On the Logic of Bunched Implications

- and its relation to separation logic

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Abstract

We study propositional and predicate logic of bunched implications (BI), a new substructural logic, and clarify the presentations of the semantics given so far. In particular we give an elaborate description of Day's construction for presheaves as well as for Grothendieck sheaves. We present a notion of predicate BI and show how it can be modeled soundly and completely by BI-hyperdoctrines. Furthermore we show how separation logic fits into this notion of predicate BI.

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Chapter 1

Introduction

The purpose of this Master's Thesis is to study propositional and predicate logic of bunched implications (BI), a new substructural logic, and clarify the presentations of the semantics given so far. In particular Day's construction, which gives rise to a class of models of BI, will be studied in detail. Furthermore, it is the aim to study the relation between separation logic and BI.

We assume that the reader is familiar with basic category theory and some categorical logic.

It is well known how intuitionistic logic can be modeled in a topos by what is called a subobject semantics. This semantics can also be defined in terms of a forcing relation known as Kripke-Joyal semantics. The (propositional) logic of bunched implications is intuitionistic logic equipped with two new connectives and logical rules for these. Consequently, it is natural to ask how the semantics for BI fits into these frameworks. This question is partly answered in [Yan02] and in [Pym02], which provide a Kripke semantics for BI in presheaves, sheaves over topological spaces, and in one specific Grothendieck sheaf category. In this thesis we will explore the subobject semantics for BI and its relation to the Kripke semantics for BI. We will also clarify the link between BI and separation logic, in particular in what sense separation logic is a predicate BI logic.

David Pym's monograph: "The Semantics and Proof Theory of the Logic of Bunched Implications", [Pym02] contains a suggestion of a proof theory and a Kripke semantics for predicate BI, which, in David Pym's own words is sketchy. Since it is also the only attempt to define predicate BI, which has been made so far, we believe that this is a subject which deserves some attention. One object for this thesis has thus been to give a more precise presentation of Pym's predicate BI and to generalize the Kripke semantics that is given for it. This turned out to be much more problematic than expected. The author and her supervisors have spent quite some time trying to understand this sketchy proof theory and David Pym himself has not been able to help us on this matter. We believe that we have clarified some of it, and this clarification reveals some serious problems. In particular we found examples showing that the sequent calculus does not preserve well-formed sequents. An in depth examination of these problems can be found in Appendix A.

We now briefly present two examples to illustrate the motivation for the logic of bunched implications.

Resources. The logic of bunched implications (BI) belongs to a family of logics known as *substructural logics.* A substructural logic is a logic which lacks one or more of the so called structural rules, which are rules such as

$$\frac{\Gamma \vdash q}{\Gamma, p \vdash q}\; Weakening \qquad \frac{\Gamma, p, p \vdash q}{\Gamma, p \vdash q}\; Contraction.$$

Weakening means that whenever some proposition q is provable under the assumptions of Γ, then q is also provable if we make additional assumptions. Contraction says that it does not matter how many times we make the same assumption. A logic that does not allow these rules is sensitive to the number of times an assumption is used – it is resource sensitive.

A particularly famous substructural logic is Girard's linear logic, which BI resembles. They both have an additive part $(\vee, \wedge, \rightarrow, \top, \bot)$, which is intuitionistic logic and a multiplicative part $(*, -\!\!*, I)$, which is substructural. The essential difference between the two logics is the modal operator "!" in linear logic, which takes a formula ϕ and makes as many copies of it as we want $!\phi$. Intuitionistic implication can be defined in terms of this operator as $\phi \rightarrow \psi :=!\phi -\!\!* \psi$. The operator ! is interpreted as a functor, which is not present in all models of BI (this is shown in [Pym02] and [Yan02]), showing that the two logics are indeed different. Consider the following example, inspired by [Amb91], that illustrates the difference between the additive conjunction, $\wedge$, and the multiplicative conjunction $*$: Let p, q, r denote the following propositions:

p = to have €1
q = to have a packet of Camels
r = to have a packet of Marlboro

A proof of the implication $p \rightarrow q$ is given by the possibility of spending €1 and buying a packet of Camels. A proof of the multiplicative implication $p -\!\!* q$ is given by actually spending €1 and buying a packet of Camels. The proposition $(q \wedge r)$ represents *the possibility of having a packet of Camels* and *the possibility of having a packet of Marlboro* whereas $(q * r)$ represents *actually having both packets at the same time.* One euro is enough to buy a packet of Camels and it is also enough to buy a packet of Marlboro, so $p \rightarrow q \wedge r$ is provable. But €1 is not enough to buy both a packet of Camels and a packet of Marlboro so $p \rightarrow q*r$ is not provable. Instead we have $p * p \rightarrow q * r$, since $(p * p)$ means to have €1 and to have (another) €1. Another way to put it is to say that in $q * r$, q and r have access to disjoint resources. The multiplicative implication $-\!\!*$ can also be given a resource interpretation: $a -\!\!* b$ is a function that has access to disjoint resources from its argument, for example the following is provable

$$p \vdash p -\!\!* (q * r),$$

Since if we have €1, we can make the functions argument p true. If we have another €1, we can show that $-\!\!*$ is a function going from p to $(q * r)$ (or equivalently that $p -\!\!* (q * r)$ holds). The combined resources p and p makes $(q * r)$ true. The additive implication $a \rightarrow b$ is a function that have access to the same resources as its argument.

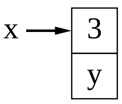
Separation logic. Separation logic, which is a very active research field, provides a good example of how BI can be used to reason about resources in systems. Separation logic is used for reasoning about low-level programs that use shared mutable data structure, and the resources in consideration are memory cells. Separation logic illustrates how the intuitionistic part of BI considers stable truth, whereas the substructural part depends on the internal state of a dynamic system. The following is from a paper by John C. Reynolds [Rey02].

> The use of shared mutable data structures, i.e., of structures where an updatable field can be referenced from more than one point, is widespread in areas as diverse as systems programming and artificial intelligence. Approaches to reasoning about this technique have been studied for three decades, but the results has been methods that suffer from either limited applicability or extreme complexity [...].

Separation logic uses the notions *heap* and *store*. One can think of the store as a description of the contents of registers and the heap as a description of the contents of an (active) addressable memory. Values are integers and we require Addresses $\subseteq$ Values. A heap is a finite, partial function h from Addresses to Values $\times$ Values. And $\text{Store}_V : V \rightarrow \text{Values}$, where V is a finite set of variables. The basic predicate is the "points-to" relation, which has the form $x \mapsto v, w$, where x is a variable in V and $v, w \in \text{Values} \times \text{Values}$, asserting that the heap contains only one active cell, at address $\text{Store}_V(x)$ (i.e., $\text{dom}(h) = \{\text{Store}_V(x)\}$) with contents v, w. We give a few illustrative examples:

1. $x \mapsto 3, y$ asserts that x points to an adjacent pair of cells containing 3 and y (i.e., the store maps x and y into some values α and β, α is an address, and the heap maps α into 3 and $\alpha + 1$ into β).

1.



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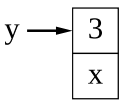

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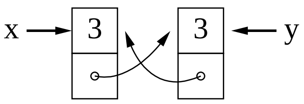

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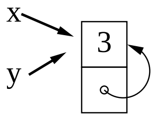

2. $y \mapsto 3, x$ asserts that y points to an adjacent pair of cells containing 3 and x.

3. $x \mapsto 3, y * y \mapsto 3, x$ asserts that situations (1) and (2) hold for separate parts of the heap.

4. $x \mapsto 3, y \wedge y \mapsto 3, x$ asserts that situations (1) and (2) hold for the same heap, which can only happen if the values for x and y are the same (because the heap has only one active cell).

Traditionally, aliasing complicates reasoning about low-level imperative programs, because changing the value of a single memory cell may affect the values of many syntactically unrelated expressions. In separation logic we solve this problem by reasoning locally about the contents of the store. To reason about how programs affect the memory we use a Hoare logic: For predicates p, q and a command c, $\{p\}c\{q\}$ reads: if p holds before c is executed, and if c terminates, then q holds after c has been executed. For example

$$\{x \mapsto 3\}[x] := 4\{x \mapsto 4\}.$$

Since this sort of reasoning has a local nature we need a rule of the kind

$$\frac{\{p\}c\{q\}}{\{p \wedge r\}c\{q \wedge r\}}$$

where no variable occurring free in r is modified by c. However, this rule is not sound for separation logic. For example, the conclusion of the instance

$$\frac{\{x \mapsto -\}[x] := 4\{x \mapsto 4\}}{\{x \mapsto - \wedge y \mapsto 3\}[x] := 4\{x \mapsto 4 \wedge y \mapsto 3\}}$$

is not valid, since its precondition does not preclude the aliasing that will occur if $x = y$. Using the multiplicative conjunction instead gives the *frame rule*

$$\frac{\{p\}c\{q\}}{\{p * r\}c\{q * r\}}$$

where no variable occurring free in r is modified by c. This rule is sound for separation logic. John Reynolds puts it this way in [Rey02]:

> By using the frame rule, one can extend a local specification, involving only the variables and *part of the heap* that are actually used by c (...), by adding arbitrary predicates about variables and parts of the heap that are not modified or mutated by c.

1.1 Overview and contributions

This section contains a detailed overview of the contents of each chapter. The chapters 2-4 contains a thorough presentation of categorical concepts which will be needed subsequently. Chapter 5 considers various notions of models and semantics for propositional intuitionistic logic including soundness and completeness results. Chapter 6 has the same contents as chapter 5, only now the logic has been extended to BI. Chapter 7 gives, jointly with Lars Birkedal and Noah Torp-Smith, a presentation of the separation logic in the frame of first order hyperdoctrines. Finally, the Appendix A contains a discussion and criticism of an attempt to define predicate BI.

The seven chapters should be readable by any graduate student who is familiar with basic category theory and has some knowledge of mathematical logic. To understand what is going on in Appendix A one has to be acquainted with David Pym's monograph: "The Semantics and Proof Theory of the Logic of Bunched Implications", [Pym02].

Chapter 2 This chapter provides a brief presentation of standard results of topos theory, in particular regarding presheaf and Grothendieck sheaf categories, which are used subsequently.

Only proofs and examples that are considered relevant or instructive are included. More details, proofs and examples are available in the literature.

Literature: [MLM94], [LS86], [ML98], [Oos], [Win01].

Chapter 3 This is a detailed introduction to dinatural transformations, ends and coends, including main results of this subject such as Fubini Theorem and Density. There are no new results, but the content of this chapter is less standard. The chapter is mainly based on [Win01] which contains a good introduction to ends and coends.

Literature: [Win01], [ML98].

Chapter 4 In Section 4.1 we define symmetric monoidal closed categories and doubly closed categories. Section 4.2 contains an elaborate (and original) description of Day's construction including original proofs of the following observations:

- The Yoneda embedding preserves the monoidal closed structure of Day's construction, and
- Day's tensor product does not preserve monos.

Section 4.3 explores Day's construction in the category of sheaves, and provides several new contributions some of which contradict earlier claims stated in [Pym02] and [Yan02]. In particular it is shown that Day's construction for presheaves induces a monoidal structure on the subcategory of sheaves, and that this tensor product does not in general have a right adjoint in the category of sheaves, i.e., Day's construction for presheaves does not in general induce a doubly closed structure on the subcategory of sheaves. It is then shown that under certain conditions we do get a doubly closed structure on the category of sheaves.

The new result is negative in the sense that the doubly closed structure which ensure that propositional BI can be soundly interpreted is not always present in the category of sheaves. However, the next Section 4.4 presents results that are adequate to repair this flaw, namely that for certain sheaf categories, the structure of Sub(1) is a BI algebra. All results of sections 4.3 and 4.4 are new.

Literature: [Pym02] and [Yan02].

Chapter 5 We present standard soundness and completeness results for propositional intuitionistic logic. The purpose of treating the intuitionistic and substructural parts separately is to point out the fact that Grothendieck topologies are needed in order to get a completeness result for intuitionistic propositional logic, and also to clarify how standard methods can be used to show soundness and completeness for propositional BI.

Section 5.1 presents the notion of algebraic models for propositional intuitionistic logic and soundness and completeness results for these. Section 5.2 presents two notions of categorical models for propositional intuitionistic logic: a model of provability (also known as subobject semantics) specialized to propositional logic, and a model of proofs (propositions as types) specialized to propositional logic. The latter is included because this kind of model is used in the presentation of propositional BI given in [Yan02] and in [Pym02] which we would like to comment on, since one aim of this thesis is to clarify that presentation.

For the categorical models of provability we give a completeness result, and in section 5.3 we develop Kripke semantics for different classes of the models presented in 5.2.

Literature: [LS86], [MLM94]

Chapter 6 In this chapter we give an original presentation of soundness and completeness results for propositional BI.

The chapter is built up exactly like the previous chapter in order to clarify how the standard ideas presented in chapter 5 can be used to derive similar results for propositional BI.

Section 6.1 presents the notion of algebraic models for propositional BI and soundness and completeness results for these following the presentation given in [Yan02]. Section 6.2 presents two notions of categorical models of propositional BI and completeness results for these. We conclude that a completeness result for the substructural or multiplicative part of the logic is obtained by the observation made in chapter 4 that the Yoneda embedding preserves the monoidal closed structure. In particular we get completeness for the multiplicative part alone without the use of sheaves. Sheaves are necessary only because the Yoneda embedding does not preserve $\vee$ and $\perp$ for presheaves. In Section 6.3 we derive a Kripke semantics for BI which is more general than the ones presented in [Yan02] and [Pym02]. We also derive Kripke semantics for the more specific classes of models living in presheaf and sheaf categories, and conclude that these correspond to those of [Yan02] and [Pym02].

In [Yan02] and [Pym02] the Kripke semantics is used to define the models and completeness is shown for these directly. The presentation we give of propositional BI uses an approach that differs from this since we use the subobject semantics to define the notion of a model and show soundness and completeness for these; Kripke semantics (and completeness for this) is then derived. This approach follows the lines of [LS86] and [MLM94] and has the advantage that the role of the Yoneda embedding becomes clearer. It also shows how the Kripke semantics is actually calculated using Day's construction on subobjects of 1, (so one does not even need any bright or lucky ideas to cook up a Kripke semantics for the new connectives).

Literature: [LS86], [MLM94],[Yan02] and [Pym02]

Chapter 7 This chapter presents joint work with Lars Birkedal and Noah Torp-Smith. We first recall the notion of first order hyperdoctrines which models first order predicate logic. Then we define first order BI-hyperdoctrines and show that they model first order predicate BI. It should be noted that this is not predicate BI in the sense that Pym has suggested in [Pym02] (in particular we do not have the new quantifiers). Finally, and most importantly, we show how separation logic can be viewed as predicate BI with a particular signature, and the pointer model as an interpretation of separation logic in a particular kind of BI-hyperdoctrine.

Literature: [Pit02] and [Yan02].

Appendix A This appendix contains some clarifications and comments to the part of David Pym's monograph [Pym02] that treats predicate BI and it will only make sense to those who are acquainted with this.

The main conclusion of the appendix is that we have not been able to understand the proof theory and semantics of predicate BI suggested by Pym in [Pym02]. In particular we provide examples that shows that the deduction system does not preserve well-formed sequents.

Literature: [Pym02] and [Amb91].

Summary of contributions

- Construction of a monoidal tensor product on any Grothendieck sheaf category using Day's construction and the associated sheaf functor.

- A counter example showing that the monoidal tensor product on sheaves over topological spaces does not in general have a right adjoint. Thus contradicting [Yan02] and [Pym02].
- Under certain conditions there is a right adjoint to the tensor for sheaves.
- For a topos $\mathcal{E}$ which is symmetric monoidal closed, $\mathrm{Sub}_{\mathcal{E}}(1)$ is a BI algebra.
- For any Grothendieck topos over a cover preserving monoidal category, Sub(1) in the category of sheaves is a BI algebra (even though the Grothendieck topos is not symmetric monoidal closed).
- An original and more general presentation of results presented in [Yan02] regarding the semantics of propositional BI.
- Jointly with Lars Birkedal and Noah Torp-Smith a presentation of separation logic and the pointer model in the context of BI-hyperdoctrines.
- Discussion and criticism of the presentation of predicate BI given in [Pym02].

In addition this thesis contains

- An elaborate description of Day's construction.
- Proofs of the observations that
  - The Yoneda embedding preserves the monoidal closed structure of Day's construction, and
  - Day's tensor product does not preserve monos.

1.2 Acknowledgments

I would like to thank my supervisor Lars Birkedal who has not only offered excellent guidance and encouragement, but has also introduced me to the enthusiastic and dynamic research environment at ITU. Carsten Butz has been co-supervisor through most of the process, and as such has offered invaluable input. Noah Torp-Smith, who has also been involved throughout the process, has helped me understand the relevance and potential of separation logic. The weekly meetings at ITU with Lars, Carsten and Noah, which were always conducted in a good spirit, have been a great source of inspiration.

In April 2004 I got the chance to meet with Hongseok Yang, who was invited to ITU. I found this meeting very inspiring.

I would also like to thank Marie Bjerrum for always having time to discuss mathematical problems, and for sharing her strong opinions on category theory; hopefully, with time, I will learn to appreciate the French way. Finally, thanks to the people from the lunch- and coffee club in S15, who have had to listen to many of our incomprehensible discussions.

Chapter 2

Toposes

[Literature: [MLM94], [LS86], [Oos] and [Win01].]

In this chapter we introduce some basic definitions and results of topos theory for reference, in particular the categories of presheaves and Grothendieck sheaves will be treated. Many results will be stated without proof since proofs of these are standard in many books on topos theory (see references above).

Definition 2.0.1 (Subobject classifier). *In a category $\mathcal{C}$ with a terminal object 1, a subobject classifier Ω is an object in $\mathcal{C}$ together with an arrow $\top : 1 \to \Omega$ such that for every mono $m : Y \rightarrowtail X$, there is a unique arrow $\chi_m : X \to \Omega$ in $\mathcal{C}$ such that the diagram*

$$\begin{tikzcd}
Y \arrow[r, tail, "m"] \arrow[d, "1_Y"'] & X \arrow[d, "\chi_m"] \\
1 \arrow[r, "\top"'] & \Omega
\end{tikzcd}$$

is a pullback.

The arrow χ_m is called the characteristic morphism of m, this is due to the situation in Set, where $\Omega = \{0, 1\}$, a mono $m : X \rightarrowtail Y$ corresponds to a subset of Y and χ_m is the characteristic morphism of m.

We are now able to define the notion of a topos.

Definition 2.0.2 (Topos). [1] *An (elementary)* topos *is a Cartesian closed category (ccc), which is finitely complete and has a subobject classifier $\Omega, \top : 1 \to \Omega$.*

When Ω exists it is unique up to isomorphism.

Remark 2.0.3. *We now list some basic properties of elementary toposes. Proofs can be found in e.g. [MLM94].*

Fact 1. *A topos has all finite colimits. [MLM94, IV.5]*

Fact 2. *For any object B in a topos $\mathcal{E}$, the slice category $\mathcal{E}/B$ is also a topos. [MLM94, IV.7]*

---

[1] In [MLM94] an elementary topos has a definition that differs from this one, but they are equivalent.

The change-of-base functor *Let $k : A \to B$ be a morphism in a category $\mathcal{C}$ with pullbacks, then k induces a change-of-base or pullback functor k^* between the slice categories*

$$k^* : \mathcal{C}/B \to \mathcal{C}/A$$

by pullback along k.

Fact 3. *For any $k : A \to B$ in a topos $\mathcal{E}$, the change-of-base functor $k^* : \mathcal{E}/B \to \mathcal{E}/A$ has both a left adjoint $\sum_k$, given by composition with k, and a right adjoint $\prod_k$. [MLM94, IV.7]*

Fact 4. *Let $\mathcal{T}$ be a topos and C an object of $\mathcal{T}$, there is an arrow $C \to 0$ iff $C \cong 0$. [MLM94, IV.7]*

Fact 5. *In a topos, every arrow $0 \to B$, where 0 is the initial object is mono. [MLM94, IV.7]*

Fact 6. *Epis are stable under pullback, and pullback preserve coproducts. [MLM94, IV.7]*

2.1 Subobjects

For an object X of a category $\mathcal{C}$, we define a partial order $\mathrm{Sub}(X)$ which can be thought of as the external categorical correspondence of the powerset of a set X. $\mathrm{Sub}(X)$ is defined as follows: Elements are equivalence classes of monos $m : Y \rightarrowtail X$ with codomain X. $m : Y \rightarrowtail X$ and $m' : Y' \rightarrowtail X$ are equivalent if there is an iso $f : Y \to Y'$ such that $m'f = m$. Such an equivalence class is called a *subobject* of X. We define $m \leq m'$ if there is an arrow $f : Y \to Y'$ not necessarily iso (but it will actually always be mono) satisfying $m'f = m$. It is not hard to check that this defines a partial order.

In the category Set we have a canonical choice of representatives, namely the inclusion of the subset to which the subobject corresponds. Such a choice is not available in general. However, in a presheaf category $\widehat{\mathcal{C}}$ we do actually have a correspondence to the inclusion, as we shall see.

Remark 2.1.1. *By usual abuse of language we will often refer to $m : Y \rightarrowtail X$ as a subobject of X, when we really mean the equivalence class that m represents. In categories such as* Set *where we have a canonical choice of representatives we may even identify a subobject of X with just an object $Y \hookrightarrow X$.*

Lemma 2.1.2. *In a category $\mathcal{C}$ with finite limits, each pair of elements of* $\mathrm{Sub}(X)$ *has a greatest lower bound. Moreover,* $\mathrm{Sub}(X)$ *has a largest element.*

Proof: The largest element of $\mathrm{Sub}(X)$ is the identity $\mathrm{id}_X : X \rightarrowtail X$.

If $m : Y \rightarrowtail X$ and $m' : Y' \rightarrowtail X$ are two subobjects of X, then the pullback $Y \wedge Y' \to X$, which is mono, represents the greatest lower bound.

$$\begin{tikzcd}
Y \wedge Y' \arrow[r, tail] \arrow[d] & Y \arrow[d, "m"] \\
Y' \arrow[r, tail, "m'"'] & X
\end{tikzcd}$$

□

The subobject functor Given a topos $\mathcal{E}$,

$$\mathrm{Sub}(-) : \mathcal{E}^{op} \to \mathrm{Set}$$

is functorial. For an arrow $k : A \to B$ in $\mathcal{E}$, $\mathrm{Sub}(k) = k^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$, i.e., the change-of-base functor, where we require that the arrows are monos. The fact that the pullback of a mono is mono makes it well-defined. For each $k : A \to B$ in $\mathcal{E}$ the morphism $k^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ preserves the order.

The following explains the name of the subobject classifier Ω, since it shows how Ω relates to the subobjects in a topos.

Proposition 2.1.3. *In a category with finite limits and a subobject classifier there is an isomorphism*

$$\mathrm{Sub}(X) \cong \mathrm{Hom}(X, \Omega)$$

natural in X. That is, the subobject functor is representable.

2.2 Image factorization

Proposition 2.2.1. *In a topos[2], every arrow $f : X \to Y$ can be factored as $f = me : X \xrightarrow{e} E \xrightarrow{m} Y$ where e is epi and m is mono. The object E is called the image of f and is denoted* $\mathrm{Im}(f)$*. This kind of factorization is known as* image factorization *or* epi-mono factorization*.*

Moreover, for any commuting diagram of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "e'"'] & M \arrow[d, "m'"] \\
E \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

with e epi and m mono there is a unique arrow from E to M making both triangles commute. In particular epi-mono factorizations are unique up to isomorphism.

Proof: In a topos there are two ways to construct the epi-mono factorization.

(i) The kernel pair of f are two arrows $p_0, p_1 : Z \to X$ such that the diagram

$$\begin{tikzcd}
Z \arrow[r, "p_0"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

is pullback. Let e be the coequalizer of p_0, p_1, since $fp_1 = fp_0$ there is a unique $m : E \to Y$ such that $f = me$, as illustrated by the diagram

$$\begin{tikzcd}
& & Y \\
Z \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[ur, "f"] \arrow[r, two heads, "e"'] & E. \arrow[u, dotted, "m"']
\end{tikzcd}$$

We claim without proof that m is mono.

---

[2] The proposition actually holds for regular, categories which have much less structure than toposes.

(ii) Epi-mono factorization can also be constructed dually by taking the cokernel pair of f, which is a pushout diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "x_1"] \\
Y \arrow[r, "x_0"'] & Z.
\end{tikzcd}$$

Let $m : E \rightarrowtail Y$ be the equalizer of x_0, x_1, then, since $x_0 f = x_1 f$ there is a unique arrow $e : X \to E$ such that $f = me$. It can be shown that e is an epi.

We now show that the second statement of the proposition is equivalent to saying that epi-mono factorizations are unique up to isomorphism. (For a proof that epi-mono factorizations are unique up to isomorphism see [Oos].) Suppose that for all commutative diagrams of the form

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "e"'] \arrow[dr, "f"] & M \arrow[d, "m"] \\
E \arrow[r, "b"'] & Y
\end{tikzcd} \tag{2.1}$$

there is a unique arrow $E \to M$ making the triangles commute, now any two epi-mono factorizations will fit into such a diagram, therefore we have arrows $u : E \to M$ and $u' : M \to E$ making the diagram commute, and it easy to see that u and u' are each other's inverses.

On the other hand, suppose we know that epi-mono factorizations are unique up to isomorphism, and we have a commuting diagram like 2.1, then we can factorize a and b into an epi followed by a mono and then use the isomorphism of epi-mono factorizations of f to get the desired arrow. ☐

2.3 The subobject lattice

Definition 2.3.1 (Lattice). *A lattice is a partial order L with a least and a greatest element 0 and 1, and with binary meets $x \wedge y$ and joins $x \vee y$ for all $x, y \in L$.*

Categorically this corresponds to finite products and finite coproducts.

Definition 2.3.2 (Complete lattice). *A lattice is complete when, regarded as a category, it has all (small) limits and colimits.*

Definition 2.3.3 (Heyting algebra). *A Heyting algebra H is a lattice with greatest and least elements in which the meet $a \wedge b$ is residuated, which is to say that there is an implication operator, $\to$, satisfying*

$$a \wedge b \leq c \textit{ iff } a \leq b \to c$$

Categorically this corresponds to a ccc with finite coproducts. Note that for a Heyting algebra the underlying lattice is always distributive, which is to say that for all $a, b, c \in H$,

$$c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b).$$

To see why, just note that saying $\wedge$ is residuated is the same as saying that it has a right adjoint, i.e., $\wedge$ preserves all colimits, which means that it preserves $\vee$.

Definition 2.3.4 (Frame). *A frame is a complete lattice that satisfies the infinite distributive law:*

$$\bigvee_{i\in I}(b \wedge a_i) = b \wedge \bigvee_{i\in I} a_i.$$

Any frame is a Heyting algebra, since we can define the implication $x \to y$ by

$$\bigvee_{l\in L'} l, \qquad \text{where } l \in L' \text{ iff } l \wedge x \leq y.$$

Definition 2.3.5 (Complete Heyting algebra). *A complete Heyting algebra is a Heyting algebra which is complete as a lattice.*

This corresponds categorically to a ccc with all small limits and colimits. It follows that the infinite distributive law holds because $\wedge$ has a right adjoint so it preserves all colimits, i.e., it commutes with $\bigvee$, so a complete Heyting algebra is also a frame.

Definition 2.3.6 (Boolean algebra). *A Boolean algebra is a distributive lattice with elements* 0 *and* 1 *such that every element* x *has a complement* $\neg x$*; thus,* $x \wedge \neg x = 0$ *and* $x \vee \neg x = 1$*.*

Lemma 2.3.7. *In a topos*[3] $\mathcal{E}$ *each pair of elements of* $\mathrm{Sub}(X)$ *has a least upper bound.*

Proof: Let $m : Y \to X$ and $m' : Y' \to X$ be two subobjects of X, $m \vee m' : Y \vee Y' \to X$ is the image factorization of the coproduct arrow $[m, m'] : Y + Y' \to X$:

$$\begin{tikzcd}
Y + Y' \arrow[dr, two heads, "e"] & & Y \arrow[ll] \arrow[dd, tail, "m"] \\
& Y \vee Y' \arrow[dr, tail, "m \vee m'"] & \\
Y' \arrow[uu] \arrow[rr, tail, "m'"'] & & X
\end{tikzcd}$$

Clearly $m \leq m \vee m'$ and $m' \leq m \vee m'$. Suppose there is a subobject $k : Z \rightarrowtail X$ such that $m, m' \leq k$, then by the universal property of the coproduct there is a unique arrow u from $Y + Y'$ to Z, such that

$$\begin{tikzcd}
Y + Y' \arrow[dr, "u"] & & Y \arrow[ll] \arrow[dd, tail, "m"] \\
& Z \arrow[dr, tail, "k"] & \\
Y' \arrow[uu] \arrow[rr, tail, "m'"'] & & X
\end{tikzcd}$$

commutes. This means that ku is a factorization of $[m, m']$ and since $(m \vee m')e$ is the least such factorization, there is an arrow $s : Y \vee Y' \to Z$ such that $ks = m \vee m'$ which shows that $m \vee m' \leq k$. ☐

Combined with Lemma 2.1.2 this shows that in a topos, for every object X, the partial order $\mathrm{Sub}(X)$ is a lattice, with least element $0 \rightarrowtail X$, which is always mono in a topos.

---

[3] The proposition holds even for coherent categories.

Corollary 2.3.8. *In a topos $\mathcal{E}$ which has all small limits and colimits, the subobject lattices are complete.*

Proof: Least upper bounds and greatest lower bounds over arbitrary index sets are constructed be means of limits and colimits. ☐

Lemma 2.3.9. *In any topos $\mathcal{E}$,* $\text{Sub}_{\mathcal{E}}(1)$ *is a Heyting algebra.*

Proof: We have already shown that Sub(1) is a lattice. It remains to be shown that it has implications. Let $U \rightarrowtail 1, V \rightarrowtail 1$ be subobjects of 1. The topos is ccc so it has exponentials $(-)^U$ and these preserve limits since exponentiation is right adjoint to product. Therefore the arrow

$$(V^U) \overset{(1_V)^U}{\rightarrowtail} (1^U) \cong 1$$

is mono and $(1^U) \cong 1$ follows from 1 being a limit and $(-)^U$ preserves limits, so 1^U must also be the terminal object. ☐

Remark 2.3.10. *For future reference we record that for any topos $\mathcal{E}$, and $A \rightarrowtail 1, B \rightarrowtail 1$ in* Sub(1)*, the Heyting structure on* $\text{Sub}_{\mathcal{E}}(1)$ *is given by:*

$\top$	$=$	id_1	
$\bot$	$=$	$0 \rightarrowtail 1$	*the unique arrow from the initial object.*
$A \wedge B$	$=$	$A \times_{\mathcal{E}} B$	*the pullback of A and B (see Lemma 2.1.2).*
$A \vee B$	$=$	$\mathrm{Im}(A + B)$	*see Lemma 2.3.7.*
$A \to B$	$=$	$B^A \rightarrowtail 1$	*see Lemma 2.3.9.*

In fact the following holds:

Theorem 2.3.11. *For any object X in a topos $\mathcal{E}$, the partially ordered set* $\text{Sub}(X)$ *is a Heyting algebra.*

Proof: By Lemma 2.3.9 above, in any topos $\mathcal{E}, \text{Sub}_{\mathcal{E}}(1)$ is a Heyting algebra. Use the identity $\text{Sub}_{\mathcal{E}}(X) \cong \text{Sub}_{\mathcal{E}/X}(1)$ and the fact that $\mathcal{E}/X$ is a topos (by Remark 2.0.3) to conclude that $\text{Sub}_{\mathcal{E}}(X)$ is a Heyting algebra. ☐

2.4 The internal Heyting algebra

The Heyting algebra structure on the subobject lattices Sub(X) in a topos $\mathcal{E}$ induces an internal Heyting algebra[4] on Ω via the isomorphism

$$\text{Sub}(X) \cong \mathcal{E}(X, \Omega).$$

For example to define the arrow $\wedge : \Omega \times \Omega \to \Omega$ we consider the commutative diagram

$$\begin{tikzcd}
\text{Sub}(X) \times \text{Sub}(X) \arrow[r, "\cap"] \arrow[d, equal, "\sim"] & \text{Sub}(X) \arrow[dd, "\sim"] \\
\text{Hom}(X,\Omega) \times \text{Hom}(X,\Omega) \arrow[d, equal, "\sim"] & \\
\text{Hom}(X,\Omega \times \Omega) \arrow[r, "\wedge_X"'] & \text{Hom}(X,\Omega)
\end{tikzcd}$$

---

[4] The exact definition can be found in [MLM94]

natural in X. Put $X = \Omega \times \Omega$ and follow the identity round the diagram to get the arrow

$$\wedge : \Omega \times \Omega \to \Omega.$$

By the Yoneda Lemma, which we show in the next section,

$$\widehat{\mathcal{E}}(\mathbf{y}(\Omega \times \Omega), \mathbf{y}(\Omega)) \cong \mathbf{y}(\Omega)(\Omega \times \Omega) = \mathcal{E}(\Omega \times \Omega, \Omega),$$

so the natural transformation $(\wedge_X)_{X \in \mathcal{E}}$ is uniquely determined by the arrow $\wedge \in \mathcal{E}(\Omega \times \Omega, \Omega)$ by composition:

$$\begin{tikzcd}
X \arrow[d, "f"] & & X \arrow[d, "f"] \\
\Omega \times \Omega & \overset{\wedge_X}{\mapsto} & \Omega \times \Omega \arrow[d, "\wedge"] \\
 & & \Omega.
\end{tikzcd}$$

Given subobjects $a \rightarrowtail X, b \rightarrowtail X$ with characteristic maps χ_a, χ_b, this implies that the characteristic map for $a \cap b \rightarrowtail x$, $\chi_{a \cap b}$ is the arrow $\wedge \circ \langle \chi_a, \chi_b \rangle$. The latter is often written $\chi_a \wedge \chi_b$.

In a similar fashion one can derive arrows $\vee, \rightarrow : \Omega \times \Omega \to \Omega$. The arrows $\top, \bot : 1 \to \Omega$ are induced by the top element $\mathrm{id}_1 : 1 \to 1$ of Sub(1) and the bottom element $0 \rightarrowtail 1$ of Sub(1), which is mono in a topos.

2.5 Presheaves

Definition 2.5.1 (Locally Small). *A category $\mathcal{C}$ is called* locally small *if for all objects X, Y in $\mathcal{C}$, the collection $\mathcal{C}(X, Y)$ is a set.*

Definition 2.5.2 (Small). *A locally small category is* small *if the collection of objects is a set.*

Given a small category $\mathcal{C}$, a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ is called a presheaf. The functor category $\mathrm{Set}^{\mathcal{C}^{op}}$ which is also denoted $\widehat{\mathcal{C}}$, has presheaves as objects and natural transformations as arrows. That is, a map $\alpha : F \to G$ in $\widehat{\mathcal{C}}$ is a family of maps $\langle \alpha_C \rangle_{C \in \mathrm{Obj}(\mathcal{C})}$ such that for each arrow $f : C \to D$

$$\begin{tikzcd}
FD \arrow[r, "\alpha_D"] \arrow[d, "F(f)"'] & GD \arrow[d, "G(f)"] \\
FC \arrow[r, "\alpha_C"'] & GC
\end{tikzcd}$$

commutes.

Lemma 2.5.3. *If $\mathcal{C}$ is small then $\widehat{\mathcal{C}}$ is locally small.*

Let $(M, \leq)$ be a preorder (i.e., reflexive and transitive), $(M, \leq)$ defines a category $\mathcal{M}$ with the elements of M as objects and with $\mathrm{Hom}(m, n) = \{*\}$ if $m \leq n$, $\mathrm{Hom}(m, n) = \emptyset$ otherwise. A presheaf F over $\mathcal{M}$ can then be viewed as an M-indexed family of sets $\langle F(m) \rangle_{m \in M}$ such that for each $m \leq n$ there is a map $F_{mn} : F(n) \to F(m)$. Satisfying the functor laws

$$F_{nn} = \mathrm{id}_{F(n)}, \quad F_{mn}F_{nk} = F_{mk}$$

for all $m \leq n \leq k$ in M, that is, the diagram

$$\begin{tikzcd}
F(k) \arrow[dd, "F_{mk}"'] \arrow[dr, "F_{nk}"] & \\
& F(n) \arrow[dl, "F_{mn}"] \\
F(m) &
\end{tikzcd}$$

commutes.

