Truncated Rozansky–Witten models as extended defect TQFTs

Nils Carqueville

Universität Wien

based on joint work with **Ilka Brunner**, **Pantelis Fragkos**, **Daniel Roggenkamp**

https://carqueville.net/nils/RW.pdf

Rozansky–Witten models: (conjectured) non-semisimple 3d TQFTs

- topological twist of supersymmetric sigma models
- (conjectured) 3-category $\mathcal{RW}$
- sub-3-category $\mathcal{RW}^{\text{aff}}$ of affine target manifolds

Rozansky–Witten models: (conjectured) non-semisimple 3d TQFTs

- topological twist of supersymmetric sigma models
- (conjectured) 3-category $\mathcal{RW}$
- sub-3-category $\mathcal{RW}^{\mathrm{aff}}$ of affine target manifolds

Theorem.

- $\mathrm{Ho}_2(\mathcal{RW}^{\mathrm{aff}})$ is pivotal symmetric monoidal 2-category.
- Every object in $\mathrm{Ho}_2(\mathcal{RW}^{\mathrm{aff}})$ is fully dualisable.

Rozansky–Witten models: (conjectured) non-semisimple 3d TQFTs

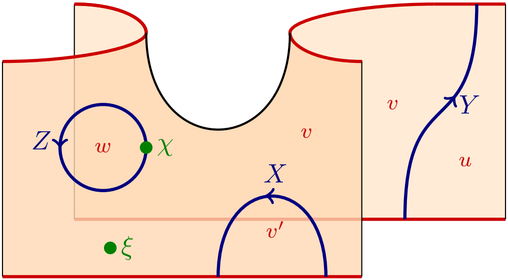
- topological twist of supersymmetric sigma models
- (conjectured) 3-category $\mathcal{RW}$
- sub-3-category $\mathcal{RW}^{\mathrm{aff}}$ of affine target manifolds

Theorem.

- $\mathrm{Ho}_2(\mathcal{RW}^{\mathrm{aff}})$ is pivotal symmetric monoidal 2-category.
- Every object in $\mathrm{Ho}_2(\mathcal{RW}^{\mathrm{aff}})$ is fully dualisable.

Application: affine RW models give truncated **extended defect TQFT**

$$\mathcal{Z} \colon \mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D}) \longrightarrow \mathrm{Ho}_2(\mathcal{RW}^{\mathrm{aff}})$$



extended

TQFT

framed extended

TQFT

Examples of symmetric monoidal 2-categories

$\mathrm{Bord}_{2,1,0}^{\mathrm{fr}}$

- objects: disjoint unions of 2-framed points $+, -$
- Hom categories: 2-framed bordisms of dimension 1 and 2

Alg (**state sum models**)

- objects: finite-dimensional $\mathbb{k}$-algebras
- Hom categories: finite-dimensional bimodules and bimodule maps

$\mathcal{V}\mathrm{ar}$ (**B-twisted sigma models**)

- objects: smooth projective varieties
- Hom categories: bounded derived categories of coherent sheaves

$\mathcal{LG}$ (**affine Landau–Ginzburg models**)

- objects: isolated singularities/potentials $W \in \mathbb{C}[x_1, \dots, x_n]$
- Hom categories: homotopy categories of matrix factorisations

$\mathrm{Ho}_2(\mathcal{RW}^{\mathrm{aff}})$ (**truncated affine Rozansky–Witten models**)

- objects: lists of variables $(x_1, \dots, x_n)$
- Hom categories: potentials and isom. classes of matrix factorisations

Pstragowski 2014, Căldăraru/Willerton 2007, Banks 2020, C/Murfet 2012, C/Montiel Montoya 2018, Kapustin/Rozansky 2009

3d graphical calculus

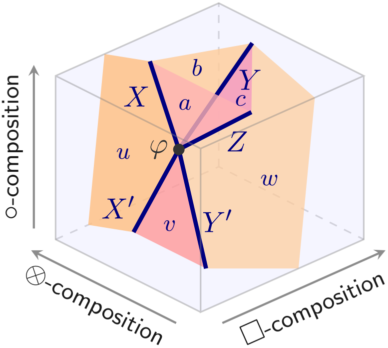
Fix symmetric monoidal 2-category with

monoidal product $\Box$

horizontal composition $\otimes$

vertical composition $\circ$

$\varphi \in \mathrm{Hom}\,(X' \otimes Y', X \otimes (Y \Box 1_a) \otimes (1_w \Box Z))$



Willerton, Barrett/Meusburger/Schaumann 2012

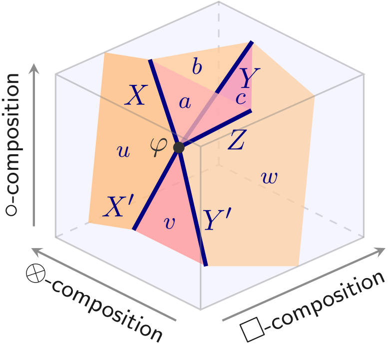
3d graphical calculus

Fix symmetric monoidal 2-category with

monoidal product $\Box$

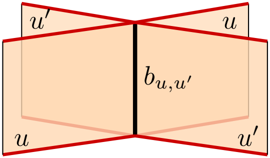
horizontal composition $\otimes$

vertical composition $\circ$



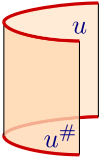
$$\varphi \in \mathrm{Hom}\left(X' \otimes Y', X \otimes (Y \,\Box\, 1_a) \otimes (1_w \,\Box\, Z)\right)$$

braiding:

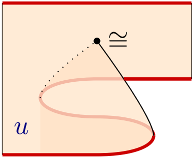


$\widehat{=}\ b_{u,u'} : u \,\Box\, u' \longrightarrow u' \,\Box\, u$

duals:



$\widehat{=}\ \widetilde{\mathrm{ev}}_u : u \,\Box\, u^{\#} \longrightarrow \mathbb{1}$



Willerton, Barrett/Meusburger/Schaumann 2012

Extended TQFT

Fix a symmetric monoidal 2-category $\mathcal{B}$. A **2d framed extended TQFT** valued in $\mathcal{B}$ is a symmetric monoidal 2-functor

$$\mathrm{Bord}^{\mathrm{fr}}_{2,1,0} \xrightarrow{\ \mathcal{Z}\ } \mathcal{B}$$

Theorem. [Framed **cobordism hypothesis** in 2d (conceptual version)]
2d framed extended TQFTs are fully dualisable objects:

$$\begin{aligned} \mathrm{Fun}^{\mathrm{sym.\,mon.}}\Big(\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}\,,\,\mathcal{B}\Big) &\xrightarrow{\ \cong\ } \big(\mathcal{B}^{\mathrm{fd}}\big)^{\times} \\ \mathcal{Z} &\longmapsto \mathcal{Z}(+) \end{aligned}$$

$$\begin{aligned} \mathcal{B}^{\mathrm{fd}} \quad &:= \quad \text{full sub-2-category of fully dualisable objects} \\ \big(\mathcal{B}^{\mathrm{fd}}\big)^{\times} \quad &:= \quad \text{maximal sub-2-groupoid of } \mathcal{B}^{\mathrm{fd}} \end{aligned}$$

Freed 1992, Baez/Dolan 1995, Lurie 2009, Schommer-Pries 2009, Pstragowski 2014

Cobordism hypothesis at work — part 1/2

Fix a symmetric monoidal 2-category $\mathcal{B}$. A **2d framed extended TQFT** valued in $\mathcal{B}$ is a symmetric monoidal 2-functor

$$\begin{array}{ccc} \mathrm{Bord}^{\mathrm{fr}}_{2,1,0} & \xrightarrow{\mathcal{Z}} & \mathcal{B} \\ + & \longmapsto & u \in \mathcal{B}^{\mathrm{fd}} \end{array}$$

Freed 1992, Baez/Dolan 1995, Lurie 2009, Schommer-Pries 2009, Pstragowski 2014

Cobordism hypothesis at work — part 1/2

Fix a symmetric monoidal 2-category $\mathcal{B}$. A **2d framed extended TQFT** valued in $\mathcal{B}$ is a symmetric monoidal 2-functor

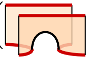
$$\mathrm{Bord}^{\mathrm{fr}}_{2,1,0} \xrightarrow{\mathcal{Z}} \mathcal{B}$$

$$+ \longmapsto u \in \mathcal{B}^{\mathrm{fd}}$$

$$= \widetilde{\mathrm{ev}}_+ \longmapsto \widetilde{\mathrm{ev}}_u$$

$$= {}^\dagger\widetilde{\mathrm{ev}}_+ \longmapsto {}^\dagger\widetilde{\mathrm{ev}}_u$$

$$= \widetilde{\mathrm{ev}}_+ \otimes {}^\dagger\widetilde{\mathrm{ev}}_+ = S^1_1 \longmapsto \widetilde{\mathrm{ev}}_u \otimes {}^\dagger\widetilde{\mathrm{ev}}_u$$

(

$$= \mathrm{ev}_{\widetilde{\mathrm{ev}}_+} : {}^\dagger\widetilde{\mathrm{ev}}_+ \otimes \widetilde{\mathrm{ev}}_+ \longrightarrow 1_{+\sqcup -}\Big) \longmapsto \mathrm{ev}_{\widetilde{\mathrm{ev}}_u}$$

(

$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} : \widetilde{\mathrm{ev}}_+ \otimes \widetilde{\mathrm{ev}}_+^\dagger \longrightarrow 1_\varnothing\Big) \longmapsto \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_u}$$

2-framing on 1-manifold M is trivialisation $TM \oplus \mathbb{R} \cong \mathbb{R}^2$, described by immersion $\iota\colon M \hookrightarrow \mathbb{R}^2$ and trivialisation of normal bundle $\nu(\iota)$; normal vectors are blue.

