which is an object of $\mathcal{T}(A)$. To see this, we must check if the intrinsic maps $\subseteq_{C,D}^{\mathcal{A}_0 U}$ are well-defined. Since $\mathcal{A}_0 U$ is a subfunctor of $L_{\widetilde{A}_{sa}}$, we have $\subseteq_{C,D}^{\mathcal{A}_0 U} = \subseteq_{C,D}^{L_{\widetilde{A}_{sa}}} |_{\mathcal{A}_0 U(C)}$ and we must check whether $\subseteq_{C,D}^{L_{\widetilde{A}_{sa}}} |_{\mathcal{A}_0 U(C)}(D_a)$ is in $(\mathcal{A}_0 U)(D) = \mathcal{A}_0(D)U(D)$. This amounts to checking whether $\subseteq_{C,D}^{L_{\widetilde{A}_{sa}}} |_{\mathcal{A}_0 U(C)}(D_a)$ satisfies property 5.14. Now since $\subseteq_{C,D}^{L_{\widetilde{A}_{sa}}}$ is a lattice homomorphism (cf. proof of theorem 5.2.3), and since applying $\bigvee$ to a finite set gives the join of all the elements of that set, we find that $\subseteq_{C,D}^{L_{\widetilde{A}_{sa}}} |_{\mathcal{A}_0 U(C)}(D_a)$ is covered by $\subseteq_{C,D}^{L_{\widetilde{A}_{sa}}} |_{\mathcal{A}_0 U(C)}(V_0)$, where V_0 is a finite subset of $U(C)$ covering D_a. So we have successfully defined our map $\mathcal{A}_0 : Sub(L_{\widetilde{A}_{sa}}) \to Sub(L_{\widetilde{A}_{sa}})$, by

$$(\mathcal{A}_0 U)(C) := \mathcal{A}_0(C)U(C).$$

The following lemma will be useful for later computations.

Lemma 5.3.2. $\mathcal{A}_0$ *is idempotent.*

Proof: $\mathcal{A}_0$ is idempotent if and only if $\mathcal{A}_0(C)$ is idempotent for every $C \in \mathcal{C}(A)$. So we must show that if U is a subfunctor of $L_{\widetilde{A}_{sa}}$, then

$$\mathcal{A}_0(C)\mathcal{A}_0(C)U(C) = \mathcal{A}_0(C)U(C).$$

The inclusion $\supseteq$ is obvious, since every element of $\mathcal{A}_0(C)U(C)$ is covered by itself. Conversely, suppose $D_a \in \mathcal{A}_0(C)\mathcal{A}_0(C)U(C)$. Then we must show that for every $q \in \mathbb{Q}^+$ there exists a finite subset V_0 of $U(C)$ such that $D_{a-q} \leq \bigvee V_0$. By the assumption $D_a \in \mathcal{A}_0(C)\mathcal{A}_0(C)U(C)$ there exists a finite subset V_0' of $\mathcal{A}_0(C)U(C)$ such that $D_{a-\frac{q}{2}} \leq \bigvee V_0'$. Now if $D_b \in V_0'$ then there exists a finite subset $V_{0,b}$ of $U(C)$ such that $D_{b-\frac{q}{2}} \leq \bigvee V_{0,b}$. Now we have

$$D_{a-q} \leq \bigvee \left\{ D_{b-\frac{q}{2}} \mid D_b \in V_0' \right\} \leq \bigvee_b \bigvee V_{0,b},$$

where the double supremum on the right hand side of the equality is the supremum of a finite set. $\square$

5.3.2 The morphism $\mathcal{A}$

Now we will define a morphism $\mathcal{A} : \Omega^{L_{\widetilde{A}_{sa}}} \to \Omega^{L_{\widetilde{A}_{sa}}}$, which is the internal version of $\mathcal{A}_0$. Before we proceed we introduce an important lemma.

Lemma 5.3.3. *Let $\mathcal{P}$ be a categorical preorder and let A be an object of $\textbf{Set}^{\mathcal{P}}$. Then all intrinsic morphisms of Ω^A are surjective maps of sets.*

Proof: In this proof, let C, D, E and F be objects of the preorder $\mathcal{P}$. Recall that Ω^A was defined as

$$\Omega^A(C) = \mathbf{Hom}_{\mathbf{Set}^{\mathcal{P}}}(\mathbf{y}(C) \times A, \Omega),$$

where we have written $\mathbf{y}(C)$ for the covariant Yoneda-functor $\mathbf{Hom}_{\mathcal{P}}(C, -)$. Notice that

$$\mathbf{y}(C)(E) = \begin{cases} \{1\} & \text{if } C \subseteq E; \\ \emptyset & \text{if } C \not\subseteq E; \end{cases}$$

and remark that

$$\mathbf{y}(C)(E) \times A(E) = \begin{cases} A(E) & \text{if } C \subseteq E; \\ \emptyset & \text{if } C \not\subseteq E. \end{cases}$$

Now suppose $C \subseteq D$ (denote this as a morphism by $\subseteq_{C,D}$) and $\tau \in \mathbf{Hom}_{\mathbf{Set}^{\mathcal{P}}}(\mathbf{y}(D) \times A, \Omega)$. We define a natural transformation $\sigma \in \mathbf{Hom}_{\mathbf{Set}^{\mathcal{P}}}(\mathbf{y}(C) \times A, \Omega)$ by

$$\begin{array}{rclcll} \sigma(E) & = & \tau(E) & & & \text{if } D \subseteq E \text{ or } C \not\subseteq E \\ \sigma(E) & : & A(E) & \to & \Omega(E) & \\ & & a & \mapsto & \left\{ F | D \subseteq F,\ (\tau(F))(\{1\} \times (\subseteq^A_{E,F})(a)) = \text{m.c. on } F \right\} & \text{if } C \subseteq E \text{ and } D \not\subseteq E, \end{array}$$

where m.c. is short for 'maximal cosieve'. It is not too hard to check that $\sigma : \mathbf{y}(C) \times A \to \Omega$ is indeed a natural transformation. We prove that $(\subseteq^*_{C,D} \times Id_A)^* \sigma = \tau$. We separate three cases:

1. First, suppose $D \subseteq E$. Then

$$\begin{array}{rcl} ((\subseteq^*_{C,D} \times Id_A)^* \sigma)(E) & = & (\sigma \circ (\subseteq^*_{C,D} \times Id_A))(E) \\ & = & \sigma(E) \circ (\subseteq^*_{C,D} \times Id_A)(E) \\ & = & \tau(E). \end{array}$$

   The last equation follows from the fact that $\subseteq^*_{C,D}(E) : \mathbf{y}(D)(E) \to \mathbf{y}(C)(E)$ is the identity $Id_{\{1\}}$.

2. Now, suppose $C \not\subseteq E$. Then again

$$\begin{array}{rcl} ((\subseteq^*_{C,D} \times Id_A)^* \sigma)(E) & = & (\sigma \circ (\subseteq^*_{C,D} \times Id_A))(E) \\ & = & \sigma(E) \circ (\subseteq^*_{C,D} \times Id_A)(E) \\ & = & \tau(E). \end{array}$$

   The last equation follows from the fact that $\subseteq^*_{C,D}(E) : \mathbf{y}(D)(E) \to \mathbf{y}(C)(E)$ is the identity $Id_{\emptyset}$.

3. Suppose $C \subseteq E$ and $D \nsubseteq E$. Note that $((\subseteq^*_{C,D} \times Id_A)^*\sigma)(E) : \emptyset = \mathbf{y}(D)(E) \times A(E) \to \Omega(E)$ is the unique morphism from the initial object to $\Omega(E)$. Hence it must equal $\tau(E)$.