2.5.1 Yoneda Lemma

Given a locally small category $\mathcal{C}$ [5], we can define the Yoneda functor $\mathbf{y} : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{op}}$ which plays a central role in category theory. It is defined as follows

$$\begin{tikzcd}
C \arrow[dd, "g"] & & \mathbf{y}C = \mathcal{C}(-, C) \arrow[dd, "\mathbf{y}g=\langle g\circ -\rangle_{C\in\mathcal{C}}"] \\
& \longmapsto & \\
D & & \mathbf{y}D = \mathcal{C}(-, D).
\end{tikzcd}$$

There is also a contravariant version $\mathbf{y}^\circ : \mathcal{C}^{op} \to \mathrm{Set}^{\mathcal{C}}$ of the Yoneda functor, which works as follows

$$\begin{tikzcd}
C \arrow[dd, "g"] & & \mathbf{y}^\circ C = \mathcal{C}(C, -) \\
& \longmapsto & \\
D & & \mathbf{y}^\circ D = \mathcal{C}(D, -). \arrow[uu, "\mathbf{y}^\circ(g)=\langle -\circ g\rangle_{C\in\mathcal{C}}"']
\end{tikzcd}$$

Theorem 2.5.4 (Yoneda Lemma). *For $C \in \mathcal{C}$ and $F \in \widehat{\mathcal{C}}$ there is an isomorphism*

$$\widehat{\mathcal{C}}(\mathbf{y}C, F) \overset{\theta_{C,F}}{\cong} FC$$

natural in C and F.

The theorem states that there is a bijection between natural transformations from $\mathbf{y}C$ to F and elements of the set FC. Let $\alpha : \mathbf{y}C \to F$ be a natural transformation, then $\theta_{C,F}$ sends it to the element

$$\check{\alpha} := \alpha_C(\mathrm{id}_C) \in FC.$$

On the other hand if $x \in FC$ then we get a corresponding natural transformation $\hat{x} : \mathbf{y}C \to F$ defined by

$$\hat{x}_D = F(-)(x)$$

which is easily shown to be a natural transformation. The $\hat{-}$ and the $\check{-}$ operations are each others inverses. There is a dual result for the contravariant Yoneda functor:

---

[5] The Yoneda functor and Yoneda Lemma can be generalized to **Class** instead of Set, so that $\mathcal{C}$ does not have to be a small category. For this reason, many authors (including this one) allow themselves to be sloppy with regard to this requirement.

Theorem 2.5.5 (Contravariant Yoneda Lemma). *For $C \in \mathcal{C}$ end $F \in \mathrm{Set}^{\mathcal{C}}$, there is an isomorphism*

$$\mathrm{Set}^{\mathcal{C}}(\mathbf{y}^\circ C, F) \overset{\theta_{C,F}}{\cong} FC$$

natural in C and F.

The isomorphism θ is defined in the exact same way as for the covariant case.

Corollary 2.5.6. *The Yoneda functor is full and faithful, i.e., given $C, D \in \mathcal{C}$, the Yoneda functor defines a bijection between $\mathcal{C}(C, D)$ and $\widehat{\mathcal{C}}(\mathbf{y}C, \mathbf{y}D)$.*

Proposition 2.5.7. *For $C, D \in \mathcal{C}$ we have $C \cong D$ in $\mathcal{C}$ iff $\mathbf{y}C \cong \mathbf{y}D$ in $\widehat{\mathcal{C}}$.*

And dually:

Proposition 2.5.8. *For $C, D \in \mathcal{C}$ we have $C \cong D$ in $\mathcal{C}$ iff $\mathbf{y}^\circ C \cong \mathbf{y}^\circ D$ in $\widehat{\mathcal{C}}$.*

Definition 2.5.9 (Representables). *A representation for a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ consists of an object $C \in \mathcal{C}$ and a natural isomorphism*

$$\mathbf{y}C = \mathcal{C}(-, C) \cong F.$$

If there exists a representation for F we say that F is representable.

If $G : \mathcal{C} \to \mathrm{Set}$ is a covariant functor, we have $\mathcal{C} = \mathcal{C}^{op^{op}}$ so a representation for G is an object C together with an iso such that

$$\mathcal{C}^{op}(-, C) \cong G$$

which is the same as saying

$$\mathcal{C}(C, -) \cong G$$

In other words, when G is covariant a representation (also called a *corepresentation*) consists of an object $C \in \mathcal{C}$ and a natural isomorphism

$$\mathbf{y}^\circ C = \mathcal{C}(C, -) \cong G.$$

2.5.2 Limits and colimits in functor categories

Limits and colimits are computed pointwise in a presheaf category, that is, if $F : \mathbb{I} \to \widehat{\mathcal{C}}$ is a diagram from a small indexing category $\mathbb{I}$ to $\widehat{\mathcal{C}}$, the limit $\lim_{\mathbb{I}} F$ is given by

$$(\lim_{I \in \mathbb{I}} F(I))(C) = \lim_{I \in \mathbb{I}} (F(I)(C)).$$

Colimits are defined similarly. For example the product of two functors F, G is the functor defined (on objects) by $(F \times G)(C) = FC \times GC$. The terminal object of a presheaf category is the constant functor 1, defined by $1(C) = \{*\}$ for each object $C \in \mathcal{C}$. Since Set has all small limits and colimits, and the limits and colimits of $\widehat{\mathcal{C}}$ are computed in Set, we get the following

Proposition 2.5.10. *The category $\widehat{\mathcal{C}}$ has all small limits and colimits.*

The category $\widehat{\mathcal{C}}$ also has exponentials (i.e., a right adjoint to the product functor), using the Yoneda Lemma we find that for $F, G : \mathcal{C}^{op} \to \mathrm{Set}$ exponentiation must satisfy

$$G^F(C) \cong \mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathbf{y}C, G^F) \cong \mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathbf{y}C \times F, G),$$

so we are led to define

$$G^F(C) = \mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathbf{y}C \times F, G).$$

The unit $\varepsilon_{G,F} : G^F \times F \to G$ is given explicitly by

$$\varepsilon_C(\alpha, c) = \alpha_C(\mathrm{id}_C, c)$$

for $\alpha \in G^F(C), c \in FC$. For $\beta : H \times F \to G$ the transposed $\tilde{\beta} : H \to G^F$ is defined by

$$(\tilde{\beta}_A(a))_C(f, c) = \beta_C(H(f)(a), c)$$

for $f \in \mathrm{Hom}_{\mathcal{C}}(C, A), a \in HA, c \in FC$. It is now routine to verify that $\tilde{\beta}$ is the unique arrow that makes

$$\begin{tikzcd}
H \times F \arrow[rr, "\beta"] \arrow[dr, "\tilde{\beta}\times F"'] & & G \\
& G^F \times F \arrow[ur, "\varepsilon_{G,F}"'] &
\end{tikzcd}$$

commute.

We have shown that

Proposition 2.5.11. *The category $\widehat{\mathcal{C}}$ is Cartesian closed.*

For a presheaf P we define the *category of elements* $\int P$. Objects are pairs (C, p), where $C \in \mathrm{Obj}(\mathcal{C})$ and $p \in PC$. Each arrow $f : D \to C$ in $\mathcal{C}$ and each $p \in PC$ induces a map $(D, p \restriction f) \to (C, p)$, where $p \restriction f$ stands for $P(f)(p)$. There is a canonical functor

$$\mathbf{y} \circ \pi : \int P \to \widehat{\mathcal{C}}$$

that maps (C, p) to $\mathbf{y}(C)$. Using the category of elements as index category we have:

Proposition 2.5.12. *Each presheaf P is the colimit of representables $P \cong \mathrm{colim}_{\int P} \mathbf{y} \circ \pi$.*

Corresponding to the notion of subsets in the category Set we have the notion of subfunctors in the category $\widehat{\mathcal{C}}$.

Definition 2.5.13 (Subfunctor). *A functor $A : \mathcal{C}^{op} \to \mathrm{Set}$ is a* subfunctor *of a functor $B : \mathcal{C}^{op} \to \mathrm{Set}$ iff*

1. *$A(C) \subseteq B(C)$ for all $C \in \mathrm{Obj}(\mathcal{C})$, and*
2. *$A(f) : A(C) \to A(D)$ is the restriction of $B(f)$ to $A(C)$ for any arrow $f : D \to C$.*

It is not hard to see that for each subobject $\alpha : G \to F$ of an object $F \in \widehat{\mathcal{C}}$, there is a unique subfunctor $F' \hookrightarrow F$ that represents it, so $F' \hookrightarrow F$ is a canonical representative.

Definition 2.5.14 (Sieve). [6] *A* sieve *is a collection of arrows S in a category $\mathcal{C}$, closed under right composition, i.e., satisfying: if $h \in S$ and $\operatorname{cod}(g) = \operatorname{dom}(h)$ then $hg \in S$.*

Of particular interest are subfunctors of representables $\mathbf{y}C$. Such a subfunctor can be characterized by a collection of arrows called a *C-sieve*.

Definition 2.5.15 (Sieve). *For an object C of a category $\mathcal{C}$, a C-sieve (or an ideal on C) is a collection S of arrows with codomain C, such that S is a sieve.*

For any object A of a category $\mathcal{C}$ there are two trivial examples of sieves, the empty set and the set of all arrows with codomain A, the latter is called the maximal sieve and is denoted $\mathcal{C}/A$.

If $\mathcal{M}$ is a preorder (viewed as a category) then an m-sieve on $\mathcal{M}$ is a set $S \subseteq \mathcal{M}$ of elements that are all smaller than m and such that if $n \in S$ and $k \leq n$ then $k \in S$. We know that in a preorder a hom-set $\operatorname{Hom}(n, m)$ is either the empty set or a singleton, so given a codomain, we can identify an arrow with its domain.

Lemma 2.5.16. *There is a bijective correspondence between subfunctors of $\mathbf{y}C$ and C-sieves.*

Proof: A subfunctor G of $\mathbf{y}C$ is defined by the sets $GA \subseteq \operatorname{Hom}(A, C)$ for $A \in \mathcal{C}$, and the union of these sets defines a sieve on C. On the other hand, given a C-sieve S, we can define a subfunctor G of $\mathbf{y}C$ by putting $GA = \{f \in S \mid \operatorname{dom}(f) = A\}$. □

Sieves can be used to characterize the subfunctors of the terminal object 1.

Proposition 2.5.17. *There is a bijective correspondence between subfunctors of 1 and sieves on $\mathcal{C}$.*

Proof: Let S be a sieve and consider the set of objects

$$O_S = \{C \in \mathcal{C} \mid C = \operatorname{dom}(f) \text{ and } f \in S\}$$

consisting of all the domains of the arrows from S.

Given a subfunctor $F \rightarrowtail 1$, we define a sieve by

$$\tilde{F} = \{C \to 1 \mid FC = \{*\}\}$$

and from a sieve I we get a subfunctor by

$$\hat{I}(C) = \begin{cases} \{*\} & \text{if } C \in O_S \\ \emptyset & \text{otherwise} \end{cases}$$

□

We have seen in Remark 2.3.10 how to define the Heyting algebra structure of Sub(1) in terms of limits, colimits and exponentials. In view of the proposition above, we now give a definition of the Heyting algebra structure in terms of sieves, or rather the set of objects O_S corresponding to a sieve S. [7]

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[6] This is what an algebraist will usually refer to as an ideal.

[7] This actually corresponds to regarding the category $\mathcal{C}$ as a preorder with $A \leq B$ iff there exists an arrow from A to B. The sieves of a preorder are downwards closed sets.

Corollary 2.5.18. *Suppose* I, J *are sieves represented by the domains of the arrows they contain, then the complete Heyting algebra on* Sub(1) *is defined by*

$$\begin{array}{lll} \top & = & \mathrm{Obj}(\mathcal{C}) \\ \bot & = & \emptyset \\ I \vee J & = & I \cup J \\ I \wedge J & = & I \cap J \\ I \to J & = & \bigcup\{W \mid W \cap I \leq J\}, \end{array}$$

order is inclusion.

Proof: Use the definition of the Heyting algebra structure of Sub(1) given in Remark 2.3.10 together with the correspondence between sieves and subfunctors of 1 and the calculations of limits and colimits in functor categories. For example consider $I \wedge J$. Let $\hat{I}$ and $\hat{J}$ be the corresponding subfuntors of 1, $\hat{I} \wedge \hat{J}$ is the pullback, which is characterized by

$$\hat{I} \wedge \hat{J}(C) = \begin{cases} * & \text{if } * \in \hat{I}(C) \text{ and } * \in \hat{J}(C) \\ \emptyset & \text{otherwise.} \end{cases}$$

We translate this back to a sieve, and get

$$\begin{array}{rcl} I \wedge J = \overline{\hat{\mathrm{I}} \wedge \hat{\mathrm{J}}} & = & \{C \mid C \in I \text{ and } C \in J\} \\ & = & I \cap J. \end{array}$$

□

2.5.3 A category of presheaves is a topos

We have shown that $\widehat{\mathcal{C}}$ is Cartesian closed. To be a topos it must also have a subobject classifier Ω, which we now define.

The requirement $\mathrm{Sub}(F) \cong \mathrm{Hom}(F, \Omega)$ for all $F \in \widehat{\mathcal{C}}$, together with the Yoneda Lemma leads to the following:

$$\Omega(A) \cong \mathrm{Hom}(\mathbf{y}A, \Omega) \cong \mathrm{Sub}(\mathbf{y}A).$$

So we define $\Omega(A)$ to be the set of all subfunctors of $\mathbf{y}A$. By Lemma 2.5.16 this is the same as the set of all A-sieves. For $f : A \to B$, $\Omega(f) : \Omega(B) \to \Omega(A)$ is given by

$$\Omega(f)(S) = \bigcup_{C \in \mathcal{C}} \{g : C \to A \mid fg \in S\}$$

for any B-sieve S. This is clearly an A-sieve.

The natural transformation $\top : \mathbf{1} \to \Omega$ is given by

$$\top_A(*) = \mathcal{C}/A$$

It can be verified that $\mathrm{Sub}(F) \cong \mathrm{Hom}(F, \Omega)$ for all $F \in \widehat{\mathcal{C}}$.

Consider the category $\widehat{\mathcal{M}}$ of presheaves over a preorder $\mathcal{M}$. The subfunctors of $\mathbf{y}m$ are the m-sieves, so if we put $\downarrow(m) = \{n \in \mathcal{M} \mid n \leq m\}$ the definition of Ω becomes

$$\Omega(m) = \{S \subseteq \downarrow(m) \mid \text{ if } n \leq k \text{ and } k \in S \text{ then } n \in S\}$$

and for $m \leq n$,

$$\Omega_{mn}(S) = S \cap \downarrow(m)$$

and

$$\top_m(*) = \downarrow(m).$$

2.6 Grothendieck sheaves

Definition 2.6.1. *A Grothendieck topology on a category $\mathcal{C}$ is a function J which assigns to each object $C \in \mathcal{C}$ a collection $J(C)$ of sieves on C, in such a way that*

1. *the maximal sieve $\mathcal{C}/C = \{f \mid \mathrm{cod}(f) = C\}$ is in $J(C)$:*
2. *(stability) if $S \in J(C)$, then $h^*(S) = \{g \mid \mathrm{cod}(g) = D, hg \in S\} \in J(D)$ for any arrow $h : D \to C$:*
3. *(transitivity) if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C$ in S, then $R \in J(C)$.*

Definition 2.6.2. *A* site *is a pair $(\mathcal{C}, J)$ consisting of a small category $\mathcal{C}$ and a Grothendieck topology J on $\mathcal{C}$. If $S \in J(C)$, then S is called a* covering sieve*.*

Grothendieck topologies generalize topological spaces as the following example shows.

Example 2.6.3 (Topological space). *Let X be a topological space (a set with open subsets specified by $\mathcal{O}(X) \subseteq \mathcal{P}(X)$) and consider the corresponding category: objects are the open subsets of X, that is, the elements of $\mathcal{O}(X)$, there is an arrow $U \to V$ iff $U \subseteq V$. A sieve on U is a family S of open subsets of U with the property that for all $V \in S$ if $V' \subseteq V$ then $V' \in S$ (it is downwards closed). A sieve S on U covers U iff U is the union of the sets in S.*

The maximal sieve on U is the family of all open subsets of U, which covers U since U itself is in the maximal sieve.

Stability: Suppose $S = \{C_i \subseteq C \mid i \in I\}$ is a sieve such that $\bigcup_{i\in I} C_i = C$ and $D \subseteq C$. We must show that $D^*(S) = \{B \in \mathcal{O}(X) \mid B \subseteq D, B \in S\}$ covers D. To see this just note that $D^*(S) = \{C_i \cap D \mid C_i \in S\}$ and $\bigcup_{i\in I}(C_i \cap D) = \bigcup_{i\in I} C_i \cap D = D$.

Transitivity: Let $S = \{C_i \subseteq C \mid i \in I\}$ be a cover of C and $R = \{R_j \subseteq C \mid j \in J\}$ any sieve on C such that for all $C_i \in S$, $\bigcup_{j\in J}(R_j \cap C_i) = C_i$. Then $\bigcup_{j\in J} R_j = \bigcup_{j\in J} R_j \cap C = \bigcup_{j\in J} R_j \cap \bigcup_{i\in I} C_i = \bigcup_{i\in I}\bigcup_{j\in J}(R_j \cap C_i) = \bigcup_{i\in I} C_i = C$.

This is also referred to as the open cover topology.

Recall that when working with topological spaces, it is often enough to consider the open sets of a *basis* for the topology. The notion of a basis can be generalized to Grothendieck topologies if $\mathcal{C}$ is a category with pullbacks.

Definition 2.6.4 (Basis for Grothendieck topology). *A basis for a Grothendieck topology on a category $\mathcal{C}$ with pullbacks is a function K which assigns to each object C a collection $K(C)$ consisting of families of morphisms with codomain C, such that*

(1') *if $f : C' \to C$ is an isomorphism, then $\{f : C' \to C\} \in K(C)$:*

(2') *if $\{f_i : C_i \to C \mid i \in I\} \in K(C)$, then for any morphism $g : D \to C$, the family of pullbacks $\{\pi_2 : C_i \times_C D \to D \mid i \in I\}$ is in $K(D)$;*

(3') *if $\{f_i : C_i \to C \mid i \in I\} \in K(C)$, and if for each $i \in I$ one has a family $\{g_{ij} : D_{ij} \to C_i \mid j \in I_i\} \in K(C_i)$, then the family of composites $\{f_i \circ g_{ij} : D_{ij} \to C \mid i \in I, j \in I_i\}$ is in $K(C)$.*

Note that the families of $K(C)$ are not necessarily sieves but they generate sieves in an obvious way. The notion of a basis can even be defined for categories without pullbacks, by substituting (2') with

(2") If $\{f_i : C_i \to C \mid i \in I\} \in K(C)$, then for any morphism $g : D \to C$ there exists a cover $\{h_j : D_j \to D \mid j \in I'\} \in K(D)$ such that for each j, gh_j factors through some f_i.

Example 2.6.5 (The finite sup topology). *Let H be a distributive lattice regarded as a category, then H can be equipped with a basis for a Grothendieck topology K, given by*

$$\{a_i \mid i \in I\} \in K(c) \text{ iff } \bigvee_{i\in I} a_i = c.$$

where the index set I must be finite.

(1'): To have an iso from c' to c in this category means that $c = c'$, clearly $c \in K(c)$.

(2'): The pullback of two arrows $c_i \leq c$ and $d \leq c$ is $c_i \wedge d \leq c$. Suppose $\bigvee_I c_i = c$ and $d \leq c$, we must show that $\bigvee_I d \wedge c_i = d$. Since $d \leq c$ we have $d \wedge c = d$, now $d \wedge c = d \wedge \bigvee_I c_i = \bigvee_I d \wedge c_i$.

(3'): If $c = \bigvee_I c_i$ and for each $i \in I$, $c_i = \bigvee_J d_{ij}$ then $c = \bigvee_{I,J} d_{ij}$.

Example 2.6.6 (The sup topology). *Let H be a complete Heyting algebra regarded as a category, then H can be equipped with a basis for a Grothendieck topology K, given by*

$$\{a_i \mid i \in I\} \in K(c) \text{ iff } \bigvee_{i\in I} a_i = c,$$

where I is any index set.

Example 2.6.7 (The atomic topology). *Let $\mathcal{C}$ be a category that satisfies: for any two morphisms $f : D \to C$ and $g : E \to C$ with a common codomain C, there exists a commutative square of the form*

$$\begin{tikzcd}
\bullet \arrow[r, dotted] \arrow[d, dotted] & D \arrow[d, "f"] \\
E \arrow[r, "g"'] & C.
\end{tikzcd}$$

This is a condition which is much weaker than existence of pullbacks. The condition is necessary to ensure that the stability axiom is satisfied for the atomic topology, *which is defined by:*

$$S \in J(C) \text{ iff the sieve } S \text{ is nonempty.}$$

As an example consider the category $\mathcal{I}^{op}$, where $\mathcal{I}$ is the category of finite sets and injective functions. This category satisfies the above condition since for $f : D \to C, g : E \to C$ in $\mathcal{I}^{op}$ we must show in $\mathcal{I}$ that there is a commutative square

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, dotted] \\
E \arrow[r, dotted] & \bullet.
\end{tikzcd}$$

Now, f, g are injective so we can put $\bullet = C \uplus (E \setminus \mathrm{Im}(g)) \uplus (D \setminus \mathrm{Im}(f))$ (the disjoint union) and define $f' : D \to \bullet$ by

$$f'(d) = \begin{cases} \text{the unique } (C, c) \text{ such that } f(c) = d & \text{if } d \in \mathrm{Im}(f) \\ (D, d) & \text{otherwise.} \end{cases}$$

$g' : E \to \bullet$ is defined in a similar way. The arrows so defined are injective. Both composites $f'f$ and $g'g$ are the inclusion $c \mapsto (C, c)$, showing that the square commutes. Thus, the atomic topology is well-defined for the category $\mathcal{I}^{op}$. Actually the square we have constructed is a pullback in the category $\mathcal{I}$ (it is also a pushout in the category Set, but not in $\mathcal{I}^{op}$, since the mediating arrow is not always injective). We shall return to this example later.

Sheaves on a site. Let $(\mathcal{C}, J)$ be a site. Consider a presheaf $P : \mathcal{C}^{op} \to \text{Set}$ and a covering sieve $S \in J(C)$.

Definition 2.6.8 (Matching family). *A* matching family *for S of elements of P is a function which assigns to each element $f : D \to C$ of S an element $x_f \in P(D)$, in such a way that*

$$x_f \restriction g = x_{fg} \qquad \text{for all } g : E \to D \text{ in } \mathcal{C}.$$

Here fg is again an element of S, because S is a sieve, and $x_f \restriction g$ stands for $P(g)(x_f)$.

Definition 2.6.9 (Amalgamation). *An* amalgamation *for such a matching family is a single element $x \in P(C)$ with*

$$x \restriction f = x_f \qquad \text{for all } f \in S.$$

Definition 2.6.10 (Sheaf). *A presheaf P is a sheaf (for J) precisely when for every cover S of an object $\mathcal{C}$, every matching family has a unique amalgamation.*

Given a site $(\mathcal{C}, J)$, the category of sheaves over the site is denoted $\text{Sh}(\mathcal{C}, J)$ (arrows are natural transformations), the category of sheaves over a topological space $\mathcal{O}(X)$ is denoted $\text{Sh}(X)$. The category of sheaves over the site $\text{Sh}(\mathcal{C}, J)$ is called the associated Grothendieck topos, and it is a full subcategory of $\widehat{\mathcal{C}}$.

A sieve S on C is the same as a subfunctor of $\mathbf{y}C$, and a matching family is the same as a natural transformation $\alpha : S \to P$. An amalgamation is then a unique extension of α to $\mathbf{y}C$. The condition that P has to satisfy to be a sheaf can then be expressed as: for any object C, subfunctor S of $\mathbf{y}C$ and natural transformation $\alpha : S \to P$, there is a unique natural transformation that makes the diagram

$$\begin{tikzcd}
S \arrow[r, "\alpha"] \arrow[d, hook] & P \\
\mathbf{y}C \arrow[ur, dotted] &
\end{tikzcd}$$

commute.

Remark 2.6.11. *The sheaf condition can also be expressed w.r.t. a basis K: Let $\mathcal{C}$ be a category with pullbacks, and K a basis for a topology on $\mathcal{C}$. If $R = \{f_i : C_i \to C \mid i \in I\}$ is a K-cover of C, a family of elements $x_i \in P(C_i)$ is said to be matching for R iff*

$$x_i \restriction \pi_i = x_j \restriction \pi_j \qquad \text{for all } i, j \in I,$$

where π_i, π_j are the projections from the pullback, as in

$$\begin{tikzcd}
C_i \times_C C_j \arrow[r, "\pi_j"] \arrow[d, "\pi_i"'] & C_j \arrow[d, "f_j"] \\
C_i \arrow[r, "f_i"'] & C.
\end{tikzcd}$$

An amalgamation for $\{x_i\}_I$ *is an element* $x \in P(C)$ *with the property that* $x \upharpoonright f_i = x_i$ *for all* $i \in I$.

Example 2.6.12 (Sheaves over a topological space). *Let* X *be a set with an open cover topology* $\mathcal{O}(X)$ *and let* $F : \mathcal{O}(X)^{op} \to \mathrm{Set}$. *Whenever* $V \subseteq U$ *for open sets* V, U, *there is an arrow* $F_{VU} : FU \to FV$. *For an element* $s \in FU$ *(s is called a* section *over* U*),* $F_{VU}(s)$ *is called the restriction of* s *to* V *and is denoted* $s \upharpoonright_V$. *Given a cover* $\bigcup_i U_i$ *of a set* U *a matching family is a set of pairwise compatible sections* $\{s_i \in F(U_i)\}$ *such that*

$$s_i \upharpoonright_{U_i \cap U_j} = s_j \upharpoonright_{U_i \cap U_j}$$

for all $i, j \in I$. *And an amalgamation is a section over* U, $s \in FU$, *such that* $s \upharpoonright_{U_i} = s_i$ *for all* $i \in I$.

An example of a sheaf over a topological space where the so called restriction really is a restriction map is given by the sheaf of continuous real valued functions on X:

$$F(V) = \mathbf{Cont}(V, \mathbb{R}),$$

which is the set of continuous functions from V *to* $\mathbb{R}$.

The trivial topology, defined by $S \in J(C)$ iff S is the maximal sieve, makes every presheaf a sheaf.

Definition 2.6.13 (Subcanonical). *A Grothendieck topology* J *on a category* $\mathcal{C}$ *is called subcanonical if for every object* C *in* $\mathcal{C}$ *the hom-functor* $\mathbf{y}C$ *is a sheaf.*

Example 2.6.14. *As an example consider again a topological space* X. *For open sets* U, C *we have*

$$\mathbf{y}(C)(U) = \mathrm{Hom}(U, C) = \begin{cases} \{*\} & \text{if } U \subseteq C \\ \emptyset & \text{otherwise.} \end{cases}$$

To show that this is a sheaf suppose $\bigcup_{i \in I} U_i$ *is a cover of* U *and* $\{x_i \in \mathrm{Hom}(U_i, C)\}$ *a matching family (then* $x_i = *$*), that is,* $U_i \subseteq C$ *for all* $i \in I$, *which implies* $U = \bigcup_{i \in I} U_i \subseteq C$, *so that* $* \in \mathrm{Hom}(U, C)$ *which is exactly what we need to have an amalgamation. This shows that the open cover topology is subcanonical.*

Consider the site $(\mathcal{I}^{op}, J)$ where $\mathcal{I}$ is the category of finite sets and injective functions and J the atomic topology. Sheaves $F : \mathcal{I} \to \mathrm{Set}$ over this site are called atomic sheaves over $\mathcal{I}^{op}$.

Proposition 2.6.15. *Let* $\mathcal{I}$ *be the category of finite sets and injective functions. A presheaf* $P : \mathcal{I} \to \mathrm{Set}$ *is an atomic sheaf over* $\mathcal{I}^{op}$ *iff* P *preserves pullbacks.*

Proof: Suppose the presheaf $P : \mathcal{I} \to \mathrm{Set}$ preserves pullbacks. That P preserves pullbacks means that for any pullback square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

in $\mathcal{I}$, there is an isomorphism $PA \cong \{(x, y) \in PB \times PC \mid x \upharpoonright f = y \upharpoonright g\}$, such that for each (x, y) on the right hand side there is a unique $a \in PA$ satisfying $a \upharpoonright h = x$ and $a \upharpoonright k = y$.

Let $S = \{f_i : C \to C_i \mid i \in I\}$ be a cover of C and $\{x_i\}_I$, $x_i \in PC_i$ a matching family. Suppose $f_i, f_j \in S$, then for any commuting square

$$\begin{tikzcd} C \arrow[r, "f_j"] \arrow[d, "f_i"'] & C_j \arrow[d, "s_j"] \\ C_i \arrow[r, "s_i"'] & D \end{tikzcd} \tag{2.2}$$

we have $x_i \restriction s_i = x_j \restriction s_j$, by the matching family property. In particular this holds if the above square is pullback, and we have seen how we can always construct a pullback in $\mathcal{I}$ given arrows f_i, f_j as above. By the assumption that P is pullback preserving, this implies that there is a unique $x \in PC$ such that $x \restriction f_i = x_i$ and $x \restriction f_j = x_j$. This x is a unique amalgamation: For any $k \in I$ construct the pullback in $\mathcal{I}$ with f_i, f_k as above. Then we get a unique x' such that $x' \restriction f_i = x_i$ and $x' \restriction f_k = x_k$, but then $x' \restriction f_i = x_i = x \restriction f_i$ which implies that $x' = x$ because $P(f_i)$ is mono (by the assumption that P preserves pullbacks).

For the converse, suppose P is a sheaf and 2.2 is a pullback in I. Observe that P preserves monos, since any arrow $f : C \to D$ of $\mathcal{I}$ is a cover of C, and a matching family for such a cover consists of one element $x_f \in PD$, so there is a one-one correspondence between elements of PD and elements of PC (unique amalgamations), showing that $P(f)$ is iso.

Now the arrow $s_i f_i = s_j f_j : C \to D$ is a cover of C. Suppose $x_i \in PC_i, x_j \in PC_j$ satisfies $x_i \restriction s_i = x_j \restriction s_j = x_D$, then there is a unique amalgamation (for the matching family x_D) $x \in PC$ such that $x \restriction s_i f_i = x_D$. To see that we have $x \restriction f_i = x_i$ and $x \restriction f_j = x_j$ as required, we calculate: $x \restriction s_i f_i = P(s_i f_i)(x) = x_D = P(s_i)(x_i)$ which implies $P(f_i)(x) = x_i$, since $P(s_i)$ is mono. □

2.6.1 The associated sheaf functor

Let $(\mathcal{C}, J)$ be a fixed site. The inclusion functor

$$i : \mathrm{Sh}(\mathcal{C}, J) \hookrightarrow \widehat{\mathcal{C}}$$

has a left adjoint

$$\mathbf{a} : \widehat{\mathcal{C}} \to \mathrm{Sh}(\mathcal{C}, J)$$

called the associated sheaf functor. The functor **a** commutes with finite limits (it is left exact). Moreover, the composite

$$\mathbf{a}i : \mathrm{Sh}(\mathcal{C}, J) \to \mathrm{Sh}(\mathcal{C}, J)$$

is naturally isomorphic to the identity functor.

Limits in $\mathrm{Sh}(\mathcal{C}, J)$ are calculated pointwise as in $\widehat{\mathcal{C}}$, and the category $\mathrm{Sh}(\mathcal{C}, J)$ is closed under limits, meaning that a limit of sheaves is a sheaf. In particular, $\mathrm{Sh}(\mathcal{C}, J)$ has all small limits. Note that since the terminal object 1 of the presheaf category $\widehat{\mathcal{C}}$ is the empty limit, it is a sheaf.

Colimits in $\mathrm{Sh}(\mathcal{C}, J)$ are calculated using the associated sheaf functor: For sheaves F_j,

$$\mathrm{colim}_{\mathrm{Sh}} F_j = \mathbf{a}(\mathrm{colim}_{\widehat{\mathcal{C}}} i F_j).$$

Where the subscripts indicate in which category the colimit is calculated. As a consequence, $\mathrm{Sh}(\mathcal{C}, J)$ has all small colimits.

Image factorizations in $\widehat{\mathcal{C}}$ are constructed by means of finite limits and colimits, so if $f : P \to Q$ is a morphism in $\widehat{\mathcal{C}}$ with image factorization

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, two heads, "e"'] & Q \\ \mathrm{Im}(f) \arrow[ur, tail, "m"'] & \end{tikzcd}$$

the arrow $\mathbf{a}(f) : \mathbf{a}P \to \mathbf{a}Q$ has image factorization

$$\begin{tikzcd} \mathbf{a}P \arrow[r, "\mathbf{a}(f)"] \arrow[d, two heads, "\mathbf{a}(e)"'] & \mathbf{a}Q. \\ \mathbf{a}\,\mathrm{Im}(f) \arrow[ur, tail, "\mathbf{a}(m)"'] & \end{tikzcd}$$

Representables for sheaves. Consider the Yoneda embedding $\mathbf{y} : \mathcal{C} \to \widehat{\mathcal{C}}$. If the topology on $\mathcal{C}$ is subcanonical, then $\mathbf{y}(C)$ is a sheaf, if it is not subcanonical we can, however, still define a canonical functor by composing Yoneda with the associated sheaf functor to get a sheaf:

$$\mathbf{ay} : \mathcal{C} \to \mathrm{Sh}(\mathcal{C}, J).$$

For any sheaf F, the Yoneda Lemma states that $\widehat{\mathcal{C}}(\mathbf{y}(C), iF) \cong FC$. Using the adjunction $\mathbf{a} \dashv i$ this implies

$$\mathrm{Sh}(\mathbf{ay}(C), F) \cong FC.$$

Every presheaf is a colimit of representables, so a sheaf F regarded as a presheaf iF can be written as $iF \cong \mathrm{colim}_i \mathbf{y}(C_i)$, which implies

$$F \cong \mathbf{a}i(F) \cong \mathbf{a}\,\mathrm{colim}_i \mathbf{y}(C_i) \cong \mathrm{colim}_{\mathrm{Sh}} \mathbf{ay}(C_i).$$

Proposition 2.6.16. *A family of morphisms $\{f_i : C_i \to C\}_{i \in I}$ covers C iff the induced morphism*

$$\coprod_{i \in I} \mathbf{ay}(C_i) \to \mathbf{ay}(C)$$

is epi.

2.6.2 Subsheaves

A subsheaf $A \rightarrowtail F$ of a sheaf F is a subfunctor of F which is also a sheaf. This can be given by the following definition.

Definition 2.6.17. *A functor $A : \mathcal{C}^{op} \to \mathrm{Set}$ is a* subsheaf *of a sheaf F iff A is a subfunctor of F and for each object C of $\mathcal{C}$ and each cover S of C, and each element $x \in FC$,*

$$x \restriction f \in A(D) \text{ for every } f : D \to C \text{ in } S \text{ implies } x \in A(C).$$

Since a mono is a limit, the monos in $\mathrm{Sh}(\mathcal{C}, J)$ are the monos of $\widehat{\mathcal{C}}$, that is, a mono in $\mathrm{Sh}(\mathcal{C}, J)$ is a natural transformation which is pointwise injective in Set. Hence, every element (equivalence class) of $\mathrm{Sub}(F)$, where F is a sheaf, can be canonically represented by a subsheaf of F.

The initial object 0 in $\mathrm{Sh}(\mathcal{C}, J)$ is not always the empty functor as it is in $\widehat{\mathcal{C}}$. The subfunctor $0 \hookrightarrow 1$ is not always a sheaf; suppose an object C has an empty cover, then by the definition above, we must have a unique amalgamation $* \in 0(C)$. The definition of 0 in $\mathrm{Sh}(\mathcal{C}, J)$ then becomes

$$0(C) = \begin{cases} \{*\} & \text{if } \emptyset \in J(C) \\ \emptyset & \text{otherwise.} \end{cases}$$

For subfunctors of 1, the sheaf condition is particularly simple: A subfunctor $A \hookrightarrow 1$ is a subsheaf of 1 if for each object C of $\mathcal{C}$ and for every cover S of C,

$$A(D) \neq \emptyset \text{ for every } f : D \to C \text{ in } S \text{ implies } A(C) \neq \emptyset.$$

As in the presheaf category, we can characterize the subobjects of 1 in the sheaf category:

Proposition 2.6.18. *In a category* $\mathrm{Sh}(X)$ *of sheaves over a topological space there is a bijective correspondence between subfunctors of* 1 *and representables* $\mathbf{y}U$, $U \in \mathcal{O}(X)$.