Freed 1992, Baez/Dolan 1995, Lurie 2009, Schommer-Pries 2009, Pstragowski 2014

Cobordism hypothesis at work — part 2/2

$$\begin{aligned}(\mathcal{B}^{\mathrm{fd}})^{\times} &\xrightarrow{\cong} \text{Coherent Full Duality Data}\,(\mathcal{B}) \\ u &\longmapsto \big(u, u^{\#}, \widetilde{\mathrm{ev}}_u, \widetilde{\mathrm{coev}}_u, S_u, S_u^{-1}, c_{\mathrm{l}}^u, c_{\mathrm{r}}^u, \mathrm{ev}_{\widetilde{\mathrm{ev}}_u}, \mathrm{coev}_{\widetilde{\mathrm{ev}}_u}, \mathrm{ev}_{\widetilde{\mathrm{coev}}_u}, \mathrm{coev}_{\widetilde{\mathrm{coev}}_u}, \phi, \psi\big)\end{aligned}$$

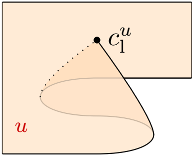
Cobordism hypothesis at work — part 2/2

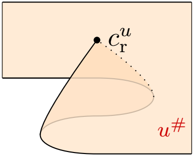
$$(\mathcal{B}^{\mathrm{fd}})^\times \xrightarrow{\cong} \text{Coherent Full Duality Data}\,(\mathcal{B})$$

$$u \longmapsto \left(u, u^\#, \widetilde{\mathrm{ev}}_u, \widetilde{\mathrm{coev}}_u, S_u, S_u^{-1}, c_{\mathrm{l}}^u, c_{\mathrm{r}}^u, \mathrm{ev}_{\widetilde{\mathrm{ev}}_u}, \mathrm{coev}_{\widetilde{\mathrm{ev}}_u}, \mathrm{ev}_{\widetilde{\mathrm{coev}}_u}, \mathrm{coev}_{\widetilde{\mathrm{coev}}_u}, \phi, \psi\right)$$

where

$$S_u := \left(1_u \,\Box\, \widetilde{\mathrm{ev}}_u\right) \otimes \left(b_{u,u} \,\Box\, 1_{u^\#}\right) \otimes \left(1_u \,\Box\, \widetilde{\mathrm{ev}}_u^\dagger\right)$$ (unique up to 2-isomorphism)

$c_{\mathrm{l}}^u =$ 

$c_{\mathrm{r}}^u =$ 

$\phi\colon S_u^{-1} \circ S_u \xrightarrow{\cong} 1_u$ $\qquad$ $\psi\colon S_u \circ S_u^{-1} \xrightarrow{\cong} 1_u$

Pstragowski 2014/2022

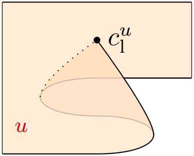
Cobordism hypothesis at work — part 2/2

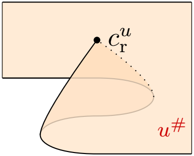
$$\left(\mathcal{B}^{\mathrm{fd}}\right)^{\times} \xrightarrow{\cong} \text{Coherent Full Duality Data}\,(\mathcal{B})$$

$$u \longmapsto \left(u, u^{\#}, \widetilde{\mathrm{ev}}_u, \widetilde{\mathrm{coev}}_u, S_u, S_u^{-1}, c_{\mathrm{l}}^u, c_{\mathrm{r}}^u, \mathrm{ev}_{\widetilde{\mathrm{ev}}_u}, \mathrm{coev}_{\widetilde{\mathrm{ev}}_u}, \mathrm{ev}_{\widetilde{\mathrm{coev}}_u}, \mathrm{coev}_{\widetilde{\mathrm{coev}}_u}, \phi, \psi\right)$$

where

$$S_u := \left(1_u \,\Box\, \widetilde{\mathrm{ev}}_u\right) \otimes \left(b_{u,u} \,\Box\, 1_{u^{\#}}\right) \otimes \left(1_u \,\Box\, \widetilde{\mathrm{ev}}_u^{\dagger}\right)$$ (unique up to 2-isomorphism)

$c_{\mathrm{l}}^u =$ 

$c_{\mathrm{r}}^u =$ 

$\phi\colon S_u^{-1} \circ S_u \xrightarrow{\cong} 1_u \qquad \psi\colon S_u \circ S_u^{-1} \xrightarrow{\cong} 1_u$

$$\Longrightarrow \qquad \mathrm{ev}_u := \widetilde{\mathrm{ev}}_u \otimes b_{u^{\#},u}$$

$$\mathrm{coev}_u := b_{u^{\#},u} \otimes \widetilde{\mathrm{coev}}_u$$

$$\widetilde{\mathrm{ev}}_u^{\dagger} \cong \left(S_u \,\Box\, 1_{u^{\#}}\right) \otimes \mathrm{coev}_u$$

$${}^{\dagger}\widetilde{\mathrm{ev}}_u \cong \left(S_u^{-1} \,\Box\, 1_{u^{\#}}\right) \otimes \mathrm{coev}_u$$ etc.

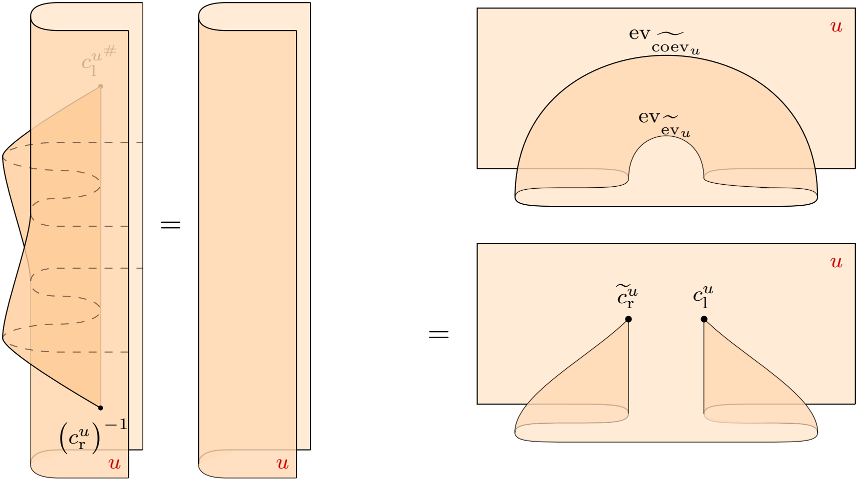
Pstragowski 2014/2022

Cobordism hypothesis at work — part 2/2

$$(\mathcal{B}^{\mathrm{fd}})^{\times} \xrightarrow{\cong} \text{Coherent Full Duality Data}\,(\mathcal{B})$$

$$u \longmapsto \left(u, u^{\#}, \widetilde{\mathrm{ev}}_u, \widetilde{\mathrm{coev}}_u, S_u, S_u^{-1}, c_{\mathrm{l}}^u, c_{\mathrm{r}}^u, \mathrm{ev}_{\widetilde{\mathrm{ev}_u}}, \mathrm{coev}_{\widetilde{\mathrm{ev}_u}}, \mathrm{ev}_{\widetilde{\mathrm{coev}_u}}, \mathrm{coev}_{\widetilde{\mathrm{coev}_u}}, \phi, \psi\right)$$

such that



Pstragowski 2014/2022

Extended framed TQFT

$$\begin{array}{ccc} \left(\mathcal{B}^{\mathrm{fd}}\right)^{\times} & \xrightarrow{F} & \text{Coherent Full Duality Data}\,(\mathcal{B}) \\ u & \longmapsto & \left(u, u^{\#}, \widetilde{\mathrm{ev}}_u, \widetilde{\mathrm{coev}}_u, S_u, S_u^{-1}, c_{\mathrm{l}}^u, c_{\mathrm{r}}^u, \mathrm{ev}_{\widetilde{\mathrm{ev}}_u}, \mathrm{coev}_{\widetilde{\mathrm{ev}}_u}, \mathrm{ev}_{\widetilde{\mathrm{coev}}_u}, \mathrm{coev}_{\widetilde{\mathrm{coev}}_u}, \phi, \psi\right) \end{array}$$