We have shown that $(\subseteq^*_{C,D} \times Id_A)^*\sigma = \tau$, hence $\subseteq^{\Omega^A}_{C,D} = (\subseteq^*_{C,D} \times Id_A)^*$ is surjective. $\square$

Corollary 5.3.4. *Suppose in lemma 5.3.3 that $\mathcal{P}$ has a bottom element $\perp$. Then, for any object B of $\boldsymbol{Set}^{\mathcal{P}}$, any morphism $\mathcal{A} : \Omega^A \to B$ is completely determined by the morphism $\mathcal{A}(\perp) : \Omega^A(\perp) \to B(\perp)$.*

Proof: Suppose $C \in \mathcal{P}$; we must define $\mathcal{A}(C) : \Omega^A(C) \to B(C)$. If $x \in \Omega^A(C)$, then there exists a $y \in \Omega^A(\perp)$ such that $\subseteq^{\Omega^A}_{\perp,C}(y) = x$. Hence

$$(\mathcal{A}(C))(x) = (\mathcal{A}(C))(\subseteq^{\Omega^A}_{\perp,C}(y)) = (\subseteq^B_{\perp,C})(\mathcal{A}(\perp))(y). \tag{5.15}$$

$\square$

Now observe that a subfunctor U of $L_{\widetilde{\bar{A}}_{sa}}$ corresponds to a morphism $\chi_U : L_{\widetilde{\bar{A}}_{sa}} \to \Omega$, which in turn corresponds to a morphism $\hat{\chi_U} : \mathbf{1} \to \Omega^{L_{\widetilde{\bar{A}}_{sa}}}$ by cartesian closedness. So the completion $\mathcal{A}_0$ corresponds to a morphism

$$(\hat{\chi}_U : \mathbf{1} \to \Omega^{L_{\widetilde{\bar{A}}_{sa}}}) \mapsto (\hat{\chi}_{\mathcal{A}_0 U} : \mathbf{1} \to \Omega^{L_{\widetilde{\bar{A}}_{sa}}}).$$

Since each morphism $\mathbf{1} \to \Omega^{L_{\widetilde{\bar{A}}_{sa}}}$ is determined by $\mathbf{1}(\perp) \to \Omega^{L_{\widetilde{\bar{A}}_{sa}}}(\perp)$ (noting that the intrinsic maps of $\mathbf{1}$ are surjective), this corresponds to a morphism

$$\Omega^{L_{\widetilde{\bar{A}}_{sa}}}(\perp) \to \Omega^{L_{\widetilde{\bar{A}}_{sa}}}(\perp),$$

which in turn induces a unique morphism $\mathcal{A}$ by equation (5.15) in corollary 5.3.4:

$$\mathcal{A} : \Omega^{L_{\widetilde{\bar{A}}_{sa}}} \to \Omega^{L_{\widetilde{\bar{A}}_{sa}}}.$$

Finally, we must check if $\mathcal{A}$ commutes with the intrinsic maps of $\Omega^{L_{\widetilde{\bar{A}}_{sa}}}$. Suppose $x = \subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{\perp,C} y$, then:

$$\begin{aligned}
(\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{C,D})(\mathcal{A}(C))(x) &= (\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{C,D})(\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{\perp,C})(\mathcal{A}(\perp))(y) \\
&= (\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{\perp,D})(\mathcal{A}(\perp))(y) \\
&= (\mathcal{A}(D))(\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{\perp,D})(y) \\
&= (\mathcal{A}(D))(\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{C,D})(\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{\perp,C})(y) \\
&= (\mathcal{A}(D))(\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{C,D})(x).
\end{aligned}$$

Hence $\mathcal{A}$ is well-defined. We next prove the counterpart of lemma 5.3.2 for $\mathcal{A}$.

Lemma 5.3.5. *$\mathcal{A}$ is idempotent.*

Proof: $\mathcal{A}$ is idempotent if and only if $\mathcal{A}(C)$ is idempotent for all $C \in \mathcal{C}(A)$. The lemma follows from the following computation. Suppose $x = \subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{\perp,C} y$, then

$$\begin{aligned}
(\mathcal{A}(C))(x) &= (\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{\perp,C})(\mathcal{A}(\perp))y \\
&= (\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{\perp,C})(\mathcal{A}(\perp))(\mathcal{A}(\perp))y \\
&= (\mathcal{A}(C))(\mathcal{A}(C))(x).
\end{aligned}$$

The first and last equations hold by definition. The second equation follows from lemma 5.3.2 and the correspondence between $\mathcal{A}_0$ and $\mathcal{A}(\perp)$. $\square$

We conclude this section by giving the definition of the spectrum as well as a proposition which is a consequence of lemma 5.3.2 and lemma 5.3.5.

Definition 5.3.6. *The* ***spectrum*** Σ *of an internal unital commutative C*-algebra* $\bar{A}$ *in the topos* $\mathcal{T}(A)$ *is defined as*

$$\Sigma = \{U \,|\mathcal{A}U = U\} \subseteq \Omega^{L_{\widetilde{\bar{A}}_{sa}}}.$$

As we said in the beginning of this chapter Mulvey and Banaschewski proved the duality

$$\mathbf{cCStar} \leftrightarrow \mathbf{KRegLoc},$$

in any topos. The construction above starting from the internal C*-algebra $\bar{A}$ and ending with the spectrum Σ is precisely the functor $\mathbf{cCStar} \to \mathbf{KRegLoc}$ in the topos $\mathcal{T}(A)$, as proved in [7]. The morphism $\mathbf{cCStar} \leftarrow \mathbf{KRegLoc}$ then is $\Sigma \mapsto \mathbf{Loc}(\Sigma, \mathbb{C})$.
The following theorem allows us to compute the spectrum once we have computed $\Sigma(\perp)$.

Theorem 5.3.7. *The maps* $\subseteq^{\Sigma}_{\perp,C}\colon \Sigma(\perp) \to \Sigma(C)$ *are surjective. Hence* $\Sigma(C)$ *is the image set of the the intrinsic map* $\subseteq^{\Sigma}_{\perp,C}$.

Proof: Suppose $U_C \in \Sigma(C)$ and $U_C = (\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{\perp,C})(U_\perp)$. Then

$$\begin{aligned} U_C &= (\mathcal{A}(C))U_C \\ &= (\subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{\perp,C})(\mathcal{A}(\perp))U_\perp \\ &= (\subseteq^{\Sigma}_{\perp,C})(\mathcal{A}(\perp))U_\perp. \end{aligned}$$

For the last equality, notice that lemma 5.3.5 gives $(\mathcal{A}(\perp))(\mathcal{A}(\perp))U_\perp = (\mathcal{A}(\perp))U_\perp$, hence $(\mathcal{A}(\perp))U_\perp \in \Sigma(\perp)$. $\square$

The correspondences between objects in this section may seem very abstract. However, the correspondences are actually quite straight forward. Suppose that we have computed the lattice $L_{\widetilde{\bar{A}}_{sa}}$ and suppose we know which subfunctors U of $L_{\widetilde{\bar{A}}_{sa}}$ have the property that $\mathcal{A}_0 U = U$. Then in order to compute the spectrum Σ, we have the following correspondences:

1. Any subfunctor U of $L_{\widetilde{\bar{A}}_{sa}}$ corresponds to a characteristic morphism $\chi_U : L_{\widetilde{\bar{A}}_{sa}} \to \Omega$ by theorem 3.1.4.

2. Any characteristic morphism χ_U in turn corresponds to a morphism $\hat{\chi}_U : \mathbf{1} \to \Omega^{L_{\widetilde{\bar{A}}_{sa}}}$ by cartesian closedness of the topos $\mathcal{T}(A)$.

3. Since $\mathcal{C}(A)$ has a bottom element $\perp$ the morphism $\hat{\chi}_U : \mathbf{1} \to \Omega^{L_{\widetilde{\bar{A}}_{sa}}}$ is completely determined by the element $(\hat{\chi}_U(\perp)(*')) \in \Sigma(\perp)$ and the computation:

$$\hat{\chi}_U(C)(*) = \hat{\chi}_U(C)(\subseteq^{\mathbf{1}}_{\perp,C})(*') = \subseteq^{\Omega^{L_{\widetilde{\bar{A}}_{sa}}}}_{\perp,C} (\hat{\chi}_U(\perp)(*')),$$

   where $*$ and $*'$ denote the unique elements of $\mathbf{1}(C)$ and $\mathbf{1}(\perp)$ respectively. Hence morphisms $\hat{\chi}_U : \mathbf{1} \to \Omega^{L_{\widetilde{\bar{A}}_{sa}}}$ correspond one-on-one to elements of $\Omega^{L_{\widetilde{\bar{A}}_{sa}}}(\perp)$. By the definition of the correspondence between $\mathcal{A}$ and $\mathcal{A}_0$, the subobjects U of $L_{\bar{A}_{sa}}^{\sim}$ such that $\mathcal{A}_0 U = U$ correspond exactly to the elements x of $\Omega^{L_{\bar{A}_{sa}}^{\sim}}(\perp)$ such that $\mathcal{A}(\perp)x = x$.