Proof: In example 2.6.14 we have shown that every representable is a sheaf w.r.t. the open cover topology. Suppose F is a subsheaf of 1, then

$$F \cong \mathbf{y}U \text{ where } U = \bigcup\{V \mid FV = \{*\}\}.$$

$FU \neq \emptyset$ because by definition U is covered by the sets V such that $FV \neq \emptyset$. Since F is a presheaf it follows that $V \subseteq U$ iff $FV \neq \emptyset$. This shows that $F \cong \mathbf{y}U$. □

Definition 2.6.19 (Ideal). *An* ideal *on a site* $(\mathcal{C}, J)$ *is a set of objects* $I \subseteq \mathrm{Obj}(\mathcal{C})$ *satisfying*

1. *If* $C \in I$ *and there exists an arrow* $D \to C$ *then* $D \in I$.
2. *For any object* C *of* $\mathcal{C}$ *and for any cover* $S \in J(C)$, *if for every* $f : C' \to C \in S$, $C' \in I$ *then* $C \in I$.

The first condition makes I a subpresheaf of 1, and the second makes it a sheaf.

Proposition 2.6.20. *There is a bijective correspondence between subsheaves of* 1 *and ideals.*

Proof: Let I be an ideal, we define a subfunctor $\hat{I}$ of 1 by

$$\hat{I}(C) = \begin{cases} \{*\} & \text{iff } C \in I \\ \emptyset & \text{otherwise.} \end{cases}$$

It is not hard to see that the sheaf condition of definition 2.6.17 corresponds exactly to the second condition of being an ideal. On the other hand let $F \hookrightarrow 1$ be a subsheaf of 1, then

$$\overline{\mathrm{F}} = \{C \in \mathrm{Obj}(\mathcal{C}) \mid F(C) = \{*\}\}$$

clearly defines an ideal. □

Corollary 2.6.21. *The complete Heyting algebra on* $\mathrm{Sub}_{\mathrm{Sh}(\mathcal{C},J)}(1)$ *can be defined in terms of ideals as follows:*

$$\begin{array}{lcl} \top & = & \mathrm{Obj}(\mathcal{C}) \\ \bot & = & \{C \mid \emptyset \in J(C)\} \\ I \vee J & = & \{C \mid \textit{ there exists a cover } S \textit{ of } C \textit{ such that for all } C_i \in S, C_i \in I \textit{ or } C_i \in J\} \\ I \wedge J & = & I \cap J \\ I \to J & = & \bigcup\{W \mid W \cap I \leq J\} \end{array}$$

Proof: Again, this is a calculation from the definition of limits, colimits and the construction of the Heyting structure in $\mathrm{Sub}_{\mathrm{Sh}(\mathcal{C},J)}(1)$. Note that $I \vee J$ and $\bot$ differs from definition in presheaf categories. This is due to the fact that these are calculated by the means of colimits, and a colimit in the sheaf category is calculated by taking the colimit in $\widehat{\mathcal{C}}$ and then using the associated sheaf functor. □

2.6.3 A category of sheaves is a topos

We have already seen that the category of sheaves have all small limits and colimits, we now show that if it has exponentials G^F for sheaves G, F, then we must have $i(G^F) \cong i(G)^{i(F)}$. To see why, let $P \in \widehat{\mathcal{C}}$ be any presheaf, and consider the following string of isomorphisms

$$\begin{array}{lll} & \widehat{\mathcal{C}}(P, i(G^F)) & \\ \cong & \mathrm{Sh}(\mathbf{a}(P), G^F) & \text{since } \mathbf{a} \dashv i \\ \cong & \mathrm{Sh}(\mathbf{a}(P) \times F, G) & \text{by the assumption that } G^F \text{ is exponential} \\ \cong & \mathrm{Sh}(\mathbf{a}(P) \times \mathbf{a}i(F), G) & \mathbf{a}i \cong \mathrm{id} \\ \cong & \mathrm{Sh}(\mathbf{a}(P \times iF), G) & \text{because } \mathbf{a} \text{ commutes with finite limits} \\ \cong & \widehat{\mathcal{C}}(P \times iF, iG) & \text{again by } \mathbf{a} \dashv i \\ \cong & \widehat{\mathcal{C}}(P, i(G)^{i(F)}) & \text{by exponentiation in } \widehat{\mathcal{C}}. \end{array}$$

All the above isomorphisms are natural in $P \in \widehat{\mathcal{C}}$, so because the Yoneda embedding is full and faithful (actually we are using Proposition 2.5.7), we conclude that $i(G^F) \cong i(G)^{i(F)}$. We still do not know that it is a sheaf, though. This follows from the following proposition.

Proposition 2.6.22. *Let* $P, F \in \widehat{\mathcal{C}}$, *if* F *is a sheaf, then so is the exponential* $(iF)^P$.

A proof of this can found in [MLM94, p. 136]. We have shown the following:

Corollary 2.6.23. *The category of sheaves over a cite,* $\mathrm{Sh}(\mathcal{C}, J)$ *is Cartesian closed.*

To see that the category of sheaves over a cite is a in fact a topos it must be shown that it has a subobject classifier. The requirement $\mathrm{Sub}(F) \cong \mathrm{Hom}(F, \Omega)$ leads to

$$i\Omega(A) \cong \widehat{\mathcal{C}}(\mathbf{y}A, i\Omega) \cong \mathrm{Sh}(\mathbf{ay}A, \Omega) \cong \mathrm{Sub}(\mathbf{ay}A),$$

so $\Omega(A)$ is the set of subsheaves of $\mathbf{ay}A$. For sheaves over a topological space X with the usual open cover topology these are characterized by the *principal sieves* $\downarrow(V)$ on A, where $\downarrow(V) = \{V' \mid V' \subseteq V\}$, i.e., the subobject classifier of $\mathrm{Sh}(X)$ is defined as follows:

$$\begin{array}{rcl} \Omega(U) & = & \{\downarrow(V) \mid V \subseteq U\} \\ & = & \{\mathbf{y}(V) \mid V \subseteq U\} \end{array}$$

Being a principal sieve is equivalent to being a sieve which is closed under arbitrary unions of its elements, i.e., for a sieve S, and for any open $W \subseteq U$, if S covers W then $W \in S$. The notion of principal sieve can be generalized to arbitrary sites, this is done in [MLM94], which also gives a proof of the fact that Ω is indeed a subobject classifier. For the record:

Corollary 2.6.24. *Every Grothendieck sheaf category is a topos.*

Chapter 3

Ends and coends

Literature: [Win01], [Cac03] and [ML98]

This chapter provides a detailed review of ends and coends for dinatural transformations and the main results regarding these, which will be needed for Day's construction.

3.1 Dinatural transformations

A dinatural transformation $\alpha : F \ddot{\rightarrow} G$ between functors $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ consists of a family $\langle \alpha_U : F(U,U) \to G(U,U) \rangle_{U \in \mathcal{C}}$ of arrows in $\mathcal{D}$ such that for every arrow $f : V \to U$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd}
 & F(U,U) \arrow[r, "\alpha_U"] & G(U,U) \arrow[dr, "G(f,U)"] & \\
F(U,V) \arrow[ur, "F(U,f)"] \arrow[dr, "F(f,V)"'] & & & G(V,U) \\
 & F(V,V) \arrow[r, "\alpha_V"] & G(V,V) \arrow[ur, "G(V,f)"'] &
\end{tikzcd}$$

commutes. Here $F(U,f)$ means $F(\mathrm{id}_U, f)$. Every natural transformation

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D}
\end{tikzcd}$$

gives rise to a dinatural transformation $\langle \eta_{U,U} : F(U,U) \to G(U,U) \rangle_{U \in \mathrm{Obj}(\mathcal{C})}$, $\langle \eta_{U,U} \rangle_U$ for short. This is verified by the commutativity of the following diagram:

$$\begin{tikzcd}
V \arrow[d, "f"'] & & F(U,U) \arrow[rr, "\eta_{U,U}"] \arrow[dr, phantom, "(1)"] & & G(U,U) \arrow[dr, "G(f,U)"] & \\
U & F(U,V) \arrow[ur, "F(U,f)"] \arrow[rr, "\eta_{U,V}"] \arrow[dr, "F(f,V)"'] & & G(U,V) \arrow[ur, "G(U,f)"] \arrow[dr, "G(f,V)"'] \arrow[rr, phantom, "(3)"] & & Gg(V,U) \\
 & & F(V,V) \arrow[rr, "\eta_{V,V}"'] \arrow[ur, phantom, "(2)"] & & G(V,V) \arrow[ur, "G(V,f)"'] &
\end{tikzcd}$$

where (1) and (2) commute because of naturality of η and (3) commutes because G is a bifunctor. Dinatural transformations compose with natural transformations:

Lemma 3.1.1. *Given* $\alpha : F' \Rightarrow F, \beta : F \ddot{\rightarrow} G$ *and* $\gamma : G \Rightarrow G'$, *where* F, F', G, G' *are all functors from* $\mathcal{C}^{op} \times \mathcal{C}$ *to* $\mathcal{D}$, *the following composites are dinatural transformations.*

1. $\beta \circ \langle \alpha_{U,U} \rangle_U$
2. $\langle \gamma_{U,U} \rangle_U \circ \beta$
3. $\langle \gamma_{U,U} \rangle_U \circ \beta \circ \langle \alpha_{U,U} \rangle_U$.

The composition is defined componentwise.

Proof: 3. follows directly from 1. and 2. To show 1. consider the following diagram

$$\begin{tikzcd}
V \arrow[d, "f"'] & & F'(U,U) \arrow[rr, "\alpha_{U,U}"] \arrow[d, phantom, "(2)"] & & F(U,U) \arrow[r, "\beta_U"] & G(U,U) \arrow[dr, "G(f,U)"] & \\
U & F'(U,V) \arrow[ur, "F'(U,f)"] \arrow[rr, "\alpha_{U,V}"] \arrow[dr, "F'(f,V)"'] & & F(U,V) \arrow[ur, "F(U,f)"] \arrow[dr, "F(f,V)"'] & (1) & & G(V,U) \\
& & F'(V,V) \arrow[rr, "\alpha_{V,V}"'] \arrow[u, phantom, "(3)"] & & F(V,V) \arrow[r, "\beta_V"'] & G(V,V) \arrow[ur, "G(V,f)"'] &
\end{tikzcd}$$

Diagram (1) commutes by dinaturality of β, (2) and (3) by naturality of α, so the whole diagram commutes. 2. is proved similarly. □

As with natural transformations, dinaturality can be verified at each component independently:

Lemma 3.1.2. *Let* $H, K : \mathcal{A}^{op} \times \mathcal{A} \times \mathcal{B}^{op} \times \mathcal{B} \rightarrow \mathcal{C}$ *be functors. A family*

$$\langle \alpha_{a,b} : H(a,a,b,b) \rightarrow K(a,a,b,b) \rangle_{a \in \mathcal{A}, b \in \mathcal{B}}$$

is dinatural if and only if the induced families

$$\alpha_{a,-} : H(a,a,-,-) \rightarrow K(a,a,-,-)$$

and

$$\alpha_{-,b} : H(-,-,b,b) \rightarrow K(-,-,b,b)$$

are dinatural.

For a proof see [Cac03, p.29].

A *wedge* is a special case of a dinatural transformation were one of the functors is a constant functor ΔX. For instance, consider a dinatural transformation $\alpha : \Delta X \ddot{\rightarrow} G$. The commuting diagram

$$\begin{tikzcd}
& X \arrow[r, "\alpha_U"] & G(U,U) \arrow[dr, "G(f,U)"] & \\
X \arrow[ur, "\mathrm{id}_X"] \arrow[dr, "\mathrm{id}_X"'] & & & G(V,U) \\
& X \arrow[r, "\alpha_V"'] & G(V,V) \arrow[ur, "G(V,f)"'] &
\end{tikzcd}$$

can be redrawn as

$$\begin{tikzcd}
 & G(U,U) \arrow[dr, "G(f,U)"] & \\
X \arrow[ur, "\alpha_U"] \arrow[dr, "\alpha_V"'] & & G(V,U). \\
 & G(V,V) \arrow[ur, "G(V,f)"'] &
\end{tikzcd}$$

The dinatural transformation α is called a wedge from X to G.

3.2 Ends and coends

Definition 3.2.1. *An end of a functor $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is a universal wedge ω from a constant E to F.*

That is, for every wedge $\beta : Y \to F$ there is a unique arrow $h : Y \to E$ such that $\beta_C = \omega_C h$ for all $C \in \mathrm{Obj}(\mathcal{C})$. Thus for each arrow $f : B \to C$ of $\mathcal{C}$ there is a diagram

$$\begin{tikzcd}
 & & F(B,B) \arrow[dr, "F(B,f)"] & \\
Y \arrow[r, dotted, "h"] \arrow[urr, bend left, "\beta_B"] \arrow[drr, bend right, "\beta_C"'] & E \arrow[ur, "\omega_B"] \arrow[dr, "\omega_C"] & & F(B,C). \\
 & & F(C,C) \arrow[ur, "F(f,C)"'] &
\end{tikzcd}$$

ω is called the ending wedge and the object E (by abuse of language) is called the end of F and is written $E = \int_C F(C,C)$.

Dually, a coend for F is colimiting (or couniversal) wedge κ from F to a constant $\int^C F(C,C)$, as illustrated by the following diagram:

$$\begin{tikzcd}
 & F(B,B) \arrow[dr, "\kappa_B"] \arrow[drr, bend left, "\alpha_B"] & & \\
F(C,B) \arrow[ur, "F(f,B)"] \arrow[dr, "F(C,f)"'] & & \int^C F(C,C) \arrow[r, dotted] & X. \\
 & F(C,C) \arrow[ur, "\kappa_C"] \arrow[urr, bend right, "\alpha_C"'] & &
\end{tikzcd}$$

Natural transformations provide an example of ends. Given two functors $F; G : \mathcal{C} \to \mathcal{D}$, we can define the hom-functor

$$\mathrm{Hom}_{\mathcal{D}}(F(-), G(-)) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$$

Which acts on objects and arrows by sending $\langle f, g \rangle : (C, D) \to (C', D')$ to

$$\mathrm{Hom}_{\mathcal{D}}(FC', GD) \xrightarrow{G(g)\circ - \circ F(f)} \mathrm{Hom}_{\mathcal{D}}(FC, GD')$$

(We will omit the subscript to Hom when it is clear from the context which category the hom-set lives in.)

Now if Y is any set, consider a wedge $\beta : Y \to \mathrm{Hom}(F(-), G(-))$, with components

$$\beta_C : Y \to \mathrm{Hom}(FC, GC).$$

For each arrow $f : B \to C$ of $\mathcal{C}$ we have the commuting diagram

$$\begin{tikzcd}
 & \mathrm{Hom}(FB, GB) \arrow[dr, "\mathrm{Hom}(FB,Gf)"] & \\
Y \arrow[ur, "\beta_B"] \arrow[dr, "\beta_C"'] & & \mathrm{Hom}(FB, GC) \\
 & \mathrm{Hom}(FC, GC) \arrow[ur, "\mathrm{Hom}(Ff,GC)"'] &
\end{tikzcd}$$

Chasing the arrows, we see that for all $y \in Y$, $\beta_C(y) \circ Ff = Gf \circ \beta_B(y)$, this states in fact that for each y, $\beta_{\_}(y)$ is a natural transformation from F to G.

$$\begin{tikzcd}
FB \arrow[r, "\beta_B(y)"] \arrow[d, "Ff"'] & GB \arrow[d, "Gf"] \\
FC \arrow[r, "\beta_C(y)"'] & GC.
\end{tikzcd}$$

Let $\omega_C : \mathrm{Nat}(F, G) \to \mathrm{Hom}(FC, GC)$ be the arrow that assigns to a natural transformation $\lambda : F \to G$ its component λ_C. The collection of arrows $\langle \omega_C \rangle_{C \in \mathcal{C}}$ is a wedge from $\mathrm{Nat}(F, G)$ to $\mathrm{Hom}(F(-), G(-))$. To see that it is an ending wedge, consider the following diagram

$$\begin{tikzcd}
Y \arrow[r, "\beta_C"] \arrow[d, dotted, "h"'] & \mathrm{Hom}(FC, GC) \\
\mathrm{Nat}(F, G) \arrow[ur, "\omega_C"'] &
\end{tikzcd}$$

where $h(y) = \beta_{\_}(y)$ is the unique map such that the diagram commutes, since $\omega_C(h(y)) = \beta_C(y)$. Hence the Naturality Formula

$$\mathrm{Nat}(F, G) = \int_C \mathrm{Hom}(FC, GC). \tag{3.1}$$

Letting $F, G : \mathbb{I} \times \mathbb{I}^{op} \to \mathcal{D}$, a similar proof shows that

$$\mathrm{Dinat}(F, G) = \int_I \mathrm{Hom}(F(I, I), G(I, I)) \tag{3.2}$$

which we call the Dinaturality Formula. Using the Naturality Formula and the Yoneda Lemma, we get the following identity

$$\int_C \mathrm{Hom}(\mathrm{Y}U(C), FC) = \mathrm{Nat}(\mathrm{Y}U, F) \overset{\theta_{U,F}}{\cong} FU. \tag{3.3}$$

3.2.1 Ends with parameters

Proposition 3.2.2 (End of a natural transformation). *Given a natural transformation* $\gamma : F \to F'$ *between functors* $F, F' : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ *which both have ends* ω, ω' *respectively, there is a unique arrow*

$$\int_C \gamma_{C,C} : \int_C F(C,C) \to \int_C F'(C,C)$$

in $\mathcal{D}$ *such that the following diagram commutes for every* $C \in \mathrm{Obj}(\mathcal{C})$*:*

$$\begin{tikzcd}
\int_C F(C,C) \arrow[r, "\omega_C"] \arrow[d, dotted, "\int_C \gamma_{C,C}"'] & F(C,C) \arrow[d, "\gamma_{C,C}"] \\
\int_C F'(C,C) \arrow[r, "\omega'_C"'] & F'(C,C)
\end{tikzcd}$$

Proof: By Lemma 3.1.1 the composite $\gamma \circ \omega$ defines a wedge from $\int_C F(C,C)$ to F'. Since ω' is a universal wedge of F', the arrow $\int_C \gamma_{C,C}$ exists and is unique. □

The arrow $\int_C \gamma_{C,C}$ is called the end of the natural transformation γ. Composing γ with another natural transformation $\gamma' : F' \to F''$ yields the rule

$$\int_C (\gamma'\gamma)_{C,C} = (\int_C \gamma'_{C,C}) \circ (\int_C \gamma_{C,C}) \tag{3.4}$$

by uniqueness of $\int_C (\gamma'\gamma)_{C,C}$.

Theorem 3.2.3 (Parameter Theorem for Ends and Limits). *Let* $G : \mathcal{P} \times \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ *be a functor such that* $G(P,-,-)$ *for each object* $P \in \mathrm{Obj}(\mathcal{P})$ *has an end*

$$\omega_P : \int_C G(P,C,C) \ddot{\to} G(P,-,-)$$

in $\mathcal{D}$*. Then there is a unique functor* $U : \mathcal{P} \to \mathcal{D}$ *with object function* $UP = \int_C G(P,C,C)$ *such that the components of the wedges* ω_P *for each* $C \in \mathrm{Obj}(\mathcal{C})$ *define a transformation*

$$(\omega_P)_C : UP \to G(P,C,C)$$

natural in P*.*

Proof: The arrow function of U must be defined such that for each arrow $f : P \to Q$ of $\mathcal{P}$, and for every $C \in \mathrm{Obj}(\mathcal{C})$, we have a commuting diagram

$$\begin{tikzcd}
UP \arrow[r, "(\omega_P)_C"] \arrow[d, "Uf"'] & G(P,C,C) \arrow[d, "G(f,C,C)"] \\
UQ \arrow[r, "(\omega_Q)_C"'] & G(Q,C,C).
\end{tikzcd}$$

Now each $f : P \to Q$ actually defines a natural transformation $G(f,-,-) : G(P,-,-) \to G(Q,-,-)$ so by Proposition 3.2.2, the unique choice for Uf is $\int_C G(f,C,C)$. The composition rule 3.4 shows that this definition does indeed determine a functor. □

As the notation suggests, the functor U is the end of the functor $\lambda C, C'.G(-, C, C') : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}^{\mathcal{P}}$ regarding U as an object of the functor category $\mathcal{D}^{\mathcal{P}}$. This follows from the facts that $(\omega_{-})_C : U \Rightarrow G(-, C, C)$ is a natural transformation for each $C \in \mathcal{C}$, and that $\omega_P : UP \to G(P, -, -)$ is an ending wedge (essentially we are exploiting that limits are computed pointwise in functor categories). We write $U = \int_C G(-, C, C)$.

There is a dual result that if $G(P, -, -)$ has a coend for each object $P \in \mathrm{Obj}(\mathcal{P})$, then there is a unique functor

$$\int^C G(-, C, C) : \mathcal{P} \to \mathcal{D}.$$

3.2.2 Ends are limits

Let $F : \mathbb{I}^{op} \times \mathbb{I} \to \mathcal{D}$ be a functor that has an end. We construct a category $\mathbb{I}^{\S}$ and a functor $F^{\S} : \mathbb{I}^{\S} \to \mathcal{D}$ such that

$$\lim_{J \in \mathrm{Obj}(\mathbb{I}^{\S})} F^{\S}(J) = \int_{I \in \mathrm{Obj}(\mathbb{I})} F(I, I).$$

- $\mathrm{Obj}(\mathbb{I}^{\S}) = \mathrm{Obj}(\mathbb{I}) \cup \mathrm{arr}(\mathbb{I})$
- $\mathrm{arr}(\mathbb{I}^{\S})$ = the collection of identities and for every $f : U \to V$ in $\mathbb{I}$ the arrows

$$\begin{tikzcd}
U \arrow[dr] & \\
& f. \\
V \arrow[ur] &
\end{tikzcd}$$

The only meaningful compositions are with identities. The category $\mathbb{I}^{\S}$ is called the subdivision category of $\mathbb{I}$. The functor $F^{\S} : \mathbb{I}^{\S} \to \mathcal{D}$ is defined by the following assignments

$$\begin{tikzcd}
U \arrow[dr] & & & F(U,U) \arrow[dr, "F(U,f)"] & \\
& f & \overset{F^{\S}}{\longmapsto} & & F(U,V). \\
V \arrow[ur] & & & F(V,V) \arrow[ur, "F(f,V)"'] &
\end{tikzcd}$$

Now a limit for $F^{\S}$ is an object L of $\mathcal{D}$ together with a collection of arrows $\langle \kappa_J \rangle_{J \in \mathrm{Obj}(\mathbb{I}^{\S})}$, $L \xrightarrow{\kappa_J} F^{\S}(J)$ such that

$$\begin{tikzcd}
& F(U,U) = F^{\S}(U) \arrow[dr, "F(U,f)"] & \\
L \arrow[ur, "\kappa_U"] \arrow[rr, "\kappa_f"] \arrow[dr, "\kappa_V"'] & & F(U,V) = F^{\S}(f) \\
& F(V,V) = F^{\S}(V) \arrow[ur, "F(f,V)"'] &
\end{tikzcd}$$

commutes. One readily sees that the collection $\langle \kappa_J \rangle_{J \in \mathrm{Obj}(\mathbb{I}^\S)}$ together with L defines an end for F. Dually a coend for F is the same as a colimit for the functor $F^\# : (\mathbb{I}^\S)^{op} \to \mathcal{D}$, where $F^\#$ is defined in the same way $F^\S$ is, with the following modifications: For $f : U \to V$ in $\mathbb{I}$

$$\begin{tikzcd}
 & U & & & F(U,U) \\
f \arrow[ur] \arrow[dr] & & \overset{F^\#}{\longmapsto} & F(V,U) \arrow[ur, "F(f,U)"] \arrow[dr, "F(V,f)"'] & \\
 & V & & & F(V,V)
\end{tikzcd}$$

As a consequence, if $\mathcal{D}$ has all (co)limits, then $\mathcal{D}$ has all (co)ends.

Example 3.2.4. *As an example we consider the concrete definition of a coend in* Set*:*

$$\int^I F(I,I) = \mathrm{colim}_{I \in \mathrm{Obj}(\mathbb{I}^\S)} F^\#(I) = \biguplus_{I \in \mathrm{Obj}(\mathbb{I}^\S)} F^\#(I)/\simeq$$

Where $\simeq$ is the least equivalence relation on $\biguplus_{I \in \mathrm{Obj}(\mathbb{I}^\S)} F^\#(I) \times \biguplus_{I \in \mathrm{Obj}(\mathbb{I}^\S)} F^\#(I)$ such that

$$(f,x) \simeq (I,y) \quad \Leftrightarrow \quad \exists u : f \to I \in \mathrm{arr}((\mathbb{I}^\S)^{op}).F^\#(u)(x) = y. \tag{3.5}$$

(Recall that $F^\#(u)(x) = F(f,I)(x)$). The couniversal wedge is the collection of injection arrows $x \mapsto [I,x]_\simeq$, where $I \in \mathrm{Obj}(\mathbb{I}^\S)$, i.e., I is an object or an arrow of $\mathbb{I}$, and $x \in F^\#(I)$. Now from this definition we deduce an equivalence relation $\sim$ on $\biguplus_{I \in \mathrm{Obj}(\mathbb{I})} F(I,I) \times \biguplus_{I \in \mathrm{Obj}(\mathbb{I})} F(I,I)$ such that

$$\int^I F(I,I) = \biguplus_{I \in \mathrm{Obj}(\mathbb{I})} F(I,I)/\sim .$$

$\sim$ is the least equivalence relation such that

$$\begin{array}{ll}
 & (I,x) \sim (J,y) \\
\Leftrightarrow & (I,x) \simeq (J,y) \\
\Leftrightarrow & \exists (f,z).(I,x) \simeq (f,z) \simeq (J,y) \\
\Leftrightarrow & \exists f : I \to J \in \mathrm{arr}(\mathbb{I}).\exists z \in F(J,I).(x = F(f,I)z \wedge y = F(J,f)z),
\end{array}$$

and the couniversal wedge is the collection of injection arrows $x \mapsto [I,x]_\sim$ with $x \in F(I,I)$, $I \in \mathrm{Obj}(\mathbb{I})$.

The following example will be useful.

Example 3.2.5. *Consider a functor $T : \mathcal{P} \times \mathcal{C}^{op} \times \mathcal{C} \to$ Set such that for each $P \in \mathcal{P}$, $T(P,-,-)$ has a coend*

$$\kappa_P : T(P,-,-) \overset{\cdot\cdot}{\to} \int^C T(P,C,C)$$

in Set. *Then by the Parameter Theorem 3.2.3, there is a unique functor* $\int^C T(-,C,C) : \mathcal{P} \to$ Set. *To objects it assigns the collection of equivalence classes just defined in 3.2.4. For an arrow* $s : Q \to P$, *the arrow* $\int^C T(s,C,C)$ *must make the following diagram commute:*

$$\begin{tikzcd}
T(P,U,U) \arrow[r, "(\kappa_P)_U"] & \int^C T(P,C,C) \\
T(Q,U,U) \arrow[u, "T(s,U,U)"] \arrow[r, "(\kappa_Q)_U"'] & \int^C T(Q,C,C) \arrow[u, "\int^C T(s,C,C)"']
\end{tikzcd}$$

for all U. *But since* $(\kappa_P)_U$ *and* $(\kappa_Q)_U$ *are the injections* $x \mapsto [P,U,x]_\sim$ *and* $x \mapsto [Q,U,x]_\sim$, *the only choice for* $\int^C T(s,C,C)$ *is*

$$(\int^C T(s,C,C))[Q,U,x]_\sim = [P,U,T(s,U,U)(x)]_\sim.$$

It is easily verified that this is well-defined. This completes the description of the functorial action of $\int^C T(-,C,C) : \mathcal{P} \to$ Set.

3.2.3 Abstract definition of end and coend

We now give more abstract definitions of ends and coends, which in some cases will be easier to work with. An end for a functor $F : \mathbb{I}^{op} \times \mathbb{I} \to \mathcal{D}$ is a representation for the functor $\mathrm{Dinat}(\Delta(-), F) : \mathcal{D}^{op} \to$ Set:

$$\mathcal{D}(-, \int_I F(I,I)) \overset{\phi}{\cong} \mathrm{Dinat}(\Delta(-), F)$$

Dually a coend for $F : \mathbb{I}^{op} \times \mathbb{I} \to \mathcal{D}$ is a representation:

$$\mathcal{D}(\int^I F(I,I), -) \overset{\psi}{\cong} \mathrm{Dinat}(F, \Delta(-))$$

for the functor $\mathrm{Dinat}(F, \Delta(-)) : \mathcal{D} \to$ Set. We can recover the concrete definition with $\kappa = \psi_{\int^I F}(\mathrm{id}_{\int^I F})$ as the couniversal wedge (see [Win01]). Here $\int^I F$ is an abbreviation of $\int^I F(I,I)$. The action of ψ is, for $h : \int^I F(I,I) \to X$ an arrow of $\mathcal{D}$:

- $\psi_X(h) = h \circ \kappa$, and
- $\psi_X^{-1}(\alpha) = U_{\int^I F}(\alpha)$, where $U_{\int^I F}$ is the function that sends α to the unique mediating arrow from the coend $\int^I F(I,I)$ to X:

$$\begin{tikzcd}
 & F(U,U) \arrow[dr, "\kappa_U"] \arrow[drr, bend left, "\alpha_U"] & & \\
F(U,V) \arrow[ur, "F(U,f)"] \arrow[dr, "F(f,V)"'] & & \int^I F(I,I) \arrow[r, dotted, "U_{\int^I F}(\alpha)"] & X \\
 & F(V,V) \arrow[ur, "\kappa_V"] \arrow[urr, bend right, "\alpha_V"'] & &
\end{tikzcd}$$

In view of the Dinaturality Formula 3.2 we can now express coends in terms of ends:

$$\int_I \mathrm{Hom}_{\mathcal{D}}(F(I,I),D) \cong \mathrm{Dinat}(F,\Delta D) \cong \mathcal{D}(\int^I F(I,I),D). \tag{3.6}$$

Since $\mathrm{Dinat}(\Delta D,F) \cong \mathcal{D}(D,\int_I F(I,I))$ we also get

$$\int_I \mathrm{Hom}_{\mathcal{D}}(D,F(I,I)) \cong \mathcal{D}(D,\int_I F(I,I)). \tag{3.7}$$

Another way to obtain these results is to exploit the fact that the hom-functor preserves and reverses limits.

Example 3.2.6. *Given a functor* $F : \mathbb{I}^{op} \times \mathbb{I} \to \mathrm{Set}$*, we know that*

$$\int^I F(I,I) = \biguplus_{I\in\mathbb{I}} F(I,I)/\sim$$

as defined in 3.2.4, and the couniversal wedge is the collection of injection arrows $x \mapsto [I,x]_\sim$*. The unique mediating arrow of* $\alpha \in \mathrm{Dinat}(F,\Delta X)$ *is then given by*

$$[U,x]_\sim \mapsto \alpha_U(x),$$

where $[U,x]_\sim \in \biguplus_{I\in\mathbb{I}} F(I,I)/\sim$*.*

3.2.4 Parameterized representability

Most isomorphisms in category theory are required to be natural, but more often than not the proof of naturality is omitted, this is probably due to the fact that such a proof usually is equivalent to some tedious diagram chase. The following theorem is extremely useful in order to proof naturality results without diagram chasing.

Theorem 3.2.7. *Let* $F : \mathcal{A} \times \mathcal{B}^{op} \to \mathrm{Set}$ *be a bifunctor such that for every* $A \in \mathrm{Obj}(\mathcal{A})$ *there exists a representation* $(G[A],\theta^A)$ *for the functor* $F(A,-) : \mathcal{B}^{op} \to \mathrm{Set}$*. Then there is a unique extension of the mapping* $A \mapsto G[A]$ *to a functor* $G : \mathcal{A} \to \mathcal{B}$ *such that*

$$\mathcal{B}(B,G(A)) \overset{\theta^A_B}{\cong} F(A,B) \tag{3.8}$$

is natural in $A \in \mathrm{Obj}(\mathcal{A})$ *and* $B \in \mathrm{Obj}(\mathcal{B})$*.*

Proof: Since $(G[A],\theta^A)$ is a representation, 3.8 is by definition natural in B. To see that it is natural in A, let $f : A \to A'$ be an arrow and consider the commuting diagram

$$\begin{tikzcd}
\mathcal{B}(-,G[A]) \arrow[r, phantom, "\overset{\theta^A}{\cong}"] \arrow[d, "(\theta^{A'})^{-1}\circ F(f,-)\circ\theta^A"'] & F(A,-) \arrow[d, "F(f,-)"] \\
\mathcal{B}(-,G[A']) \arrow[r, phantom, "\overset{\theta^{A'}}{\cong}"] & F(A',-).
\end{tikzcd} \tag{3.9}$$

Since the Yoneda functor is full and faithful there must exist a unique arrow, $G(f)$ say, such that

$$\mathcal{B}(-,G(f)) = (\theta^{A'})^{-1} \circ F(f,-) \circ \theta^A.$$

It is routine to verify that this definition gives a functor. The concrete definition of $G(f), f : A \to B$ is found by chasing $\mathrm{id}_{G[A]}$ round the diagram 3.9:

$$G(f) = ((\theta^B)^{-1} \circ F(f, G[A]) \circ \theta^A)(\mathrm{id}_{G[A]})$$

□

Corollary 3.2.8. *If the functor F of the theorem is a hom-functor $\mathcal{B}(B, H(A))$ for some functor $H : \mathcal{A} \to \mathcal{B}$, then there is an isomorphism*

$$G(A) \cong H(A)$$

which is natural in A.

Proof: By the full and faithfulness of the Yoneda functor, we have for each $A \in \mathcal{A}$ that $GA \cong HA$ in $\mathcal{B}$, but since it is only pointwise we do not get naturality in A for free, so to see this consider

$$\begin{tikzcd}
\mathcal{B}(B, GA) \arrow[r, phantom, "\overset{\theta^A_B}{\cong}"] \arrow[d, "Gf\circ-"'] & \mathcal{B}(B, HA) \arrow[d, "Hf\circ-"] \\
\mathcal{B}(B, GC) \arrow[r, phantom, "\overset{\theta^C_B}{\cong}"] & \mathcal{B}(B, HC),
\end{tikzcd}$$

where $f : A \to C$. Letting $B = GA$ and following the identity round the diagram we get the following commutative diagram

$$\begin{tikzcd}
GA \arrow[r, "\theta^A_{GA}(\mathrm{id}_{GA})"] \arrow[dr, "\theta^C_{GA}(Gf)"'] & HA \arrow[d, "Hf"] \\
 & HC
\end{tikzcd}$$

to see that we also have a commuting diagram

$$\begin{tikzcd}
GA \arrow[d, "Gf"'] \arrow[dr, "\theta^C_{GA}(Gf)"] & \\
GC \arrow[r, "\theta^C_{GC}(\mathrm{id}_{GC})"'] & HC
\end{tikzcd}$$

just follow id_{GC} round the following commuting diagram

$$\begin{tikzcd}
\mathcal{B}(GA, GC) \arrow[r, "\theta^C_{GA}"] & \mathcal{B}(GA, HC) \\
\mathcal{B}(GC, GC) \arrow[u, "-\circ Gf"] \arrow[r, "\theta^C_{GC}"'] & \mathcal{B}(GC, HC). \arrow[u, "-\circ Gf"']
\end{tikzcd}$$

□

A few examples:

Proposition 3.2.9. *Let $F : \mathcal{B} \times \mathbb{I}^{op} \times \mathbb{I} \to \mathcal{D}$ be a functor such that for every $B \in \mathcal{B}$ the coend $\int^I F(B, I, I)$ exists in $\mathcal{D}$. Then the function $B \mapsto \int^I F(B, I, I)$ uniquely extends to a functor*

$$\int^I F(-, I, I) : \mathcal{B} \to \mathcal{D}$$

such that

$$\mathcal{D}(\int^I F(B, I, I), D) \overset{\psi^B_D}{\cong} \mathrm{Dinat}(F(B, -, -), \Delta D) \tag{3.10}$$

natural in B, D.