Freed 1992, Baez/Dolan 1995, Lurie 2009, Schommer-Pries 2009, Pstragowski 2014/2022

Extended framed TQFT

$$(\mathcal{B}^{\mathrm{fd}})^{\times} \xrightarrow{F} \text{Coherent Full Duality Data}\,(\mathcal{B})$$

$$u \longmapsto \Big(u, u^{\#}, \widetilde{\mathrm{ev}}_u, \widetilde{\mathrm{coev}}_u, S_u, S_u^{-1}, c_{\mathrm{l}}^u, c_{\mathrm{r}}^u, \mathrm{ev}_{\widetilde{\mathrm{ev}_u}}, \mathrm{coev}_{\widetilde{\mathrm{ev}_u}}, \mathrm{ev}_{\widetilde{\mathrm{coev}_u}}, \mathrm{coev}_{\widetilde{\mathrm{coev}_u}}, \phi, \psi\Big)$$

Theorem. [Framed **cobordism hypothesis** in 2d (explicit version)]

$$(\mathcal{B}^{\mathrm{fd}})^{\times} \xrightarrow{\cong} \mathrm{Fun}^{\mathrm{sym.\,mon.}}\Big(\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}\,,\,\mathcal{B}\Big)$$

$$u \longmapsto \Big(\text{bordism} \longmapsto \text{graphical calculus of } F(u)\Big)$$

"Simply interpret bordisms in graphical calculus of $\mathcal{B}$."

Freed 1992, Baez/Dolan 1995, Lurie 2009, Schommer-Pries 2009, Pstragowski 2014/2022

(Non-)semisimple framed extended TQFTs

Theorem. Every *separable* (hence semisimple) $A \in \mathrm{Alg}$ gives TQFT

$$\mathrm{Bord}^{\mathrm{fr}}_{2,1,0} \longrightarrow \mathrm{Alg}$$

$$+ \longmapsto A$$

$$- \longmapsto A^{\mathrm{op}}$$

$$= \widetilde{\mathrm{ev}}_+ \longmapsto {}_{\Bbbk}A_{A \otimes_{\Bbbk} A^{\mathrm{op}}}$$

$$= \mathrm{coev}_+ \longmapsto {}_{A \otimes_{\Bbbk} A^{\mathrm{op}}}A_{\Bbbk}$$

$$= \widetilde{\mathrm{ev}}_+ \otimes \mathrm{coev}_+ = S^1_0 \longmapsto A \otimes_{A \otimes_{\Bbbk} A^{\mathrm{op}}} A = \mathrm{HH}_0(A)$$

Lurie 2009, Schommer-Pries 2009

(Non-)semisimple framed extended TQFTs

Theorem. Every *separable* (hence semisimple) $A \in \mathrm{Alg}$ gives TQFT

$$\begin{aligned}
\mathrm{Bord}^{\mathrm{fr}}_{2,1,0} &\longrightarrow \mathrm{Alg} \\
+ &\longmapsto A \\
- &\longmapsto A^{\mathrm{op}}
\end{aligned}$$

$$= \widetilde{\mathrm{ev}}_+ \longmapsto {}_{\Bbbk}A_{A\otimes_{\Bbbk}A^{\mathrm{op}}}$$

$$= \mathrm{coev}_+ \longmapsto {}_{A\otimes_{\Bbbk}A^{\mathrm{op}}}A_{\Bbbk}$$

$$= \widetilde{\mathrm{ev}}_+ \otimes \mathrm{coev}_+ = S^1_0 \longmapsto A \otimes_{A\otimes_{\Bbbk}A^{\mathrm{op}}} A = \mathrm{HH}_0(A)$$

Theorem. Every $W \in \mathcal{LG}$ gives extended TQFT:

$$\begin{aligned}
\mathrm{Bord}^{\mathrm{fr}}_{2,1,0} &\longrightarrow \mathcal{LG} \\
+ &\longmapsto W
\end{aligned}$$

$$= \widetilde{\mathrm{ev}}_+ \otimes {}^{\dagger}\widetilde{\mathrm{ev}}_+ = S^1_1 \longmapsto \mathrm{Jac}_W = \mathbb{C}[\underline{x}]/(\partial W)$$

$$= 1_{\widetilde{\mathrm{ev}}_+} \otimes \mathrm{ev}_{\widetilde{\mathrm{ev}}_+} \otimes 1_{{}^{\dagger}\widetilde{\mathrm{ev}}_+} \longmapsto \text{multiplication in } \mathrm{Jac}_W$$

Lurie 2009, Schommer-Pries 2009, Carqueville/Montiel Montoya 2018 oriented extended

TQFT

Oriented cobordism hypothesis

"Rotating frames" gives rise to SO_2-homotopy action on $\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}$:

$$\begin{array}{rcl} \Pi_{\leqslant 2}(\mathrm{SO}_2) & \longrightarrow & \mathrm{Aut}\Big(\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}\Big) \\ \pi_0(\mathrm{SO}_2) \cong \{*\} \ni * & \longmapsto & \mathrm{Id} \\ \pi_1(\mathrm{SO}_2) \cong \mathbb{Z} \ni -1 & \longmapsto & \big(S\colon \mathrm{Id} \longrightarrow \mathrm{Id}\big)\,, \quad S_+ = \end{array}$$

Lurie 2009, Douglas/Schommer-Pries/Snyder 2013

Oriented cobordism hypothesis

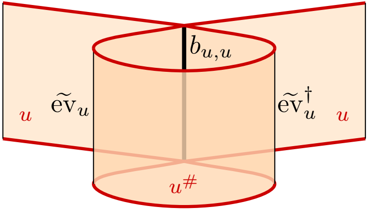
"Rotating frames" gives rise to SO_2-homotopy action on $\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}$:

$$\begin{array}{rcl} \Pi_{\leqslant 2}(\mathrm{SO}_2) & \longrightarrow & \mathrm{Aut}\Big(\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}\Big) \\ \pi_0(\mathrm{SO}_2) \cong \{*\} \ni * & \longmapsto & \mathrm{Id} \\ \pi_1(\mathrm{SO}_2) \cong \mathbb{Z} \ni -1 & \longmapsto & \big(S\colon \mathrm{Id} \longrightarrow \mathrm{Id}\big)\,, \quad S_+ = \end{array}$$

For any $u \in \mathcal{B}^{\mathrm{fd}}$, have **Serre automorphism**

$$S_u := \big(1_u \,\square\, \widetilde{\mathrm{ev}}_u\big) \otimes \big(b_{u,u} \,\square\, 1_{u^\#}\big) \otimes \big(1_u \,\square\, \widetilde{\mathrm{ev}}_u^\dagger\big)$$

$=$



Lurie 2009, Douglas/Schommer-Pries/Snyder 2013

Oriented cobordism hypothesis

"Rotating frames" gives rise to SO_2-homotopy action on $\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}$:

$$\begin{array}{rcl} \Pi_{\leqslant 2}(\mathrm{SO}_2) & \longrightarrow & \mathrm{Aut}\Big(\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}\Big) \\ \pi_0(\mathrm{SO}_2) \cong \{*\} \ni * & \longmapsto & \mathrm{Id} \\ \pi_1(\mathrm{SO}_2) \cong \mathbb{Z} \ni -1 & \longmapsto & (S\colon \mathrm{Id} \longrightarrow \mathrm{Id}), \quad S_+ = \end{array}$$

Theorem. [Oriented cobordism hypothesis in 2d]
2d oriented extended TQFTs are SO_2-homotopy fixed points:

$$\mathrm{Fun}^{\mathrm{sym.\,mon.}}\Big(\mathrm{Bord}^{\mathrm{or}}_{2,1,0}\,,\,\mathcal{B}\Big) \xrightarrow{\cong} \Big[(\mathcal{B}^{\mathrm{fd}})^{\times}\Big]^{\mathrm{SO}_2}$$

Such TQFTs $\mathcal{Z}$ are classified by objects $u := \mathcal{Z}(+) \in \mathcal{B}^{\mathrm{fd}}$ together with **trivialisation of Serre automorphism**, $\lambda_u\colon S_u \xrightarrow{\cong} 1_u$.