4. By the above correspondences we can compute $\Sigma(\perp)$ from the set of subfunctors U of $L_{\bar{A}_{sa}}^{\sim}$ that satisfy $\mathcal{A}_0 U = U$. Then theorem 5.3.7 shows that we can compute each $\Sigma(C)$ and hence the functor Σ from $\Sigma(\perp)$, resulting in the spectrum of our initial C*-algebra A.

5.4 Final remarks

We conclude this chapter by summarizing the construction of the spectrum of a C*-algebra A and the most important theorems for computations.

1. First, one tries to find all unital C*-subalgebras of A, resulting in the preorder $\mathcal{C}(A)$, the topos $\mathcal{T}(A)$ and the tautological functor $\bar{A}$. In the next chapter we will prove some theorems to compute $\mathcal{C}(A)$.
2. Next, one computes the distributive lattice freely generated by $\bar{A}_{sa}$ subject to relations (5) - (9). Theorem 5.2.6 shows that this can be done locally, i.e. one computes the distributive lattices freely generated by B_{sa} subject to relations (5) - (9) in **Set**, where B is a subalgebra of A. These local freely generated distributive lattices can be combined into the functor $L_{\bar{A}_{sa}}^{\sim}$.
3. Finally, one computes the subfunctors U of $L_{\bar{A}_{sa}}^{\sim}$ such that $\mathcal{A}_0 U = U$. In the next chapter we will see that if A is finite, this can be done by finding the subfunctors U such that $U(C)$ is a down-set for every object $C \in \mathcal{C}(A)$. By the correspondences stated at the end of paragraph 5.3 one then directly finds $\Sigma(\perp)$. Finally, theorem 5.3.7 can be applied to find the spectrum Σ. In the next chapter we will introduce a theorem (see 6.1.3) that allows us to compute Σ easily from the subfunctors U of $L_{\bar{A}_{sa}}^{\sim}$ such that $\mathcal{A}_0 U = U$.

The duality by Mulvey and Banaschewski

$$\mathbf{cCStar} \leftrightarrow \mathbf{KRegLoc},$$

is proved by a general construction for the morphism $\mathbf{cCStar} \to \mathbf{KRegLoc}$ for any topos $\mathcal{T}$. This general construction introduced by Mulvey and Banaschewski (cf. [12], [13] and [14]) and reformulated by Coquand and Spitters [4] is quite similar. Again, one first computes the distributive lattice freely generated by the self-adjoint part of an (arbitrary) internal C*-algebra $\bar{A}$ subject to (5) - (9). Then, one turns this into a frame that satisfies the relation

$$D_a \leq \bigvee_{q \in \mathbf{Q}^+} D_{a-q}.$$

The construction of this frame can be done by introducing a general morphism $\mathcal{A}$, which involves technical definitions of finite objects of a topos and infinite joins. For brevity, we have avoided the direct definition of $\mathcal{A}$ in $\mathcal{T}(A)$ and chosen to introduce the (less complicated) map $\mathcal{A}_0$ first. In the next chapter we will do some computations and fortunately, with a good understanding of the definition of $\mathcal{A}_0$ the spectrum can be computed.

Chapter 6

Computations of spectra

"Can you do addition?" the White Queen asked. "What's one and one and one and one and one and one and one and one and one and one?" "I don't know," said Alice. "I lost count." - Lewis Carroll

In this chapter we compute some examples of Gelfand spectra. Our main examples will be $\mathbb{C}^2$ and $M(2,\mathbb{C})$. We have chosen these examples since $\mathbb{C}^2$ is the first C*-algebra whose spectrum is nontrivial, whereas $M(2,\mathbb{C})$ is the simplest noncommutative algebra. Furthermore, it turns out that $\mathbb{C}$ has a trivial spectrum, and that $\mathbb{C}^n$ is computable, but quite complicated for $n \geq 3$. Further examples that would be interesting are $C(X)$, with X a compact Hausdorff space, and $B(H)$, the bounded operators on a Hilbert-space. However, whenever a C*-algebra has a commutative C*-subalgebra of dimension 3 or higher, the spectrum turns out to be hard to compute by hand. Nevertheless we will spend a few words on $C(X)$.

In the following examples A will be the C*-algebra that we study. We recapitulate that the spectrum can be computed in three steps:

1. The computation of $\mathcal{C}(A)$ and hence the corresponding topos $\mathcal{T}(A)$.
2. The computation of the free distributive lattice generated by $\bar{A}_{sa}$ subject to the relations:

$$
\begin{array}{llcll}
\models & D_1 & = & \top; & (5)\\
\models \forall a & D_a \wedge D_{-a} & = & \bot; & (6)\\
\models \forall a & D_{-a^2} & = & \bot; & (7)\\
\models \forall a \forall b & D_{a+b} & \leq & D_a \vee D_b; & (8)\\
\models \forall a \forall b & D_{ab} & \leq & (D_a \wedge D_b) \vee (D_{-a} \wedge D_{-b}). & (9)
\end{array}
$$

3. The computation of the spectrum Σ, via $\Sigma(\bot)$ as described at the end of section 5.3.

6.1 The finite, commutative case: $\mathbb{C}^n$

First we study the commutative C*-algebra $A = \mathbb{C}^n$. It turns out that we can compute the spectrum Σ of $\mathbb{C}^n$ for every n. However, for $n = 3$ things become quite complicated and for $n \geq 4$ one would rather use a computer. Here we will make the structure precise for the special cases $n = 1$ and $n = 2$, avoiding the extensive amount of computations for $n \geq 3$.

C*-subalgebra and topos

The first question we should answer is what the unital C*-subalgebras of $\mathbb{C}^n$ look like. Recall that we can regard $\mathbb{C}^n$ as the C*-algebra of functions from an n-point space with discrete topology to $\mathbb{C}$. Suppose Π is a partition of these points. We prove that the commutative unital C*-subalgebras are the ones that consist of functions that are constant on the equivalence classes of some partition using elementary linear algebra. In section 6.3 we will give another proof using the Gelfand isomorphism, which shortens the case of $\mathbb{C}^n$, but requires more technical tools.

Theorem 6.1.1. *Let A be the C*-algebra $\mathbb{C}^n$ and regard it as the space of functions on the topological space X with n points and discrete topology. Let Π be a partition of the points of X and let A_Π be the set of functions that are constant on every equivalence class of the partition. Then:*

1. *A_Π is a C*-subalgebra.*
2. *Every C*-subalgebra of A is equal to A_Π for a unique partition of n points called Π.*

Proof:

1. Trivial.
2. Given any C*-subalgebra B of A, we must find the pertinent partition Π. Let $e_{p_i} : B \to \mathbb{C}$ be the evaluation map at the i^{th} point of X. Now define a relation on the points of X by $p_i \sim_X p_j$ if and only if $\mathrm{Ker}(e_{p_i} - e_{p_j}) = B$. It is straightforward that this relation is an equivalence relation on X. Let Π be the corresponding partition.
   If $f \in B$ is a function on X, then it is constant on equivalence classes of Π by definition of $\sim_X$. Hence $f \in A_\Pi$, so that $B \subseteq A_\Pi$.
   Conversely, let $f \in A_\Pi$. Let R be a set of representatives of Π. If we can find a $g \in B$ that equals f on R we may conclude $A_\Pi \subseteq B$. By induction (to i) we show that for any set of i different points of R and for any $f \in A_\Pi$ we can construct a function $g \in B$ that equals f at these i different points.
   - For $i = 1$ this is trivial, since B contains the identity. If the partition has only one equivalence class, we are finished.
   - For $i = 2$ note that B contains at least one function h that has different values at two given representatives $r_{k_1}, r_{k_2} \in R$. Since the vectors $(h(r_{k_1}), h(r_{k_2}))$ and $(Id(r_{k_1}), Id(r_{k_2}))$ are independent vectors, there is a linear combination of h and the identity whose values at r_{k_1} and r_{k_2} equal f.
   - For $i + 1$, take elements $r_1, r_2, \ldots, r_{i+1} \in R$. Let $f : R \to \mathbb{C}$ be a function and suppose we chose a point $r_k \in \{r_1, r_2, \ldots, r_{i+1}\}$. Then there is, by induction, a function $h \in B$ such that h equals f at all points $r_1, r_2, \ldots, r_{i+1}$ except for maybe r_k.
     - Suppose $i + 1$ is odd. Then construct functions $h_1, \ldots, h_{i+1}$ that satisfy