Proof: Consider the functor $\lambda B, D.\, \mathrm{Dinat}(F(B, -, -), \Delta D) : \mathcal{B}^{op} \times \mathcal{D} \to \mathrm{Set}$. Since for every $B \in \mathcal{B}$ the coend $\int^I F(B, I, I)$ exists, we have a representation $(\psi^B, \int^I F(B, -, -))$ for the functor $\lambda D.\, \mathrm{Dinat}(F(B, -, -), \Delta D)$. So by parameterized representability there is a unique extension of $B \mapsto \int^I F(B, I, I)$ to a functor $\int^I F(-, I, I) : \mathcal{B}^{op} \to \mathcal{D}^{op}$, such that $\mathcal{D}^{op}(D, \int^I F(B, I, I)) = \mathcal{D}(\int^I F(B, I, I), D) \overset{\psi^B_D}{\cong} \mathrm{Dinat}(F(B, -, -), \Delta D)$. The definition on arrows can be calculated as follows: For $F : A \to B$,

$$\int^I F(f, I, I) : \int^I F(A, I, I) \to \int^I F(B, I, I)$$

is

$$\begin{array}{rcl}
\int^I F(f, I, I) & = & (\psi^A_{\int F(B,I,I)})^{-1} \circ \mathrm{Dinat}(F(f, -, -), \Delta \int^I F(B, I, I)) \circ \psi^B_{\int F(B,I,I)}(\mathrm{id}_{\int F(B,I,I)}) \\
 & = & (\psi^A_{\int F(B,I,I)})^{-1} \circ \mathrm{Dinat}(F(f, -, -), \Delta \int^I F(B, I, I))(\kappa^B) \\
 & = & (\psi^A_{\int F(B,I,I)})^{-1}(\kappa^B \circ F(f, -, -)) \\
 & = & U_{\int F(A,I,I)}[\kappa^B \circ F(f, -, -)]
\end{array}$$

Where κ^B is the couniversal wedge from $F(B, -, -)$ to $\int^I F(B, I, I)$. Notice that we have rediscovered the functor defined in Theorem 3.2.3. □

Proposition 3.2.10 (Coend functor). *If a category $\mathcal{D}$ has all coends of type $\mathbb{I}$, we can define the coend functor*

$$\mathrm{Coend}(-) : [\mathbb{I} \times \mathbb{I}^{op} \to \mathcal{D}] \to \mathcal{D}$$

by

$$\begin{array}{ccc}
F & \overset{\alpha}{\Rightarrow} & G \\
 & \mapsto & \\
\int^I F(I, I) & \overset{\int^I \alpha_{I,I}}{\longrightarrow} & \int^I G(I, I)
\end{array}$$

where $\int^I \alpha_{I,I} = U_{\int^I F}[\kappa^G \circ \alpha]$ and κ^G is the couniversal wedge for G. Moreover, it follows that the formula

$$\mathcal{D}(\int^I F(I, I), D) \overset{\psi^F_D}{\cong} \mathrm{Dinat}(F, \Delta D), \tag{3.11}$$

is natural in F as well as in D.

Proof: This follows from parameterized representability with respect to the functor

$$\lambda F.\lambda D.\,\mathrm{Dinat}(F, \Delta D) : [\mathbb{I} \times \mathbb{I}^{op} \to \mathcal{D}]^{op} \times \mathcal{D} \to \mathrm{Set}.$$

From Theorem 3.2.7 we also get the natural isomorphism

$$\mathcal{D}(\int^I F(I,I), D) \overset{\psi^F_D}{\cong} \mathrm{Dinat}(F, \Delta D)$$

and the definition on arrows is found using the formula given in Theorem 3.2.7:

$$\begin{aligned} \textstyle\int^I \alpha_{I,I} &= (\psi^F_{\int G})^{-1} \circ \mathrm{Dinat}(\alpha, \Delta \textstyle\int^I F(I,I)) \circ \psi^G_{\int G}(\mathrm{id}_{\int G}) \\ &= (\psi^F_{\int G})^{-1} \circ \mathrm{Dinat}(\alpha, \Delta \textstyle\int^I F(I,I))(\kappa^G) \\ &= (\psi^F_{\int G})^{-1}(\kappa^G \circ \alpha) \\ &= U_{\int^I F}[\kappa^G \circ \alpha] \end{aligned}$$

Where κ^G is the couniversal wedge corresponding to G. □

3.2.5 Fubini for coends

In this section and the next we state and proof results about coends which all dualize to ends. We shall also see plenty of justification for the notation $\int F$ for ends and coends.

Proposition 3.2.11. *Let $G : \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{B}$ be a functor. If for every pair c_1, c_2 in $\mathcal{C}$ the coend $\int^y G(c_1, c_2, y, y)$ exists in $\mathcal{B}$, then there is an isomorphism*

$$\int^{(x,y)} G(x,x,y,y) \cong \int^x \int^y G(x,x,y,y)$$

natural in G. Meaning that if one side exists so does the the other, and then they are isomorphic.

Proof: We claim that there is an isomorphism

$$\mathrm{Dinat}(\lambda c_1, c_2. \int^y G(c_1, c_2, y, y), \Delta b) \overset{\gamma}{\cong} \mathrm{Dinat}(G, \Delta b)$$

natural in b, where $(\gamma(\alpha))_{c,d}$ is the composite

$$G(c,c,d,d) \xrightarrow{\varepsilon^c_d} \textstyle\int^y G(c,c,y,y) \xrightarrow{\alpha_c} b$$

and $\varepsilon^c : \lambda d_1, d_2. G(c,c,d_1,d_2) \stackrel{\cdot\cdot}{\to} \int^y G(c,c,y,y)$ is the couniversal wedge. We now show that this family is a wedge. In view of Lemma 3.1.2 this can be verified for each component apart. Fixing $c \in \mathcal{C}$, the family

$$\langle \alpha_c \circ \varepsilon^c_d \rangle_{d \in \mathcal{D}}$$

is a wedge since ε^c is a wedge and for an arrow $f : d' \to d$, the diagram

$$\begin{tikzcd}
 & G(c,c,d,d) \arrow[dr, "\varepsilon^c_d"] & & \\
G(c,c,d,d') \arrow[ur, "{G(c,c,d,f)}"] \arrow[dr, "{G(c,c,f,d')}"'] & & \int^y G(c,c,y,y) \arrow[r, "\alpha_c"] & b \\
 & G(c,c,d',d') \arrow[ur, "\varepsilon^c_{d'}"'] & &
\end{tikzcd}$$

commutes. By fixing $d \in \mathcal{D}$ and for $g : c' \to c$, we get

$$\begin{tikzcd}
& G(c,c,d,d) \arrow[rrr, "\varepsilon_d^c"] & & & \int^y G(c,c,y,y) \arrow[ddrr, "\alpha_c"] & & \\
& & \text{(1)} & & & & \\
G(c,c',d,d) \arrow[uur, "{G(c,g,d,d)}"] \arrow[rrr, "\varepsilon_d"] \arrow[ddr, "{G(g,c',d,d)}"'] & & & \int^y G(c,c',y,y) \arrow[uur, "{\int^y G(c,g,y,y)}"] \arrow[ddr, "{\int^y G(g,c',y,y)}"'] & & \text{(3)} & b \\
& & \text{(2)} & & & & \\
& G(c',c',d,d) \arrow[rrr, "\varepsilon_d^{c'}"'] & & & \int^y G(c',c',y,y) \arrow[uurr, "\alpha_{c'}"'] & &
\end{tikzcd}$$

(1) commutes since by definition $\int^y G(c,g,y,y)$ is the mediating arrow defined by the wedge $\varepsilon^c \circ G(c,g,-,-)$ (by Proposition 3.2.9).

(2) commutes for the same reason as (1) but applied to the arrow $\int^y G(g,c',y,y)$.

(3) commutes because α is a wedge.

The inverse is defined by

$$(\gamma^{-1}(\beta))_c = U_{\int^y G(c,c,y,y)}[\beta_{c,-}] : \int^y G(c,c,y,y) \to b.$$

That is: fix c to get a wedge from $G(c,c,-,-)$ to b then take the mediating arrow defined by this wedge. We must verify that this collection defines a wedge. By Lemma 3.1.2 we have that the family $\langle \beta_{c,d} \rangle_{d \in \mathcal{D}}$ is a wedge for any fixed c, and $\langle \beta_{c,d} \rangle_{c \in \mathcal{C}}$ is a wedge for any fixed d. Let $f : c \to c'$ be any arrow in $\mathcal{C}$ then $G(c,f,d,d) : G(c,c',d,d) \Rightarrow G(c,c,d,d)$ is a natural transformation, and the family of composites δ_d defined by

$$G(c,c',d,d) \xrightarrow{G(c,f,d,d)} G(c,c,d,d) \xrightarrow{\beta_{c,d}} b$$

is a wedge from $G(c,c',-,-)$ to b. Since, by the remarks above, we have a commuting diagram

$$\begin{tikzcd}
& G(c,c,d,d) \arrow[dr, "\beta_{c,d}"] & \\
G(c,c',d,d) \arrow[ur, "{G(c,f,d,d)}"] \arrow[dr, "{G(f,c',d,d)}"'] & & b, \\
& G(c',c',d,d) \arrow[ur, "\beta_{c',d}"'] &
\end{tikzcd}$$

δ_d is equal to $\beta_{c',d} \circ G(f,c',d,d)$. Now, let $g : d \to d'$, we have the following commuting diagram

$$\begin{tikzcd}
& G(c,c',d,d) \arrow[dr, "\omega_d"] \arrow[drr, bend left, "\delta_d"] & & \\
G(c,c',d',d) \arrow[ur] \arrow[dr] & & \int^y G(c,c',y,y) \arrow[r, dotted, "u"] & b \\
& G(c,c',d',d') \arrow[ur, "\omega_{d'}"] \arrow[urr, bend right, "\delta_{d'}"'] & &
\end{tikzcd}$$

where u is the unique mediating arrow, and ω the couniversal wedge. Consider the following diagram

$$\begin{tikzcd}
 & & & G(c,c,d,d) \arrow[dr, "\kappa_d"] \arrow[ddrr, bend left, "\beta_{c,d}"] & & \\
 & & G(c,c',d,d) \arrow[ur, "{G(c,f,d,d)}"] \arrow[dr, "\omega_d"] & (1) & \int^y G(c,c,y,y) \arrow[dr, "(\gamma^{-1}(\beta))_c"] & \\
 & G(c,c',d,d') \arrow[ur] \arrow[dr] & & \int^y G(c,c',y,y) \arrow[ur, "{\int^y G(c,f,y,y)}"] \arrow[dr, "{\int^y G(f,c',y,y)}"'] \arrow[rr, dotted, "(3)\ u"] & & b \\
 & & G(c,c',d',d') \arrow[ur, "\omega_{d'}"] \arrow[dr, "{G(f,c',d',d')}"'] & (2) & \int^y G(c',c',y,y) \arrow[ur, "(\gamma^{-1}(\beta))_{c'}"] & \\
 & & & G(c',c',d',d') \arrow[ur, "\kappa_{d'}"] \arrow[uurr, bend right, "\beta_{c',d'}"'] & &
\end{tikzcd}$$

where (1) and (2) commute by the Parameter Theorem 3.2.3. Recall that $\delta_d = \beta_{c,d} \circ G(c,f,d,d)$ and $\delta_{d'} = \beta_{c',d'} \circ G(f,c',d',d')$, now since u is unique, (3) commutes as required.

Moreover, γ so defined is natural in b, so we get

$$\begin{aligned}
\mathcal{B}(\textstyle\int^x \int^y G(x,x,y,y),-) \quad &\overset{\psi^{\lambda c_1,c_2.\int^y G(c_1,c_2,y,y)}}{\cong} \quad \mathrm{Dinat}(\lambda c_1,c_2.\textstyle\int^y G(c_1,c_2,y,y),\Delta-) \\
&\overset{\gamma}{\cong} \quad \mathrm{Dinat}(G,\Delta-)
\end{aligned}$$

(ψ is defined in Proposition 3.2.10). In other words $(\int^x \int^y G(x,x,y,y), \gamma \circ \psi^{\lambda c_1,c_2.\int^y G(c_1,c_2,y,y)})$ is a representation for the covariant functor $\mathrm{Dinat}(G,\Delta-)$. Consider the functor $\lambda G,b.\,\mathrm{Dinat}(G,\Delta b) : [\mathcal{D}^{\mathbb{I}^{op}\times\mathbb{I}}]^{op} \times \mathcal{B} \to \mathrm{Set}$. By parameterized representability we have

$$\mathcal{B}(\int^x \int^y G(x,x,y,y),b) \overset{\gamma^G_b \circ \psi_b^{\lambda c_1,c_2.\int^y G(c_1,c_2,y,y)}}{\cong} \mathrm{Dinat}(G,\Delta b)$$

is natural in G. Since by Proposition 3.2.10

$$\mathcal{B}(\int^{(x,y)} G(x,x,y,y),b) \overset{\psi^G_b}{\cong} \mathrm{Dinat}(G,\Delta b)$$

we get an isomorphism

$$\mathcal{B}(\int^x \int^y G(x,x,y,y),b) \overset{(\psi^{-1})^G_b \gamma^G_b \psi_b^{\lambda c_1,c_2.\int^y G(c_1,c_2,y,y)}}{\cong} \mathcal{B}(\int^{(x,y)} G(x,x,y,y),b)$$

natural in b, G. By Corollary 3.2.8 we then have

$$\int^{(x,y)} G(x,x,y,y) \overset{\delta_G}{\cong} \int^x \int^y G(x,x,y,y)$$

natural in G, where by theorem 3.2.7

$$\begin{array}{rcl}
\delta_G & = & ((\psi^{-1})^G\gamma^G\psi^{\lambda c_1,c_2.\int^y G(c_1,c_2,y,y)})_{\int^x\int^y G}(\mathrm{id}_{\int^x\int^y G}) \\
& = & ((\psi^{-1})\gamma)^G_{\int^x\int^y G}(\kappa^{\lambda c_1,c_2.\int^y G(c_1,c_2,y,y)}) \\
& = & (\psi^{-1})^G_{\int^x\int^y G}(\kappa^{\lambda c_1,c_2.\int^y G(c_1,c_2,y,y)}\circ\varepsilon) \\
& = & U_{\int^{(x,y)} G}[\kappa^{\lambda c_1,c_2.\int^y G(c_1,c_2,y,y)}\circ\varepsilon].
\end{array}$$

The following diagram may help illustrate.

$$\begin{tikzcd}
G(c,c,d,d) \arrow[r, "\varepsilon^c_d"] \arrow[dr, "\kappa^G_{c,d}"'] & \int^y G(c,c,y,y) \arrow[dr, "\kappa_c^{\lambda c_1,c_2.\int^y G(c_1,c_2,y,y)}"'] & \\
& \int^{(x,y)} G(x,x,y,y) \arrow[r, dotted, "\delta_G"'] & \int^x\int^y G(x,x,y,y)
\end{tikzcd}$$

□

Example 3.2.12. *If G is a set valued functor,*

$$\begin{array}{rcl}
\delta_G([\langle c,d\rangle,x]_\sim) & = & \kappa_c^{\lambda c_1,c_2.\int^y G}\circ\varepsilon^c_d(x) \\
& = & \kappa_c^{\lambda c_1,c_2.\int^y G}([d,x]_\sim) \\
& = & [c,[d,x]_\sim]_\sim
\end{array}$$

for $x \in G(c,c,d,d)$.

Analogously, if for every pair $d_1, d_2 \in \mathcal{D}$ the coend $\int^x G(x,x,d_1,d_2)$ exists in $\mathcal{B}$, then there is an isomorphism

$$\int^y\int^x G(x,x,y,y) \overset{\phi_G}{\cong} \int^{(x,y)} G(x,x,y,y)$$

natural in G, defined by

$$\phi_G = (\psi^{-1})^{\lambda d_1,d_2.\int^x G(x,x,d_1,d_2)}_{\int^{(x,y)} G} \circ \delta^{-1} \circ \psi^G_{\int^{(x,y)} G}(\mathrm{id}_{\int^{(x,y)} G}).$$

Example 3.2.13. *If G is set valued, $\phi_G([d,[c,x]_\sim]_\sim) = [\langle c,d\rangle,x]_\sim$.*

Corollary 3.2.14 (Fubini). *There is an isomorphism*

$$\int^x\int^y G(x,x,y,y) \cong \int^y\int^x G(x,x,y,y)$$

meaning one side exists if and only if the other side does, and it is natural in G.

Example 3.2.15. *If G is set valued, $\delta_G \circ \phi_G([d,[c,x]_\sim]_\sim) = [c,[d,x]_\sim]_\sim$.*

Proposition 3.2.16. *Let $\mathcal{B}$ be a ccc and F, G functors $F : \mathcal{C}^{op} \to \mathcal{B}$, $G : \mathcal{C} \times \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{B}$. If for every $c \in \mathcal{C}$ the coend $\int^y Fc \times G(c,y,y)$ exists in $\mathcal{B}$, then there is a natural isomorphism*

$$Fc \times \int^y G(c',y,y) \overset{\phi_{c,c'}}{\cong} \int^y Fc \times G(c',y,y)$$

Proof: Since $\mathcal{B}$ is ccc, the product functor $Fc \times - : \mathcal{B} \to \mathcal{B}$ has a right adjoint and therefore preserves colimits ([Oos] p.49). We have seen that a coend can be expressed as a colimit using the category $\mathcal{D}^{\#}$, so for each c, c',

$$\begin{array}{rcl} Fc \times \int^y G(c', y, y) & = & Fc \times \mathrm{colim}_{\mathcal{D}^{\#}}(G^{\#}(c', -, -)) \\ & \cong & \mathrm{colim}_{\mathcal{D}^{\#}}(Fc \times G^{\#}(c', -, -)) \\ & = & \mathrm{colim}_{\mathcal{D}^{\#}}((Fc \times G(c', -, -))^{\#}) \\ & = & \int^y Fc \times G(c', y, y). \end{array}$$

This means that for each c, c' we have a representation

$$\lambda D.\mathcal{D}(Fc \times \int^y G(c', y, y), D) \overset{\psi_{c,c'}}{\cong} \lambda D.\,\mathrm{Dinat}(Fc \times G(c', -, -), \Delta D).$$

By parameterized representability 3.2.7 this isomorphism is natural in c, c'. Since by Proposition 3.2.9 we also have an isomorphism

$$\mathcal{D}(\int^y Fc \times G(c', y, y), D) \overset{\psi^D_{c,c'}}{\cong} \mathrm{Dinat}(Fc \times G(c', -, -), \Delta D)$$

natural in c, c', D, using Corollary 3.2.8 we find that there is an isomorphism $\phi_{c,c'}$ which is natural in c, c' as claimed. ☐

Corollary 3.2.17. *For $F : \mathcal{C}^{op} \to \mathcal{B}$ and $G : \mathcal{C} \times \mathcal{D}^{op} \times \mathcal{D} \to B$, and $\mathcal{B}$ ccc there is an isomorphism*

$$\int^x \int^y Fx \times G(x, y, y) \cong \int^x Fx \times \int^y G(x, y, y)$$

natural in F, G.

Proof: By the proposition above, we have an isomorphism

$$\mathrm{Dinat}(\lambda c_1, c_2. \int^y Fc_1 \times G(c_2, y, y), \Delta b) \overset{\gamma_b = - \circ \phi}{\cong} \mathrm{Dinat}(\lambda c_1, c_2. Fc_1 \times \int^y G(c_2, y, y), \Delta b)$$

natural in b, since composing a natural transformation with a dinatural transformation yields a dinatural transformation. We have the following string of isomorphisms

$$\begin{array}{lcr} \mathcal{B}(\int^x \int^y Fx \times G(x, y, y), b) & \overset{\psi_b^{\lambda c_1, c_2. \int^y Fc_1 \times G(c_2, y, y)}}{\cong} & \mathrm{Dinat}(\lambda c_1, c_2. \int^y Fc_1 \times G(c_2, y, y), \Delta b) \\ & \overset{\gamma_b}{\cong} & \mathrm{Dinat}(\lambda c_1, c_2. Fc_1 \times \int^y G(c_2, y, y), \Delta b) \\ & \overset{(\psi^{-1})_b^{Fc_1 \times \int^y G(c_2, y, y)}}{\cong} & \mathcal{B}(\int^x Fx \times \int^y G(x, y, y), b). \end{array}$$

By parameterized representability we get an isomorphism $\int^x \int^y Fx \times G(x, y, y) \cong \int^x Fx \times \int^y G(x, y, y)$ that is natural in F, G. ☐

Example 3.2.18. *For set valued functors F, G the isomorphism is defined by the mapping*

$$[U, [V, (a, b)]_\sim]_\sim \mapsto [U, (a, [V, b]_\sim)]_\sim$$

for $a \in FU, b \in G(U, V, V)$.

3.2.6 Density

Proposition 3.2.19 (Density). *Let $F \in \mathrm{Set}^{\mathcal{C}^{op}}$. There is an isomorphism*

$$FU \cong \int^{W} FW \times \mathcal{C}(U,W)$$

natural in F, U.

Proof: For any $G \in \mathrm{Set}^{\mathcal{C}^{op}}$

$$\begin{array}{lll}
 & \mathrm{Set}^{\mathcal{C}^{op}}(\int^{W} FW \times \mathbf{y}W, G) & \\
= & \int_U \mathrm{Hom}((\int^{W} FW \times \mathbf{y}W)U, GU) & \text{by naturality formula 3.1} \\
= & \int_U \mathrm{Hom}(\int^{W} FW \times \mathcal{C}(U,W), GU) & \text{as coends are computed point-wise.} \\
\cong & \int_U \int_W \mathrm{Hom}(FW \times \mathcal{C}(U,W), GU) & \text{by 3.6} \\
\cong & \int_U \int_W \mathrm{Hom}(FW, \mathrm{Hom}(\mathcal{C}(U,W), GU)) & \text{by currying} \\
\cong & \int_W \int_U \mathrm{Hom}(FW, \mathrm{Hom}(\mathcal{C}(U,W), GU)) & \text{by Fubini} \\
\cong & \int_W \mathrm{Hom}(FW, \int_U \mathrm{Hom}(\mathcal{C}(U,W), GU)) & \text{by 3.7} \\
\cong & \int_W \mathrm{Hom}(FW, GW) & \text{by 3.3} \\
\cong & \mathrm{Set}^{\mathcal{C}^{op}}(F, G) & \text{by naturality formula 3.1.}
\end{array}$$

All natural in F, G. The iso $F \cong \int^{W} FW \times \mathbf{y}W$ follows from Prop.2.5.8 and is found by following $\mathrm{id}_{\int^{W} FW \times \mathbf{y}W}$ through the equations. It is given by $\tilde{\kappa}^{U}_{U}(-)(\mathrm{id}_W)$, where $\tilde{\kappa}^{U}$ is the curried version of the couniversal wedge $\kappa^{U} : F \times \mathcal{C}(U,-) \mathrel{\ddot{\rightarrow}} \int^{W} FW \times \mathcal{C}(U,W)$ (which is just the inclusion), so $\tilde{\kappa}^{U}_{U}(x)(\mathrm{id}_U) = [U,(x,\mathrm{id}_U)]_{\sim}$ for $x \in FU$, and the inverse is given by $[V,(y,f_{U\to V})]_{\sim} \mapsto F(f)(y)$. □

Chapter 4

Doubly closed categories

Literature: [Pym02]

This chapter presents an original and detailed description of Day's construction for presheaves, followed by original proofs of the properties that the Yoneda embedding preserves the monoidal closed structure (this is important for the completeness proof of propositional BI) and that Day's tensor product does not preserve monos and pullbacks. The latter is an interesting result since a soundness proof of predicate BI[1] presumably requires preservation of pullbacks.

In section 4.3 we study Day's construction for sheaves. It has been claimed (in [Pym02] and [Yan02]) that Day's construction works for sheaves over topological spaces, we conjecture that Day's tensor product does not restrict to sheaves and present a counter example showing that whether or not the conjecture holds, there can be no right adjoint in the general case. A monoidal tensor product for Grothendieck sheaves is then constructed using Day's tensor together with the associated sheaf functor, and it is shown that under certain conditions this tensor has a right adjoint in the category of sheaves.

To interpret predicate logic in a topos one uses the Heyting algebra structure on each subobject lattice to interpret the logic. It is therefore relevant to ask whether a topos with a doubly closed structure induces a BI algebra structure on each subobject lattice. Propositional logic is modeled in Sub(1) so this subobject lattice will be of particular interest. In section 4.4 we present evidence that there is not a BI structure on every subobject lattice of a doubly closed topos, and prove that in many cases, even when the topos is not doubly closed, Sub(1) is a BI algebra. These results are necessary for the proofs of soundness and completeness that will be given in the next chapter.

4.1 Symmetric monoidal closed categories

Definition 4.1.1. *A category $\mathcal{C}$ is said to be monoidal if there is a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $e \in Obj(\mathcal{C})$, and natural isomorphisms*

$$\alpha_{a,b,c}\colon a \otimes (b \otimes c) \to (a \otimes b) \otimes c, \qquad \lambda_a\colon e \otimes a \to a, \qquad \rho_a\colon a \otimes e \to a,$$

---

[1] Here we mean predicate BI as suggested by Pym, see Appendix A.

so that

$$\begin{tikzcd}
a\otimes(b\otimes(c\otimes d)) \arrow[rr, "\alpha"] \arrow[d, "1\otimes\alpha"] & & (a\otimes b)\otimes(c\otimes d) \arrow[d, "\alpha"] \\
a\otimes((b\otimes c)\otimes d) \arrow[dr, "\alpha"] & & ((a\otimes b)\otimes c)\otimes d \\
& (a\otimes(b\otimes c))\otimes d \arrow[ur, "\alpha\otimes 1"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
a\otimes(e\otimes b) \arrow[dr, "1\otimes\lambda"] \arrow[dd, "\alpha"'] & \\
& a\otimes b \\
(a\otimes e)\otimes b \arrow[ur, "\rho\otimes 1"'] &
\end{tikzcd}$$

commute. $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho)$ *is then a monoidal category.*

Definition 4.1.2. *A monoidal category* $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho)$ *is said to be symmetric if there is a natural isomorphism*

$$\gamma_{a,b} \colon a \otimes b \to b \otimes a$$

so that

$$\begin{tikzcd}
a\otimes b \arrow[rr, "id"] \arrow[dr, "\gamma"'] & & a\otimes b \\
& b\otimes a \arrow[ur, "\gamma"'] &
\end{tikzcd}
,\qquad
\begin{tikzcd}
a\otimes e \arrow[rr, "\rho"] \arrow[dr, "\gamma"'] & & a \\
& e\otimes a \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
a\otimes(b\otimes c) \arrow[r, "\alpha"] \arrow[d, "1\otimes\gamma"'] & (a\otimes b)\otimes c \arrow[dr, "\gamma"] & \\
a\otimes(c\otimes b) \arrow[dr, "\alpha"'] & & c\otimes(a\otimes b) \arrow[d, "\alpha"] \\
& (a\otimes c)\otimes b \arrow[r, "\gamma\otimes 1"'] & (c\otimes a)\otimes b
\end{tikzcd}$$

commute. $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ *is then a symmetric monoidal category.*

A preordered monoid considered as a category, i.e., with an arrow $a \to b$ iff $a \leq b$ is an example of a monoidal category (this is probably where the name comes from). If the preordered monoid is commutative, the category is symmetric monoidal.

In a category with finite products $(\times, 1)$ is a monoidal tensor product. The same is true for coproducts with the initial object as the unit.

Definition 4.1.3. *A symmetric monoidal category* $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ *is called closed if the functor*

$$- \otimes b \colon \mathcal{C} \to \mathcal{C}$$

has a right adjoint $b \multimap -$, *for all* $b \in Obj(\mathcal{C})$. $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho, \gamma, (b \multimap -)_{b \in Obj(\mathcal{C})})$ *is then a symmetric monoidal closed category (smcc).*

A Cartesian closed category is in particular smcc.

Definition 4.1.4 (DCC, CDCC). *A doubly closed category (DCC) is a category equipped with two monoidal closed structures. A DCC is Cartesian (CDCC) if one of the closed structures is Cartesian and the other is symmetric monoidal and bi-Cartesian (bi-CDCC) if it also has finite coproducts.*

In the next section we shall see how to construct bi-CDCC's from a class of well known ccc's.

4.2 Day's construction

We now show that if $\mathcal{C}$ is small, monoidal but not necessarily closed, then $\mathrm{Set}^{\mathcal{C}^{op}}$ is monoidal closed, and if $\mathcal{C}$ is symmetric monoidal then so is $\mathrm{Set}^{\mathcal{C}^{op}}$. Put together with the result that $\mathrm{Set}^{\mathcal{C}^{op}}$ is bi-Cartesian closed we get a whole class of bi-CDCC's.

In the following let $(\mathcal{C}, \cdot, e, \overline{\alpha}, \overline{\lambda}, \overline{\rho})$ be a small, monoidal category. For objects E, F in $\widehat{\mathcal{C}}$, we define

$$(E \otimes F)x = \int^{y,y'} Ey \times Fy' \times \mathcal{C}(x, y \cdot y').$$

The coend exists since $\mathcal{C}$ is cocomplete and by Proposition 3.2.9 it does indeed define a functor, so $E \otimes F$ is an object of $\widehat{\mathcal{C}}$. By example 3.2.4 we have the following concrete definition of $(E \otimes F)(x)$

$$\int^{y,y'} Ey \times Fy' \times \mathcal{C}(x, y \cdot y') = \biguplus_{(y,y' \in \mathrm{Obj}(\mathcal{C} \times \mathcal{C}))} Ey \times Fy' \times \mathcal{C}(x, y \cdot y')/\sim$$

where $\sim$ is the symmetric, reflexive transitive closure of $\simeq$ defined by:

$$\begin{aligned} & ((y,y'), a, b, h_{x \to y \cdot y'}) \simeq ((z,z'), a', b', h'_{x \to z \cdot z'}) \\ \overset{def}{\Leftrightarrow}\ & \exists (f,g) : (z,z') \to (y,y') \in \mathrm{arr}(\mathcal{C} \times \mathcal{C}).\, E(f)(a) = a' \wedge F(g)(b) = b' \wedge (f \cdot g) \circ h' = h \end{aligned}$$

$$\begin{tikzcd}
(y, & y', & a, \arrow[d, "E(f)"] & b, \arrow[d, "F(g)"] & h_{x \to y \cdot y'}) \\
(z, \arrow[u, "f"'] & z', \arrow[u, "g"'] & a', & b', & h'_{x \to z \cdot z'}) \arrow[u, "f \cdot g"']
\end{tikzcd}$$

Definition 4.2.1. *If two elements $((y,y'), a, b, h_{x \to y \cdot y'})$ and $((z,z'), a', b', h'_{x \to z \cdot z'})$ are related by $\simeq$, i.e., they are in the generating relation, we say that they are* atomic equivalent*.*

For any $\bar{y} \in Ey \times Fy' \times \mathcal{C}(x, y \cdot y')$ and $\bar{z} \in Ez \times Fz' \times \mathcal{C}(x, z \cdot z')$ such that $\bar{y} \sim \bar{z}$ there is a finite sequence $\bar{x}_1, \ldots, \bar{x}_n$ of atomic equivalences such that

$$\bar{y} \simeq \bar{x}_1 \simeq \bar{x}_2 \simeq \cdots \simeq \bar{x}_n \simeq \bar{z}.$$

Example 4.2.2. *Suppose $(\mathcal{M}, \cdot, e)$ is a preordered commutative monoid and $E, F \in \widehat{\mathcal{M}}$, an element $x \in En \times Fn' \times \mathcal{M}(m, n \cdot n')$ has the form $x = (x_n, x_{n'}, m \leq n \cdot n')$. x is atomic equivalent to an element $y = (y_s, y_{s'}, m \leq s \cdot s')$ if and only if $s \leq n$ and $s' \leq n'$ (or vice versa) and $E_{sn}(x_n) = y_s$ and $F_{s'n'}(x_{n'}) = y_{s'}$. Thus, two elements $x, y \in (E \otimes F)(m)$ are equivalent iff there exists a finite string of such atomic equivalences all sitting above m.*

If $(\mathcal{M}, \cdot, 1)$ is the preorder with only two elements $0 \leq 1$, and $\cdot$ is multiplication, then we have a preordered commutative monoid, and given functors $F, G : \mathcal{M}^{op} \to \mathrm{Set}$ the equivalence classes are simply

$$(F \otimes G)(0) = F(0) \times G(0) \qquad (F \otimes G)(1) = F(1) \times G(1).$$

Lemma 4.2.3. *There is an isomorphism*

$$\mathcal{C}(x, y \cdot w) \overset{\psi_x}{\cong} \int^{y'} \mathcal{C}(y', w) \times \mathcal{C}(x, y \cdot y')$$

natural in X.