Lurie 2009, Schommer-Pries 2009, Hesse/Schweigert/Valentino 2016, Hesse/Valentino 2017

Oriented cobordism hypothesis at work

Theorem. [Oriented **cobordism hypothesis** in 2d (explicit version)]

$$\Big[(\mathcal{B}^{\mathrm{fd}})^{\times}\Big]^{\mathrm{SO}_2} \xrightarrow{\cong} \mathrm{Fun}^{\mathrm{sym.\,mon.}}\Big(\mathrm{Bord}^{\mathrm{or}}_{2,1,0}\,,\,\mathcal{B}\Big)$$

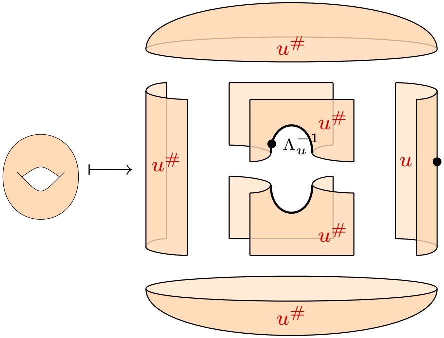
$$\Big(u,\, S_u \xrightarrow[\cong]{\lambda_u} 1_u\Big) \longmapsto \Big(\text{bordism} \longmapsto \text{graphical calculus of } F(u)\ \&\ \lambda_u\Big)$$

Oriented cobordism hypothesis at work

Theorem. [Oriented **cobordism hypothesis** in 2d (explicit version)]

$$\Big[(\mathcal{B}^{\mathrm{fd}})^{\times}\Big]^{\mathrm{SO}_2} \xrightarrow{\cong} \mathrm{Fun}^{\mathrm{sym.\,mon.}}\Big(\mathrm{Bord}^{\mathrm{or}}_{2,1,0}\,,\,\mathcal{B}\Big)$$

$$\Big(u,\, S_u \xrightarrow[\cong]{\lambda_u} 1_u\Big) \longmapsto \Big(\text{bordism} \longmapsto \text{graphical calculus of } F(u)\ \&\ \lambda_u\Big)$$



$\Lambda_u := (\lambda_u^\dagger \cdot \lambda_u \,\square\, 1_{1_{u^{\#}}}) \otimes 1_{\mathrm{coev}_u} : {}^\dagger\widetilde{\mathrm{ev}}_u \xrightarrow{\cong} \widetilde{\mathrm{ev}}_u^\dagger$

$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_u} \cdot \Big[1_{\widetilde{\mathrm{ev}}_u} \otimes \Big(\mathrm{ev}_{\widetilde{\mathrm{ev}}_u} \cdot [\Lambda_u^{-1} \otimes 1_{\widetilde{\mathrm{ev}}_u}] \cdot \widetilde{\mathrm{coev}}_{\widetilde{\mathrm{ev}}_u}\Big) \otimes \Lambda_u\Big] \cdot \mathrm{coev}_{\widetilde{\mathrm{ev}}_u}$$

Oriented & spin extended TQFTs

Theorem. Every separable *symmetric Frobenius* algebra $A \in \mathrm{Alg}$ gives oriented extended TQFT $\mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathrm{Alg}$.

Lurie 2009, Schommer-Pries 2009

Oriented & spin extended TQFTs

Theorem. Every separable *symmetric Frobenius* algebra $A \in \mathrm{Alg}$ gives oriented extended TQFT $\mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathrm{Alg}$.

Theorem. Every $W(x_1, \ldots, x_{2n}) \in \mathcal{LG}$ gives oriented extended TQFT

$$\mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathcal{LG}$$

$$+ \longmapsto W$$

 $\longmapsto$ Jac_W

 $\longmapsto$ multiplication

 $\longmapsto$ $\mathrm{Res}\left[\frac{(-)\,\mathrm{d}x}{\partial_{x_1}W \ldots \partial_{x_{2n}}W}\right]$

Lurie 2009, Schommer-Pries 2009, Carqueville/Montiel Montoya 2018

Oriented & spin extended TQFTs

Theorem. Every separable *symmetric Frobenius* algebra $A \in \mathrm{Alg}$ gives oriented extended TQFT $\mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathrm{Alg}$.

Theorem. Every $W(x_1, \dots, x_{2n}) \in \mathcal{LG}$ gives oriented extended TQFT

$$\mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathcal{LG}$$

$$+ \longmapsto W$$

 $\longmapsto \mathrm{Jac}_W$

 $\longmapsto$ multiplication

 $\longmapsto \mathrm{Res}\left[\frac{(-)\,\mathrm{d}x}{\partial_{x_1} W \dots \partial_{x_{2n}} W}\right]$

Theorem. *Every* $W \in \mathcal{LG}$ gives **spin** extended TQFT

$$\mathrm{Bord}^{\mathrm{spin}}_{2,1,0} \longrightarrow \mathcal{LG}$$

Lurie 2009, Schommer-Pries 2009, Carqueville/Montiel Montoya 2018, Carqueville/Szegedy 2021 truncated affine

Rozansky–Witten

models

Rozansky–Witten models

– rigorously constructed 3d TQFTs = Reshetikhin–Turaev models

Rozansky/Witten 1996, Kapranov 1997, Kontsevich 1997, Roberts/Willerton 2006, Kapustin/Rozansky/Saulina 2008, ...

Rozansky–Witten models

– rigorously constructed 3d TQFTs = Reshetikhin–Turaev models
– **RW models**: conjecturally 3d TQFTs from non-semisimple data
  - twisted 3d $\mathcal{N} = 4$ sigma model with holomorphic symplectic target
  - reduction on S^1 gives 2d B-model
  - "has local observables"
  - participate in 3d mirror symmetry

Rozansky/Witten 1996, Kapranov 1997, Kontsevich 1997, Roberts/Willerton 2006, Kapustin/Rozansky/Saulina 2008, ...

Rozansky–Witten models

- rigorously constructed 3d TQFTs = Reshetikhin–Turaev models
- **RW models**: conjecturally 3d TQFTs from non-semisimple data
  - twisted 3d $\mathcal{N} = 4$ sigma model with holomorphic symplectic target
  - reduction on S^1 gives 2d B-model
  - "has local observables"
  - participate in 3d mirror symmetry
- Kapustin–Rozansky(–Saulina) propose defect 3-category $\mathcal{RW}$:
  - objects: holomorphic symplectic manifolds X
  - k-cells: "deformed Landau–Ginzburg models fibred over Lagrangians"

Rozansky/Witten 1996, Kapranov 1997, Kontsevich 1997, Roberts/Willerton 2006, Kapustin/Rozansky/Saulina 2008, ...

Rozansky–Witten models

- rigorously constructed 3d TQFTs = Reshetikhin–Turaev models
- **RW models**: conjecturally 3d TQFTs from non-semisimple data
  - twisted 3d $\mathcal{N} = 4$ sigma model with holomorphic symplectic target
  - reduction on S^1 gives 2d B-model
  - "has local observables"
  - participate in 3d mirror symmetry
- Kapustin–Rozansky(–Saulina) propose defect 3-category $\mathcal{RW}$:
  - objects: holomorphic symplectic manifolds X
  - k-cells: "deformed Landau–Ginzburg models fibred over Lagrangians"
- **affine case** $X = T^*\mathbb{C}^n$
  - related to Chern–Simons theory for $\mathrm{psl}(1|1)$
  - related to free $\mathcal{N} = 4$ hypermultiplet
  - 3-category $\mathcal{RW}^{\mathrm{aff}}$ under explicit control

Rozansky/Witten 1996, Kapranov 1997, Kontsevich 1997, Roberts/Willerton 2006, Kapustin/Rozansky/Saulina 2008, ...

Rozansky–Witten models

- rigorously constructed 3d TQFTs = Reshetikhin–Turaev models
- **RW models**: conjecturally 3d TQFTs from non-semisimple data
  - twisted 3d $\mathcal{N} = 4$ sigma model with holomorphic symplectic target
  - reduction on S^1 gives 2d B-model
  - "has local observables"
  - participate in 3d mirror symmetry
- Kapustin–Rozansky(–Saulina) propose defect 3-category $\mathcal{RW}$:
  - objects: holomorphic symplectic manifolds X
  - k-cells: "deformed Landau–Ginzburg models fibred over Lagrangians"
- **affine case** $X = T^*\mathbb{C}^n$
  - related to Chern–Simons theory for $\mathrm{psl}(1|1)$
  - related to free $\mathcal{N} = 4$ hypermultiplet
  - 3-category $\mathcal{RW}^{\mathrm{aff}}$ under explicit control

Upshot:
Construct RW models as **extended defect TQFTs** valued in $\mathcal{C} := \mathrm{Ho}_2(\mathcal{RW}^{\mathrm{aff}})$.

Rozansky/Witten 1996, Kapranov 1997, Kontsevich 1997, Roberts/Willerton 2006, Kapustin/Rozansky/Saulina 2008, ...

Basic idea

There is a 2-category $\mathcal{C}$ with

$$\begin{aligned} \text{objects} &\approx \text{variables} \\ \text{1-morphisms} &\approx \text{polynomials} \\ \text{2-morphisms} &\approx \text{matrix factorisations} \end{aligned}$$

Theorem.
$\mathcal{C}$ is pivotal symmetric monoidal, every object is fully dualisable.