$$\begin{pmatrix} h_1(r_1) & h_1(r_2) & \ldots & h_1(r_i) & h_1(r_{i+1}) \\ h_2(r_1) & h_2(r_2) & \ldots & h_2(r_i) & h_2(r_{i+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_i(r_1) & h_i(r_2) & \ldots & h_i(r_i) & h_i(r_{i+1}) \\ h_{i+1}(r_1) & h_{i+1}(r_2) & \ldots & h_{i+1}(r_i) & h_{i+1}(r_{i+1}) \end{pmatrix} = \begin{pmatrix} 1 & a_1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & a_2 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & a_i \\ a_{i+1} & 0 & 0 & \ldots & 0 & 1 \end{pmatrix},$$

so every indexed function h is specified at precisely i points and its value is unknown at only one point. These unknown values are denoted by $a_1, \ldots, a_{i+1}$ in the matrix above. Since $i+1$ is odd, the determinant of this matrix is $1+a_1 \cdot \ldots \cdot a_{i+1}$. If this is non-zero, the functions are independent. Hence for any $g \in A_\Pi$ there is a linear combination of functions in B such that this linear combination equals g at the $i+1$ points. If the determinant is zero, note that the determinant of the matrix obtained by evaluating the squares of $h_1, \ldots, h_{i+1}$ at $r_1, \ldots, r_{i+1}$ is $1 + (a_1 \cdot \ldots \cdot a_{i+1})^2 = 2$ which leads to the same conclusion.

– Suppose $i+1$ is even. Then consider

$$\begin{pmatrix} h_1(r_1) & h_1(r_2) & \ldots & h_1(r_i) & h_1(r_{i+1}) \\ h_2(r_1) & h_2(r_2) & \ldots & h_2(r_i) & h_2(r_{i+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_i(r_1) & h_i(r_2) & \ldots & h_i(r_i) & h_i(r_{i+1}) \\ h_{i+1}(r_1) & h_{i+1}(r_2) & \ldots & h_{i+1}(r_i) & h_{i+1}(r_{i+1}) \end{pmatrix} = \begin{pmatrix} 1 & a_1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & a_2 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_i & 0 & 0 & \ldots & 1 & 0 \\ a_{i+1} & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

A simple calculation gives that the determinant of this matrix is $1 + a_1 \cdot \ldots \cdot a_i$ and the induction hypothesis is proved by the same reasoning as in the case where $i+1$ is odd.

□

The theorem above tells us that $\mathcal{C}(\mathbb{C}^n)$ is the partial order of partitions of n points, where the order is given by refinements of partitions. As special cases we have the Hasse diagrams:

$$\mathcal{C}(\mathbb{C}^1) = \mathbb{C}; \qquad \mathcal{C}(\mathbb{C}^2) = \begin{pmatrix} \mathbb{C}^2 \\ \uparrow \\ \mathbb{C} \end{pmatrix}; \qquad \mathcal{C}(\mathbb{C}^3) = \left(\begin{tikzcd} & \mathbb{C}^3 & \\ A_{\{\{1\},\{2,3\}\}} \arrow[ur] & A_{\{\{2\},\{1,3\}\}} \arrow[u] & A_{\{\{3\},\{1,2\}\}} \arrow[ul] \\ & \mathbb{C} \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd} \right).$$

These diagrams also represent the tautological functor $\bar{A}$ as an object of $\mathcal{T}(\mathbb{C}^n) = \mathbf{Set}^{\mathcal{C}(\mathbb{C}^n)}$. Note that if $n = 1$, $\mathcal{T}(\mathbb{C})$ is the category **Set**. If $n = 2$ we have $\mathcal{T}(\mathbb{C}^2) = \mathbf{Set}^{\mathcal{C}_\mathbf{2}}$.

The lattice

In theorem 5.2.6 we showed that we may compute the lattice generated by $\bar{A}_{sa}$ subject to relations (5) - (9) locally. The local lattices are generated by $\mathbb{R}^m$, $m \leq n$ subject to the given relations. Theorem 5.2.12 provides the means to compute the lattice generated by $\mathbb{R}^m$ subject to the relations. We restate the theorem for the special case $\mathbb{C}^m$.

Theorem 6.1.2. *Let $B = \mathbb{C}^m$. The lattice generated by B_{sa} subject to relations (5) - (9) consists of the elements $\left\{D_{(\pm 1, \pm 1, \ldots, \pm 1)}\right\}$, where $D_a \leq D_b$ if and only if $a \leq b$, i.e. $a_i \leq b_i$ for all coordinates i.*

Proof: Define a function *sign*:

$$\begin{array}{rccll} sign: & \mathbb{R} & \to & \mathbb{R} & ; \\ & x & \mapsto & 1 & \text{if } x > 0; \\ & x & \mapsto & -1 & \text{if } x \leq 0. \end{array}$$

For $a \in B_{sa}$, theorem 5.2.12 gives $D_a = D_{a^+} = D_{(sign(a_1),\ldots,sign(a_m))}$. Hence every element of the lattice generated by B_{sa} subject to (5) - (9) is of the form $D_{(\pm 1,\ldots,\pm 1)}$. By definition of $\sim_+$, none of the elements of $\left\{D_{(\pm 1,\pm 1,\ldots,\pm 1)}\right\}$ is equivalent to any other. The statement about the partial order follows from theorem 5.2.12. □

We could express theorem 6.1.2 geometrically by saying that the lattice $\mathbb{C}^m$ produces is the m-dimensional hypercube. We will try to visualize $L^{\sim}_{\widetilde{A}_{sa}}$ for the cases $n = 1$, $n = 2$ and $n = 3$. The case $n = 1$ looks like:

$$\begin{pmatrix} D_1 \\ \Big| \\ D_{-1} \end{pmatrix}.$$

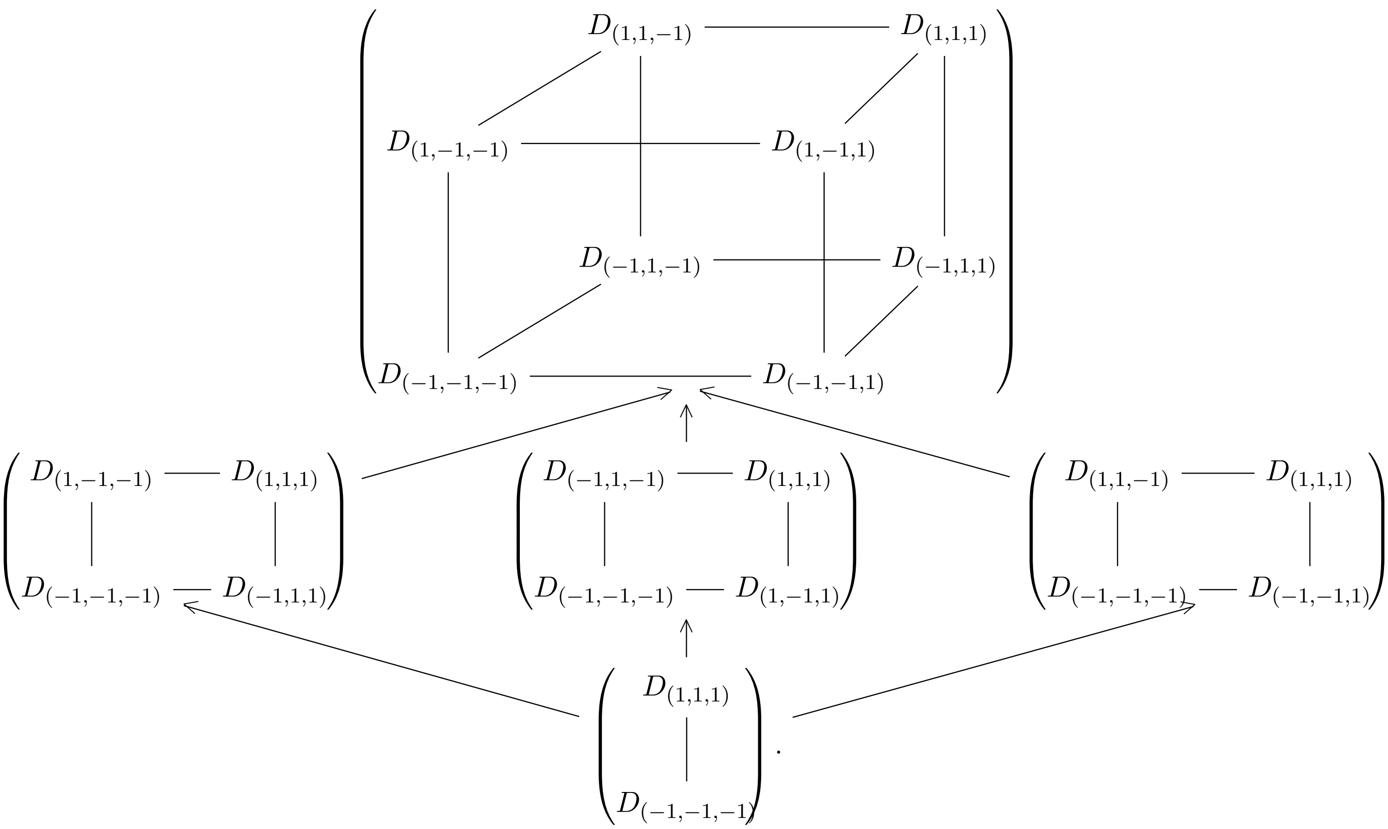
The case $n = 2$ has the form:

$$\begin{pmatrix} & D_{(1,1)} & \\ \diagup & & \diagdown \\ D_{(-1,1)} & & D_{(1,-1)} \\ \diagdown & & \diagup \\ & D_{(-1,-1)} & \end{pmatrix}$$

$$\uparrow$$

$$\begin{pmatrix} D_{(1,1)} \\ \Big| \\ D_{(-1,-1)} \end{pmatrix}.$$

The case $n = 3$ takes the form:



The spectrum

Now we compute Σ. We will do this for $n = 1$ and $n = 2$. For $n = 3$ the situation starts to become too complicated to calculate by hand, as we will see.

First we recall the definition of a down-set. If L is a lattice and $x \in L$, then the down-set $x\downarrow$ of x is defined by

$$x\downarrow = \{y \mid y \leq x\}\,.$$

Now, let us recall the definition of $\mathcal{A}_0$. Let U be a subfunctor of $L_{\widetilde{A}_{sa}}$ and $C \in \mathcal{C}(A)$. Then

$$\begin{aligned}
\mathcal{A}_0(C)U(C) &= \left\{D_a \in L_{\widetilde{A}_{sa}}(C) \mid D_a \lhd U(C)\right\} \\
&= \left\{D_a \in L_{\widetilde{A}_{sa}}(C) \mid \text{For every } q \in \mathbb{Q}^+ :\ D_{a-q} \leq \bigvee U(C)\right\} \\
&= \left\{D_a \in L_{\widetilde{A}_{sa}}(C) \mid D_a \leq \bigvee U(C)\right\}
\end{aligned}$$

The first equation is the definition. The second follows from the fact that $U(C)$ is already finite in the case of $\mathbb{C}^n$. The last equation follows from the fact that $D_{a-q} = D_a$ for small q, which follows from theorem 6.1.2. The last equation says that $\mathcal{A}_0(C)U(C)$ is the down-set of the meet of $U(C)$. Conversely, every down-set is the completion under $\mathcal{A}_0$ of itself. Recall from section 5.3 that $\Sigma(\perp)$ as a lattice in **Set** may be represented by the subfunctors U of $L_{\widetilde{A}_{sa}}$ such that $\mathcal{A}_0 U = U$. We find that these subfunctors are the ones that are locally down-sets.

$n = 1$: We find that $\Sigma(\perp)$ is equal to the following lattice:

$$\begin{tikzcd} D_1\downarrow \arrow[d, dash] \\ D_{-1}\downarrow . \end{tikzcd}$$

Since $\mathcal{C}(A)$ consists only of one element, this determines Σ globally.

$n = 2$: The following subfunctors of $L^{\sim}_{\tilde{A}_{sa}}$ are the ones that are locally down-sets and hence these subfunctors will form the lattice $\Sigma(\perp)$:

$$\begin{tikzcd} & & & D_{(1,1)} \arrow[dl, dash] \arrow[dr, dash] & \\ A_{1,1} \;= & \begin{array}{c} D_{(1,1)} \\ \big| \\ D_{(-1,-1)} \end{array} \arrow[r] & D_{(-1,1)} \arrow[dr, dash] & & D_{(1,-1)} \arrow[dl, dash] \\ & & & D_{(-1,-1)} & \end{tikzcd}$$

$$\begin{tikzcd} & & & D_{(1,1)} \arrow[dl, dash] \arrow[dr, dash] & \\ A_{2,1} \;= & \begin{array}{c} 0 \\ \big| \\ D_{(-1,-1)} \end{array} \arrow[r] & D_{(-1,1)} \arrow[dr, dash] & & D_{(1,-1)} \arrow[dl, dash] \\ & & & D_{(-1,-1)} & \end{tikzcd}$$

$$\begin{tikzcd} & & & 0 \arrow[dl, dash] \arrow[dr, dash] & \\ A_{2,2} \;= & \begin{array}{c} 0 \\ \big| \\ D_{(-1,-1)} \end{array} \arrow[r] & 0 \arrow[dr, dash] & & D_{(1,-1)} \arrow[dl, dash] \\ & & & D_{(-1,-1)} & \end{tikzcd}$$

$$\begin{tikzcd} & & & 0 \arrow[dl, dash] \arrow[dr, dash] & \\ A_{2,3} \;= & \begin{array}{c} 0 \\ \big| \\ D_{(-1,-1)} \end{array} \arrow[r] & D_{(-1,1)} \arrow[dr, dash] & & 0 \arrow[dl, dash] \\ & & & D_{(-1,-1)} & \end{tikzcd}$$

$$
\begin{tikzcd}
 & & & & 0 \arrow[dl, dash] \arrow[dr, dash] & \\
A_{2,4} \;=\; & 0 \arrow[d, dash] \arrow[rr] & & 0 \arrow[dr, dash] & & 0 \arrow[dl, dash] \\
 & D_{(-1,-1)} & & & D_{(-1,-1)} &
\end{tikzcd}
$$

Here the 0 indicates that the element is not included in the subfunctor. The lattice structure of $\Sigma(\perp)$ is then given by

$$
\begin{tikzcd}
 & A_{1,1} \arrow[d, dash] & \\
 & A_{2,1} \arrow[dl, dash] \arrow[dr, dash] & \\
A_{2,2} \arrow[dr, dash] & & A_{2,3} \arrow[dl, dash] \\
 & A_{2,4} & .
\end{tikzcd}
$$

We now compute $\Sigma(\mathbb{C}^2) = \Sigma(\top)$, which by theorem 5.3.7 and the subsequent discussion equals $\mathrm{Im}(\subseteq^{\Sigma}_{\perp,\top})$. Recall that images can be taken point-wise. We compute:

$$
\begin{aligned}
(\subseteq^{*}_{\perp,\top} \times Id_{L^{\sim}_{\tilde{A}_{sa}}})(\top) &= Id_{\{\mathbf{1}\times L^{\sim}_{\tilde{A}_{sa}}\}}; \\
(\subseteq^{*}_{\perp,\top} \times Id_{L^{\sim}_{\tilde{A}_{sa}}})(\perp) &= !,
\end{aligned}
$$

where $! : \mathbf{0} \to L^{\sim}_{\tilde{A}_{sa}}$ is the unique arrow from the initial object of **Set**. It follows from these equations that $\subseteq^{\Sigma}_{\perp,\top} \chi_U = \chi_U(\subseteq^{*}_{\perp,\top} \times Id_{L^{\sim}_{\tilde{A}_{sa}}}) = \chi_V(\subseteq^{*}_{\perp,\top} \times Id_{L^{\sim}_{\tilde{A}_{sa}}}) = \subseteq^{\Sigma}_{\perp,\top} \chi_V$ if and only if $U(\top) = V(\top)$. Hence $\Sigma(\top)$ has the form

$$
\begin{tikzcd}
 & A_{2,1} \simeq A_{1,1} \arrow[dl, dash] \arrow[dr, dash] & \\
A_{2,2} \arrow[dr, dash] & & A_{2,3} \arrow[dl, dash] \\
 & A_{2,4} & .
\end{tikzcd}
$$

We will restate the argument above for the computation of $\Sigma(\top)$ in the more general theorem 6.1.3 below.