Proof: The proof is very much like the the proof of density. Let G be any object in $\mathrm{Set}^{\mathcal{C}^{op}}$, then

$$\begin{array}{lll}
 & \mathrm{Set}^{\mathcal{C}^{op}}(\int^{y'} \mathcal{C}(y', w) \times \mathcal{C}(-, y \cdot y'), G) & \\
= & \int_u \mathrm{Hom}((\int^{y'} \mathcal{C}(y', w) \times \mathcal{C}(-, y \cdot y'))u, Gu) & \text{by naturality formula 3.1} \\
= & \int_u \mathrm{Hom}(\int^{y'} \mathcal{C}(y', w) \times \mathcal{C}(u, y \cdot y'), Gu) & \text{since coends are computed pointwise} \\
\cong & \int_u \int_{y'} \mathrm{Hom}(\mathcal{C}(y', w) \times \mathcal{C}(u, y \cdot y'), Gu) & \text{by 3.6} \\
\cong & \int_u \int_{y'} \mathrm{Hom}(\mathcal{C}(y', w), \mathrm{Hom}(\mathcal{C}(u, y \cdot y'), Gu)) & \text{by currying} \\
\cong & \int_{y'} \int_u \mathrm{Hom}(\mathcal{C}(y', w), \mathrm{Hom}(\mathcal{C}(u, y \cdot y'), Gu)) & \text{by Fubini} \\
\cong & \int_{y'} \mathrm{Hom}(\mathcal{C}(y', w), \int_u \mathrm{Hom}(\mathcal{C}(u, y \cdot y'), Gu)) & \text{by 3.7} \\
= & \int_{y'} \mathrm{Hom}(\mathcal{C}(y', w), \mathrm{Set}^{\mathcal{C}^{op}}(\mathbf{y}(y \cdot y'), G)) & \text{by naturality formula 3.1} \\
\cong & \int_{y'} \mathrm{Hom}(\mathcal{C}(y', w), G(y \cdot y')) & \text{by the Yoneda Lemma} \\
\cong & \mathrm{Set}^{\mathcal{C}^{op}}(\mathbf{y}w, G(y \cdot -)) & \text{by naturality formula 3.1} \\
\cong & G(y \cdot w) & \text{by the Yoneda Lemma} \\
\cong & \mathrm{Set}^{\mathcal{C}^{op}}(\mathbf{y}(y \cdot w), G) & \text{by the Yoneda Lemma}
\end{array}$$

All natural in G. Since Yoneda is full and faithful and $\widehat{\mathcal{C}}$ is locally small, we have $\int^{y'} \mathcal{C}(y', w) \times \mathcal{C}(-, y \cdot y') \cong \mathbf{y}(y \cdot w)$ in $\widehat{\mathcal{C}}$, which means that there is a natural isomorphism between them. As usual the isomorphism is found by putting $G = \int^{y'} \mathcal{C}(y', w) \times \mathcal{C}(-, y \cdot y')$ and following the identity on G through the equations. □

Remark 4.2.4. *We will need the concrete definition of the isomorphism ψ. First notice that for all $[u, (f_{u \to w}, g_{x \to y \cdot u})]_\sim \in \int^{y'} \mathcal{C}(y', w) \times \mathcal{C}(x, y \cdot y')$ we have*

$$(u, (f, g)) \simeq (w, (\mathrm{id}_w, (\mathrm{id}_y \cdot f) \circ g))$$

which implies

$$(u, (f, g)) \sim (v, (s, t)) \textit{ iff } (\mathrm{id}_y \cdot f) \circ g = (\mathrm{id}_y \cdot s) \circ t.$$

Now it is clear that the following is well-defined

$$\psi_x(h) = [w, (\mathrm{id}_w, h)]$$

and

$$\psi_x^{-1}([u, (f, g)]_\sim) = (\mathrm{id}_y \cdot f) \circ g.$$

Similarly we have

Lemma 4.2.5.

$$\mathcal{C}(x, w \cdot y') \overset{\psi'_x}{\cong} \int^{y} \mathcal{C}(y, w) \times C(x, y \cdot y')$$

with

$$\psi'_x(h) = [w, (\mathrm{id}_w, h)]_\sim$$

and

$$\psi'^{-1}_x([u, (f, g)]_\sim) = (f \cdot \mathrm{id}_{y'}) \circ g.$$

The unit of the tensor product $\otimes$ is $\mathcal{C}(-,e)$, Yoneda taken on the unit of the monoidal structure on $\mathcal{C}$. Let us show that it actually works, i.e., that we have a natural isomorphism

$$\mathcal{C}(-,e)\otimes F \overset{\lambda_F}{\cong} F.$$

$$\begin{array}{rcll}
\mathcal{C}(-,e)\otimes F & = & \lambda x.\int^{y,y'}\mathcal{C}(y,e)\times Fy'\times\mathcal{C}(x,y\cdot y') & \\
& \cong & \lambda x.\int^{y}\int^{y'}\mathcal{C}(y,e)\times Fy'\times\mathcal{C}(x,y\cdot y') & \text{by Prop. 3.2.11}\\
& \cong & \lambda x.\int^{y'}\int^{y}\mathcal{C}(y,e)\times Fy'\times\mathcal{C}(x,y\cdot y') & \text{by Fubini}\\
& \cong & \lambda x.\int^{y'}Fy'\times\int^{y}\mathcal{C}(y,e)\times\mathcal{C}(x,y\cdot y') & \text{by 3.2.17}\\
& \cong & \lambda x.\int^{y'}Fy'\times\mathcal{C}(x,e\cdot y') & \text{by Lemma 4.2.5}\\
& \cong & \lambda x.\int^{y'}Fy'\times\mathcal{C}(x,y') & \text{by the monoid structure on } \mathcal{C}\\
& \cong & F & \text{by Density.}
\end{array}$$

All the isomorphisms above are natural in F. Each component λ_F is a natural transformation

$$\lambda_F : (\mathbf{y}e\otimes F)\Rightarrow F.$$

Since we know all the isomorphisms involved, we can calculate the concrete action of λ_F. The elements of the set $(\mathbf{y}e\otimes F)x$ are equivalence classes $[u,u',f_{u\to e},b\in Fu',g_{x\to u\cdot u'}]_\sim$. λ_F^x sends this element to $\tilde{b} := F(\overline{\lambda}_{u'}\circ(f\cdot\mathrm{id}_{u'})\circ g)(b)\in Fx$, where $\overline{\lambda}_{u'} : e\cdot u'\to u'$ comes from the monoid structure on $\mathcal{C}$. The inverse is defined by

$$(\lambda_F^x)^{-1}(\tilde{b}) = [e,x,\mathrm{id}_e,\tilde{b},\overline{\lambda}_x^{-1}]_\sim.$$

To see that it is well-defined, note that $[u,u',f_{u\to e},b\in Fu',g_{x\to u\cdot u'}]_\sim = [e,x,\mathrm{id}_e,\tilde{b},\overline{\lambda}_x^{-1}]_\sim$.

To see that we have an isomorphism

$$F\otimes\mathcal{C}(-,e) \overset{\rho_F}{\cong} F$$

natural in F, consider the following equations

$$\begin{array}{rcll}
F\otimes\mathcal{C}(-,e) & = & \lambda x.\int^{y,y'}Fy\times\mathcal{C}(y',e)\times\mathcal{C}(x,y\cdot y') & \\
& \cong & \lambda x.\int^{y}\int^{y'}Fy\times\mathcal{C}(y',e)\times\mathcal{C}(x,y\cdot y') & \text{by Prop. 3.2.11}\\
& \cong & \lambda x.\int^{y}Fy\times\int^{y'}\mathcal{C}(y',e)\times\mathcal{C}(x,y\cdot y') & \text{by Corollary 3.2.17}\\
& \cong & \lambda x.\int^{y}Fy\times\mathcal{C}(x,y\cdot e) & \text{by Lemma 4.2.3}\\
& \cong & \lambda x.\int^{y}Fy\times\mathcal{C}(x,y) & \text{by } \overline{\rho}_y : y\cdot e\to y\\
& \cong & F & \text{by Density.}
\end{array}$$

All the above isomorphisms are natural in F. ρ_F^x is defined by

$$\rho_F^x([u,u',a\in Fu,s_{u'\to e},g_{x\to u\cdot u'}]_\sim) = F(\overline{\rho}_u\circ(\mathrm{id}_u\cdot s)\circ g)(a)$$

well-definedness can be verified like in the case for λ. Finally, the associativity part, a natural isomorphism

$$\alpha_{E,F,G} : E\otimes(F\otimes G)\to(E\otimes F)\otimes G$$

comes from

$$\begin{array}{lll}
 & E \otimes (F \otimes G) & \\
= & \lambda x. \int^{y,y'} Ey \times (\int^{z,z'} Fz \times Gz' \times \mathcal{C}(y', z \cdot z')) \times \mathcal{C}(x, y \cdot y') & \\
\cong & \lambda x. \int^{y,y'} \int^{z,z'} Ey \times Fz \times Gz' \times \mathcal{C}(y', z \cdot z') \times \mathcal{C}(x, y \cdot y') & \text{by Corollary 3.2.17} \\
\cong & \lambda x. \int^{z,z'} \int^{y,y'} Ey \times Fz \times Gz' \times \mathcal{C}(y', z \cdot z') \times \mathcal{C}(x, y \cdot y') & \text{by Fubini} \\
\cong & \lambda x. \int^{z,z'} \int^{y} Ey \times Fz \times Gz' \times \int^{y'} \mathcal{C}(y', z \cdot z') \times \mathcal{C}(x, y \cdot y') & \text{by 3.2.11 and 3.2.17} \\
\cong & \lambda x. \int^{z,z'} \int^{y} Ey \times Fz \times Gz' \times \mathcal{C}(x, y \cdot (z \cdot z')) & \text{by Lemma 4.2.3} \\
\cong & \lambda x. \int^{z,z'} \int^{y} Ey \times Fz \times Gz' \times \mathcal{C}(x, (y \cdot z) \cdot z') & \text{using } \overline{\alpha} \\
\cong & \lambda x. \int^{z,z'} \int^{y} Ey \times Fz \times Gz' \times \int^{y'} \mathcal{C}(y', y \cdot z) \times \mathcal{C}(x, y' \cdot z') & \text{by Lemma 4.2.5} \\
\cong & \lambda x. \int^{y',z'} (\int^{y,z} Ey \times Fz \times \mathcal{C}(y', y \cdot z)) \times Gz' \times \mathcal{C}(x, y' \cdot z') & \text{by 3.2.17, 3.2.11 and Fubini} \\
= & (E \otimes F) \otimes G. &
\end{array}$$

Given points $a, b, c \in Eu, Fv, Gw$, we can associate an equivalence class $[[a, b], c] \in ((E \otimes F) \otimes G)((u \cdot v) \cdot w)$ by putting

$$[[a, b], c] := [u \cdot v, w, [u, v, a, b, \mathrm{id}_{u \cdot v}], c, \mathrm{id}_{(u \cdot v) \cdot w}],$$

$[a, b] = [u, v, a, b, \mathrm{id}_{u \cdot v}]$ is known as *Day's pairing* of a and b. Now we can define α:

$$\begin{array}{rcr}
 & \alpha^x_{E,F,G}([u, u', a, [v, v', b, c, g_{u' \to v \cdot v'}], f_{x \to u \cdot u'}]) & (4.1) \\
= & ((E \otimes F) \otimes G)(\overline{\alpha} \circ (\mathrm{id}_u \cdot g) \circ f)([[a, b], c]) & (4.2) \\
= & [u \cdot v, v'[u, v, a, b, \mathrm{id}_{u \cdot v}], c, \overline{\alpha} \circ \mathrm{id}_u \cdot g \circ f] & (4.3)
\end{array}$$

4.2.1 Coherence laws

Showing the coherence laws is not possible without "getting the fingers dirty". Let us show that for all objects x of $\mathcal{C}$, we have a commuting triangle

$$\begin{tikzcd}
(E \otimes (\mathcal{C}(-, e) \otimes F))x \arrow[d, "\alpha^x"'] \arrow[dr, "(E \otimes \lambda)^x"] & \\
 & (E \otimes F)x\,. \\
((E \otimes \mathcal{C}(-, e)) \otimes F)x \arrow[ur, "(\rho \otimes F)^x"'] &
\end{tikzcd}$$

Let $[u, u', a, [v, v', h, c, g_{u' \to v \cdot v'}], f_{x \to u \cdot u'}]$ be an element of $(E \otimes (\mathcal{C}(-, e) \otimes F))x$. Then

$$\begin{array}{ll}
[u, u', a, [v, v', h, c, g_{u' \to v \cdot v'}], f_{x \to u \cdot u'}] & \overset{\alpha^x}{\longmapsto} \\
[u \cdot v, v', [u, v, a, h, \mathrm{id}_{u \cdot v}], c, \overline{\alpha} \circ \mathrm{id}_u \cdot g \circ f] & \overset{(\rho \otimes F)^x}{\longmapsto} \\
[u \cdot v, v', E(\overline{\rho}_u \circ \mathrm{id}_u \cdot h)(a), c, \overline{\alpha} \circ \mathrm{id}_u \cdot g \circ f] &
\end{array}$$

and

$$\begin{array}{ll}
[u, u', a, [v, v', h, c, g], f] & \overset{(E \otimes \lambda)^x}{\longmapsto} \\
[u, u', a, F(\overline{\lambda}_{v'} \circ (h \cdot \mathrm{id}_{v'}) \circ g)(c), f] &
\end{array}$$

In order for the triangle to commute we need

$$[u\cdot v, v', \tilde{a} = E(\overline{\rho}_u \circ \mathrm{id}_u \cdot h)(a), c, \overline{\alpha} \circ \mathrm{id}_u \cdot g \circ f] = [u, u', a, \tilde{c} = F(\overline{\lambda}_{v'} \circ (h \cdot \mathrm{id}_{v'}) \circ g)(c), f]$$

which can be shown in two steps: With $(\mathrm{id}_u, \overline{\lambda}_{v'} \circ (h\cdot \mathrm{id}_{v'}) \circ g) : (u, u') \to (u, v')$ we get

$$(u, u', a, \tilde{c}, f) \simeq (u, v', a, c, \mathrm{id}_u \cdot (\overline{\lambda}_{v'} \circ (h \cdot \mathrm{id}_{v'}) \circ g) \circ f),$$

and with $(\overline{\rho}_u \circ \mathrm{id}_u \cdot h, \mathrm{id}_{v'}) : (u\cdot v, v') \to (u, v')$ we get

$$(u\cdot v, v', \tilde{a}, c, \overline{\alpha} \circ \mathrm{id}_u \cdot g \circ f) \simeq (u, v', a, c, \mathrm{id}_u \cdot (\overline{\lambda}_{v'} \circ (h \cdot \mathrm{id}_{v'}) \circ g) \circ f),$$

to see this, we use the coherence laws for the monoidal category $\mathcal{C}$, more specifically that

$$\mathrm{id}_u \cdot \overline{\lambda}_{v'} = \overline{\rho}_u \cdot \mathrm{id}_{v'} \circ \overline{\alpha}. \tag{4.4}$$

We need to show that

$$\begin{array}{rcll}
\mathrm{id}_u \cdot (\overline{\lambda}_{v'} \circ (h \cdot \mathrm{id}_{v'}) \circ g) \circ f & = & (\overline{\rho}_u \circ \mathrm{id}_u \cdot h) \cdot \mathrm{id}_{v'} \circ \overline{\alpha} \circ \mathrm{id}_u \cdot g \circ f, & \text{i.e.,} \\
(\mathrm{id}_u \cdot \overline{\lambda}_{v'} \circ \mathrm{id}_u) \cdot (h \cdot \mathrm{id}_{v'}) & = & \overline{\rho}_u \cdot \mathrm{id}_{v'} \circ (\mathrm{id}_u \cdot h) \cdot \mathrm{id}_{v'} \circ \overline{\alpha} & \\
 & = & (\overline{\rho}_u \cdot \mathrm{id}_{v'} \circ \overline{\alpha}) \circ \mathrm{id}_u \cdot (h \cdot \mathrm{id}_{v'}) & \text{by naturality of } \overline{\alpha} \\
 & = & \mathrm{id}_u \cdot \overline{\lambda}_{v'} \circ \mathrm{id}_u \cdot (h \cdot \mathrm{id}_{v'}) & \text{by 4.4 above.}
\end{array}$$

Similarly, the coherence law for α can be shown using the naturality and coherence law for $\overline{\alpha}$.

4.2.2 Symmetry

Proposition 4.2.6. *Suppose the monoidal category $(\mathcal{C}, \cdot, e)$ is also symmetric with $\overline{\gamma}_{a,b} : a\cdot b \to b \cdot a$, then the monoidal structure on $\mathrm{Set}^{\mathcal{C}^{op}}$ is symmetric too.*

Proof: We must show that there is an isomorphism

$$\gamma_{E,F} : E \otimes F \xrightarrow{\cong} F \otimes E$$

natural in E, F. This follows from

$$\begin{array}{rcll}
E \otimes F & = & \lambda x. \int^{y,y'} Ey \times Fy' \times \mathcal{C}(x, y\cdot y') & \\
 & \cong & \lambda x. \int^{y,y'} Fy' \times Ey \times \mathcal{C}(x, y\cdot y') & \text{by “swap”} \\
 & \cong & \lambda x. \int^{y,y'} Fy' \times Ey \times \mathcal{C}(x, y'\cdot y) & \text{since } (\mathcal{C}, \cdot, e) \text{ is symmetric} \\
 & \cong & \lambda x. \int^{y',y} Fy' \times Ey \times \mathcal{C}(x, y'\cdot y) & \text{by Fubini} \\
 & = & F \otimes E &
\end{array}$$

all natural in E, F. The action of γ on $[u, v, a, b, f_{x\to u\cdot v}]$ is $[v, u, b, a, \overline{\gamma} \circ f]$. It follows that

$$\begin{tikzcd}
E \otimes F \arrow[rr, "\mathrm{id}"] \arrow[dr, "\gamma_{E,F}"'] & & E \otimes F \\
 & F \otimes E \arrow[ur, "\gamma_{F,E}"'] &
\end{tikzcd}$$

commute. To see that

$$\begin{tikzcd}
E \otimes \mathcal{C}(-,e) \arrow[rr, "\rho_E"] \arrow[dr, "\gamma"'] & & E \\
& \mathcal{C}(-,e) \otimes E \arrow[ur, "\lambda_E"'] &
\end{tikzcd}$$

commute just calculate and then use the corresponding coherence law for $\overline{\gamma}, \overline{\rho}, \overline{\lambda}$ and the naturality of γ. The last coherence law we need to show in order to confirm symmetry (see Definition 4.1.2) can also be confirmed by a tedious calculation using the concrete definitions of the isomorphisms α and γ. ☐

4.2.3 Closed structure

Like the Cartesian product, the tensor product extends to a functor

$$- \otimes - : \widehat{\mathcal{C}} \times \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}.$$

We are already familiar with the object part, and for natural transformations $\alpha : E \to E'$ and $\beta : F \to F'$,

$$(\alpha \otimes \beta)_c : (E \otimes F)c \to (E' \otimes F')c$$

maps $[x, y, a \in Ex, b \in Fy, f]$ to $[x, y, \alpha_x(a), \beta_y(b), f]$. It is now a straight forward matter to verify that $\otimes$ satisfies the functor laws.

For each functor $F \in \widehat{\mathcal{C}}$ the functor

$$- \otimes F : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$$

has a right adjoint

$$F \multimap - : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$$

given by the formula

$$\begin{aligned}
(F \multimap T)(x) &= \int_y \mathrm{Set}(Fy, T(x \cdot y)) && (4.5)\\
&= \widehat{\mathcal{C}}(F(-), T(x \cdot -)) && (4.6)
\end{aligned}$$

where the latter equality follows from Naturality Formula 3.1. The morphism part is that of a hom-functor with the obvious modifications. To see that there is an isomorphism

$$\widehat{\mathcal{C}}(S \otimes F, T) \cong \widehat{\mathcal{C}}(S, F \multimap T)$$

natural in S, T, consider the following:

$$\begin{aligned}
& \widehat{\mathcal{C}}(S \otimes F, T) \\
= \;& \widehat{\mathcal{C}}(\textstyle\int^{y,y'} Sy \times Fy' \times \mathbf{y}(y \cdot y'), T) \\
\cong \;& \textstyle\int_{y,y'} \widehat{\mathcal{C}}(Sy \times Fy' \times \mathbf{y}(y \cdot y'), T) && \text{by 3.6} \\
\cong \;& \textstyle\int_{y,y'} \mathrm{Hom}(Sy \times Fy', \widehat{\mathcal{C}}(\mathbf{y}(y \cdot y'), T)) && \text{by currying} \\
\cong \;& \textstyle\int_{y,y'} \mathrm{Hom}(Sy \times Fy', T(y \cdot y')) && \text{by the Yoneda Lemma} \\
\cong \;& \mathrm{Nat}(S \times F, T(- \cdot -)) && \text{by Naturality Formula 3.1} \\
\cong \;& \mathrm{Nat}(\lambda y.Sy, \lambda y.\,\mathrm{Hom}_{\widehat{\mathcal{C}}}(F, T(y \cdot -))) && \text{by currying} \\
= \;& \widehat{\mathcal{C}}(S, F \multimap T).
\end{aligned}$$

We have shown:

Theorem 4.2.7. *If* $(\mathcal{C}, \cdot, e)$ *is a small, (symmetric) monoidal category, then* $(\widehat{\mathcal{C}}, \otimes, \mathbf{y}e, \multimap)$ *is a (symmetric) monoidal closed category.*

And, since the presheaf category $\widehat{\mathcal{C}}$ is bi-Cartesian:

Corollary 4.2.8. *If* $(\mathcal{C}, \cdot, e)$ *is a small, symmetric monoidal category, then* $(\widehat{\mathcal{C}}, \otimes, \mathbf{y}e, \multimap)$ *is a bi-CDCC.*

4.2.4 Properties of Day's construction

In this section we present some observations that are useful for working with the tensor product. Consider a small, monoidal closed category $(\mathcal{C}, \cdot, e, \twoheadrightarrow\!\!\ast)$.

Proposition 4.2.9. *The Yoneda functor* $\mathbf{y} : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{op}}$ *preserves the monoidal closed structure given by Day's construction.*

Proof: We have $I = \mathbf{y}e$ by definition. To see that $\mathbf{y}(m \cdot n) = \mathbf{y}m \otimes \mathbf{y}n$ consider the following equations

$$\begin{array}{rcll} \mathbf{y}m \otimes \mathbf{y}n & = & \lambda x. \int^{y,y'} \mathcal{C}(y,m) \times \mathcal{C}(y',n) \times \mathcal{C}(x, y \cdot y') & \\ & \cong & \lambda x. \int^{y} \mathcal{C}(y,m) \times \int^{y'} \mathcal{C}(y',n) \times \mathcal{C}(x, y \cdot y') & \text{by 3.2.17} \\ & \cong & \lambda x. \int^{y} \mathcal{C}(y,m) \times \mathcal{C}(x, y \cdot n) & \text{by 4.2.3} \\ & \cong & \lambda x. \mathcal{C}(x, m \cdot n) & \text{by 4.2.3} \\ & = & \mathbf{y}(m \cdot n). & \end{array}$$

Preservation of the closed structure follows from

$$\begin{array}{rcll} \mathbf{y}m \multimap \mathbf{y}n & = & \lambda x. \widehat{\mathcal{C}}(\mathbf{y}m, \mathrm{Hom}_{\mathcal{C}}(x \cdot -, n)) & \\ & \cong & \lambda x. \mathrm{Hom}_{\mathcal{C}}(x \cdot m, n) & \text{by the Yoneda Lemma} \\ & \cong & \lambda x. \mathrm{Hom}_{\mathcal{C}}(x, m \twoheadrightarrow\!\!\ast\, n) & \\ & = & \mathbf{y}(m \twoheadrightarrow\!\!\ast\, n) & \end{array}$$

□

With Day's construction the Yoneda functor gives us a way of embedding a symmetric monoidal closed category into a bi-CDCC.

Definition 4.2.10 (Topological monoid). *Suppose* $(X, \cdot, e)$ *is a (commutative) monoid, and* $\mathcal{O}(X)$ *a (open cover) topology on* X*, such that the pointwise defined maps*

$$\otimes : \mathcal{O}(X) \times \mathcal{O}(X) \to \mathcal{O}(X),$$

defined by $U \otimes V = \{u \cdot v \mid u \in U, v \in V\}$*, and*

$$I : (1) \to \mathcal{O}(X)$$

defined by $I(*) = \{e\}$ *are open. We then have a preordered (commutative) monoid which we call a* topological monoid $(\mathcal{O}(X), \otimes, \{e\})$ *where the order is inclusion.*

As an example, consider a monoid $(X, \cdot, e)$; the power set $\mathcal{P}(X)$ is a topological monoid, since all maps are open.

One thing that should be noted about this definition is that we require the maps to be defined pointwise using the composition in X. It is possible to define a monoid structure on a topological space which is not pointwise (just take as monoid composition a set operation like e.g. intersection), but the above definition gives us a host of *cover preserving* topologies.

Definition 4.2.11 (cover preserving). *Let $(\mathcal{C}, *, e)$ be a monoidal category with a basis for a Grothendieck topology K, then $*$ is called* cover preserving *if, for all covers $\{f_i : C_i \to C\} \in K(C)$ and $\{g_j : D_j \to D\} \in K(D)$, we have*

$$\{f_i * g_j : C_i * D_j \to C * D\} \in K(C * D).$$

If we have a topology J instead, and sieves $S_1 \in J(C), S_2 \in J(D)$, then J is cover preserving if $(S_1 * S_2) \in J(C * D)$, where $(S_1 * S_2)$ is the sieve generated by $S_1 * S_2$.

In [Yan02] and [Pym02] cover preserving is called the continuity property.

Lemma 4.2.12. *For a topological monoid $(\mathcal{O}(X), *, \{e\})$, $*$ is cover preserving.*

Proof: Let $\bigcup_{i\in I} U_i = U$ and $\bigcup_{j\in J} V_j = V$.

$$\bigcup_{i\in I}\bigcup_{j\in J}(U_i * V_j) = \bigcup_{i\in I} U_i * \bigcup_{j\in J} V_j = U * V.$$

□

Proposition 4.2.13. *Day's tensor product does* not *preserve monos nor pullbacks.*

Proof: It is enough to show that monos are not preserved, since preservation of pullbacks implies preservation of monos. Consider the monoid of natural numbers $(\mathbb{N}, +, 0)$, it induces a topological monoid on the power set $(\mathcal{P}(\mathbb{N}), +, \{0\})$ ordered by inclusion. Let $B : \mathcal{P}(\mathbb{N})^{op} \to$ Set be the constant presheaf defined by $B(S) = \{a, b\}$, were $a \neq b$. And let A be the subfunctor of B defined by

$$A(S) = \begin{cases} \{a, b\} & \text{if } S = \emptyset \text{ or } S = \{n\}, n \in \mathbb{N} \\ \{a\} & \text{otherwise.} \end{cases}$$

The inclusion $\iota : A \hookrightarrow B$ is a mono in $\widehat{\mathcal{P}(\mathbb{N})}$, but $\iota \otimes \mathrm{id}_B : A \otimes B \to B \otimes B$ is not:

We have $x := [b, b, \{2\} \subseteq \{1\} + \{0, 1\}] = [b, b, \{2\} \subseteq \{2\} + \{0, 1\}] =: y$ in $(B \otimes B)(\{2\})$ because

$$\begin{tikzcd}
 & & {[b,b,} & \{2\}\subseteq & \{1,2\}+ & \{0,1\}] & & \\
{[b,b,} & \{2\}\subseteq & \{1\}+ \arrow[urr] & \{0,1\}] \arrow[urr] & {[b,b,} & \{2\}\subseteq & \{2\}+ \arrow[ull] & \{0,1\}] \arrow[ull]
\end{tikzcd}$$

and $B(\{1\} \subseteq \{1, 2\})(b) = b$ and $B(\{2\} \subseteq \{1, 2\})(b) = b$ but $b \notin A(\{1, 2\})$. To see that $\iota \otimes \mathrm{id}_B$ is not mono we must show that $x \neq y$ in $(A \otimes B)(\{2\})$.

Suppose $x = y$ in $(A \otimes B)(\{2\})$ then there must be some finite string of atomic equivalences in $(A \otimes B)(\{2\})$ that has the form

$$\begin{tikzcd}
 & x_1 & & & & & x_n & \\
x \arrow[ur] & & x_2 \arrow[ul] & \cdots & \cdots & x_{n-1} \arrow[ur] & & y \arrow[ul]
\end{tikzcd} \tag{4.7}$$

The element x_1 must have either the form $[b, b, \{2\} \subseteq \{1\} + T_1]$ where $\{0, 1\} \subseteq T_1$ or it is an element of the form $[b, b, \{2\} \subseteq S_1 + T_1]$ where $\{1\} \subset S_1$ and $b \in A(S_1)$ and $\{0, 1\} \subseteq T_1$. The latter is not possible since $A(S_1) = \{a\}$ when S_1 is not a singleton or the empty set. x_2 must then be either $[b, b, \{2\} \subseteq \{1\} + T_2]$ where $T_2 \subseteq T_1$ or $[b, b, \{2\} \subseteq S_2 + T_2]$ where $T_2 \subseteq T_1$ and $S_2 \subset \{1\}$. The latter is not possible since $S_2 \subset \{1\}$ implies $S_2 = \emptyset$ and $\emptyset + T_2 = \emptyset \not\supseteq \{2\}$, so we must have $x_2 = [b, b, \{2\} \subseteq \{1\} + T_2]$. It follows that we must have $x_n = [b, b, \{2\} \subseteq \{1\} + T_n]$ for some $T_n \supseteq \{0, 1\}$, but there is no map between $\{1\}$ and $\{2\}$, which means that there can not be an atomic equivalence between x_n and y, so we conclude that $x \neq y$. □

Remark 4.2.14. *Let $(\mathcal{C}, \cdot, e)$ be a small monoidal category. Day's tensor product can also be constructed as a Kan extension. The monoid composition*

$$\sigma : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

defined by $\sigma(a, b) = a \cdot b$ defines a functor from $\widehat{\mathcal{C}}$ to $\widehat{\mathcal{C} \times \mathcal{C}}$ by precomposition. By a general result this functor has a left adjoint $\Sigma_\sigma : \widehat{\mathcal{C} \times \mathcal{C}} \to \widehat{\mathcal{C}}$ (the left Kan extension) and a right adjoint (the right Kan extension). This left adjoint can be used to define Day's tensor as follows: Let $\pi, \pi' : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{C}^{op}$ be the projections, and $A, B : \mathcal{C}^{op} \to \mathrm{Set}$, then the compositions $A\pi, B\pi'$ are functors from $(\mathcal{C} \times \mathcal{C})^{op}$ to Set, and the tensor of A and B is then

$$A \otimes B = \Sigma_\sigma A\pi \times B\pi',$$

and Day's pairing is the unit of this adjunction:

$$\eta : A\pi \times B\pi' \to (\Sigma_\sigma A\pi \times B\pi')\sigma = (A \otimes B)(- \cdot -).$$

4.3 Day's construction on sheaves

As the category of sheaves inherits a lot of structure from the presheaf category, it is natural to ask whether Day's construction gives rise to a monoidal closed structure on the category of sheaves. It turns out that we always have a monoidal tensor product while to get a closed structure (a right adjoint to the tensor) we need a stronger requirement. Demanding the tensor product of the source category to be both pullback preserving and cover preserving turns out to be adequate to ensure existence of the closed structure.

There are two (obvious) ways to try to define a tensor product $\otimes^{\mathrm{Sh}}$ on $\mathrm{Sh}(\mathcal{C}, J)$ using the tensor product $\otimes$ that we already have for $\widehat{\mathcal{C}}$. For sheaves E, F we can try

1. $E \otimes^{\mathrm{Sh}} F := iE \otimes iF$, or
2. $E \otimes^{\mathrm{Sh}} F := \mathbf{a}(iE \otimes iF)$,

For the former to be well-defined we must show that $iE \otimes iF$ is a sheaf. For the latter to work we must show that $\mathbf{a}(iE \otimes iF)$ actually defines a monoidal tensor product. Note that if the former is a sheaf then the two definitions are equivalent. We will show (Proposition 4.3.5) that if $E \otimes^{\mathrm{Sh}} F := \mathbf{a}(iE \otimes iF)$ has a right adjoint then it must be the one we have by Day's construction in the presheaf category. This means that in order to have a closed structure (whether we use the first or the second definition for $\otimes^{\mathrm{Sh}}$) we need $iE \multimap iF$ to be a sheaf whenever E and F are.

Lemma 4.3.1. *We have, for $P, Q \in \widehat{\mathcal{C}}$, an isomorphism*

$$\Theta_1 : \mathbf{a}(i\mathbf{a}P \otimes Q) \cong \mathbf{a}(P \otimes Q)$$

natural in P and Q. The inverse is defined by $\Theta_1^{-1} = \mathbf{a}(\eta_P \otimes Q)$, where η is the unit of the adjunction $\mathbf{a} \vdash i$. Moreover, if the presheaf P has the form iG for some sheaf G, then $\Theta_1 = \mathbf{a}(i\varepsilon_G \otimes Q)$, where ε is the counit.

Proof: By definition of Day's tensor,

$$\begin{array}{rcll}
\mathbf{a}(i\mathbf{a}P \otimes Q) & = & \mathbf{a}(\int_{\widehat{\mathcal{C}}}^{n,n'} i\mathbf{a}P(n) \times Q(n') \times \mathbf{y}(n \cdot n')) & \\
& \cong & \int_{\mathrm{Sh}}^{n,n'} \mathbf{a}i\mathbf{a}P(n) \times \mathbf{a}Q(n') \times \mathbf{a}\mathbf{y}(n \cdot n') & \mathbf{a} \text{ preserves colimits and} \\
& & & \text{commutes with finite limits} \\
& \cong & \int_{\mathrm{Sh}}^{n,n'} \mathbf{a}P(n) \times \mathbf{a}Q(n') \times \mathbf{a}\mathbf{y}(n \cdot n') & \mathbf{a}i \cong \mathrm{id}_{\mathrm{Sh}} \\
& \cong & \mathbf{a}(\int_{\widehat{\mathcal{C}}}^{n,n'} P(n) \times Q(n') \times \mathbf{y}(n \cdot n')) & \\
& = & \mathbf{a}(P \otimes Q) &
\end{array}$$

all natural in P, Q. Here the subscripts $\mathrm{Sh}, \widehat{\mathcal{C}}$ indicates in which category the coends are calculated.

Consider the following naturality square

$$\begin{tikzcd}
\mathbf{a}(\int_{\widehat{\mathcal{C}}}^{n,n'} i\mathbf{a}P(n) \times Q(n') \times \mathbf{y}(n \cdot n')) \arrow[r, leftrightarrow, "\sim"] \arrow[d, dotted, shift left, "\Theta_1"] & \int_{\mathrm{Sh}}^{n,n'} \mathbf{a}i\mathbf{a}P(n) \times \mathbf{a}Q(n') \times \mathbf{a}\mathbf{y}(n \cdot n') \arrow[d, dotted, shift right, "\int_{\mathrm{Sh}}^{n,n'} (\varepsilon_{\mathbf{a}P})_n \times \mathbf{a}Q(n') \times \mathbf{a}\mathbf{y}(n \cdot n')"'] \\
\mathbf{a}(\int_{\widehat{\mathcal{C}}}^{n,n'} P(n) \times Q(n') \times \mathbf{y}(n \cdot n')) \arrow[u, shift left, "\mathbf{a}(\eta_P \otimes Q)"] \arrow[r, leftrightarrow, "\sim"] & \int_{\mathrm{Sh}}^{n,n'} \mathbf{a}P(n) \times \mathbf{a}Q(n') \times \mathbf{a}\mathbf{y}(n \cdot n'). \arrow[u, shift right, "\int_{\mathrm{Sh}}^{n,n'} \mathbf{a}(\eta_P)_n \times \mathbf{a}Q(n') \times \mathbf{a}\mathbf{y}(n \cdot n')"']
\end{tikzcd}$$

By general properties for adjunctions we have $\varepsilon_{\mathbf{a}P} \circ \mathbf{a}(\eta_P) = \mathrm{id}_{\mathbf{a}P}$, which implies

$$(\int_{\mathrm{Sh}}^{n,n'} (\varepsilon_{\mathbf{a}P})_n \times \mathbf{a}Q(n') \times \mathbf{a}\mathbf{y}(n \cdot n')) \circ (\int_{\mathrm{Sh}}^{n,n'} \mathbf{a}(\eta_P)_n \times \mathbf{a}Q(n') \times \mathbf{a}\mathbf{y}(n \cdot n')) = \mathrm{id}$$

using the fact that $\int_{\mathrm{Sh}}$ is functorial. This shows that $\Theta_1^{-1} = \mathbf{a}(\eta \otimes Q)$.

To see that $\Theta_1 = \mathbf{a}(i\varepsilon_G \otimes Q)$ for presheaves of the form iG, where G is a sheaf, just use the identity $\eta_{iG} \circ i\varepsilon_G = \mathrm{id}$.

□

Similarly we have

$$\Theta_2 : \mathbf{a}(P \otimes i\mathbf{a}Q) \to \mathbf{a}(P \otimes Q)$$

with $\Theta_2^{-1} = \mathbf{a}(P \otimes \eta)$.

Theorem 4.3.2. *Let $\mathrm{Sh}(\mathcal{C}, J)$ be a category of sheaves over a Grothendieck topology, and $E, F \in \mathrm{Sh}(\mathcal{C}, J)$. Define*

$$E \otimes^{\mathrm{Sh}} F := \mathbf{a}(iE \otimes iF),$$

then $(\otimes^{\mathrm{Sh}}, \mathbf{a}\mathbf{y}e)$ is a monoidal tensor product on the category of sheaves.