Kapustin/Rozansky 2009, Brunner/Carqueville/Roggenkamp 2022, Brunner/Carqueville/Fragkos/Roggenkamp 2023

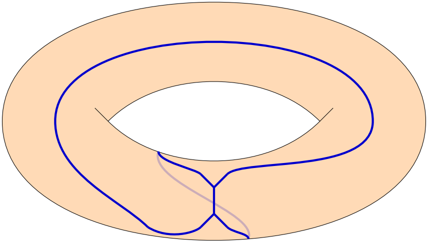
Basic idea

There is a 2-category $\mathcal{C}$ with

$$\begin{aligned} \text{objects} &\approx \text{variables} \\ \text{1-morphisms} &\approx \text{polynomials} \\ \text{2-morphisms} &\approx \text{matrix factorisations} \end{aligned}$$

Theorem.
$\mathcal{C}$ is pivotal symmetric monoidal, every object is fully dualisable.

$\mathcal{C}$ computes state spaces (with defects) of affine RW models.



Kapustin/Rozansky 2009, Brunner/Carqueville/Roggenkamp 2022, Brunner/Carqueville/Fragkos/Roggenkamp 2023

Truncated affine Rozansky–Witten theory

There is a 2-category $\mathcal{C}$ with:

- **objects** are lists of variables $\underline{x} := (x_1, x_2, \dots, x_n)$, $n \in \mathbb{Z}_{\geqslant 0}$
- **1-cells** $\underline{x} \longrightarrow \underline{y}$ are pairs $(\underline{a}; W)$ with $W \in \mathbb{C}[\underline{a}, \underline{x}, \underline{y}]$:

- **horizontal composition**:

$$\big(\underline{b};\, V(\underline{b}, \underline{y}, \underline{z})\big) \circ \big(\underline{a};\, W(\underline{a}, \underline{x}, \underline{y})\big) = \big(\underline{a}, \underline{b}, \underline{y};\, V(\underline{b}, \underline{y}, \underline{z}) + W(\underline{a}, \underline{x}, \underline{y})\big)$$

- $1_{\underline{x}} = \big(\underline{a};\, \underline{a} \cdot (\underline{x}' - \underline{x})\big)$, where $\underline{a} \cdot (\underline{x}' - \underline{x}) := \sum_{i=1}^{n} a_i \cdot (x_i' - x_i)$

Kapustin/Rozansky 2009, Oblomkov/Rozansky 2018, Brunner/Carqueville/Roggenkamp 2022

Matrix factorisations

– **Matrix factorisation** of $f \in \mathbb{C}[\underline{x}]$ is (X, d_X), where
  - $X = X^0 \oplus X^1$ is free $\mathbb{Z}_2$-graded $\mathbb{C}[\underline{x}]$-module
  - $d_X \colon X \longrightarrow X$ is odd $\mathbb{C}[\underline{x}]$-linear module map with $d_X^2 = f \cdot 1_X$

Example: $f = y^4 - x^3, \quad X = \mathbb{C}[x,y]^2 \oplus \mathbb{C}[x,y]^2,$

$$d_X = \begin{pmatrix} 0 & 0 & -y^2 & -x \\ 0 & 0 & x^2 & y^2 \\ -y^2 & -x & 0 & 0 \\ x^2 & y^2 & 0 & 0 \end{pmatrix}$$

Matrix factorisations

- **Matrix factorisation** of $f \in \mathbb{C}[\underline{x}]$ is (X, d_X), where
  - $X = X^0 \oplus X^1$ is free $\mathbb{Z}_2$-graded $\mathbb{C}[\underline{x}]$-module
  - $d_X : X \longrightarrow X$ is odd $\mathbb{C}[\underline{x}]$-linear module map with $d_X^2 = f \cdot 1_X$

Matrix factorisations

- **Matrix factorisation** of $f \in \mathbb{C}[\underline{x}]$ is (X, d_X), where
  - $X = X^0 \oplus X^1$ is free $\mathbb{Z}_2$-graded $\mathbb{C}[\underline{x}]$-module
  - $d_X : X \longrightarrow X$ is odd $\mathbb{C}[\underline{x}]$-linear module map with $d_X^2 = f \cdot 1_X$
- For $p_i, q_i \in \mathbb{C}[\underline{x}]$, $i \in \{1, \ldots, k\}$, have **Koszul matrix factorisation** of $f = \sum_i p_i \cdot q_i$:

$$\left[\underline{p},\, \underline{q}\right] := \left(K(\underline{p},\underline{q}),\, d_{K(\underline{p},\underline{q})}\right), \quad K(\underline{p},\underline{q}) = \bigwedge\left(\bigoplus_{i=1}^k \mathbb{C}[\underline{x}] \cdot \theta_i\right)$$

$$d_{K(\underline{p},\underline{q})} = \sum_{i=1}^k \left(p_i \cdot \theta_i + q_i \cdot \theta_i^*\right)$$

- With $\partial_{[i]}^{x',x} f := \frac{f(x_1,\ldots,x_{i-1},x'_i,\ldots x'_n) - f(x_1,\ldots,x_i,x'_{i+1},\ldots x'_n)}{x'_i - x_i}$ have

$$I_f := \left[\underline{\partial^{x',x} f},\, \underline{x}' - \underline{x}\right]$$

Matrix factorisations

– With $\partial_{[i]}^{x',x} f := \frac{f(x_1,\dots,x_{i-1},x'_i,\dots x'_n) - f(x_1,\dots,x_i,x'_{i+1},\dots x'_n)}{x'_i - x_i}$ have

$$I_f := \left[\underline{\partial^{x',x} f},\ \underline{x}' - \underline{x}\right]$$

– **homotopy category of matrix factorisations** $\mathrm{HMF}(\mathbb{C}[\underline{x}], f)$ has as morphisms even cohomology classes of differential

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}[\underline{x}]}(X, X') &\longrightarrow \mathrm{Hom}_{\mathbb{C}[\underline{x}]}(X, X') \\ \zeta &\longmapsto d_{X'} \circ \zeta - (-1)^{|\zeta|} \zeta \circ d_X \end{aligned}$$

– $\mathrm{hmf}(\mathbb{C}[\underline{x}], f)^\omega :=$ idempotent completion of finite-rank objects

– **Knörrer periodicity**:

$$\mathrm{hmf}\left(\mathbb{C}[\underline{x}], f\right)^\omega \cong \mathrm{hmf}\left(\mathbb{C}[\underline{x}, u, v], f + uv\right)^\omega$$

(used for unitors in $\mathcal{C}$)

Truncated affine Rozansky–Witten theory

There is a 2-category $\mathcal{C}$ with

- **objects** are lists of variables $\underline{x} := (x_1, x_2, \dots, x_n)$, $n \in \mathbb{Z}_{\geqslant 0}$

- **1-cells** $\underline{x} \longrightarrow \underline{y}$ are pairs $(\underline{a}; W)$ with $W \in \mathbb{C}[\underline{a}, \underline{x}, \underline{y}]$:

- **horizontal composition**:

$$\big(\underline{b};\, V(\underline{b}, \underline{y}, \underline{z})\big) \circ \big(\underline{a};\, W(\underline{a}, \underline{x}, \underline{y})\big) = \big(\underline{a}, \underline{b}, \underline{y};\, V(\underline{b}, \underline{y}, \underline{z}) + W(\underline{a}, \underline{x}, \underline{y})\big)$$

- $1_{\underline{x}} = \big(\underline{a};\, \underline{a} \cdot (\underline{x}' - \underline{x})\big)$, where $\underline{a} \cdot (\underline{x}' - \underline{x}) := \sum_{i=1}^{n} a_i \cdot (x_i' - x_i)$

Kapustin/Rozansky 2009, Oblomkov/Rozansky 2018, Brunner/Carqueville/Roggenkamp 2022

Truncated affine Rozansky–Witten theory

There is a 2-category $\mathcal{C}$ with

- **objects** are lists of variables $\underline{x} := (x_1, x_2, \dots, x_n)$, $n \in \mathbb{Z}_{\geqslant 0}$
- **1-cells** $\underline{x} \longrightarrow \underline{y}$ are pairs $(\underline{a}; W)$ with $W \in \mathbb{C}[\underline{a}, \underline{x}, \underline{y}]$:

- **horizontal composition**:

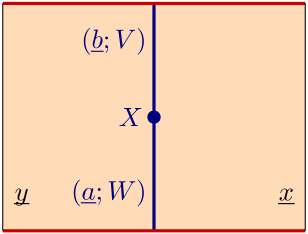
$$\big(\underline{b};\, V(\underline{b}, \underline{y}, \underline{z})\big) \circ \big(\underline{a};\, W(\underline{a}, \underline{x}, \underline{y})\big) = \big(\underline{a}, \underline{b}, \underline{y};\, V(\underline{b}, \underline{y}, \underline{z}) + W(\underline{a}, \underline{x}, \underline{y})\big)$$

- $1_{\underline{x}} = \big(\underline{a};\, \underline{a} \cdot (\underline{x}' - \underline{x})\big)$, where $\underline{a} \cdot (\underline{x}' - \underline{x}) := \sum_{i=1}^{n} a_i \cdot (x_i' - x_i)$
- Let $(\underline{a}; W), (\underline{b}; V) \colon \underline{x} \longrightarrow \underline{y}$. A **2-cell** $(\underline{a}; W) \longrightarrow (\underline{b}; V)$ is an isomorphism class X of objects in $\operatorname{hmf}(\mathbb{C}[\underline{a}, \underline{b}, \underline{x}, \underline{y}], V - W)^{\omega}$.