$n \geq 3$: For this case one proceeds in exactly the same way. One has to compute all subfunctors of $L_{\widetilde{A}_{sa}}^{\sim}$ such that locally the subfunctor is a down-set. This can easily be done by a computer. In the case $n = 3$ one finds 96 such functors, which form $\Sigma(\perp)$. The following theorem shows what the images of the intrinsic maps $\subseteq_{\perp,C}^{\Sigma}$ look like.

Theorem 6.1.3. *Let U and V be subfunctors of $L_{\widetilde{A}_{sa}}^{\sim}$. Then:*

(a) The corresponding elements of U in V in $\Omega^{L_{\widetilde{A}_{sa}}^{\sim}}(\perp) = \boldsymbol{Hom}_{\mathcal{T}(A)}(\boldsymbol{y}(\perp)\times A, \Omega) = \boldsymbol{Hom}_{\mathcal{T}(A)}(A, \Omega)$ are the natural transformations χ_U and χ_V.

(b) For $C \in \mathcal{C}(A)$, one has $\subseteq_{\perp,C}^{\Sigma} \chi_U = \subseteq_{\perp,C}^{\Sigma} \chi_V$ if and only if $U(D) = V(D)$ for all objects $C \subseteq D$.

Proof:

(a) This is obvious, being exactly the definition of exponentiation (see theorem 1.2.10).

(b) Recall that $\subseteq_{\perp,C}^{\Sigma} = (\subseteq_{\perp,C}^{*} \times Id_{L_{\widetilde{A}_{sa}}^{\sim}})^*$. The second assertion is a consequence of the following computation of the map $(\subseteq_{\perp,C}^{*} \times Id_{L_{\widetilde{A}_{sa}}^{\sim}}) : \mathbf{y}(C) \times L_{\widetilde{A}_{sa}}^{\sim} \to \mathbf{y}(\perp) \times L_{\widetilde{A}_{sa}}^{\sim}$, which is given by

$$\begin{aligned} (\subseteq_{\perp,C}^{*} \times Id_{L_{\widetilde{A}_{sa}}^{\sim}})(D) &= Id_{\{1\}\times L_{\widetilde{A}_{sa}}^{\sim}(D)} && \text{if } C \subseteq D; \\ (\subseteq_{\perp,C}^{*} \times Id_{L_{\widetilde{A}_{sa}}^{\sim}})(D) &= \,! && \text{if } C \nsubseteq D, \end{aligned}$$

where $! : \mathbf{0} = (\mathbf{y}(C))(D) \times L_{\widetilde{A}_{sa}}^{\sim}(D) \to (\mathbf{y}(\perp))(D) \times L_{\widetilde{A}_{sa}}^{\sim}(D) = L_{\widetilde{A}_{sa}}^{\sim}(D)$ is the unique arrow from the initial object, i.e. the empty set. Now we have the following situation:

i. If $D \supseteq C$, then

$$\begin{aligned} ((\subseteq_{\perp,C}^{*} \times Id_{L_{\widetilde{A}_{sa}}^{\sim}})^* \chi_U)(D) &= \chi_U(D)(\subseteq_{\perp,C}^{*} \times Id_{L_{\widetilde{A}_{sa}}^{\sim}})(D); \\ &= \chi_U(D). \end{aligned}$$

ii. If $D \nsupseteq C$, then

$$\begin{aligned} ((\subseteq_{\perp,C}^{*} \times Id_{L_{\widetilde{A}_{sa}}^{\sim}})^* \chi_U)(D) &= \chi_U(D)(\subseteq_{\perp,C}^{*} \times Id_{L_{\widetilde{A}_{sa}}^{\sim}})(D); \\ &= \,! : \mathbf{0} \to \Omega(D). \end{aligned}$$

Now $U(D) = V(D)$ for all $D \supseteq C$ if and only if $\chi_U(D) = \chi_V(D)$ for all $D \supseteq C$ if and only if $(\subseteq_{\perp,C}^{\Sigma} \chi_U)(D) = (\subseteq_{\perp,C}^{\Sigma} \chi_V)(D)$ for all $D \supseteq C$.

□

From this theorem we derive a corollary that gives the localic spectrum of $\mathbb{C}^n$ by the original Gelfand theorem in **Set**.

Corollary 6.1.4. *In the case of $A = \mathbb{C}^n$, $\mathcal{C}(A)$ has a top element and the top level of the spectrum $\Sigma(\top)$ is given by the lattice $\mathcal{P}(\{1,\ldots,n\})$.*

Proof: In this case A itself is the top element of $\mathcal{C}(A)$. By theorem 6.1.2 $L_{\widetilde{A}_{sa}}^{\sim}(\top)$ is of the form $\mathcal{P}(\{1,\ldots,n\})$. Then:

(a) Clearly, if $V \subseteq L_{\widetilde{A}_{sa}}^{\sim}(\top)$ then there is a subfunctor $U_V \subseteq L_{\widetilde{A}_{sa}}^{\sim}$ such that $U_V(\top) = V$. It follows from theorem 6.1.3 that if $\subseteq_{\perp,\top}^{\Sigma} \chi_{U_V} = \subseteq_{\perp,\top}^{\Sigma} \chi_{U_{V'}}$, then $V = V'$. Hence the images $\subseteq_{\perp,\top}^{\Sigma} \chi_{U_V}$ represent different elements of $\Sigma(\top)$ for every V.

(b) From theorem 6.1.3 it immediately follows that for every subfunctor $U \subseteq L_{\widetilde{A}_{sa}}^{\sim}$ one has $\subseteq_{\perp,\top}^{\Sigma} \chi_U = \subseteq_{\perp,\top}^{\Sigma} \chi_{U_{U(\top)}}$. Hence every element of $\Sigma(\top)$ is of the form $\subseteq_{\perp,\top}^{\Sigma} \chi_{U_V}$.

From this we conclude that $\Sigma(\top) = \mathcal{P}(\{1,\ldots,n\})$. □

6.2 The finite case in general

Now that we know what the spectrum of $\mathbb{C}^n$ looks like, computing the spectrum of any finite-dimensional (non-commutative) C*-algebra becomes much easier! Once we know the structure of $\mathcal{C}(A)$, the lattice $L_{\widetilde{A}_{sa}}$ can be computed locally at C and every local lattice has the form of an n-dimensional hypercube. Then $\Sigma(\perp)$ has the form of all subfunctors such that locally the subfunctor is a down-set. Theorems 5.3.7 and 6.1.3 then give a complete description of Σ globally.

The spectrum of $M(2, \mathbb{C})$

As an example, we study $M(2, \mathbb{C})$. We first prove a proposition that gives all commutative unital C*-subalgebras of $A = M(2, \mathbb{C})$.

Proposition 6.2.1. *All nontrivial unital C*-subalgebra's of $M(2, \mathbb{C})$ are of the form*

$$B_u = \left\{ u \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} u^{-1} \mid z_1, z_2 \in \mathbb{C} \right\},$$

where u is a unitary matrix.

Proof: We denote matrices by small characters. Let B be a unital commutative C*-subalgebra of $M(2, \mathbb{C})$. Let $b \in B$, then $b = udu^{-1}$. Where d is either diagonal or of the form:

$$\begin{pmatrix} d_{1,1} & 1 \\ 0 & d_{2,2} \end{pmatrix}.$$

If d has a 1 in the upper right entry one can show that $B = M(2, \mathbb{C})$. So suppose $b = udu^{-1}$, where d is diagonal. Then $b^* = (u^*)^{-1} d^* u^*$ commutes with b, which means that b and b^* have a common set of eigenvalues. Hence $u^* = u^{-1}$, so u is unitary. Now suppose $b' \in B$, then $b' = u'd'u'^{-1}$. Since b' commutes with b, b and b' have a common set of eigenvalues, hence $u' = u$. This proves the theorem. $\square$

We thus find:

$$\mathcal{C}(M(2, \mathbb{C})) = \left(\begin{array}{ccc} \cdots & B_u & \cdots \\ & \Uparrow & \\ & \mathbb{C}1 = \left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \mid z \in \mathbb{C} \right\} & \end{array} \right),$$

where the dots stand for uncountably many C*-subalgebras running through all unitary $u \in M(2, \mathbb{C})$, between which there is no (nontrivial) inclusion relation. Then $L_{\widetilde{A}_{sa}}$ is given by the following functor, as follows from the example of $\mathbb{C}^n$ and the fact that we may compute this lattice point-wise.