Proof: Using the definition of $\otimes^{\mathrm{Sh}}$ and Lemma 4.3.1 we calculate

$$E \otimes^{\mathrm{Sh}} \mathbf{a}\mathbf{y}e = \mathbf{a}(iE \otimes i(\mathbf{a}\mathbf{y}e)) \cong \mathbf{a}(iE \otimes \mathbf{y}e) \cong \mathbf{a}(iE) \cong E$$

and

$$\begin{array}{rcll}
(E \otimes^{\mathrm{Sh}} F) \otimes^{\mathrm{Sh}} G & = & \mathbf{a}(i\mathbf{a}(iE \otimes iF) \otimes iG) & \\
& \cong & \mathbf{a}((iE \otimes iF) \otimes iG) & \text{by Lemma 4.3.1} \\
& \cong & \mathbf{a}(iE \otimes (iF \otimes iG)) & \text{since } \otimes \text{ is a monoidal tensor} \\
& \cong & \mathbf{a}(iE \otimes i\mathbf{a}(iF \otimes iG)) & \text{by Lemma 4.3.1} \\
& = & E \otimes^{\mathrm{Sh}} (F \otimes^{\mathrm{Sh}} G) &
\end{array}$$

For all sheaves G and presheaves P, by Lemma 4.3.1 there exist isomorphisms (omitting the subscripts)

$$\Theta_1 : \mathbf{a}(i\mathbf{a}(P) \otimes iG) \cong \mathbf{a}(P \otimes iG)$$

and

$$\Theta_2 : \mathbf{a}(iG \otimes i\mathbf{a}(P)) \cong \mathbf{a}(iG \otimes P)$$

natural in P, G. We use these to define the natural isomorphisms $\lambda^{\mathrm{Sh}}, \rho^{\mathrm{Sh}}, \alpha^{\mathrm{Sh}}$ as follows.

$$\begin{tikzcd}
\mathbf{a}(i\mathbf{a}\mathbf{y}(e) \otimes iG) \arrow[r, "\lambda^{\mathrm{Sh}}"] \arrow[d, "\Theta_1"'] & G \\
\mathbf{a}(\mathbf{y}(e) \otimes iG) \arrow[r, "a(\lambda)"'] & \mathbf{a}iG, \arrow[u, "\varepsilon"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{a}(iG \otimes i\mathbf{a}\mathbf{y}(e)) \arrow[r, "\rho^{\mathrm{Sh}}"] \arrow[d, "\Theta_2"'] & G \\
\mathbf{a}(iG \otimes \mathbf{y}(e)) \arrow[r, "\mathbf{a}(\rho)"'] & \mathbf{a}iG, \arrow[u, "\varepsilon"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{a}(iE \otimes i\mathbf{a}(iF \otimes iG)) \arrow[r, "\alpha^{\mathrm{Sh}}"] \arrow[d, "\Theta_2"'] & \mathbf{a}(i\mathbf{a}(iE \otimes iF) \otimes iG) \\
\mathbf{a}(iE \otimes (iF \otimes iG)) \arrow[r, "\mathbf{a}(\alpha)"'] & \mathbf{a}((iE \otimes iF) \otimes iG), \arrow[u, "\mathbf{a}(\eta \otimes iG)"']
\end{tikzcd}$$

where ε is the counit of the adjunction $\mathbf{a} \dashv i$, ε is an isomorphism because i is full and faithful. The coherence laws can be verified by taking the corresponding coherence laws for Day's construction on presheaves and using the associated sheaf functor $\mathbf{a}$ and the isomorphisms Θ_i. Let us show that $\mathrm{id} \otimes^{\mathrm{Sh}} \lambda^{\mathrm{Sh}} = \alpha^{\mathrm{Sh}} \circ \rho^{\mathrm{Sh}} \otimes^{\mathrm{Sh}} \mathrm{id}$.

We must show that the following gigantic diagram

$$\begin{tikzcd}
\mathbf{a}(iE \otimes i\mathbf{a}(i\mathbf{a}\mathbf{y}(e) \otimes iG)) \arrow[d, "E\otimes^{\mathrm{Sh}}\Theta_1"'] \arrow[d, phantom, shift left=8ex, "(1)"] \arrow[ddrr, bend left, "E\otimes^{\mathrm{Sh}}\lambda^{\mathrm{Sh}}"] \arrow[ddddd, bend right=70, "\alpha^{\mathrm{Sh}}"'] \arrow[ddddd, phantom, bend right=40, "(5)"] & & \\
\mathbf{a}(iE \otimes i\mathbf{a}(\mathbf{y}(e) \otimes iG)) \arrow[r, "E\otimes^{\mathrm{Sh}}\mathbf{a}(\lambda)"] \arrow[d, "\Theta_2"'] \arrow[d, phantom, shift left=8ex, "(2)"] & \mathbf{a}(iE \otimes i\mathbf{a}(iG)) \arrow[dr, "E\otimes^{\mathrm{Sh}}\varepsilon"] & \\
\mathbf{a}(iE \otimes (\mathbf{y}(e) \otimes iG)) \arrow[rr, "\mathbf{a}(iE\otimes\lambda)"] \arrow[d, "\mathbf{a}(\alpha)"'] \arrow[d, phantom, shift left=8ex, "(3)"] & & \mathbf{a}(iE \otimes iG) \\
\mathbf{a}((iE \otimes \mathbf{y}(e)) \otimes iG) \arrow[urr, "\mathbf{a}(\rho\otimes iG)"'] \arrow[d, "\Theta_1^{-1}"'] \arrow[d, phantom, shift left=8ex, "(4)"] & & \\
\mathbf{a}(i\mathbf{a}(iE \otimes \mathbf{y}(e)) \otimes iG) \arrow[d, "\Theta_2^{-1}\otimes^{\mathrm{Sh}}G"'] & & \\
\mathbf{a}(i\mathbf{a}(iE \otimes i\mathbf{a}\mathbf{y}(e)) \otimes iG) \arrow[uuurr, "\rho^{\mathrm{Sh}}\otimes^{\mathrm{Sh}}G"'] & &
\end{tikzcd}$$

commutes: (1) commutes by the definition of λ^{Sh}. (2) commutes by naturality of Θ_2 and because $E\otimes^{\mathrm{Sh}}\varepsilon = \Theta_2$. (3) commutes by the corresponding coherence law for $\otimes$. (4) commutes by arguments similar to those for (1) and (2). To see why (5) commutes, we first draw a diagram

$$\begin{tikzcd}
\mathbf{a}(iE \otimes i\mathbf{a}(\mathbf{y}(e) \otimes iG)) \arrow[d, dotted, "\Theta_2"] & \mathbf{a}(i\mathbf{a}(iE \otimes \mathbf{y}(e)) \otimes iG) \arrow[ddd, dotted, bend left=60, "\Theta_2^{-1} \otimes^{\mathrm{Sh}} G"] \\
\mathbf{a}(iE \otimes (\mathbf{y}(e) \otimes iG)) \arrow[r, "\mathbf{a}(\alpha)"] \arrow[d, "\mathbf{a}(iE \otimes (\eta \otimes iG))"'] & \mathbf{a}((iE \otimes \mathbf{y}(e)) \otimes iG) \arrow[u, dotted, "\Theta_1^{-1}"] \arrow[d, "\mathbf{a}((iE \otimes \eta) \otimes iG)"] \\
\mathbf{a}(iE \otimes (i\mathbf{a}\mathbf{y}(e) \otimes iG)) \arrow[r, "\mathbf{a}(\alpha)"] \arrow[d, "\Theta_2^{-1}"'] & \mathbf{a}((iE \otimes i\mathbf{a}\mathbf{y}(e)) \otimes iG) \arrow[d, "\mathbf{a}(\eta \otimes iG)"] \\
\mathbf{a}(iE \otimes i\mathbf{a}(i\mathbf{a}\mathbf{y}(e) \otimes iG)) \arrow[r, "\alpha^{\mathrm{Sh}}"] \arrow[uuu, dotted, bend left=60, "E \otimes^{\mathrm{Sh}} \Theta_1"] & \mathbf{a}(i\mathbf{a}(iE \otimes i\mathbf{a}\mathbf{y}(e)) \otimes iG).
\end{tikzcd}$$

The square commute by naturality of α and by definition of α^{Sh}. Then use naturality of Θ_2^{-1} to get

$$\Theta_2^{-1} \circ \mathbf{a}(iE \otimes (\eta \otimes iG)) = \mathbf{a}(iE \otimes i\mathbf{a}(\eta \otimes iG)) \circ \Theta_2^{-1},$$

where $\mathbf{a}(iE \otimes i\mathbf{a}(\eta \otimes iG)) = E \otimes^{\mathrm{Sh}} \mathbf{a}(\eta \otimes iG)$ is the inverse of $E \otimes^{\mathrm{Sh}} \Theta_1$. To see that $\mathbf{a}(i\mathbf{a}(iE \otimes \eta) \otimes iG) \circ \Theta_1^{-1} = \Theta_1^{-1} \circ \mathbf{a}((iE \otimes \eta) \otimes iG)$ use naturality of Θ_1^{-1} and the identity $\Theta_2^{-1} \otimes^{\mathrm{Sh}} G = \mathbf{a}(i\mathbf{a}(iE \otimes \eta) \otimes iG)$. ☐

The following conjecture contradicts Lemma 5.2 of [Pym02].

Conjecture 4.3.3. *Day's tensor does not in general preserve sheaves, not even for a category of sheaves over a topological monoid.*

To see why this is probably true, consider a topological monoid$(\mathcal{O}(X), \cdot, e)$ and sheaves F, G. We have

$$(F \otimes G)(W) = \int^{U,V} FU \times GV \times \mathcal{O}(X)(W, U \cdot V).$$

Suppose we have a cover of $W = \bigcup_i W_i$ and let $\{x_i \in (F \otimes G)(W_i)\}$ be a matching family. The x_i's are equivalence classes of the form $x_i = [U_i, V_i, a_i \in FU_i, b_i \in GV_i, W_i \subseteq U_i \cdot V_i]$, so that

$$x_i \restriction_{W_i \cap W_j} = x_j \restriction_{W_i \cap W_j}$$

means that

$$[U_i, V_i, a_i, b_i, W_i \cap W_j \subseteq U_i \cdot V_i] = [U_j, V_j, a_j, b_j, W_i \cap W_j \subseteq U_i \cdot V_i].$$

With the equivalence described in 4.7. We have $W = \bigcup_i W_i \subseteq \bigcup_i U_i \cdot V_i = \bigcup_i U_i \cdot \bigcup_i V_i = U \cdot V$, and we want to conclude that there is a unique element $x \in (F \otimes G)(W)$ such that $x \restriction_{W_i} = x_i$ for all i, i.e., an element $[U, V, a, b, W \subseteq U \cdot V]$ such that

$$[U, V, a, b, W_i \subseteq U \cdot V] = [U_i, V_i, a_i, b_i, W_i \subseteq U_i \cdot V_i].$$

If we knew that for some $a \in FU, b \in GV$, $F(U_i \subseteq U)(a) = a_i$ and $G(V_i \subseteq V)(b) = b_i$ for all i, then the above would hold. This condition corresponds to requiring that the elements $\{a_i\}$ which are part of the representatives $\{x_i\}$ and $\{b_i\}$ are matching families for F, G. Then we could use that F, G are sheaves to get a, b. There is, however, no apparent way of knowing that $a_i \restriction_{U_i \cap U_j} = a_j \restriction_{U_i \cap U_j}$, on the other hand it is quite difficult to find a counter example due to the complex nature of the equivalence relation on $(F \otimes G)(W)$.

Proposition 4.3.4. *For the topology of the pointer model, Day's tensor restricts to sheaves.*

Proof: Let $(\mathcal{M}, *, e)$ be a preordered commutative monoid. The topology of the pointer model is defined by:

$$\begin{aligned} J(\bot) &= \{\{\bot\}, \emptyset\}, \\ J(m) &= \{\{m\}\}, m \in \mathcal{M}, m \neq \bot. \end{aligned}$$

Suppose $E, F : \mathcal{M}^{op} \to \mathrm{Set}$ are sheaves, we must show that $iE \otimes iF$ is a sheaf, where $\otimes$ is Day's tensor.

Suppose $m \neq \bot$ then the only cover of m is the the sieve generated by $\{m\}$, i.e., the maximal sieve on m. Since the sieve contains the identity id_m, a matching family comes with a unique amalgamation, namely x_m.

A cover of $\bot$ is either $\{\bot\}$ or $\emptyset$. In the case $\{\bot\}$ the above argument applies, in the case $\emptyset$: a matching family for $\emptyset$ must be $\emptyset$. Now E and F are both sheaves and $\emptyset$ is a matching for those as well so we must have $E(\bot) = F(\bot) = \{*\}$ to ensure that there is a unique amalgamation for the empty family. To see that $E \otimes F(\bot) = \{*\}$ as well, note that

$$[*, *, \bot \leq \bot * \bot] = [a, b, \bot \leq U * V]$$

for any element $[a, b, \bot \leq U * V]$ of $E \otimes F(\bot)$. The two equivalence classes are equal since we always have arrows $0_U : \bot \to U, 0_V : \bot \to V$ and $E(0_U)(a) = *$, $F(0_V)(b) = *$ for any $a \in E(U)$ and $b \in F(V)$. □

Proposition 4.3.5. *If the tensor product $- \otimes^{\mathrm{Sh}} F = \mathbf{a}(i(-) \otimes iF)$ has a right adjoint in the category of sheaves, then it must be $iF \multimap i(-)$.*

Proof: Let $P \in \widehat{\mathcal{C}}$ and $F, G \in \mathrm{Sh}(\mathcal{C}, J)$. Suppose $- \otimes^{\mathrm{Sh}} F$ has a right adjoint, $F \twoheadrightarrow -$ say. We have the following string of natural isomorphisms

$$\begin{aligned} \widehat{\mathcal{C}}(P, i(F \twoheadrightarrow G)) &\cong \mathrm{Sh}(\mathbf{a}P, F \twoheadrightarrow G) \\ &\cong \mathrm{Sh}(\mathbf{a}P \otimes^{\mathrm{Sh}} F, G) \\ &\cong \mathrm{Sh}(\mathbf{a}(i\mathbf{a}P \otimes iF), G) \\ &\cong \mathrm{Sh}(\mathbf{a}(P \otimes iF), G) \qquad \text{by Lemma 4.3.1} \\ &\cong \widehat{\mathcal{C}}(P \otimes iF, iG) \\ &\cong \widehat{\mathcal{C}}(P, iF \multimap iG) \end{aligned}$$

By full and faithfulness of the Yoneda functor it follows that $i(F \twoheadrightarrow G) \cong iF \multimap iG$ in the presheaf category $\widehat{\mathcal{C}}$. □

Corollary 4.3.6. *If $iF \multimap iG$ is a sheaf for F, G sheaves, then there is a monoidal closed structure on the category of sheaves.*

A counter example. The following proposition gives a counter example to Lemma 5.3 of [Pym02].

Proposition 4.3.7. *Let $\mathcal{O}(X)$ be a topology on a set of points X with a monoidal tensor product $*$. For a presheaf $P \in \widehat{\mathcal{O}(X)}$ and sheaf $F \in \mathrm{Sh}(X)$, the presheaf $P \multimap F = \lambda U.\, \mathrm{Hom}(P, F(U * -))$ is not in general a sheaf.*

Proof: Note that

$$\lambda U.\operatorname{Hom}_{\widehat{\mathcal{O}(X)}}(\mathbf{y}C, F(U * -)) \cong \lambda U.F(U * C)$$

is an isomorphism natural in U; The isomorphism holds pointwise for each U by the Yoneda Lemma, and naturality in U is verified straight forward. This implies that $\mathbf{y}C \multimap F = \lambda U.\operatorname{Hom}_{\widehat{\mathcal{O}(X)}}(\mathbf{y}C, F(U * -))$ is a sheaf iff $F(- * C) : \mathcal{O}(X)^{op} \to \mathrm{Set}$ is a sheaf.

$(\mathbb{N}, +, 0)$ is a commutative monoid, so $(\mathcal{P}(\mathbb{N}), +, \{0\})$ is a topological monoid. Let $F : \mathcal{P}(\mathbb{N})^{op} \to \mathrm{Set}$ be the sheaf defined by $F(V) = \mathrm{Fct}(V, \mathbb{N})$, that is, all functions from the subset V to the natural numbers $\mathbb{N}$. We now show that for a fixed $U \in \mathcal{P}(\mathbb{N})$ the presheaf $G := F(- * U)$ is not a sheaf. For any sets V_1, V_2, U we have $V_1 \cap V_2 \subseteq V_1$ and $V_1 \cap V_2 \subseteq V_2$, so we always have

$$(V_1 \cap V_2) * U \subseteq V_1 * U \cap V_2 * U$$

but equality fails.

Let $V = \{1, 2, 3\}, V_1 = \{1, 2\}, V_2 = \{2, 3\}$, then $V = V_1 \cup V_2$. Let $U = \{2, 4\}$. We have the following identities

1. $V_1 \cap V_2 * U = \{2\} * \{2, 4\} = \{4, 6\}$ and
2. $V_1 * U \cap V_2 * U = \{3, 4, 5, 6\} \cap \{4, 5, 6, 7\} = \{4, 5, 6\}$.

Let $s_1 \in G(V_1) = \mathrm{Fct}(V_1 * U, \mathbb{N})$ be the partial function defined by

$$s_1(n) = \begin{cases} 0 & \text{if} & n \in \{3, 4, 6\} \\ 1 & \text{if} & n = 5 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Let $s_2 \in G(V_2)$ be the constant function $s_2(n) = 0$. We have $s_1 \restriction_{V_1 \cap V_2} = G(V_1 \cap V_2 \subseteq V_1)(s_1) = F(V_1 \cap V_2 * U \subseteq V_1 * U)(s_1)$, which is the restriction of s_1 to the set $V_1 \cap V_2 * U$, likewise, $s_2 \restriction_{V_1 \cap V_2}$ is the restriction of s_2 to $V_1 \cap V_2 * U$, and since s_1 and s_2 agree on this set we have a matching family for the cover $V_1 \cup V_2$. However, there is no function $s : V * U \to \mathbb{N}$ such that $s \restriction_{V_1} = s_1$ and $s \restriction_{V_2} = s_2$ because $5 \in V_1 * U \cap V_2 * U$ and $s_1(5) \neq s_2(5)$. So G is not a sheaf. ☐

This means that there is not in general a monoidal closed structure on the category of sheaves, using Day's construction.

Lemma 4.3.8. *For a monoidal category $(\mathcal{C}, \cdot, e)$, J a topology, F a sheaf, P a presheaf. If for each $C \in \mathcal{C}$, the presheaf $F(- \cdot C)$ is a sheaf then $P \multimap F$ is a sheaf.*

Proof: By definition, $P \multimap F = \lambda X.\operatorname{Hom}(P, F(X \cdot -))$. If $P \cong \mathbf{y}C$ for some $C \in \widehat{\mathcal{C}}$, we have

$$\lambda X.\operatorname{Hom}(\mathbf{y}C, F(X \cdot -)) \cong \lambda X.F(X \cdot C) \tag{4.8}$$

natural in X. Every presheaf is a colimit of representables so we have $P = \operatorname{colim}_{\widehat{\mathcal{C}}} \mathbf{y}C_i$, which gives

$$\begin{array}{lll} & \lambda X.\operatorname{Hom}(P, F(X \cdot -)) & \\ \cong & \lambda X.\operatorname{Hom}(\operatorname{colim}_i \mathbf{y}C_i, F(X \cdot -)) & \\ \cong & \lim_i \lambda X.\operatorname{Hom}(\mathbf{y}C_i, F(X \cdot -)) & \text{The Hom-functor reverses colimits} \\ \cong & \lim_i \lambda X.F(X \cdot C_i) & \text{by 4.8 above} \end{array}$$

The category of sheaves is closed under limits so if $\lambda X.F(X \cdot C_i)$ is a sheaf for each C_i, we have by the above calculations that $\lambda X.\widehat{\mathcal{C}}(P, F(X \cdot -)) = P \multimap F$ is a sheaf. ☐

Lemma 4.3.9. *For a monoidal category $(\mathcal{C}, \cdot, e)$ with pullbacks, if for an element H of $\mathcal{C}$, $- \cdot H$ preserves pullbacks, and if $\cdot$ is also cover preserving w.r.t. the topology J, then for a sheaf F over $(\mathcal{C}, J)$, the presheaf $F(- \cdot H)$ is a sheaf.*

Proof: That $- \cdot H$ preserves pullbacks means that for each pair of arrows $f_i : C_i \to C, f_j : C_j \to C$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} C_i \wedge C_j \cdot H \arrow[r, "\pi_i \cdot H"] \arrow[d, "\pi_j \cdot H"'] & C_i \cdot H \arrow[d, "f_i \cdot H"] \\ C_j \cdot H \arrow[r, "f_j \cdot H"'] & C \cdot H \end{tikzcd} \tag{4.9}$$

is a pullback. To see that $F(- \cdot H)$ is a sheaf, let $\{f_i : C_i \to C\}_I$ be a cover of $C \in \mathcal{C}$, and $x_{f_i} \in F(C_i \cdot H)$ a matching family for this cover. Consider the pullback diagram

$$\begin{tikzcd} C_i \wedge C_j \arrow[r, "\pi_i"] \arrow[d, "\pi_j"'] & C_i \arrow[d, "f_i"] \\ C_j \arrow[r, "f_j"'] & C. \end{tikzcd}$$

Since $f_i \pi_i = f_j \pi_j$ we have

$$x_{f_i \pi_i} = x_{f_j \pi_j}$$

which reads

$$F(\pi_i \cdot H)(x_{f_i}) = F(\pi_j \cdot H)(x_{f_j}). \tag{4.10}$$

Since we have assumed that the monoid composition $\cdot$ is cover preserving, the family $S = \{f_i \cdot H : C_i \cdot H \to C \cdot H\}_I$ is a cover of $C \cdot H$ (recall that $f_i \cdot H$ means $f_i \cdot \mathrm{id}_H$). We claim that

$$y_{f_i \cdot H} := x_{f_i} \in F(C_i \cdot H)$$

constitutes a matching family of F for the cover S. To see this, first recall that by Remark 2.6.11 it is enough to show that

$$y_{f_i \cdot H} \restriction_{C_i \cdot H \wedge C_j \cdot H} = y_{f_j \cdot H} \restriction_{C_i \cdot H \wedge C_j \cdot H}$$

for all pairs $f_i \cdot H, f_j \cdot H$ in S. We now show that this is indeed the case. By diagram 4.9 we get $y_{f_i \cdot H} \restriction_{C_i \cdot H \wedge C_j \cdot H} = y_{f_i \cdot H} \restriction_{\pi_i \cdot H}$ and $y_{f_j \cdot H} \restriction_{C_i \cdot H \wedge C_j \cdot H} = y_{f_j \cdot H} \restriction_{\pi_j \cdot H}$, so the identity that must be shown is

$$y_{f_i \cdot H} \restriction_{\pi_i \cdot H} = y_{f_j \cdot H} \restriction_{\pi_j \cdot H}$$

which, when unwinded, is exactly what is stated in equation 4.10. Since we now have a matching family for F and F is a sheaf there is a unique amalgamation $y \in F(C \cdot H)$ which obviously is an amalgamation for the family $\{x_{f_i}\}_I$ as well. □

Corollary 4.3.10. *For a monoidal category $(\mathcal{C}, \cdot, e)$ with pullbacks, if $\cdot$ preserves pullbacks and covers, the tensor product $- \otimes^{\mathrm{Sh}} F : \mathrm{Sh}(\mathcal{C}, J) \to \mathrm{Sh}(\mathcal{C}, J)$ has right adjoint $iF \multimap -$.*

Example 4.3.11. [2] *Consider the category $\mathcal{I}$ of finite sets and injective functions. For the opposite category $\mathcal{I}^{op}$ with the atomic topology, a presheaf $P : \mathcal{I} \to \mathrm{Set}$ is a sheaf iff it preserves pullbacks (Prop 2.6.15). The disjoint union $+$ of finite sets induces a monoidal functor on I as well as $\mathcal{I}^{op}$ and it preserves pullbacks in I. This implies that for each $C \in \mathcal{I}$, $P(C + -) : \mathcal{I} \to \mathrm{Set}$ is a sheaf. By Lemma 4.3.8 this means that Day's tensor induces a monoidal closed structure on the category of atomic sheaves over $\mathcal{I}^{op}$.*

These kinds of sheaves (plus an additional constraint) are used to interpret types of SCI+ in [O'H03].

Since $\mathcal{I}$ has pullbacks, the atomic topology is well-defined for $\mathcal{I}$ also. Moreover, $+$ is cover preserving: take a nonempty sieve S on C and a nonempty sieve T on D then the sieve generated by $S+T$ is a nonempty sieve on $C+D$. Using the corollary 4.3.10 we get that Day's tensor induces a monoidal closed structure on the category of atomic sheaves ($P : \mathcal{I}^{op} \to \mathrm{Set}$) over $\mathcal{I}$.

4.4 Subobjects in DCC's

The algebraic counterpart of a bi-CDCC is called a BI-algebra.

Definition 4.4.1 (BI algebra). *A BI algebra B is a Heyting algebra with an additional residuated commutative monoid structure $(*, e, -\!\!*)$, such that $*$ is monotone with respect to order on B. That is, $(B, *, e)$ is a commutative monoid and (monotonicity)*

$$a \leq a' \text{ and } b \leq b' \text{ implies } a * a' \leq b * b'$$

and (residuated)

$$a * b \leq c \text{ iff } a \leq b -\!\!* c$$

Definition 4.4.2 (Complete BI algebra). *A* Complete BI algebra (cBIa) *is a BI algebra which is complete as a Heyting algebra.*

We have seen that in a topos $\mathcal{T}$ the partial order $\mathrm{Sub}(A)$, for $A \in \mathrm{Obj}(\mathcal{T})$ is a Heyting algebra. If the topos is also a DCC, we have the following:

Proposition 4.4.3. *In a topos $\mathcal{T}$ with a symmetric monoidal closed structure $(\otimes, I, \multimap)$, the partial order* $\mathrm{Sub}(1)$ *on the terminal object is a BI algebra.*[3]

Proof: We already know that in a topos, $\mathrm{Sub}(1)$ is a Heyting algebra. Let $U \rightarrowtail 1$ and $V \rightarrowtail 1$ be subobjects of 1, and define

- $U * V = Im(U \otimes V)$, which is the image factorization

$$\begin{tikzcd}
U \otimes V \arrow[rr, "u \otimes v"] \arrow[dr, two heads] & & 1 \otimes 1 \arrow[r, "1_{1 \otimes 1}"] & 1 \\
& Im(U \otimes V) \arrow[urr, tail] & &
\end{tikzcd}$$

---

[2] I am grateful to H. Yang for pointing out this example.

[3] In fact the proposition holds if $\mathcal{T}$ a bi-CDCC which is also regular, i.e., has epi-mono factorizations.

- $e = Im(I)$,

$$\begin{tikzcd}
I \arrow[r, "1_I"] \arrow[d, two heads] & 1 \\
e \arrow[ur, tail] &
\end{tikzcd}$$

- $U \mathrel{-\!\!*} V = (U \multimap V)$, as in

$$\begin{tikzcd}
(U \multimap V) \arrow[r, tail, "U \multimap 1_V"] & (U \multimap 1) \cong 1.
\end{tikzcd}$$

$(U \mathrel{-\!\!*} V)$ is well-defined since $U \multimap - : \mathcal{T} \to \mathcal{T}$ is right adjoint to the tensor so it preserves limits. A mono is a limit, so $(U \multimap V) \xrightarrow{U \multimap 1_V} (U \multimap 1)$ is mono. Now 1 is also a limit so $U \multimap 1$ must be the terminal object as well. Commutativity: $U * V = Im(U \otimes V) = Im(V \otimes U) = V * U$. Associativity: Since $\otimes$ is associative, we have $U \otimes (V \otimes W) = (U \otimes V) \otimes W$. Consider the following commutative diagram

$$\begin{tikzcd}
U \otimes (V \otimes W) \arrow[d, two heads, "U \otimes e_1"'] & = & (U \otimes V) \otimes W \arrow[d, two heads, "e_2 \otimes W"] \\
U \otimes Im(V \otimes W) \arrow[d, two heads] & & (Im(U \otimes V) \otimes W) \arrow[d, two heads] \\
Im(U \otimes Im(V \otimes W)) \arrow[dr, tail] & & Im(Im(U \otimes V) \otimes W) \arrow[dl, tail] \\
& 1 &
\end{tikzcd}$$

Where e_1, e_2 are the epis corresponding to the epi-mono factorization of $V \otimes W \to 1$ and $U \otimes V \to 1$, and $\otimes$ preserves epis since it has a right adjoint. The diagram gives two epi-mono factorizations of the arrow $U \otimes V \otimes W \to 1$, by uniqueness of such factorizations, $Im(U \otimes Im(V \otimes W)) = Im(Im(U \otimes V) \otimes W)$. Unit: We must show that $U * e = U$, by definition of e and $*$ this means showing $Im(U \otimes Im(I)) \cong U$.

$$\begin{tikzcd}
U \otimes I \arrow[d, two heads, "U \otimes s"'] & = & U \arrow[dd, two heads, "\mathrm{id}_U"] \\
U \otimes Im(I) \arrow[d, two heads] & & \\
Im(U \otimes Im(I)) \arrow[dr, tail] & & U \arrow[dl, tail] \\
& 1, &
\end{tikzcd}$$

where $s : I \twoheadrightarrow Im(I)$ is the epi part of the image factorization of $1_I : I \to 1$. By uniqueness of image factorizations $Im(U \otimes Im(I)) \cong U$. Monotonicity: Assume that $U \leq V$ and $U' \leq V'$ for subobjects U, U', V, V' of 1. In particular this means that there is an arrow $U \to V$ and an arrow $U' \to V'$, $\otimes$ is a covariant bi-functor, so we get an arrow $U \otimes U' \to V \otimes V'$. We then have a commuting diagram

$$\begin{tikzcd}
U \otimes U' \arrow[r] \arrow[d, two heads] & V \otimes V' \arrow[d, two heads] \\
Im(U \otimes U') \arrow[r, dotted, "u"] \arrow[d, tail] & Im(V \otimes V') \arrow[dl, tail] \\
1 &
\end{tikzcd}$$

The outer square commutes because there is only one arrow from $U \otimes U'$ to 1, and u exists, making the diagram commute, by the universal property of image factorization (see Proposition 2.2.1). The diagram shows that in Sub(1) we have $Im(U \otimes U') \leq Im(V \otimes V')$, i.e., $U * U' \leq V * V'$. Residuated: Suppose $U * V \leq W$, that is

$$\begin{tikzcd}
U \otimes V \arrow[d, two heads] \arrow[dr, dotted] & \\
U * V \arrow[r] \arrow[d, tail] & W \arrow[dl, tail] \\
1 &
\end{tikzcd}$$

so there is an arrow $U \otimes V \to W$; by the adjunction in $\mathcal{T}$ this corresponds to an arrow $U \to (V \multimap W)$, which is precisely what we need to have $U \leq V \mathrel{-\!\!*} W$. On the other hand, assuming $U \leq V \mathrel{-\!\!*} W$, we have an arrow $U \to (V \multimap W)$ which by adjunction gives as arrow $U \otimes V \to W$ (this is the dotted arrow in the diagram above). Using Proposition 2.2.1 again, we get an arrow $U * V \to W$ as needed. ☐

Example 4.4.4. *As an example consider the presheaf category $\widehat{\mathcal{C}}$ where $(\mathcal{C}, \cdot, e)$ is a small symmetric monoidal category. $\widehat{\mathcal{C}}$ is a topos with a symmetric monoidal closed structure, so by the proposition we get a BI algebra on* Sub(1)*. Recall that there is a one-one correspondence between subfunctors of* 1 *and sieves on $\mathcal{C}$ (it is only necessary to consider the domains of the arrows in the sieve, see Proposition 2.5.17), for a presheaf $P \in \mathrm{Sub}(1)$ it is given by*

$$\overline{P} = \{C \mid * \in PC\}$$

and from a sieve I we get a subfunctor of 1 *by*

$$\hat{I}(C) = \begin{cases} \{*\} & \textit{if } C \in I \\ \emptyset & \textit{otherwise} \end{cases}$$

From the proposition above we know what the definition of $P * Q$ is as a subfunctor, we now calculate the corresponding operation on sieves. Let P, Q be subfunctors of 1*, then*

$$\begin{array}{rcl} P \otimes Q(C) & = & \int^{Y,Y'} PY \times QY' \times \mathcal{C}(C, Y \cdot Y') \\ & = & \bigcup_{Y,Y'}(Y, Y', *)/\sim \quad \textit{such that } * \in PY, * \in QY', C \leq Y \cdot Y'. \end{array}$$

Image factorization in the presheaf category is calculated in Set*, therefore*

$$P * Q = \begin{cases} \{*\} & \textit{if there exists } Y, Y' \textit{ such that } C \leq Y \cdot Y', * \in PY, * \in QY', \\ \emptyset & \textit{otherwise.} \end{cases}$$

This implies that for sieves I, J,

$$\begin{aligned} I * J &= \overline{\hat{I} * \hat{J}} \\ &= \{A \mid \textit{there exists}\, Y \in J,\, Y' \in I.\, A \leq Y \cdot Y'\} \\ &= \downarrow\{Y \cdot Y' \mid Y \in I, Y' \in J\}. \end{aligned}$$

We now show that

$$I \twoheadrightarrow\!\!* \; J = \{A \mid \ \textit{for all}\ C \in I, A \cdot C \in J\}.$$

By definition we have $I \twoheadrightarrow\!\!* \; J := \overline{\hat{I} \multimap \hat{J}}$, *where* $(\hat{I} \multimap \hat{J})(A) = \widehat{\mathcal{C}}(\hat{I}, \hat{J}(A \cdot -))$ *and we want to determine whether this set is empty or not. If* $\alpha \in \widehat{\mathcal{C}}(\hat{I}, \hat{J}(A \cdot -))$ *we have a commuting square*

$$\begin{tikzcd} \hat{I}(C) \arrow[r, "\alpha_C"] \arrow[d] & \hat{J}(A \cdot C) \arrow[d] \\ \hat{I}(B) \arrow[r, "\alpha_B"'] & \hat{J}(A \cdot B) \end{tikzcd}$$

for each $B \leq C$ *in* $\mathcal{C}$ *(i.e., there exists an arrow* $B \to C$*). Clearly such an* α *exists if and only if* $\hat{I}(C) = \{*\}$ *implies* $\hat{J}(A \cdot C) = \{*\}$ *for all* C.

The unit in $\widehat{\mathcal{C}}$ *is* $\mathbf{y}(e)$, *which corresponds to the sieve* $\downarrow(e)$.

Remark 4.4.5. *Let* $(\mathcal{M}, \cdot, e)$ *be a preordered commutative monoid. It does not seem to be the case that this monoidal structure induces a monoidal structure on* Sub(A) *for all objects* $A \in \widehat{\mathcal{M}}$.

Because of the isomorphism Sub$(A) \cong \widehat{\mathcal{M}}(A, \Omega)$, *it is equivalent to say that* Ω *is not an internal monoid.* Ω *is an internal monoid iff there exists an arrow* $* : \Omega \times \Omega \to \Omega$ *and an arrow* $e : 1 \to \Omega$ *which render commutative the diagrams which express associative, commutative and unit laws, for example the composite*

$$\Omega \cong \Omega \times 1 \xrightarrow{\mathrm{id} \times e} \Omega \times \Omega \xrightarrow{*} \Omega$$

must be the identity on Ω.

Such arrows do in fact exist since both $(\vee, \bot)$ *and* $(\wedge, \top)$ *satisfies the monoid laws, but these are not induced by "$\cdot$".*

$\Omega(m)$ *is the set of* m*-sieves. So the natural way to define a monoid would be pointwise using* $\cdot$ *and then making this an* m*-sieve, i.e., for* $S_1, S_2 \in \Omega(m)$,

$$S_1 * S_2 = \downarrow(S_1 \cdot S_2) \cap \downarrow m = \{x \leq m \mid \textit{there exists}\ s_1 \in S_1, s_2 \in S_2 . x \leq s_1 \cdot s_2\}.$$

But this is not necessarily a natural transformation from $\Omega \times \Omega$ *to* Ω. *Showing naturality amounts to showing that for* $S_1, S_2 \in \Omega(m)$, *and* $n \leq m$

$$(\downarrow(S_1 \cap \downarrow n) \cdot (S_2 \cap \downarrow n)) \cap \downarrow n = \downarrow(S_1 \cdot S_2) \cap \downarrow n.$$

Suppose $x \not\leq n, y \not\leq n$ *and* $x \in S_1, y \in S_2$ *and* $x \cdot y \leq n$ *then* $x \cdot y$ *is in the RHS but not in the LHS.*

Another way to see that $\cdot$ *does not (in a natural way) induce a monoidal structure on* Sub(A), *for any object* $A \in \widehat{\mathcal{C}}$, *is by the following argument: Given* $M \rightarrowtail A, N \rightarrowtail A$, *we can* *construct a subobject $M * N \rightarrowtail A \otimes A$ by using Day's tensor $\otimes$ and image factorization, but there is no map*

$$A \to A \otimes A \text{ or } A \otimes A \to A,$$

which means that there is no evident way to construct a subobject of A given a subobject of $A \otimes A$.

Example 4.4.6. *For any discretely ordered monoid $(M, *)$, the powerset $\mathcal{P}(M)$ is a complete (Boolean) BI algebra. This can be shown in several ways:*

- *$\mathcal{P}(M)$ corresponds to sieves on M because the order is discrete so any subset of M is downwards closed. Now, sieves on M are in bijective correspondence with $\mathrm{Sub}_{\widehat{M}}(1)$, which we have shown (Proposition 4.4.3, and Corollary 2.3.8) is a (complete) BI algebra. It is Boolean since $\mathcal{P}(M)$ is so.*
- *$\mathcal{P}(M)$ is a topological monoid. A topological space is in particular a complete Heyting algebra (and it is Boolean), together with the fact that a topological monoid is cover preserving, a right adjoint to $*$ can be defined by*

  $$U \mathrel{-\!\!*} V := \bigcup \{W \subseteq M \mid (W * U) \subseteq V\},$$

  and this make $\mathcal{P}(M)$ a cBIa.