Kapustin/Rozansky 2009, Oblomkov/Rozansky 2018, Brunner/Carqueville/Roggenkamp 2022

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

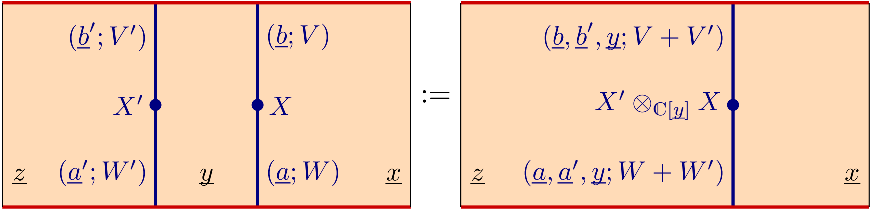
Let $(\underline{a}; W), (\underline{b}; V) \colon \underline{x} \longrightarrow \underline{y}$. A 2-cell $(\underline{a}; W) \longrightarrow (\underline{b}; V)$ is an isomorphism class X of objects in $\mathrm{hmf}(\mathbb{C}[\underline{a}, \underline{b}, \underline{x}, \underline{y}], V - W)^{\omega}$. $1_{(\underline{a}; W)} := I_W$.

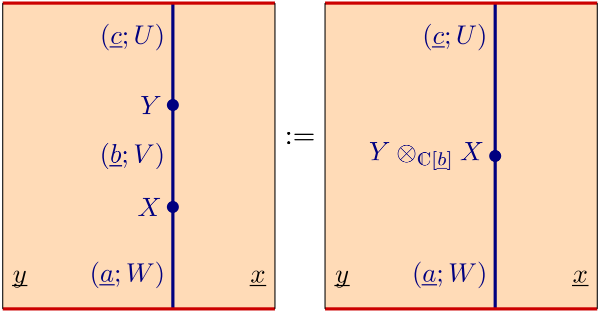


Kapustin/Rozansky 2009, Oblomkov/Rozansky 2018, Brunner/Carqueville/Roggenkamp 2022

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Let $(\underline{a}; W), (\underline{b}; V) \colon \underline{x} \longrightarrow \underline{y}$. A 2-cell $(\underline{a}; W) \longrightarrow (\underline{b}; V)$ is an isomorphism class X of objects in $\mathrm{hmf}(\mathbb{C}[\underline{a}, \underline{b}, \underline{x}, \underline{y}], V - W)^{\omega}$. $1_{(\underline{a};W)} := I_W$.





Kapustin/Rozansky 2009, Oblomkov/Rozansky 2018, Brunner/Carqueville/Roggenkamp 2022

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

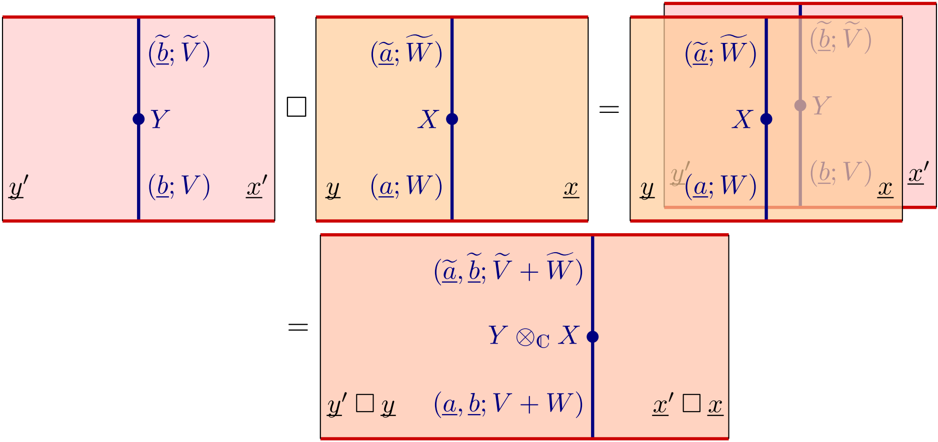
Monoidal product $\Box\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$,

$$(x_1, \dots, x_n) \,\Box\, (y_1, \dots, y_m) := (x_1, \dots, x_n, y_1, \dots, y_m)$$

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Monoidal product $\Box\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$,

$$(x_1, \dots, x_n) \Box (y_1, \dots, y_m) := (x_1, \dots, x_n, y_1, \dots, y_m)$$



Monoidal unit $= \varnothing$

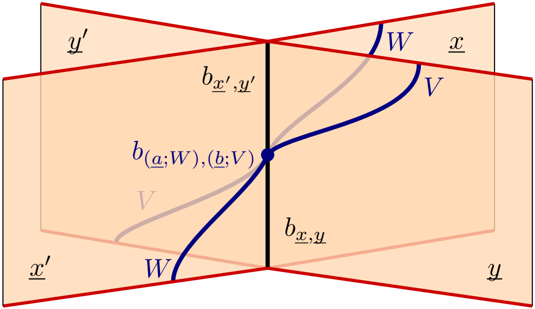
(structure 2-cells explicit and unsurprising)

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Theorem. $\mathcal{C}$ is symmetric monoidal 2-category with braiding

$$b_{\underline{x},\underline{y}} := \Big(\underline{c}, \underline{d};\ \underline{d} \cdot (\underline{y}' - \underline{y}) + \underline{c} \cdot (\underline{x}' - \underline{x})\Big) \colon \underline{x} \,\square\, \underline{y} \longrightarrow \underline{y} \,\square\, \underline{x} \equiv \underline{y}' \,\square\, \underline{x}'$$

$b_{(\underline{a};W),(\underline{b};V)} :=$

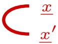


$\equiv 1_{(\underline{a},\underline{b};V+W)}$

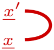
Brunner/Carqueville/Roggenkamp 2022

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

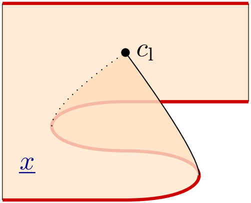
Lemma. Every $\underline{x} \in \mathcal{C}$ has **dual** $\underline{x}^{\#} := \underline{x}$ with



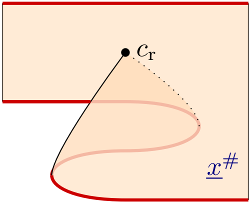
$$= \widetilde{\mathrm{ev}}_{\underline{x}} := \big(\underline{a};\, \underline{a} \cdot (\underline{x}' - \underline{x})\big) \colon \underline{x} \,\square\, \underline{x}^{\#} = (\underline{x}, \underline{x}') \longrightarrow \varnothing$$



$$= \widetilde{\mathrm{coev}}_{\underline{x}} := \big(\underline{a};\, \underline{a} \cdot (\underline{x} - \underline{x}')\big) \colon \varnothing \longrightarrow \underline{x}^{\#} \,\square\, \underline{x} = (\underline{x}', \underline{x})$$



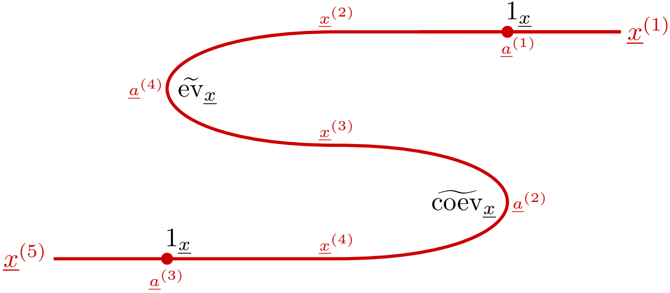
$$= c_{\mathrm{l}} \colon \big(\widetilde{\mathrm{ev}}_{\underline{x}} \,\square\, 1_{\underline{x}}\big) \circ \big(1_{\underline{x}} \,\square\, \widetilde{\mathrm{coev}}_{\underline{x}}\big) \xrightarrow{\cong} 1_{\underline{x}}$$



$$= c_{\mathrm{r}} \colon \big(1_{\underline{x}^{\#}} \,\square\, \widetilde{\mathrm{ev}}_{\underline{x}}\big) \circ \big(\widetilde{\mathrm{coev}}_{\underline{x}} \,\square\, 1_{\underline{x}^{\#}}\big) \xrightarrow{\cong} 1_{\underline{x}^{\#}}$$

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

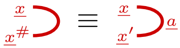
Proof.