$$
\begin{array}{c}
\cdots\left(\begin{tikzcd}[column sep=small]
 & D_{Id_2} \arrow[dl, dash] \arrow[dr, dash] & \\
D_{ua_{-+}u^{-1}} \arrow[dr, dash] & & D_{ua_{+-}u^{-1}} \arrow[dl, dash] \\
 & D_{-Id_2} &
\end{tikzcd}\right)\cdots . \\
\Uparrow \\
\left(\begin{tikzcd}
D_{Id_2} \arrow[d, dash] \\
D_{-Id_2}
\end{tikzcd}\right)
\end{array}
$$

Here

$$a_{-+} = -a_{+-} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now what are the subfunctors that are locally down-sets? First, there is $L^{\sim}_{\bar{A}_{sa}}$ itself. For any other subfunctor $S \subseteq L^{\sim}_{\bar{A}_{sa}}$ we have $D_{Id_2} \notin S(\perp)$. Then for every unitary matrix u we have four choices for down-sets of $L^{\sim}_{\bar{A}_{sa}}(B_u)$, namely $D_{Id_2}\downarrow, D_{ua_{-+}u^{-1}}\downarrow, D_{ua_{+-}u^{-1}}\downarrow$ and $D_{-Id_2}\downarrow$. For each B_u we are free to chose any of these four down-sets, as each defines a subfunctor of $L^{\sim}_{\bar{A}_{sa}}$. Hence we conclude that $\Sigma(\perp)$ is the lattice

$$
\begin{tikzcd}
L^{\sim}_{\bar{A}_{sa}} \arrow[d, dash] \\
\mathcal{P}(\{0,1\})^{\mathbb{R}},
\end{tikzcd}
$$

where $\mathbb{R}$ indicates the cardinal number of $\{u \mid u \text{ is a unitary matrix}\}$. It follows straight from theorem 6.1.3 that

$$
\Sigma(B_u) = L^{\sim}_{\bar{A}_{sa}}(B_u) = \left(\begin{tikzcd}[column sep=small]
 & D_{Id_2}\downarrow \arrow[dl, dash] \arrow[dr, dash] & \\
D_{ua_{-+}u^{-1}}\downarrow \arrow[dr, dash] & & D_{ua_{+-}u^{-1}}\downarrow \arrow[dl, dash] \\
 & D_{-Id_2}\downarrow &
\end{tikzcd}\right),
$$

where the map $\subseteq^{\Sigma}_{\perp,B_u}$ maps a subfunctor $U \subseteq L^{\sim}_{\bar{A}_{sa}}$, regarded as an element of $\Sigma(\perp)$, to $U(B_u)$, regarded as an element of $\Sigma(B_u)$.

6.3 Some remarks on $C(X)$

In this paragraph we study $\mathcal{C}(A)$ for $A = C(X)$, the continuous functions on a compact Hausdorff space X. For convenience we first assume $X = [0,1] \subseteq \mathbb{R}$, commenting the general case afterwards. Let Π be a *closed partition* of $[0,1]$, that is, a partition of $[0,1]$ such that every equivalence class is closed in $[0,1]$ (with the euclidian topology). One can prove that the cardinality of the number of equivalence classes of Π is not countable. If the number of equivalence classes is finite, then the partition is trivial. Now, as in the situation of $\mathbb{C}^n$, we introduce the algebra C_Π of all continuous functions on $[0,1]$ that are constant on equivalence classes. There are three natural questions to ask:

1. Is C_Π a C*-algebra?
2. For two closed partitions Π_1 and Π_2, does the following implication hold: $C_{\Pi_1} = C_{\Pi_2} \Rightarrow \Pi_1 = \Pi_2$?
3. Is every unital C*-subalgebra of $C([0,1])$ of the form C_Π for some closed partition Π?

The first question

The answer to the first question is obviously yes. From this we make a short remark: we could, on the one hand consider the category of partitions of $[0,1]$ with the refinements of partitions as morphisms, and on the other hand the category of unital C*-subalgebras of $C([0,1])$ with the inclusions. Then $\Pi \to C_\Pi$ is a functor between these categories.

The second question

The answer to the second question is no. This can be illustrated by the following example. Let ϕ be a function from the open inteval $(0,1)$ to itself having the following properties:

1. ϕ is a bijection.
2. For every open subinterval of $(0,1)$ the image of this subinterval under ϕ is dense in $(0,1)$.

Such a function exists as we shall prove later. Let

$$\varphi(x) = \frac{1}{2} + \frac{1}{2}\phi(2x) : (0, \frac{1}{2}) \to (\frac{1}{2}, 1),$$

and define $\sim_\varphi$ to be the smallest equivalence relation on $[0,1]$ such that if $y = \varphi(x)$ then $y \sim_\varphi x$. Note that each of $0, \frac{1}{2}$ and 1 is only equivalent to itself and that every number in $(0, \frac{1}{2})$ is equivalent to exactly two numbers, namely itself and the image of φ (which lies in $(\frac{1}{2}, 1)$). Call the corresponding partition $\Pi_{\sim_\varphi}$ and call the trivial partition consisting of only one equivalence class Π_0. We claim that that $C_{\Pi_{\sim_\varphi}} = C_{\Pi_0}$. Note that C_{Π_0} only contains the constant functions. Hence we would have proved the claim if every function in $C_{\Pi_{\sim_\varphi}}$ is constant. Let $f \in C_{\Pi_{\sim_\varphi}}$. By definition of the equivalence relation and the continuity of f we are finished if we can prove f to be constant on the interval $(0, \frac{1}{2})$. Suppose it is not. Then there exist $p, q \in (0, \frac{1}{2})$ such that $f(p) \neq f(q)$. Since f is continuous we can find a δ such that if $x \in (p - \delta, p + \delta)$ then

$$|f(x) - f(p)| < |f(p) - f(q)|/2.$$

Now remark that $f(\varphi(x)) = f(x)$ and $\varphi((p - \delta, p + \delta))$ is dense in the interval $(\frac{1}{2}, 1)$. From this and the continuity of f we may conclude $f(x)$ takes values between $f(p) \pm |f(p) - f(q)|/2$ for every $x \in (\frac{1}{2}, 1)$. Then

$$f(q) = f(\varphi(q)) \in (f(p) - |f(p) - f(q)|/2, f(p) + |f(p) - f(q)|/2),$$

which is a contradiction. Hence f is constant.

It remains to prove the existence of the function ϕ. Let $a_{k,n} = \frac{k}{2^n}$ for natural numbers $2 \leq n$, $1 \leq k < 2^n$ and $2 \not| k$. Let $b_{k,n,i}$ be a sequence in $\mathbb{Q}$ converging to $a_{k,n}$ such that $b_{k,n,i} = b_{k',n',i'} \Rightarrow k = k', n = n', i = i'$. Such sequences $b_{k,n,i}$ exist. Define:

$$B_{k,n} = \{b_{n,k,i} | i \in \mathbb{N}\}.$$

For p prime, define:

$$D_p = \left\{ \frac{k}{p^n} | k, n \in \mathbb{N}, 2 \leq n, 1 \leq k < p^n, p \not| k \right\}.$$

Let f be a bijection between the countable sets

$$\{(k,n) | k, n \in \mathbb{N}, 2 \leq n, 1 \leq k < 2^n, 2 \not| k\},$$

and the set of all primes. Note that both the sets $B_{k,n}$ and D_p are countable. Hence there exist bijections

$$g_{k,n} : B_{k,n} \to D_{f(k,n)}.$$

Let g be a bijection between the uncountable sets $(0,1) \backslash \bigcup_{k,n} B_{k,n}$ and $(0,1) \backslash \bigcup_p D_p$. Finally, define

$$\begin{array}{llll} \phi: & x & \mapsto & g_{k,n}(x) \quad x \in B_{k,n}; \\ & x & \mapsto & g(x) \qquad x \in (0,1) \backslash \bigcup_{k,n} B_{k,n}. \end{array}$$

We claim this function to be the desired ϕ . It is obvious that this function is a bijection. Now suppose $I \subseteq (0, \frac{1}{2})$ is an open subinterval. It at least contains one point $a_{n,k}$. Fix n and k for now. Then there exists a $N \in \mathbb{N}$ such that for all $i > N$, $b_{k,n,i} \in I$. Then $\phi(I)$ contains $D_{f(k,n)}$ except for a finite subset of $D_{f(k,n)}$. From this it follows $\phi(I)$ is dense in $(0, \frac{1}{2})$.