Recall that a subsheaf of 1 in $\mathrm{Sh}(\mathcal{C}, J)$ is the same as an ideal on $\mathcal{C}$, where I is an ideal iff

1. If $C \in I$ and there exists an arrow $D \to C$ then $D \in I$.
2. For any object C of $\mathcal{C}$ and for any cover $S \in J(C)$, if for every $f : C' \to C \in S$, $C' \in I$ then $C \in I$.

The monoidal structure on $\mathrm{Sh}(\mathcal{C}, J)$ (which is there by 4.3.2) induces a monoidal structure on $\mathrm{Sub}_{\mathrm{Sh}(\mathcal{C},J)}(1)$ by

$$A * B = \mathrm{Im}(A \otimes^{\mathrm{Sh}} B) = \mathrm{Im}(\mathbf{a}(iA \otimes iB)) \cong \mathbf{a}(\mathrm{Im}(iA \otimes iB))$$

where $A \rightarrowtail 1, B \rightarrowtail 1$ are subsheaves of 1. If we translate to ideals this becomes

$$J * K = \mathbf{a} \downarrow \{j \cdot k \mid j \in J, k \in K\}.$$

The unit is defined by

$$\mathrm{I} = \mathrm{Im}(\mathbf{ay}(e)) \cong \mathbf{a}(\mathrm{Im}(\mathbf{y}(e))) \cong \mathbf{ay}(e).$$

In other words

$$\begin{aligned} c \in J * K &\text{ iff there exists a cover } S \in J(c) \\ &\text{such that for all } f_i : c_i \to c \in S, c_i \in \downarrow \{j * k \mid j \in J, k \in K\}. \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &c \in \bar{\mathrm{I}} \text{ iff } (\mathbf{ay}(e))(c) \neq \emptyset \\ &\text{iff there exists a cover } S \in J(c) \text{ such that for all } f_i : c_i \to c \in S, \mathbf{y}(e)(c_i) \neq \emptyset. \end{aligned} \tag{4.12}$$

Lemma 4.4.7. *Let $\cdot$ be a cover preserving, symmetric monoidal tensor product on $\mathcal{C}$. If F is a subsheaf of* 1 *and P a subpresheaf of* 1 *then $P \multimap F$ is a subsheaf of* 1.

Proof: Let $\overline{\mathrm{P}}, \overline{\mathrm{F}}$ be the ideals (sieves) corresponding to P, F. If $\overline{\mathrm{P} \multimap \mathrm{F}}$ is an ideal, then $P \multimap F$ is a subsheaf of 1.

$$\overline{\mathrm{P} \multimap \mathrm{F}} = \overline{\mathrm{P}} \mathrel{-\!\!*} \overline{\mathrm{F}} = \{a \mid p \in \overline{\mathrm{P}} \Rightarrow p \cdot a \in \overline{\mathrm{F}}\}$$

We have already seen that this is a sieve, to see that it is an ideal, suppose $\{f_i : d_i \to d\}_{i \in I} \in J(d)$ and $d_i \in \overline{\mathrm{P}} \mathrel{-\!\!*} \overline{\mathrm{F}}$ for all i. We must show that $d \in \overline{\mathrm{P}} \mathrel{-\!\!*} \overline{\mathrm{F}}$. Let $p \in \overline{\mathrm{P}}$ then $p \cdot d_i \in \overline{\mathrm{F}}$ for all i. By the cover preserving property, $\{p \cdot d_i \to p \cdot d\}_{i \in I} \in J(p \cdot d)$, so, $\overline{\mathrm{F}}$ being an ideal, $p \cdot d \in \overline{\mathrm{F}}$. ☐

Proposition 4.4.8. *Whenever $(\mathcal{C}, \cdot, e)$ is a symmetric monoidal category and $\cdot$ is cover preserving for the topology J, $\mathrm{Sub}_{\mathrm{Sh}(\mathcal{C},J)}(1)$ is a BI algebra. Moreover, for ideals J, K in $\mathrm{Sub}_{\mathrm{Sh}(\mathcal{C},J)}(1)$, the BI structure is given by*

$$\begin{array}{lcl} J * K & = & \mathbf{a}\downarrow\{j \cdot k \mid j \in J, k \in K\}, \\ J \mathrel{-\!\!*} K & = & \{a \mid j \in J \Rightarrow j \cdot a \in K\}, \end{array}$$

and the unit is given by $\mathbf{ay}(e) \cong \mathbf{a}(\downarrow(e))$.

Proof: We know that (by Proposition 4.4.3) for a subpresheaf $Q \rightarrowtail 1$, we have the adjunction

$$\mathrm{Im}(- \otimes Q) \dashv Q \multimap -$$

in $\mathrm{Sub}_{\widehat{\mathcal{C}}}(1)$, so since $\mathbf{a} \vdash i$, for a subsheaf $F \rightarrowtail 1$, we have

$$\mathbf{a}(\mathrm{Im}(- \otimes F)) \dashv iF \multimap i(-)$$

in $\mathrm{Sub}_{\mathrm{Sh}(\mathcal{C},J)}(1)$. Both $\mathbf{a}$ and i preserve monos so the functors are well-defined for the category $\mathrm{Sub}_{\mathrm{Sh}(\mathcal{C},J)}(1)$. By the lemma above, $iF \multimap iG$ is in $\mathrm{Sub}_{\mathrm{Sh}(\mathcal{C},J)}(1)$. ☐

Corollary 4.4.9. *If $(\mathcal{C}, *, e, \mathrel{-\!\!*})$ is a small, symmetric monoidal closed category and if J is the sup or the finite sup topology, then the partial order* $\mathrm{Sub}_{\mathrm{Sh}}(1)$ *of subobjects of the terminal object in* $\mathrm{Sh}(\mathcal{C}, J)$ *is a BI algebra.*

Proof: Since $*$ has a right adjoint it preserves colimit. Sups are coproducts so $*$ is automatically cover preserving. ☐

The (finite) sup topology is subcanonical, i.e., $\mathbf{y}e$ is a sheaf. So in these cases the unit is $\mathbf{y}e$, moreover, for the (finite) sup topology the monoidal tensor product $*$ also preserves ideals. Let I, J be ideals, then

$$I * J = \downarrow\{i * j \mid i \in I, j \in J\}$$

is an ideal: Obviously $I * J$ is downwards closed. Suppose $a_k \in I * J$ for all k in some index set K. We must show that $\bigvee_{k \in K} a_k \in I * J$. $a_k \in I * J$ iff there exists $n_k \in I$ and $m_k \in J$ such that $a_k \leq n_k * m_k$. So for all $k \in K$ we have $a_k \leq n_k * m_k$, it follows that $\bigvee_{k \in K} a_k \leq \bigvee_{k \in K}(n_k * m_k)$. I, J being ideals this implies $\bigvee_{k \in K} n_k \in I$ and $\bigvee_{k \in K} m_k \in J$. Moreover,

$$\bigvee_{k \in K} a_k \leq \bigvee_{k \in K} (n_k * m_k) = \bigvee_{k \in K} n_k * \bigvee_{k \in K} m_k$$

where the last equality is by the cover preserving property of $*$.

Chapter 5

Propositional intuitionistic logic

Literature: [LS86], [MLM94], [Pym02] and [Yan02].

In this chapter we give three kinds of models of propositional intuitionistic logic. First a class of algebraic models for which we prove soundness and completeness. The completeness result that we get is not very informative, however, since one of the models is essentially the syntax disguised as a model. It is desirably to obtain completeness for a smaller class of models than the class consisting of all Heyting algebras. The categorical models of provability, which are the propositional fragments of subobject semantics for predicate logic, provide such a completeness result. The categorical models of provability are actually just algebraic models with the property that they are Sub(1) in a topos, but the point is to narrow the class of models to get a more informative completeness theorem. The essence of the completeness proof is that the Yoneda embedding preserves the Heyting algebra structure. This is not the case for presheaves, so we are led to consider Grothendieck sheaves instead. The last model, a categorical model of proofs, is included only because this is how the categorical models (for BI) are presented in [Pym02] and [Yan02] and we are going to comment on these presentations.

Finally, in section 5.3 we derive a Kripke semantics for propositional logic in a topos, and in the special cases in which the topos is a presheaf or sheaf category. This follows standard presentations as in [LS86] and [MLM94].

5.1 Algebraic models

Propositional logic is logic without free variables. We think of propositions as statements. The language of propositional calculus consists of a set of countably infinite many propositional letters $\mathcal{L} = \{p, q, r, \ldots\}$ including two special symbols $\top, \bot$ and the logical connectives $\vee, \wedge, \rightarrow, \neg$. The set of propositions over $\mathcal{L}$, $\mathrm{Prop}(\mathcal{L})$ is given by the following grammar

$$p ::= p \mid p \vee q \mid p \wedge q \mid \neg p \mid p \rightarrow q \mid \top \mid \bot$$

Classically, a model for propositional logic is a function $\mathcal{I} : \mathrm{Prop}(\mathcal{L}) \rightarrow \{0, 1\}$ such that $\mathcal{I}(\top) = 1, \mathcal{I}(\bot) = 0, \mathcal{I}(p \wedge q) = \mathcal{I}(p) \times \mathcal{I}(q)$,

$$\mathcal{I}(p \vee q) = \begin{cases} 0 & \text{if } \mathcal{I}(p) + \mathcal{I}(q) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

$$
\begin{array}{cc}
p \vdash p & \dfrac{p \vdash q \quad q \vdash r}{p \vdash r} \\[2ex]
p \vdash \top & \bot \vdash p \\[2ex]
\dfrac{p \vdash q \quad p \vdash r}{p \vdash q \wedge r} & \dfrac{p \vdash q_1 \wedge q_2}{p \vdash q_i}\ (i = 1, 2) \\[2ex]
\dfrac{p \vdash r \quad q \vdash r}{p \vee q \vdash r} & \dfrac{p \vdash q_i}{p \vdash q_1 \vee q_2}\ (i = 1, 2) \\[2ex]
\dfrac{p \wedge q \vdash r}{p \vdash q \to r} & \dfrac{p \vdash s \to r \quad q \vdash s}{p \wedge q \vdash r}
\end{array}
$$

Table 5.1: Hilbert-type system for intuitionistic propositional logic : **HIL**

$\mathcal{I}(\neg p) = 1 - \mathcal{I}(p)$ and $\mathcal{I}(p \to q) = \mathcal{I}(\neg p \vee q)$. A (classical) model is thus an assignment of truth values to the propositional letters that respects the semantics of the logical connectives. We can also think of it as a characteristic function telling us which propositions are true. If we have a free variable x of type X and the interpretation of X is some set U it is natural to think of a formula $p(x)$ as the subset $P \subseteq U$ such that $p(u)$ holds iff $u \in P$. Since in propositional calculus we do not have free variables, such a subset can only be all or nothing, so we can replace 0 with the empty set and 1 with some fixed set U and regard a model as a map from $\mathrm{Prop}(\mathcal{L})$ to $\mathcal{P}(U)$. Then the interpretation of connectives becomes an operation on subsets of U, more specifically $\wedge$ is intersection, $\vee$ is union, $\neg$ is the complement, and $\to$ is explained in terms of $\neg$ and $\vee$. $\mathcal{P}(U)$ with these four operations is a Boolean algebra.

On the set of propositions $\mathrm{Prop}(\mathcal{L})$ we define a binary relation $\vdash$ by the closure rules given in table 5.1. A rule of the form $\frac{P_1 \qquad P_2}{Q}$, states that if P_1 and P_2 are in the relation $\vdash$, then Q is also in the relation $\vdash$, and a rule of the form P just states that P is in the relation. $p \vdash q$ can be thought of as: Under the assumption p, there is a (purely syntactical) proof that q holds.

Remark 5.1.1 (conventions). *We write $\vdash p$ for $\top \vdash p$. The system in Table 5.1 does not use the negation symbol, but this can be defined as $\neg p := p \to \bot$.*

Classical propositional logic is obtained by adding the axiom

$$\top \vdash p \vee \neg p.$$

In intuitionistic logic the equivalence of propositions: $p \to q \equiv \neg p \vee q$, is not valid since it implies law of the excluded middle $(p \vee \neg p)$. To see why this is, note that we can always derive $\vdash p \to p$. This implies that the interpretation of $p \to q$ must be different from the classical one. The algebraic system that we obtain is thus not a Boolean algebra but what is known as a Heyting algebra. The typical model is not the powerset of some set, but the set of open subsets w.r.t. some topology. The operations corresponding to $\vee$ and $\wedge$ are still union and intersection, but $U \to V$ becomes the largest open set W such that $U \cap W \subseteq V$, and $\neg U$ is the interior of the complement of U, i.e., $U \to V = (\complement U \cup V)^\circ$ and $\neg U = (\complement U)^\circ$. Note that $U \to V$ can also be characterized as $\bigcup\{W \mid W \cap U \subseteq V\}$. The constants $\top$ and $\bot$ correspond to the greatest and least elements of a Heyting algebra. So to be specific,

Definition 5.1.2 (Algebraic model of propositional intuitionistic logic). *An* algebraic model $\mathcal{A}$ *of intuitionistic propositional logic consists of a Heyting algebra* $(H, \vee_H, \wedge_H, \rightarrow_H, \bot_H, \top_H)$ *and an interpretation*

$$[\![\cdot]\!] : \mathrm{Prop}(\mathcal{L}) \to H$$

such that the structure is preserved, i.e., $[\![p \circ q]\!] = [\![p]\!] \circ_H [\![q]\!]$ *for* $\circ \in \{\vee, \wedge, \rightarrow\}$*, and* $[\![\top]\!] = \top_H, [\![\bot]\!] = \bot_H$*.*

We write $[\![p]\!]_{\mathcal{A}} \leq [\![q]\!]_{\mathcal{A}}$ if the interpretation of p is below the interpretation of q in the Heyting algebra H, part of the model $\mathcal{A}$. If $[\![p]\!]_{\mathcal{A}} \leq [\![q]\!]_{\mathcal{A}}$ for all models $\mathcal{A}$, we write $[\![p]\!] \leq [\![q]\!]$.

Theorem 5.1.3 (Soundness). *If* $p \vdash q$ *is provable in* ***HIL****, then* $[\![p]\!] \leq [\![q]\!]$*.*

Proof: By induction on the structure of proofs in **HIL**.

$p \vdash p$:	Clearly for all interpretations in all Heyting algebras $[\![p]\!] \leq [\![p]\!]$.
$\dfrac{p \vdash q \quad q \vdash r}{p \vdash r}$:	The relation $\leq_H$ is transitive.
$p \vdash \top$:	By definition, $[\![\top]\!] = \top_H$ is the greatest element in the Heyting algebra.
$\bot \vdash p$:	$[\![\bot]\!]$ is the least element in the model.
$\dfrac{p \vdash q \quad p \vdash r}{p \vdash q \wedge r}$:	$[\![q \wedge r]\!]$ is by definition the greatest lower bound of $[\![q]\!]$ and $[\![r]\!]$.
$\dfrac{p \vdash q_1 \wedge q_2}{p \vdash q_i}$:	Same reason as above.
$\dfrac{p \vdash r \quad q \vdash r}{p \vee q \vdash r}$:	$[\![p \vee q]\!]$ is the least upper bound in the model of $[\![p]\!]$ and $[\![q]\!]$.
$\dfrac{p \vdash q_i}{p \vdash q_1 \vee q_2}$:	By the same reason as above.
$\dfrac{p \wedge q \vdash r}{p \vdash q \rightarrow r}$:	In a Heyting algebra H, $[\![p]\!] \wedge_H [\![q]\!] \leq [\![r]\!]$ iff $[\![p]\!] \leq [\![q]\!] \rightarrow_H [\![r]\!]$.
$\dfrac{p \vdash s \rightarrow r \quad q \vdash s}{p \wedge q \vdash r}$:	We have $[\![p]\!] \wedge_H [\![s]\!] \leq [\![r]\!]$ and $[\![q]\!] \leq [\![s]\!]$, the result follows by transitivity of $\leq$.

□

We now define what is known as the free Heyting algebra on countably many variables HS. The elements of HS are equivalence classes of propositions in the language of propositional calculus under provability, that is

$$p \sim q \text{ iff } \vdash p \leftrightarrow q$$

where $\vdash p \leftrightarrow q$ is an abbreviation of $\vdash p \rightarrow q$ and $\vdash q \rightarrow p$. The equivalence classes form a Heyting algebra HS with

- $\top_{HS} := [\top] = \{p \mid \top \vdash p\}$
- $\bot_{HS} := [\bot] = \{p \mid p \vdash \bot\}$
- $[p] \vee [q] := [p \vee q]$
- $[p] \wedge [q] := [p \wedge q]$
- $[p] \to [q] := [p \to q]$

ordered by

$$[p] \leq [q] \text{ iff } p \vdash q.$$

Well-definedness Clearly $\sim$ defines an equivalence relation. For well-definedness of the operations $\vee, \wedge, \to$ on equivalence classes, suppose $s \sim s'$ and $t \sim t'$ then $[s] \vee [t] = [s \vee t] = \{r \mid \vdash r \leftrightarrow (s \vee t)\} = \{r \mid \vdash r \leftrightarrow (s' \vee t')\} = [s'] \vee [t']$. The others are similar.

The syntactic model consists of the Heyting algebra HS and the interpretation given by $[\![p]\!] := [p]$ for propositional letters $p \in \mathcal{L}$ extended to all formulas using the operations $\wedge, \vee, \to$ on HS as defined above.

Theorem 5.1.4 (Completeness). *For propositional formulas* p, q, $p \vdash q$ *iff* $[\![p]\!] \leq [\![q]\!]$.

Proof: This is immediate from the definitions. □

5.2 Categorical models

Though algebraic models may be adequate for some purposes, the completeness result that they provide is not very strong in the sense that the class of models is very close to the syntactic system **HIL**; it does not make much difference whether we study the logic **HIL** or the models (Heyting algebras).

The aim is to find a smaller class of models that will still provide completeness. The categorical notion of a model enables us to identify a natural class of models for which we do have completeness.

For the purpose of modeling propositional logic it is perhaps a bit exaggerated to talk about a categorical model since we are only using a small part of the category to actually model the formulas, but still, the structure that we need *is* induced by the structure on the whole category, so when dealing with matters in a categorical framework we can make use of general results about categories. Besides, to define the class of models, we will need categories.

There are (at least) two different notions of a categorical model. One that is concerned with provability of formulas only and one that also models the proofs between formulas. We shall mainly be interested in the former ones, i.e., models that are not concerned with the structure of a proof, but only with the provability of formulas. However, to compare some results with results given in [Pym02] we also need to know what categorical models of proofs are, so we give a definition of those as well.

The categorical models of provability are really just algebraic models in disguise, however, the definition arises naturally as the propositional fragment of the categorical models of predicate logic (where the whole category is needed).

Definition 5.2.1 (Categorical model of provability in HIL). *A categorical model* $\mathcal{M}$ *of provability is a category* $\mathcal{T}$ *with the property that* $\mathrm{Sub}_{\mathcal{T}}(1)$ *is a Heyting algebra* $(\mathrm{Sub}_{\mathcal{T}}(1), \vee_{\mathcal{T}}, \wedge_{\mathcal{T}}, \rightarrow_{\mathcal{T}}, \bot_{\mathcal{T}}, \top_{\mathcal{T}})$*, together with an interpretation function*

$$[\![\cdot]\!] : \mathrm{Prop}(\mathcal{L}) \rightarrow \mathrm{Sub}_{\mathcal{T}}(1)$$

which preserves the structure, i.e., $[\![p \circ q]\!] = [\![p]\!] \circ_{\mathcal{T}} [\![q]\!]$ *for* $\circ \in \{\vee, \wedge, \rightarrow\}$*, and* $[\![\top]\!] = \top_{\mathrm{Sub}_{\mathcal{T}}(1)}, [\![\bot]\!] = \bot_{\mathrm{Sub}_{\mathcal{T}}(1)}$.

We say that a formula p holds in $\mathcal{M}$, written $\mathcal{M} \models p$ iff $[\![p]\!] = \top_{\mathrm{Sub}_{\mathcal{T}}(1)}$, i.e., iff the interpretation of p is the maximal subobject of 1. If the interpretation of a formula p is below the interpretation of a formula q in $\mathrm{Sub}_{\mathcal{T}}(1)$ we write $p \models_{\mathcal{M}} q$.

Categorically a Heyting algebra is bi-ccc and since it is a preorder, there is at most one arrow from an object a to an object b. If a and b are interpretations of formulas, we have b is provable from a iff $a \leq b$ iff there is an arrow from a to b. If we are also interested in the structure of a proof $a \vdash b$, it makes sense to model the logic in a category that possibly has more than one arrow between objects a and b. This leads to the following definition.

Definition 5.2.2 (Categorical model of proofs in HIL). *A categorical model of proofs consists of a bi-ccc* $\mathcal{C}$ *and an interpretation function*

$$[\![\cdot]\!] : \mathrm{Prop}(\mathcal{L}) \rightarrow \mathrm{Obj}(\mathcal{C})$$

satisfying

$$\begin{aligned} [\![p \wedge q]\!] &= [\![p]\!] \times [\![q]\!] \\ [\![\top]\!] &= 1 \\ [\![p \vee q]\!] &= [\![p]\!] + [\![q]\!] \\ [\![\bot]\!] &= 0 \\ [\![p \rightarrow q]\!] &= [\![p]\!] \rightarrow [\![q]\!]. \end{aligned}$$

Note that models of provability are contained in the models of proofs, since a Heyting algebra considered as a category is a bi-ccc.

Consider the category $\widehat{HS}$ of presheaves over the syntactic preorder HS. Recall that the subobjects of 1 form a complete Heyting algebra $\mathrm{Sub}_{\widehat{HS}}(1)$:

- $\top_{\mathrm{Sub}_{\widehat{HS}}(1)} = HS$
- $\bot_{\mathrm{Sub}_{\widehat{HS}}(1)} = \emptyset$
- $I \vee J = I \cup J$
- $I \wedge J = I \cap J$
- $I \rightarrow J = \bigcup \{L \in \mathrm{Sub}_{\widehat{HS}}(1) \mid L \wedge I \subseteq J\}$

where I, J are sieves on HS (by Prop. 2.5.17 there is a one-one correspondence between sieves on HS and subfunctors of 1 in $\widehat{HS}$). Order is inclusion.

The idea is to define an interpretation of the propositional calculus in $\mathrm{Sub}_{\widehat{HS}}(1)$ as

$$|p| := \downarrow [p] = \{t \mid t \vdash p\}$$

for $p \in \mathcal{L}$, and extend this to all formulas $t \in \mathrm{Prop}(\mathcal{L})$ using the Heyting algebra operations on $\mathrm{Sub}_{\widehat{HS}}(1)$. Note that the map $[p] \mapsto \downarrow [p]$ is actually the Yoneda embedding, since $\downarrow [p]$ is the sieve corresponding to $\mathbf{y}[p]$. *If* this is a well-defined interpretation, i.e., if for all propositional formulas t, $|t| = \downarrow [t]$, we would get an immediate completeness proof (using the Yoneda embedding), however, it is not the case that $|t| = \downarrow [t]$ for all formulas t (it does not work for $\bot$ and $\vee$).

Proposition 5.2.3. *The map $\downarrow (-)$ from HS to $\mathrm{Sub}_{\widehat{HS}}(1)$ defined by $[p] \mapsto \downarrow [p]$ preserves $\top, \wedge, \rightarrow$ but not $\bot, \vee$.*

Proof:

$$\begin{array}{lllll}
[\top] & \mapsto & \downarrow[\top] & = & HS \\
 & & & = & \top_{\mathrm{Sub}_{\widehat{HS}}(1)} \\
 & & & & \\
[t_1 \wedge t_2] & \mapsto & \downarrow[t_1 \wedge t_2] & = & \{[s] \mid s \vdash (t_1 \wedge t_2)\} \\
 & & & = & \{[s] \mid s \vdash t_1\} \cap \{[s] \mid s \vdash t_2\} \\
 & & & = & \downarrow[t_1] \cap \downarrow[t_2] \\
 & & & & \\
[t_1 \rightarrow t_2] & \mapsto & \downarrow[t_1 \rightarrow t_2] & = & \{[s] \mid s \vdash t_1 \rightarrow t_2\} \\
 & & & = & \{[s] \mid s \wedge t_1 \vdash t_2\} \\
 & & & = & \{[s] \mid [s \wedge t_1] \leq_{HS} [t_2]\} \\
 & & & = & \{[s] \mid \downarrow[s \wedge t_1] \subseteq \downarrow[t_2]\} \quad \text{Yoneda is full and faithful} \\
 & & & = & \{[s] \mid \downarrow[s] \cap \downarrow[t_1] \subseteq \downarrow[t_2]\} \quad \text{by the previous calculation} \\
 & & & = & \bigcup\{U \mid U \cap \downarrow[t_1] \subseteq \downarrow[t_2]\} \\
 & & & = & \downarrow[t_1] \rightarrow \downarrow[t_2]
\end{array}$$

But

$$\begin{array}{lllll}
[\bot] & \mapsto & \downarrow[\bot] & = & \{[s] \mid s \vdash \bot\} \\
 & & & = & \{[\bot]\} \\
 & & & \neq & \emptyset \\
 & & & & \\
[t_1 \vee t_2] & \mapsto & \downarrow[t_1 \vee t_2] & = & \{[s] \mid s \vdash t_1 \vee t_2\} \\
 & & & \supseteq & \{[s] \mid s \vdash t_1\} \cup \{[s] \mid s \vdash t_2\}
\end{array}$$

we have $\dfrac{s \vdash t_i}{s \vdash t_1 \vee t_2}$ but not $\dfrac{s \vdash t_1 \vee t_2}{s \vdash t_i}$, so the inclusion above is not an equality. □

The interpretation is well-defined for the $(\bot, \vee)$-free fragment of the logic.

Corollary 5.2.4. *$|t| = \downarrow [t]$ for all formulas t of the $(\bot, \vee)$-free fragment of propositional intuitionistic logic.*

Proof: By induction on t using Proposition 5.2.3. □

Proposition 5.2.5 (completeness for the $(\bot, \vee)$-free fragment). *If p, q are $(\bot, \vee)$-free formulas of* **PIL, then** $p \vdash q$ **iff** $|p| \subseteq |q|$ **in** $\mathrm{Sub}_{\widehat{HS}}(1)$**.**

Proof:

$$\begin{array}{rlcclcl} p \vdash q & \text{iff} & [p] & \leq_{HS} & [q] & \text{by Proposition 5.1.4} \\ & \text{iff} & \downarrow[p] & \subseteq & \downarrow[q] & \text{Yoneda is full and faithful} \\ & \text{iff} & |p| & \subseteq & |q| & \text{by Corollary 5.2.4} \end{array}$$

□

To get a completeness result for the entire logic, we embed the syntactic model HS into the category of sheaves over the preorder HS. In chapter 2 we have seen that for any Grothendieck topology J on a category $\mathcal{C}$, $\mathrm{Sh}(\mathcal{C}, J)$ is a topos and that in any topos $\mathrm{Sub}(E)$ is a Heyting algebra for any object E.

We define a Grothendieck topos over the syntactic Heyting algebra HS. The topology is the finite sup topology (see example 2.6.5), which has the basis

$$\{a_i \mid i \in I\} \in K(c) \text{ iff } \bigvee_{i \in I} a_i = c$$

where I is finite. Recall that a subsheaf of 1 is a subfunctor S such that

If $* \in S(a_i)$ for all $i \in I$ then $* \in S(a)$ for all coverings $\bigvee_{i \in I} a_i = a$ of a.

In particular, all representables are subsheaves of 1. Recall also that there is a one-one correspondence between subsheaves of 1 and ideals (sieves that are closed under finite $\bigvee$). The Heyting structure of $\mathrm{Sub}_{\mathrm{Sh}(HS)}(1)$ is, for I, J ideals on HS:

- $\top_{\mathrm{Sub}_{\mathrm{Sh}(HS)}(1)} = HS$
- $\bot_{\mathrm{Sub}_{\mathrm{Sh}(HS)}(1)} = \{[\bot]\}$ (ideal generated by $\emptyset$)
- $I \wedge J = I \cap J$
- $I \vee J = \downarrow\{i \vee j \mid i \in I, j \in J\}$, ideal generated by $I \cup J$
- $I \to J = \bigcup\{L \in \mathrm{Sub}_{\mathrm{Sh}(HS)}(1) \mid L \wedge I \subseteq J\}$.

Proposition 5.2.6. *The map $\downarrow(-) : HS \to \mathrm{Sub}_{\mathrm{Sh}(HS)}(1)$ is a map of Heyting algebras, i.e., it preserves all the Heyting algebra structure.*

Proof: For $[\top], \wedge, \to$ we can use the arguments given in the proof of Proposition 5.2.3.

$$\begin{array}{lcl} [\bot] & \mapsto & \downarrow[\bot] & = & \{[\bot]\} \\ \\ [t_1 \vee t_2] & \mapsto & \downarrow[t_1 \vee t_2] & = & \downarrow\{[s] \mid s \vdash t_1 \vee t_2\} \\ & & & = & \downarrow\{[s] \mid s \vdash s_1 \vee s_2, \text{ where } s_1 \vdash t_1, s_2 \vdash t_2\} \\ & & & = & \downarrow\{[s_1] \vee [s_2] \mid s_1 \vdash t_1, s_2 \vdash t_2\} \\ & & & = & \downarrow[t_1] \vee \downarrow[t_2] \qquad \text{by definition of } \vee \text{ in } \mathrm{Sub}_{\mathrm{Sh}(HS)}(1) \end{array}$$

□

Interpretation of propositional formulas in $\mathrm{Sub}_{\mathrm{Sh}(HS)}(1)$ is

$$\{p\} := \downarrow[p]$$

for $p \in \mathcal{L}$, extended to formulas in the usual way. For this Heyting algebra the interpretation *is* well-defined.

Corollary 5.2.7. $\{t\} = \downarrow [t]$ *for all formulas* t *of propositional intuitionistic logic.*

Proof: By induction on t using Proposition 5.2.6. □

Theorem 5.2.8 (Completeness). *For propositional formulas* p, q, $p \vdash q$ *iff* $\{p\} \subseteq \{q\}$ *in the model* $(\mathrm{Sh}(HS), \{\cdot\})$.

Proof:

$$\begin{array}{rlrclll} p \vdash q & \text{iff} & [p] & \leq & [q] & \text{by definition of } HS \\ & \text{iff} & \downarrow [p] & \subseteq & \downarrow [q] & \text{since the Yoneda functor is full and faithful} \\ & \text{iff} & \{p\} & \subseteq & \{q\} & \text{by Corollary 5.2.7} \\ & \text{iff} & p & \models_{\mathrm{Sh}(HS)} & q. & \end{array}$$

□

This completeness result is stronger than the one in 5.1.4 because it says that if we want to show that some sequence $p \vdash q$ of **HIL** holds, it is enough to show that $[\![p]\!] \leq_{\mathcal{M}} [\![q]\!]$ for all $\mathcal{M} \in$ **Models**, where **Models** is a class of categorical models that contain the syntactic model $\mathrm{Sh}(HS)$, e.g. **Models** could be all Grothendieck sheaf toposes (together with interpretations that make each of them a model).

5.3 Kripke-Joyal semantics

In the categorical models of provability, a formula is true or holds for a model if and only if it is interpreted as the maximal subobject of the terminal object. If the model lives in a topos, this can be rephrased: We always have the pullback

$$\begin{tikzcd} {[\![p]\!]} \arrow[r, tail] \arrow[d, tail] & 1 \arrow[d, "\mathrm{char}\, p"] \\ 1 \arrow[r, "\top"'] & \Omega, \end{tikzcd} \tag{5.1}$$

where $[\![p]\!]$ is the maximal subobject of 1 iff $\mathrm{char}\, p = \top$, which in turn is the case iff the following square commutes for all objects $C \in \mathcal{T}$

$$\begin{tikzcd} C \arrow[r, "1_C"] \arrow[d, "1_C"'] & 1 \arrow[d, "\mathrm{char}\, p"] \\ 1 \arrow[r, "\top"'] & \Omega. \end{tikzcd}$$

If the above square commutes for C, we write $C \Vdash p$, and we say that p holds at stage C. So by the argument above we have:

Proposition 5.3.1. *Let* $\mathcal{T}$ *be a topos and* $\mathcal{M} = (\mathcal{T}, [\![\cdot]\!])$ *a categorical model of provability as defined in 5.2.1, then for all* $p \in \mathrm{Prop}(\mathcal{L})$, $\mathcal{M} \models p$ *iff* p *holds at all stages* C *of* $\mathcal{T}$*, that is, for all objects* C *in* $\mathcal{T}$, $C \Vdash p$.

Remark 5.3.2. *Since we are working with subobjects of* 1 *only, using the pullback property of diagram 5.1, we have* $C \Vdash p$ *iff there is an arrow from* C *to* $[\![p]\!]$.

Since there is only one arrow from A *to* 1*, we can identify a (representative) of a subobject of* 1, $A \rightarrowtail 1$ *with its domain* A.

The advantage of this reformulation of truth of a proposition is that it enables us to give an alternative inductive definition of truth in a model, as follows.

Theorem 5.3.3 (Kripke-Joyal semantics). *Let $\mathcal{T}$ be a topos and $\mathcal{M} = (\mathcal{T}, [\![\cdot]\!])$ a categorical model of provability as defined in 5.2.1. Given propositions p, q and $C \in \mathrm{Obj}(\mathcal{T})$, then*

(0) $C \Vdash p$ *iff* $\mathrm{char}(p)1_C = \top 1_C$
iff there exists an arrow $C \to [\![p]\!]$,

(1) $C \Vdash p \wedge q$ *iff* $C \Vdash p$ *and* $C \Vdash q$,

(2) $C \Vdash \top$ *always,*

(3) $C \Vdash \bot$ *iff* C *is the initial object of* $\mathcal{T}$,

(4) $C \Vdash p \vee q$ *iff there exists an epi* $[k, l] : D + E \to C$ *such that* $D \Vdash p$ *and* $E \Vdash q$,

(5) $C \Vdash p \to q$ *iff, for all* $h : D \to C$, *if* $D \Vdash p$ *then* $D \Vdash q$.

Proof: Propositions are interpreted in $\mathrm{Sub}_{\mathcal{T}}(1)$. The Heyting algebra structure of $\mathrm{Sub}_{\mathcal{T}}(1)$ is given in Remark 2.3.10.

(0) This is the definition.

(1) Suppose $C \Vdash p \wedge q$, this means that there is an arrow $C \to [\![p \wedge q]\!] = [\![p]\!] \times_{\mathcal{T}} [\![q]\!]$, where $[\![p]\!] \times_{\mathcal{T}} [\![q]\!]$ is the pullback

$$\begin{tikzcd}
C \arrow[dr] & & \\
& {[\![p]\!] \times_{\mathcal{T}} [\![q]\!]} \arrow[r] \arrow[d] & {[\![p]\!]} \arrow[d] \\
& {[\![q]\!]} \arrow[r] & 1
\end{tikzcd}$$

so there are arrows $C \to [\![p]\!]$ and $C \to [\![q]\!]$, i.e., $C \Vdash p$ and $C \Vdash q$. On the other hand, if $C \Vdash p$ and $C \Vdash q$, then there are arrows $C \to [\![p]\!]$ and $C \to [\![q]\!]$, by the pullback property we then get an arrow $C \to [\![p \wedge q]\!]$.

(2) Nothing to show.

(3) The interpretation of $\bot$ is the mono $0 \to 1$, i.e., there is a pullback

$$\begin{tikzcd}
0 \arrow[r, "{[\![\bot]\!]}"] \arrow[d] & 1 \arrow[d, "\mathrm{char}(\bot)"] \\
1 \arrow[r, "\top"'] & \Omega
\end{tikzcd}$$

so $C \Vdash \bot$ iff there is an arrow $C \to 0$ which holds (in a topos) iff $C \cong 0$ by Remark 2.0.3.