$$
\begin{aligned}
&= \underline{a}^{(1)} \cdot \big(\underline{x}^{(2)} - \underline{x}^{(1)}\big) + \underline{a}^{(2)} \cdot \big(\underline{x}^{(4)} - \underline{x}^{(3)}\big) + \underline{a}^{(3)} \cdot \big(\underline{x}^{(5)} - \underline{x}^{(4)}\big) \\
&\qquad\qquad + \underline{a}^{(4)} \cdot \big(\underline{x}^{(3)} - \underline{x}^{(2)}\big) \\
&\cong \underline{a}^{(2)} \cdot \big(\underline{x}^{(5)} - \underline{x}^{(3)}\big) + \underline{a}^{(4)} \cdot \big(\underline{x}^{(3)} - \underline{x}^{(1)}\big) \\
&= \underline{x}^{(3)} \cdot \big(\underline{a}^{(4)} - \underline{a}^{(2)}\big) + \underline{a}^{(2)} \cdot \underline{x}^{(5)} - \underline{a}^{(4)} \cdot \underline{x}^{(1)} \\
&\cong \underline{a}^{(2)} \cdot \big(\underline{x}^{(5)} - \underline{x}^{(1)}\big) \cong 1_{\underline{x}}
\end{aligned}
$$

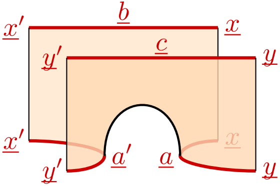
Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Theorem. Every $\underline{x} \in \mathcal{C}$ is fully dualisable:



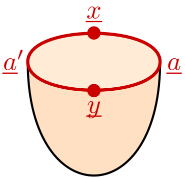
$$= \mathrm{coev}_{\underline{x}} = {}^{\dagger}\widetilde{\mathrm{ev}}_{\underline{x}} = \widetilde{\mathrm{ev}}_{\underline{x}}^{\dagger} := \big(\underline{a};\, \underline{a} \cdot (\underline{x} - \underline{x}')\big)$$

$$= \mathrm{ev}_{\underline{x}} = {}^{\dagger}\widetilde{\mathrm{coev}}_{\underline{x}} = \widetilde{\mathrm{coev}}_{\underline{x}}^{\dagger} := \big(\underline{a};\, \underline{a} \cdot (\underline{x}' - \underline{x})\big)$$



$$= \mathrm{ev}_{\widetilde{\mathrm{ev}}_{\underline{x}}} = \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{coev}}_{\underline{x}}}$$

$$:= [\underline{c} - \underline{a},\, \underline{y} - \underline{y}'] \otimes [\underline{b} - \underline{a}',\, \underline{x}' - \underline{x}] \otimes [\underline{a}' - \underline{a},\, \underline{y}' - \underline{x}]$$

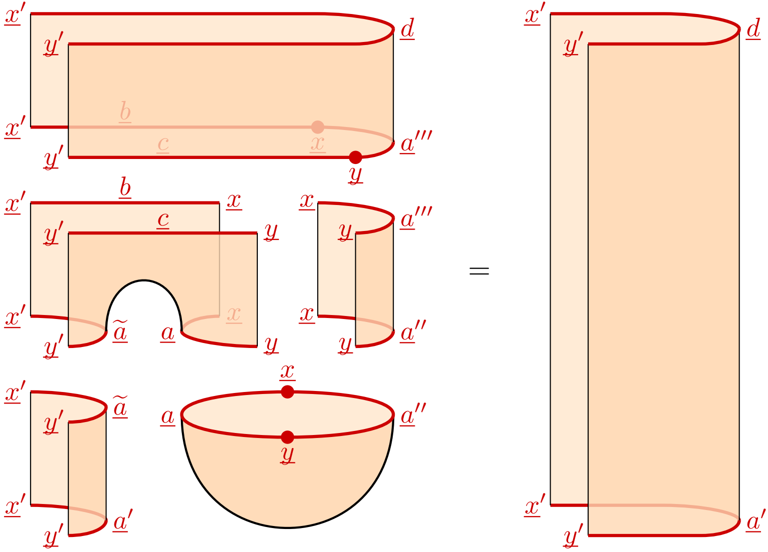


$$= \mathrm{coev}_{\widetilde{\mathrm{ev}}_{\underline{x}}} = \widetilde{\mathrm{coev}}_{\widetilde{\mathrm{coev}}_{\underline{x}}} := [\underline{a}' - \underline{a},\, \underline{x} - \underline{y}]$$

Brunner/Carqueville/Roggenkamp 2022

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Proof. Explicit computation of Zorro moves, e. g.



Brunner/Carqueville/Roggenkamp 2022

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Lemma. For all $\underline{x} \in \mathcal{C}$, there are precisely two isomorphisms

$$S_{\underline{x}} \xrightarrow{\cong} 1_{\underline{x}}$$

represented by the matrix factorisations $I_{1_{\underline{x}}}$ and $I_{1_{\underline{x}}}[1]$.

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Lemma. For all $\underline{x} \in \mathcal{C}$, there are precisely two isomorphisms

$$S_{\underline{x}} \xrightarrow{\cong} 1_{\underline{x}}$$

represented by the matrix factorisations $I_{1_{\underline{x}}}$ and $I_{1_{\underline{x}}}[1]$.

Proof.

$$\begin{aligned} S_{\underline{x}} &= (1_{\underline{x}} \,\square\, \widetilde{\mathrm{ev}}_{\underline{x}}) \circ (b_{\underline{x},\underline{x}} \,\square\, 1_{\underline{x}^\#}) \circ (1_{\underline{x}} \,\square\, \widetilde{\mathrm{ev}}_{\underline{x}}^\dagger) \\ &= \Big(\underline{a}^{(1)}, \dots, \underline{a}^{(7)}, \underline{x}^{(2)}, \dots, \underline{x}^{(7)} \,;\, \sum_{i=1}^{7} \underline{a}^{(i)} \cdot \big(\underline{x}^{(i+1)} - \underline{x}^{(i)}\big)\Big) \\ &= \Big(\underline{a}^{(1)};\; \underline{a}^{(1)} \cdot \big(\underline{x}^{(2)} - \underline{x}^{(1)}\big)\Big) \circ \Big(\underline{a}^{(2)};\; \underline{a}^{(2)} \cdot \big(\underline{x}^{(3)} - \underline{x}^{(2)}\big)\Big) \\ &\quad \circ \dots \circ \Big(\underline{a}^{(7)};\; \underline{a}^{(7)} \cdot \big(\underline{x}^{(8)} - \underline{x}^{(7)}\big)\Big) \;=\; (1_{\underline{x}})^7 \end{aligned}$$

and

$$\begin{aligned} \mathrm{hmf}\left(\mathbb{C}[\underline{a},\underline{b},\underline{x},\underline{y}], (\underline{a}-\underline{b})\cdot(\underline{x}-\underline{y})\right)^\omega &\cong \mathrm{hmf}\left(\mathbb{C}[\underline{a},\underline{b},\underline{x},\underline{y}], \underline{b}\cdot\underline{y}\right)^\omega \\ &\cong \mathrm{hmf}\left(\mathbb{C}[\underline{a},\underline{x}], 0\right)^\omega \\ &\cong \mathrm{mod}^{\mathbb{Z}_2}(\mathbb{C}[\underline{a},\underline{x}]) \end{aligned}$$

Truncated affine Rozansky–Witten models

Theorem. Every $\underline{x} = (x_1, \dots, x_n) \in \mathcal{C}$ gives unique extended TQFT:

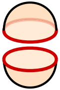
$$\mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathcal{C}$$

$$+ \longmapsto \underline{x}$$

$$= \widetilde{\mathrm{ev}}_+ \longmapsto \underline{a} \cdot (\underline{x} - \underline{x}')$$

$$= \widetilde{\mathrm{ev}}_+ \otimes {}^{\dagger}\widetilde{\mathrm{ev}}_+ = S^1 \longmapsto (\underline{a} - \underline{a}') \cdot (\underline{x} - \underline{x}')$$

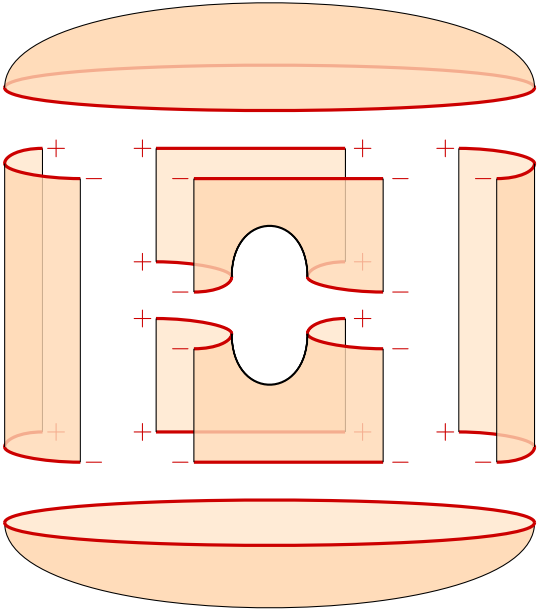
$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} \longmapsto \left[\underline{a} - \underline{a}', \underline{x} - \underline{x}'\right]$$