The third question

The answer to the third question is yes. A proof can be established from the Gelfand - Naimark theorem, some category theory and topology. Suppose B to be a unital C*-subalgebra of $C([0,1])$. We would like to prove B is equal to the continous functions on $[0,1]$ to $\mathbb{C}$ that are constant on equivalence classes of some partition Π. We first introduce some notation. Let A be the C*-algebra $C[(0,1)]$. If C is any C*-algebra, write $\Omega(C)$ for the spectrum of C, meant by the Gelfand-Naimark theorem (rather than the spectrum as defined in previous chapters). Let $i : B \to A$ denote the inclusion of B in A. Then we have:

$$\begin{array}{ccc} B & \to^i & A \\ \Omega(B) & \leftarrow^{i^*} & \Omega(A) \end{array}$$

First remark that i is monic since it is injective. The Gelfand-Naimark theorem states that there is a duality between compact Hausdorff spaces and unital commutative C*-algebras. From this one obtains i^* is epi. Now we claim i^* is surjective. Suppose i^* is not surjective. Then there is a point $p \in \Omega(B)$ such that p is not in the image of i^*. Remark that the one-point set $\{p\}$ is closed, hence compact. The image of i^* is compact since i^* is continuous and $\Omega(A)$ is compact. $\Omega(B)$ is compact Hausdorff, hence normal. Hence we can apply Urysohn's lemma giving a continuous function $f : \Omega(B) \to [0,1]$ such that $f(p) = 1$ and $f(x) = 0$ for all $x \in i^*(\Omega(A))$. Now let the function $h : \Omega(B) \to [0,1]$ be constant equal to 0. Then $f \circ i^* = h \circ i^*$ but $f \neq h$ which contradicts with the fact that i^* is epi. We conclude that i^* is surjective. Now define an equivalence relation on $\Omega(A)$ by:

$$e_p \sim_\Omega e_q \quad \text{if and only if} \quad i^*(e_p) = i^*(e_q)$$

Here $e_p, e_q \in \Omega(A)$ are the evaluations in p and q.

Then i^* factors through $(\Omega(A)/\sim_\Omega)$ by $i^* : \Omega(A) \rightarrow^{\pi} (\Omega(A)/\sim_\Omega) \rightarrow^{\bar{i^*}} \Omega(B)$, where both π and $\bar{i^*}$ are continuous. From the facts that:

- $\bar{i^*}$ is bijective and continuous,
- $(\Omega(A)/\sim_\Omega)$ is compact (since the quotient of a compact space is compact),
- $\Omega(B)$ is Hausdorff,

it follows that $\bar{i^*}$ is a homeomorphism. So we may view i^* as a map from $\Omega(A)$ to $(\Omega(A)/\sim)$ (this is actually the map π). The Gelfand-Naimark theorem tells us that

$$B \simeq C(\Omega(B)) \simeq C(\Omega(A)/\sim_\Omega) \simeq C([0,1]/\sim_0).$$

For the last isomorphism, define for $p, q \in [0,1]$:

$$p \sim_0 q \quad \text{if and only if} \quad f(p) = f(q), \forall f \in B,$$

and notice that $f(p) = f(q), \forall f \in B$ if and only if $i^*(e_p) = i^*(e_q)$, where $e_p, e_q \in \Omega(A)$ are evaluation at p and q, respectively.

Finally, we claim that $C([0,1]/\sim_0)$ is nothing else but the continuous functions from $[0,1]$ to $\mathbb{C}$ that are constant on the equivalence classes defined by $\sim_0$. It is easy to see that each continuous function on $[0,1] \to \mathbb{C}$ that is constant on equivalence classes defines a continuous function on $[0,1]/\sim_0$ and that this assignment is injective. On the other hand, if f is a function in $C([0,1]/\sim_0)$, then $f \circ \pi_0$, where $\pi_0 : [0,1] \to [0,1]/\sim_0$ is the quotient map, is a continuous function that is constant on equivalence classes, and this is the inverse assignment. Hence $C([0,1]/\sim_0)$ consist of all continuous functions that are constant on (closed) equivalence classes. This proves claim number 3.

Further remarks

In the above discussion we only used the topological properties of $[0,1]$ being compact and Hausdorff. The answers to the first and third question stay the same in the setting of any compact Hausdorff space X instead of $[0,1]$. The answer to the second question does depend on X, as we saw in the example of a discrete n-point space and in the example of $[0,1]$. The three questions amount to a representation of all unital C*-subalgebras of $C(X)$ where X is a compact Hausdorff space. Namely:

$$\begin{aligned} \mathcal{C}(C(X)) &= \{B \subseteq C(X) | B \text{ a unital C*-subalgebra of } C(X)\} \\ &\simeq \{\Pi \mid \Pi \text{ is a partition of } X\} / \sim_\Pi, \end{aligned}$$

$$\text{where } \Pi_1 \sim_\Pi \Pi_2 \quad \text{iff} \quad C_{\Pi_1}(X) = C_{\Pi_2}(X).$$

Of course the question wether two partitions are equialent with respect to $\sim_\Pi$ may become difficult to anwer, but at least this representation gives an idea what the C*-subalgebra's of $C(X)$ look like and, if the equivalence relation is neat enough, it even tells you when a C*-subalgebra is contained in one another, i.e. it gives a hint what the lattice structure looks like. For the computation of the spectrum, theorem 5.2.12 seems to be useful. However, the structure of $\mathcal{C}(A)$ is still too rough to compute the spectrum properly.

Remark: after finishing this thesis we found out that the C*-subalgebras of $C(X)$ correspond to the partitions of X that are obtained by a closed equivalence relation R. That is $R \subseteq X \times X$ should be closed in the euclidian topology of $X \times X$. Then the quotient space X/R is compact Hausdorff, as this follows from elementary topology. The space of complex functions on this quotient space is a C*-subalgebra of $C(X)$ and it can be proved that every such algebra is of this form for a unique equivalence relation R.

Further work

In this chapter we make some final remarks. We have shown that if a C*-algebra is finite dimensional, we are able to compute its spectrum once we have found its commutative subalgebras. As a special case, we computed the spectrum of $M(2,\mathbb{C})$, which was the main goal at the start of this research. For the case of $A = C(X)$, there is some research to do to find a representation of the structure of $\mathcal{C}(A)$. Also, it would be interesting if the questions stated in paragraph 6.3 can be answered in the same way as $C_0(X)$, where X only locally compact, instead of compact; i.e. in the general case of a commutative C*-algebra. Of course, $B(H)$ would be interesting too, especially since the GNS-construction shows that every C*-algebra admits an injective representation in $B(H)$.

Another generalization would be to change our topos $\mathcal{T}(A)$ to the setting of sheaves. The most natural way to do this is by remarking that $\mathbf{Set}^{\mathcal{C}(A)}$ itself can be regarded as a sheaf if one equips $\mathcal{C}(A)$ with the so-called Alexandrov topology. In this topology the open sets are defined as the up-sets of the partial order $\mathcal{C}(A)$. If one puts some other topology on this space, one can take the sheaves on this topological space as the definition of $\mathcal{T}(A)$. One can still define the free lattice subjected to relations and one can still define a completion map (maybe you will need the to take the sheafification of a presheaf once in a while). If we let $U \subseteq \mathcal{C}(A)$ be an open set and V_i an open covering of U, then the properties of a sheaf at V_i give information about the sheaf at U, which might lead to more interaction between the information of the C*-subalgebras of A and hence spectra may become different.

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Acknowledgements

After devoting five years of my life to mathematics, this Masters thesis is the final result in becoming a Master of Science. This is the right place to thank some people who have helped me over the past few years. First, I would like to thank the staff of the mathematics department for teaching me, and especially Klaas Landsman who taught me on the subjects of C*-algebra's, Hilbert spaces and Category theory and supervised my work for the past half year. His enthusiasm and helpfulness motivated me during the past two years and eventually led to this thesis. Also I would like to thank Chris Heunen and Bas Spitters for answering my questions and Mai Gehrke for spending her time to act as the second reader.

Some special words should be spent on the Nijmegen mathematics students, who are very cooperative and helped me a lot throughout the past five years. This was especially the cas in the last months, when I tried to explain what my Masters thesis was about, even though some of them had no idea what a C*-algebra or a topos was and hence possibly had no idea what I was talking about. Also, I would like to mention Desda, the students association.

Finally, I would like to thank my parents, since they have always showed interest in the things I have done and who were always prepared to help, to spend time to me and advise me. Thanks.