$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} \circ \mathrm{coev}_{\widetilde{\mathrm{ev}}_+} = S^2 \longmapsto \mathbb{C}[\underline{a}, \underline{x}]$$

Truncated affine Rozansky–Witten models

$$T^2 \quad = $$



Truncated affine Rozansky–Witten models

Theorem. Every $\underline{x} \equiv (x_1, \dots, x_n) \in \mathcal{C}$ gives unique extended TQFT:

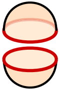
$$\mathrm{Bord}_{2,1,0}^{\mathrm{or}} \longrightarrow \mathcal{C}$$

$$+ \longmapsto (x_1, \dots, x_n)$$

$$= \widetilde{\mathrm{ev}}_+ \longmapsto \underline{a} \cdot (\underline{x} - \underline{x}')$$

$$= \widetilde{\mathrm{ev}}_+ \otimes {}^\dagger\widetilde{\mathrm{ev}}_+ = S^1 \longmapsto (\underline{a} - \underline{a}') \cdot (\underline{x} - \underline{x}')$$

$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} \longmapsto \big[\underline{a} - \underline{a}', \underline{x} - \underline{x}'\big]$$



$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} \circ \mathrm{coev}_{\widetilde{\mathrm{ev}}_+} = S^2 \longmapsto \mathbb{C}[\underline{a}, \underline{x}]$$

$$\Sigma_g \longmapsto \mathbb{C}[\underline{a}, \underline{x}] \otimes \big(\mathbb{C} \oplus \mathbb{C}[1]\big)^{\otimes 2ng}$$

Brunner/Carqueville/Roggenkamp 2022

Truncated affine Rozansky–Witten models

Theorem. Every $\underline{x} \equiv (x_1, \dots, x_n) \in \mathcal{C}$ gives unique extended TQFT:

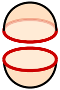
$$\mathrm{Bord}_{2,1,0}^{\mathrm{or}} \longrightarrow \mathcal{C}$$

$$+ \longmapsto (x_1, \dots, x_n)$$

$$= \widetilde{\mathrm{ev}}_+ \longmapsto \underline{a} \cdot (\underline{x} - \underline{x}')$$

$$= \widetilde{\mathrm{ev}}_+ \otimes {}^{\dagger}\widetilde{\mathrm{ev}}_+ = S^1 \longmapsto (\underline{a} - \underline{a}') \cdot (\underline{x} - \underline{x}')$$

$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} \longmapsto \left[\underline{a} - \underline{a}', \underline{x} - \underline{x}'\right]$$



$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} \circ \mathrm{coev}_{\widetilde{\mathrm{ev}}_+} = S^2 \longmapsto \mathbb{C}[\underline{a}, \underline{x}]$$

$$\Sigma_g \longmapsto \mathbb{C}[\underline{a}, \underline{x}] \otimes \big(\mathbb{C} \oplus \mathbb{C}[1]\big)^{\otimes 2ng}$$

($\lambda = I_{1_{\underline{x}}}$ and $\lambda = I_{1_{\underline{x}}}[1]$ give equivalent TQFTs.)

obtain Rozansky–Witten **state spaces** from extended TQFT

Brunner/Carqueville/Roggenkamp 2022

Further directions

Option 1. $\mathcal{C}$ symmetric monoidal $(\infty, 2)$-category
$\Longrightarrow$ obtain **mapping class group** representations (wip)

Bartlett/Douglas/Schommer-Pries/Vicary 2015, Müller/Woike 2022, Carqueville/Yang

Further directions

Option 1. $\mathcal{C}$ symmetric monoidal $(\infty, 2)$-category
$\Longrightarrow$ obtain **mapping class group** representations (wip)

Option 2.

- Encorporate **flavour and R-charge** into new 2-category $\mathcal{C}^{\mathrm{gr}}$:
- Every $\underline{x} \in \mathcal{C}^{\mathrm{gr}}$ fully dualisable, $S_{\underline{x}}$ trivialisable.
- Get extended TQFT $\mathcal{Z}_n^{\mathrm{gr}} \colon \mathrm{Bord}_{2,1,0}^{\mathrm{or}} \longrightarrow \mathcal{C}^{\mathrm{gr}}$ with (✓)

$$\mathcal{Z}_n^{\mathrm{gr}}(\Sigma_g) = \Big(\big(\mathbb{C} \oplus \mathbb{C}[1]_{\{0,\,1\}} \big)^{\otimes n} \otimes \big(\mathbb{C} \oplus \mathbb{C}[1]_{\{0,\,-1\}} \big)^{\otimes n}{}_{\{1,\,0\}} \Big)^{\otimes g} \otimes \mathbb{C}[\underline{a}, \underline{x}]_{\{-1,\,0\}}$$

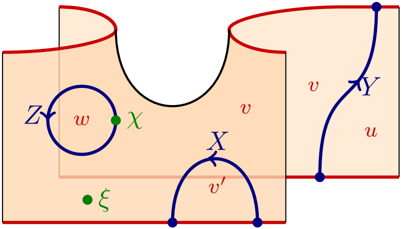
Option 3.
Construction for target $T^*\mathbb{CP}^{n-1}$ via **$U(1)$-equivariantisation**... (✓ wip)

Option 4.
Consider all Rozansky–Witten models with compact target (?)

Option 5.
Construct **extended defect TQFT** (✓)

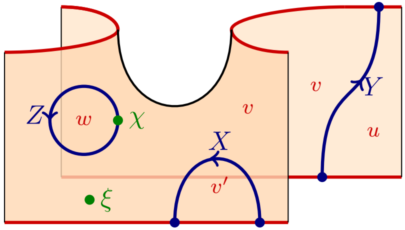
Brunner/Carqueville/Roggenkamp 2022, Brunner/Carqueville/Fragkos/Roggenkamp 2023

Extended defect TQFTs



is 2-morphism in $\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})$

Extended defect TQFTs

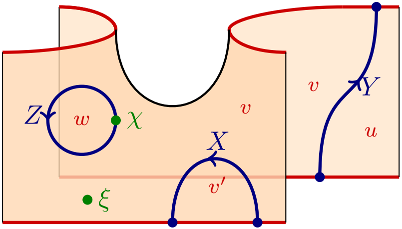


is 2-morphism in $\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})$

Oriented **cobordism hypothesis with defects** in 2d (explicit version):

$$\mathrm{Fun}^{\text{sym. mon.}}\Big(\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})\,,\,\mathcal{B}\Big) \mathrel{\widehat{=}} \begin{pmatrix}\text{graphical calculus in}\\ \text{pivotal subcategory of } \mathcal{B}^{\mathrm{fd}}\end{pmatrix}$$

Extended defect TQFTs



is 2-morphism in $\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})$

Oriented **cobordism hypothesis with defects** in 2d (explicit version):

$$\mathrm{Fun}^{\mathrm{sym.\,mon.}}\Big(\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})\,,\,\mathcal{B}\Big) \mathrel{\widehat{=}} \begin{pmatrix}\text{graphical calculus in} \\ \text{pivotal subcategory of } \mathcal{B}^{\mathrm{fd}}\end{pmatrix}$$

Theorem. $\mathcal{C} = \mathrm{Ho}_2(\mathcal{RW}^{\mathrm{aff}}) = \mathrm{Ho}_2(\mathcal{RW}^{\mathrm{aff}})^{\mathrm{fd}}$ is pivotal.

Applications:

– boundary conditions

– state spaces with defects

– implement group actions, orbifolds

– "turn on background connection"

Brunner/Carqueville/Fragkos/Roggenkamp 2023, Lurie 2009

Summary

Theorem.
Affine **Landau–Ginzburg models** give spin extended TQFTs

$$\mathrm{Bord}_{2,1,0}^{\mathrm{spin}} \longrightarrow \mathcal{LG}$$

$$+ \longmapsto W$$

 $\longmapsto \mathrm{Jac}_W$

 $\longmapsto \mathrm{Res}\Big[\frac{(-)\,\mathrm{d}x}{\partial_{x_1}W \dots \partial_{x_n}W}\Big]$

Theorem.
Affine **Rozansky–Witten models** give extended defect TQFTs

$$\mathrm{Bord}_{2,1,0}^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathcal{C} = \mathrm{Ho}_2(\mathcal{RW}^{\mathrm{aff}})$$

$$+ \longmapsto \underline{x} = (x_1, \dots, x_n)$$

$$S^1 \longmapsto (\underline{a} - \underline{a}') \cdot (\underline{x} - \underline{x}')$$

$$\Sigma_g \longmapsto \mathbb{C}[\underline{a}, \underline{x}] \otimes \big(\mathbb{C} \oplus \mathbb{C}[1]\big)^{\otimes 2ng}$$

Carqueville/Montiel Motoya 2018, Carqueville/Szegedy 2021, Brunner/Carqueville/Fragkos/Roggenkamp 2022