(4) The interpretation of $p \vee q$ is the image factorization of the arrow from the coproduct $[\![p]\!] + [\![q]\!]$ to 1.

$$\begin{tikzcd}
{[\![p]\!] + [\![q]\!]} \arrow[dr, two heads, "e"] & & {[\![p]\!]} \arrow[ll, "\iota_1"'] \arrow[dd] \\
& \mathrm{Im}([\![p]\!] + [\![q]\!]) \arrow[dr, hook] & \\
{[\![q]\!]} \arrow[uu, "\iota_2"] \arrow[rr] & & 1
\end{tikzcd}$$

Suppose there is an arrow $C \to \mathrm{Im}([\![p]\!] + [\![q]\!])$ then we can take the pullback of this arrow along $e\iota_1$ and along $e\iota_2$ to get

$$\begin{tikzcd}
D \arrow[r] \arrow[d, "k"'] & {[\![p]\!]} \arrow[d] & & E \arrow[r] \arrow[d, "l"'] & {[\![q]\!]} \arrow[d] \\
C \arrow[r] & \mathrm{Im}([\![p]\!] + [\![q]\!]) & & C \arrow[r] & \mathrm{Im}([\![p]\!] + [\![q]\!]).
\end{tikzcd}$$

Since the pullback functor preserves coproducts (it has a right adjoint) we get a pullback

$$\begin{tikzcd}
D + E \arrow[r] \arrow[d, "{[k+l]}"'] & {[\![p]\!] + [\![q]\!]} \arrow[d] \\
C \arrow[r] & \mathrm{Im}([\![p]\!] + [\![q]\!])
\end{tikzcd}$$

where $[k + l]$ is epi because pullback preserves epis in a topos.

On the other hand suppose there are maps $k : D \to C$ and $l : E \to C$ such that $[k, l] : D + E \to C$ is epi and $D \Vdash p$ and $E \Vdash q$, we then have a commuting diagram

$$\begin{tikzcd}
D + E \arrow[r, "s"] \arrow[d, "{[k,l]}"'] & {[\![p]\!] + [\![q]\!]} \arrow[r, two heads, "e"] \arrow[d] & \mathrm{Im}([\![p]\!] + [\![q]\!]) \arrow[dl, hook', "m"] \\
C \arrow[r, "1_C"'] & 1 &
\end{tikzcd}$$

Since there is only one arrow from $D + E$ to 1, we must have $1_C[k, l] = mes$. $[k, l]$ is epi and m is mono so by Proposition 2.2.1, there is an arrow $C \to \mathrm{Im}([\![p]\!] + [\![q]\!])$, i.e., $C \Vdash p \vee q$.

(5) Suppose $C \Vdash p \to q$ and $h : D \to C$ and $D \Vdash p$, then we have $D \to [\![p \to q]\!]$, where $[\![p \to q]\!] = [\![q]\!]^{[\![p]\!]}$, which again means that there is an arrow

$$D \longrightarrow [\![p]\!] \times ([\![q]\!]^{[\![p]\!]}) \xrightarrow{\varepsilon} [\![q]\!]$$

where ε is the counit of the adjunction between the Cartesian product and the exponent.

To get the other implication take any C, and pull back $[\![p]\!] \to 1$:

$$\begin{tikzcd}
C \times_{\mathcal{T}} [\![p]\!] \arrow[r] \arrow[d, "h"'] & {[\![p]\!]} \arrow[d] \\
C \arrow[r] & 1.
\end{tikzcd}$$

It is not hard to see that $C \times_{\mathcal{T}} \llbracket p \rrbracket$ must be the Cartesian product $C \times \llbracket p \rrbracket$. Now, $h : C \times \llbracket p \rrbracket \to C$ and $C \times \llbracket p \rrbracket \Vdash p$ so by assumption we have $C \times \llbracket p \rrbracket \Vdash q$, i.e., there is an arrow $C \times \llbracket p \rrbracket \to \llbracket q \rrbracket$, using the adjunction we get an arrow $C \to \llbracket q \rrbracket^{\llbracket p \rrbracket}$.

□

Proposition 5.3.4 (Kripke monotonicity). *If there exists an arrow $C \to D$ in $\mathcal{T}$ and $D \Vdash p$ then $C \Vdash p$.*

Proof: By remark 5.3.2, $D \Vdash p$ iff there is an arrow $D \to \llbracket p \rrbracket$, and then we get arrow $C \to D \to \llbracket p \rrbracket$. In fact the proposition just states that $\llbracket p \rrbracket$ is a subfunctor of 1. □

The definition of a categorical model 5.2.1 is the propositional case of what is sometimes referred to as *subobject semantics*. Here the Kripke semantics, which is less general than the subobject semantics since we require the category to be a topos, is derived from the subobject semantics. However, one can also use the Kripke semantics as the definition of the interpretation in a model, then Kripke monotonicity (and local character for sheaves) must be requirements, ensuring well-definedness, rather than propositions. Then, given interpretations of the propositional letters as subobjects of 1, each of the clauses (1)-(5) of Theorem 5.3.3 defines a subobject of 1 if and only if the Kripke monotonicity holds for the clause.

It turns out that we only need to consider a subset of the stages in $\mathcal{T}$ in order to determine whether $\mathcal{T} \models p$, namely a generating set.

Definition 5.3.5 (Generating set). *A set $\mathcal{G}$ of objects of $\mathcal{T}$ is called a* generating set *if, for any two arrows $f, g : A \rightrightarrows B$, $f = g$ iff for all $C \in \mathcal{G}$ and all arrows $h : C \to A$, $fh = gh$.*

Proposition 5.3.6. *If $\mathcal{G}$ is a generating set of objects of a topos $\mathcal{T}$, and $\mathcal{M} = (\mathrm{Sub}_{\mathcal{T}}(1), \llbracket \cdot \rrbracket)$ is a categorical model of* **HIL** *then $\mathcal{M} \models p$ iff for all stages $C \in \mathcal{G}$, $C \Vdash p$.*

Proof: By definition $\mathcal{M} \models p$ iff $\llbracket p \rrbracket = \top_{\mathrm{Sub}_{\mathcal{T}}(1)}$. Suppose $C \Vdash p$ for all $C \in \mathcal{G}$, then $\mathrm{char}(p) 1_C = \top 1_C$ for all $C \in G$, by definition of generating set this is equivalent to $\mathrm{char}(p) = \top$, i.e., $\llbracket p \rrbracket = \top_{\mathrm{Sub}_{\mathcal{T}}(1)}$. The converse is trivial. □

Sometimes the clause (4) can be strengthened:

Definition 5.3.7. *An object C is called* indecomposable *if, for all arrows $k : D \to C$ and $l : E \to C$ such that $[k, l] : D + E \to C$ is epi, either k or l is epi.*

Lemma 5.3.8. *(4') If C is indecomposable, then $C \Vdash p \vee q$ iff $C \Vdash p$ or $C \Vdash q$.*

Proof: Assume C is indecomposable and that $C \Vdash p \vee q$. By (4) of Theorem 5.3.3, there is an epi $[k, l] : D + E \to C$ such that $D \Vdash p$ and $E \Vdash q$. Since C is indecomposable, either $k : D \to C$ or $l : E \to C$ is epi. We have a commuting diagram

$$\begin{tikzcd}
D \arrow[r, "k"] \arrow[d] & C \arrow[r, two heads] \arrow[d] & \mathrm{Im}(1_C) \arrow[dl] \\
\llbracket p \rrbracket \arrow[r, tail] & 1 &
\end{tikzcd}$$

and a similar one for $l, \llbracket q \rrbracket$. If k is epi we have $\mathrm{Im}(1_C) = \mathrm{Im}(1_C k)$ which implies that there is an arrow $\mathrm{Im}(1_C) \to \llbracket p \rrbracket$ (using the universal property of image factorizations). This means, if k is epi, $C \Vdash p$. If l is epi there is a similar proof that $C \Vdash q$.

On the other hand if $C \Vdash p$ then there is an arrow

$$C \to [\![p]\!] \to [\![p]\!] + [\![q]\!] \twoheadrightarrow \mathrm{Im}([\![p]\!] + [\![q]\!]) = [\![p \vee q]\!]$$

(and similar for q if $C \Vdash q$) showing that $C \Vdash p \vee q$. □

Lemma 5.3.9. *The clause (5) of Theorem 5.3.3 also holds when the objects C, D are restricted to be in a generating set $\mathcal{G}$.*

Proof: We must show that

$$\text{for all } h : D \to C, \text{ where } D \in \mathcal{G}, D \Vdash p \text{ implies } D \Vdash q. \tag{5.2}$$

implies

$$\text{for all } h : D \to C, \text{ where } D \in \mathrm{Obj}(\mathcal{T}), D \Vdash p \text{ implies } D \Vdash q. \tag{5.3}$$

Assume 5.2 and assume that $h : D \to C$ and $D \Vdash p$ for some $D \in \mathrm{Obj}(\mathcal{T})$. To show that $D \Vdash q$ it is enough to show that for all $E \in \mathcal{G}$ and $g : E \to D$, $E \Vdash q$. Let $E \in \mathcal{G}$ and $g : E \to D$, then there is an arrow $E \to D \to [\![p]\!]$, hence $E \Vdash p$, and by 5.2 then $E \Vdash q$. □

5.3.1 Kripke-Joyal semantics in functor categories

In this section we shall formulate the Kripke-Joyal semantics for the specific classes of toposes that we have been working with.

Kripke-Joyal semantics in presheaf categories

For any small category $\mathcal{C}$, the presheaf category $\widehat{\mathcal{C}}$ is a topos, so given an interpretation such that $\mathcal{M} = (\widehat{\mathcal{C}}, [\![\cdot]\!])$ is a categorical model of provability, Theorem 5.3.3 holds. For this class of toposes we can simplify the Kripke semantics a bit. We begin by noticing

Proposition 5.3.10. *The representable functors $\mathbf{y}C$, where $C \in \mathrm{Obj}(\mathcal{C})$, form a generating set for $\widehat{\mathcal{C}}$. Moreover each representable is indecomposable.*

Proof: Suppose we have natural transformations $f, g : F \to G$. We must show that $f = g$ iff for all representables $\mathbf{y}C$ and all arrows $h : \mathbf{y}C \to F$, $fh = gh$. By the Yoneda Lemma this is the same as showing $f = g$ iff for all C and all $\check{h} \in FC$, $f_C(\check{h}) = g_C(\check{h})$.

To see that $\mathbf{y}C$ is indecomposable, suppose $[k, l] : F + G \to \mathbf{y}C$ is epi, we claim that either k or l is epi. Epis are defined pointwise, in particular $[k_C, l_C] : FC + GC \to \mathcal{C}(C, C)$ is surjective, therefore id_C must be in the image of either k_C or l_C, say the former. Then there is an element $a \in FC$ such that $k_C(a) = \mathrm{id}_C$. By the Yoneda Lemma, an element in $f \in \mathcal{C}(C, C) = \mathbf{y}C(C)$ corresponds uniquely to a natural transformation $\hat{f} : \mathbf{y}C \Rightarrow \mathbf{y}C$. In particular $\widehat{\mathrm{id}_C} = \mathrm{id}_{\mathbf{y}C}$ and $\widehat{k_C(a)} = k \circ \hat{a}$, so $k \circ \hat{a} = \mathrm{id}_{\mathbf{y}C}$, which shows that k is epi. □

Recall that subobjects of a presheaf have a canonical representative, so we can identify $F \in \mathrm{Sub}(G)$ with a subfunctor F' of the functor G.

Theorem 5.3.11. *Let $\mathcal{C}$ be a small category and $\mathcal{M} = (\widehat{\mathcal{C}}, [\![\cdot]\!])$ a categorical model of provability as defined in 5.2.1. Given propositions p, q and $C \in \mathrm{Obj}(\mathcal{C})$, then, writing C for the representable $\mathbf{y}C$,*

(0) *$C \Vdash p$ iff $[\![p]\!](C) \neq \emptyset$,*

(1) *$C \Vdash p \wedge q$ iff $C \Vdash p$ and $C \Vdash q$,*

(2) *$C \Vdash \top$ always,*

(3) *$C \Vdash \bot$ never,*

(4') *$C \Vdash p \vee q$ iff $C \Vdash p$ or $C \Vdash q$,*

(5) *$C \Vdash p \to q$ iff, for all $h : D \to C$ in $\mathcal{C}$, if $D \Vdash p$ then $D \Vdash q$.*

Proof: $\widehat{\mathcal{C}}$ is a topos so we can use Theorem 5.3.3. By Proposition 5.3.6 we only have to consider stages C of a generating set, which by Proposition 5.3.10 means all representables.

(0) By 5.3.2 $C \Vdash p$ iff there is some arrow from $\mathbf{y}C$ to $[\![p]\!]$, by the Yoneda Lemma this is equivalent to $[\![p]\!](C) \neq \emptyset$.

(1) Nothing to show.

(2) Nothing to show.

(3) The initial object of $\widehat{\mathcal{C}}$ is the empty functor and the empty functor is not a representable since $\mathbf{y}(C)(C) \neq \emptyset$ for any $C \in \mathcal{C}$.

(4') Since all representables are indecomposable.

(5) By (5) of Theorem 5.3.3 and Lemma 5.3.9, $C \Vdash p \to q$ iff, for all $h : \mathbf{y}D \to \mathbf{y}C$, if $D \Vdash p$ then $D \Vdash q$. Yoneda is full and faithful so arrows $k : D \to C$ in $\mathcal{C}$ are in bijective correspondence with arrows $\mathbf{y}(k) : \mathbf{y}D \to \mathbf{y}C$ between objects of the generating set.

□

If $\mathcal{C}$ is a preorder we get the same table except for clause (5), which then reads

$$C \Vdash p \to q \text{ iff, for all } D \leq C \text{ in } \mathcal{C}, \text{ if } D \Vdash p \text{ then } D \Vdash q.$$

Kripke-Joyal semantics in sheaf categories

Proposition 5.3.12. *The representables $\mathbf{y}U$ for $U \in \mathcal{O}(X)$ form a generating set for $\mathrm{Sh}(X)$.*

Proof: Recall that for a topological space, every representable is a sheaf, so the claim makes sense. The result follows directly from the fact that the representables form a generating set for $\widehat{\mathcal{O}(X)}$. □

There is no reason why the representables should be indecomposable in $\mathrm{Sh}(X)$. If $[k, l] : D + E \to C$ is epi in $\mathrm{Sh}(X)$, it is not necessarily the case that $i([k, l])$ is epi in the presheaf category. The inclusion functor $i : \mathrm{Sh}(X) \hookrightarrow \widehat{\mathcal{O}(X)}$ is right adjoint, hence it preserves all limits, but an epi is a colimit, so we can not know whether it remains epi or not.

$\mathrm{Sh}(X)$ is a topos so we can derive a Kripke semantics for sheaves over topological spaces. We write $U \Vdash p$ for $\mathbf{y}U \Vdash p$.

Theorem 5.3.13 (Kripke semantics for $\mathrm{Sh}(X)$). *Let $\mathcal{O}(X)$ be a topology over a set X, and let $\mathcal{M} = (\mathrm{Sh}(X), [\![\cdot]\!])$ be a categorical model of provability as defined in 5.2.1. Given propositions p, q and $U \in \mathcal{O}(X)$, then writing U for the representable $\mathbf{y}U$,*

(0) *$U \Vdash p$ iff, $[\![p]\!](U) \neq \emptyset$.*

(1) *$U \Vdash p \wedge q$ iff $U \Vdash p$ and $U \Vdash q$,*

(2) *$U \Vdash \top$ always,*

(3) *$U \Vdash \bot$ iff $U = \emptyset$,*

(4) *$U \Vdash p \vee q$ iff $U = V \cup W$ for some $V, W \in \mathcal{O}(X)$ such that $V \Vdash p$ and $W \Vdash q$,*

(5) *$U \Vdash p \to q$ iff, for all $V \subseteq U$, if $V \Vdash p$ then $V \Vdash q$.*

Proof: Clauses (0),(1),(2) and (5) holds by the arguments given in Theorem 5.3.11.

(3) $[\![\bot]\!]$ is the initial object, which is defined by

$$0(U) = \begin{cases} \{*\} & \text{if } \emptyset \in J(U) \\ \emptyset & \text{otherwise} \end{cases}$$

For a topological space the empty set is the only object with an empty cover. Hence $[\![\bot]\!](U) \neq \emptyset$ iff $U = \emptyset$.

(4) Suppose $U \Vdash p \vee q$ then by Theorem 5.3.3 there are arrows $k : D \to \mathbf{y}U, l : E \to \mathbf{y}U$ such that $[k,l] : D + E \twoheadrightarrow \mathbf{y}U$ is epi and $D \Vdash p, E \Vdash q$. By Proposition 2.6.18 there is a one-one correspondence between subobjects of 1 and representables, so in particular the image of k is a representable $\mathbf{y}V \rightarrowtail \mathbf{y}U$, and we have $V \Vdash p$, since there is an arrow $\mathbf{y}V \to [\![p]\!]$. Similar by image factorization we get a representable $\mathbf{y}W$ such that $W \Vdash q$. It remains to show that $U = V \cup W$. It is a enough to show that the coproduct arrow $[v,w] : \mathbf{y}V + \mathbf{y}W \to \mathbf{y}U$ is epi. We have

$$\begin{tikzcd} D + E \arrow[rr, two heads, "{[k,l]}"] \arrow[dr] & & \mathbf{y}U \\ & \mathbf{y}V + \mathbf{y}W \arrow[ur, "{[v,w]}"'] & \end{tikzcd}$$

and since we always have fg epi implies f epi, it follows that $[v,w]$ is epi.

Conversely, suppose $U = V \cup W$ and $V \Vdash p, W \Vdash q$. $\{V, W\}$ is a cover of U so the coproduct arrow $\mathbf{y}V + \mathbf{y}W \twoheadrightarrow \mathbf{y}U$ is epi, which by Theorem 5.3.3 means that $U \Vdash p \vee q$.

□

Proposition 5.3.14 (Local character). *If $\bigcup_i U_i = U$ is a cover of U and $p \in \mathrm{Prop}(\mathcal{L})$, if $U_i \Vdash p$ for all i, then $C \Vdash p$.*

Proof: This is just a reformulation of the fact that $[\![p]\!]$ is a subsheaf of 1. □

Kripke-Joyal semantics in Grothendieck sheaf categories

If a Grothendieck topology is not sub-canonical, the representables $\mathbf{y}C$ are not sheaves, however, we still have a canonical embedding of the category $\mathcal{C}$ into the category of sheaves:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathbf{y}"] & \widehat{\mathcal{C}} \arrow[r, shift left, "\mathbf{a}"] & \mathrm{Sh}(\mathcal{C}, J) \arrow[l, shift left, "i"] \end{tikzcd}$$

Objects of the form $\mathbf{ay}C$ are also called representables. This should not cause any confusion since, for a subcanonical topology we have $\mathbf{ay}C \cong \mathbf{y}C$.

Proposition 5.3.15. *If J is a Grothendieck topology over a category $\mathcal{C}$, then the representables $\mathbf{ay}C$ for $C \in \mathcal{C}$ form a generating set for* $\mathrm{Sh}(\mathcal{C}, J)$*.*

Proof: Suppose

$$\begin{tikzcd} \mathbf{ay}C_i \arrow[r] & F \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & G \end{tikzcd}$$

commute for all $C_i \in \mathrm{Obj}(\mathcal{C})$. By the adjunction $\mathbf{a} \dashv i$ and the Yoneda Lemma,

$$FC \cong \widehat{\mathcal{C}}(\mathbf{y}C, iF) \cong \mathrm{Sh}(\mathbf{ay}(C), F).$$

Now, $f = g$ iff for all $C \in \mathrm{Obj}(\mathcal{C})$ and all $a \in FC$, $f_C(a) = g_C(a)$ iff for all $C \in \mathrm{Obj}(\mathcal{C})$ and for all $t \in \mathrm{Sh}(\mathbf{ay}(C), F)$, $ft = gt$.

□

Theorem 5.3.16 (Kripke semantics for Grothendieck sheaves). *Let J be a Grothendieck topology over a category $\mathcal{C}$, and let $\mathcal{M} = (\mathrm{Sh}(\mathcal{C}, J), [\![\cdot]\!])$ a categorical model of provability as defined in 5.2.1. Given propositions p, q and $C \in \mathcal{C}$, then writing C for the representable $\mathbf{ay}C$,*

(0) *$C \Vdash p$ iff, $[\![p]\!](C) \neq \emptyset$.*

(1) *$C \Vdash p \wedge q$ iff $C \Vdash p$ and $C \Vdash q$,*

(2) *$C \Vdash \top$ always,*

(3) *$C \Vdash \bot$ iff $\emptyset \in J(C)$,*

(4) *$C \Vdash p \vee q$ iff there exists $S \in J(C)$ such that for any $C_i \in S$, $C_i \Vdash p$ or $C_i \Vdash q$,*

(5) *$C \Vdash p \to q$ iff, for all $h : D \to C$ in $\mathcal{C}$, if $D \Vdash p$ then $D \Vdash q$.*

Proof: Since $\mathrm{Sh}(\mathcal{C}, J)$ is a topos, we can use Theorem 5.3.3.

(0) This follows from the remark 5.3.2, the Yoneda Lemma and the adjunction $\mathbf{a} \vdash i$.

(1)-(2) Immediate by Theorem 5.3.3.

(3) The initial object of $\mathrm{Sh}(\mathcal{C}, J)$ is the functor 0 defined by

$$0(A) = \begin{cases} \{*\} & \text{if } \emptyset \in J(A) \\ \emptyset & \text{otherwise} \end{cases}$$

It follows that $\mathbf{y}C$ is the initial object iff $\emptyset \in J(C)$.

(4) For this clause it will be more simple to look at the calculations of the interpretations in $\mathrm{Sub}(1)$. $[\![p \vee q]\!]$ is interpreted as the subsheaf $[\![p]\!] \vee [\![q]\!]$, where $\vee$ is the lub in the Heyting algebra $\mathrm{Sub}(1)$, it is defined by (see Corollary 2.6.21)

$$[\![p]\!] \vee [\![q]\!] = \{C \mid \text{there exists a cover } S \text{ such that for all } C_i \in S. [\![p]\!](C_i) \neq \emptyset \text{ or} [\![q]\!](C_i) \neq \emptyset\},$$

which immediately gives clause (4).

(5) By Theorem 5.3.3 and Proposition 5.3.9

□

Proposition 5.3.17 (Local character). *If $\{f_i : C_i \to C\}$ is a cover of C such that $C_i \Vdash p$ for all i, then $C \Vdash p$.*

Proof: Again, this is just a reformulation of the fact that $[\![p]\!]$ is a subsheaf of 1. □

Chapter 6

Propositional BI

Literature: [LS86], [MLM94],[Yan02] and [Pym02] Following the structure of chapter 5 we extend the propositional logic with two new (multiplicative) connectives and a unit to get propositional logic of bunched implications. As we did for the intuitionistic (additive) part we first prove soundness and completeness for the algebraic models, then, to get a more informative completeness result we consider a smaller class of models: categorical models of provability. The core of the completeness proof is that there is a map going from a syntactic algebraic model (BS) into a categorical model of provability, which essentially is the Yoneda embedding, and which preserves all of the BI algebra structure. We show that there is such a structure preserving map when the categorical model is presheaves over BS and when it is sheaves over BS (for the finite sup topology). We conclude that to prove completeness for the multiplicative part alone we only need to consider presheaves, then to get full propositional BI we embed the syntactic model in the category of sheaves over the finite sup topology (as was done in chapter 5). It is essential that Sub(1) of this sheaf category is a BI algebra (this is shown in Proposition 4.4.9) since otherwise it would not be a well-defined model of BI.

In section 6.3 we give Kripke semantics for BI in doubly closed toposes (this a generalization of the semantics of [Pym02] and [Yan02]), presheaf and sheaf categories. We conclude that these semantics, which are derived from the subobject semantics correspond to the Kripke semantics for presheaves and sheaves given in [Pym02] and in [Yan02]. In particular we note that the interpretation of BI given in these references, though stated as a model of proofs, corresponds to a model of provability.

6.1 Algebraic models for BI

To get propositional BI we take intuitionistic propositional logic and add two new connectives $*, -\!\!*$, which we call the multiplicative connectives, and a new propositional constant I. Well-formed propositions are thus given by the grammar

$$p ::= p \mid p \vee q \mid p \wedge q \mid p \to q \mid p * q \mid p -\!\!* q \mid \top \mid \bot \mid \mathrm{I}$$

We define an entailment relation between propositions by the Hilbert-type system in table 6.1.

Definition 6.1.1 (Algebraic model of propositional BI). *An algebraic model $\mathcal{B}$ of propositional BI consists of a BI algebra $(B, \vee_B, \wedge_B, \to_B, \bot_B, \top_B, *_B, -\!\!*_B, \mathrm{I}_B)$ and an interpreta-*

$$
\begin{array}{cc}
p \vdash p & \dfrac{p \vdash q \quad q \vdash r}{p \vdash r} \\[2ex]
p \vdash \top & \bot \vdash p \\[2ex]
\dfrac{p \vdash q \quad p \vdash r}{p \vdash q \wedge r} & \dfrac{p \vdash q_1 \wedge q_2}{p \vdash q_i}\ (i = 1, 2) \\[2ex]
\dfrac{p \vdash r \quad q \vdash r}{p \vee q \vdash r} & \dfrac{p \vdash q_i}{p \vdash q_1 \vee q_2}\ (i = 1, 2) \\[2ex]
\dfrac{p \wedge q \vdash r}{p \vdash q \to r} & \dfrac{p \vdash s \to r \quad q \vdash s}{p \wedge q \vdash r} \\[2ex]
p * (q * r) \dashv\vdash (p * q) * r & p * \mathrm{I} \dashv\vdash p \dashv\vdash \mathrm{I} * p \\[2ex]
\dfrac{p \vdash q \quad r \vdash s}{p * r \vdash q * s} & p * q \vdash q * p \\[2ex]
\dfrac{p * q \vdash r}{p \vdash q \mathrel{-\!\!*} r} & \dfrac{p \vdash q \mathrel{-\!\!*} r \quad s \vdash q}{p * s \vdash r}
\end{array}
$$

Table 6.1: Hilbert-type system for propositional BI: **HBI**

tion

$$[\![\cdot]\!] : \mathrm{Prop} \to B$$

such that the structure is preserved, i.e., $[\![p \circ q]\!] = [\![p]\!] \circ_B [\![q]\!]$ *for* $\circ \in \{\vee, \wedge, \to, *, \mathrel{-\!\!*}\}$*, and* $[\![\top]\!] = \top_B, [\![\bot]\!] = \bot_B, [\![\mathrm{I}]\!] = \mathrm{I}_B$*.*

We write $[\![p]\!]_{\mathcal{B}} \leq [\![q]\!]_{\mathcal{B}}$ if the interpretation of p is below the interpretation of q in the BI algebra B, part of the model $\mathcal{B}$. If $[\![p]\!]_{\mathcal{B}} \leq [\![q]\!]_{\mathcal{B}}$ for all models $\mathcal{B}$, then we write $[\![p]\!] \leq [\![q]\!]$.

Theorem 6.1.2 (Soundness). *If* $p \vdash q$ *is provable in* ***HBI****, then* $[\![p]\!] \leq [\![q]\!]$*.*

Proof: By induction on the structure of proofs in **HBI**. This is just an unwinding of definitions, as was done for Heyting algebra in the previous section. The system **HBI** is really just a rephrasing of the definition of a BI algebra. □

As we did for intuitionistic logic in the previous chapter, we now define a syntactic BI algebra BS. The elements of BS are equivalence classes of propositions in the language of propositional BI under provability, that is

$$p \sim q \text{ iff } \vdash p \leftrightarrow q$$

where $\vdash p \leftrightarrow q$ is an abbreviation of $\vdash p \to q$ and $\vdash q \to p$. The equivalence classes form a BI algebra BS with

- $\top_{BS} := [\top] = \{p \mid \top \vdash p\}$
- $\bot_{BS} := [\bot] = \{p \mid p \vdash \bot\}$

- $[p] \vee [q] := [p \vee q]$
- $[p] \wedge [q] := [p \wedge q]$
- $[p] \to [q] := [p \to q]$
- $[p] * [q] := [p * q]$
- $[p] \mathrel{-\!\!*} [q] := [p \mathrel{-\!\!*} q]$
- $\mathrm{I}_{BS} := [\mathrm{I}] = \{p \mid\vdash p \leftrightarrow \mathrm{I}\}$

ordered by

$$[p] \leq [q] \text{ iff } p \vdash q.$$

Well-definedness. This should also be clear by the arguments given in section 5.1.

The syntactic model consists of the BI algebra BS and the interpretation given by $[\![p]\!] := [p]$ for propositional letters $p \in \mathrm{Prop}$ extended to all formulas using the operations $\wedge_{BS}, \vee_{BS}, \to_{BS}, *_{BS}, \mathrel{-\!\!*}_{BS}$ on BS.

Theorem 6.1.3 (Completeness). *For propositional formulas p, q of BI, $p \vdash q$ iff $[\![p]\!] \leq [\![q]\!]$.*

Proof: This is immediate from the definitions. □

6.2 Categorical models

As for the intuitionistic fragment we have two kinds of categorical models.

Definition 6.2.1 (Categorical model of provability in HBI). *A categorical model of provability $\mathcal{M}$ is a category $\mathcal{T}$ with the property that $\mathrm{Sub}_{\mathcal{T}}(1)$ is a BI algebra $(\mathrm{Sub}_{\mathcal{T}}(1), \vee_{\mathcal{T}}, \wedge_{\mathcal{T}}, \to_{\mathcal{T}}, \bot_{\mathcal{T}}, \top_{\mathcal{T}}, *_{\mathcal{T}}, \mathrel{-\!\!*}_{\mathcal{T}}, \mathrm{I}_{\mathcal{T}})$, together with an interpretation function*

$$[\![\cdot]\!] : \mathrm{Prop} \to \mathrm{Sub}_{\mathcal{T}}(1)$$

which preserves the structure, i.e., $[\![p \circ q]\!] = [\![p]\!] \circ_{\mathcal{T}} [\![q]\!]$ for $\circ \in \{\vee, \wedge, \to, *, \mathrel{-\!\!*}\}$, and $[\![\top]\!] = \top_{\mathcal{T}}, [\![\bot]\!] = \bot_{\mathcal{T}}, [\![\mathrm{I}]\!] = \mathrm{I}_{\mathcal{T}}$.

We say that a formula p holds in $\mathcal{M}$, written $\mathcal{M} \models p$ iff $[\![p]\!] = \top_{\mathrm{Sub}_{\mathcal{T}}(1)}$, i.e., iff the interpretation of p is the maximal subobject of 1. If the interpretation of a formula p is below the interpretation of a formula q in $\mathrm{Sub}_{\mathcal{T}}(1)$ we write $p \models_{\mathcal{M}} q$.

In section 4.4 we have shown that a topos which is DCC, has the property that Sub(1) is a BI algebra, so given a well-defined interpretation, the presheaf toposes $\widehat{\mathcal{C}}$, where $\mathcal{C}$ is symmetric monoidal is a categorical model.

Definition 6.2.2 (Categorical model of proofs in HBI). *A categorical model of proofs consists of a bi-DCC $\mathcal{C}$ with the two closed structures $(\times, 1, \to)$ and $(\otimes, \mathrm{I}, \multimap)$ and an interpretation function*

$$[\![\cdot]\!] : \mathrm{Prop} \to \mathrm{Obj}(\mathcal{C})$$

satisfying

$$\begin{array}{rclrcl} [\![p \wedge q]\!] & = & [\![p]\!] \times [\![q]\!] & & & \\ [\![\top]\!] & = & 1 & [\![p * q]\!] & = & [\![p]\!] \otimes [\![q]\!] \\ [\![p \vee q]\!] & = & [\![p]\!] + [\![q]\!] & [\![\mathrm{I}]\!] & = & \mathrm{I} \\ [\![\bot]\!] & = & 0 & [\![p \mathrel{-\!\!*} q]\!] & = & [\![p]\!] \multimap [\![q]\!] \\ [\![p \rightarrow q]\!] & = & [\![p]\!] \rightarrow [\![q]\!] & & & \end{array}$$

Note that models of provability are contained in the models of proofs, since a BI algebra considered as a category is a bi-DCC.

Consider the presheaf category $\widehat{BS}$. BS is a BI algebra, in particular it is a symmetric, monoidal preorder, so $\widehat{BS}$ is bi-CDCC, and Sub(1) of this category forms a BI algebra. The BI structure is elaborated in example 4.4.4, we just repeat the definition of the multiplicative part, for sieves J, K,

- $\mathrm{I}_{\mathrm{Sub}_{\widehat{BS}}(1)} = \downarrow [\mathrm{I}]$
- $J * K = \downarrow \{j * k \mid j \in J, k \in K\}$
- $J \mathrel{-\!\!*} K = \{a \mid \forall c.c \in J \Rightarrow a * c \in K\}$.

To get an algebraic model we must give a well-defined interpretation. As for intuitionistic logic we try with the Yoneda embedding. Interpretation of the propositional calculus in $\mathrm{Sub}_{\widehat{BS}}(1)$ is

$$|p| := \downarrow [p] = \{t \mid t \vdash p\}$$

for $p \in$ Prop. The interpretation is extended to all formulas using the BI algebra operations on $\mathrm{Sub}_{\widehat{BS}}(1)$. We have already seen in Proposition 5.2.3 that this interpretation is not well-defined for formulas that contains $\vee$ or $\bot$, what we have is the following:

Proposition 6.2.3. *The map* $\downarrow (-)$ *from* BS *to* $\mathrm{Sub}_{\widehat{BS}}(1)$ *defined by* $[p] \mapsto \downarrow [p]$ *preserves* $\top, \wedge, \rightarrow, \mathrm{I}, *, \mathrel{-\!\!*}$ *but not* $\bot, \vee$.

Proof: Most of this has already been proved in 5.2.3, we must prove the part concerning the new connectives. By Proposition 4.2.9 the Yoneda functor preserves all of the monoidal closed structure, which is exactly the new connectives. For example

$$\begin{array}{lll} & \downarrow [t_1 * t_2] & \\ \cong & \mathbf{y}[t_1 * t_2] & \\ \cong & \mathrm{Im}(\mathbf{y}[t_1 * t_2]) & \text{since any representable is a subobject of 1} \\ \cong & \mathrm{Im}(\mathbf{y}[t_1] \otimes \mathbf{y}[t_2]) & \text{by Proposition 4.2.9} \\ \cong & \downarrow [t_1] * \downarrow [t_2] & \text{since } \downarrow [t_1] * \downarrow [t_2] \text{ by definition is the sieve} \\ & & \text{corresponding to } \mathrm{Im}(\mathbf{y}[t_1] \otimes \mathbf{y}[t_2]) \ . \end{array}$$

□

Corollary 6.2.4. $|t| = \downarrow [t]$ *for all formulas* t *of the* $(\bot, \vee)$*-free fragment of BI.*

Proof: By induction on t using Proposition 6.2.3. □

Proposition 6.2.5 (completeness for the $(\bot, \vee)$-free fragment). *If p, q are $(\bot, \vee)$-free formulas of* **HBI, then** $p \vdash q$ **iff** $\downarrow[p] \leq \downarrow[q]$.

Proof:

$$\begin{array}{rcrcrl} p \vdash q & \text{iff} & [p] & \leq_{BS} & [q] & \text{by Proposition 6.1.3} \\ & \text{iff} & \downarrow[p] & \subseteq & \downarrow[q] & \text{Yoneda is full and faithful} \\ & \text{iff} & |p| & \subseteq & |q| & \text{by Corollary 6.2.4} \end{array}$$

□

As in the previous section we consider interpretation in the Grothendieck topos with the finite sup topology. By Corollary 4.4.9 there is a BI structure on $\mathrm{Sub}_{\mathrm{Sh}(BS)}(1)$, and for this particular topology the multiplicative part is calculated as for subpresheaves. So, using Proposition 5.2.6, we get:

Proposition 6.2.6. *The map $\downarrow(-) : BS \to \mathrm{Sub}_{\mathrm{Sh}(BS)}(1)$ is a map of BI algebras, i.e., it preserves all the BI algebra structure.*

Interpretation of propositional formulas in $\mathrm{Sub}_{\mathrm{Sh}(BS)}(1)$ is

$$\{p\} := \downarrow[p]$$

for $p \in \mathrm{Prop}$, extended to formulas in the usual way.

Lemma 6.2.7. $\{t\} = \downarrow[t]$ *for all formulas t of propositional BI.*

Proof: By induction on t using Proposition 6.2.6. □

Theorem 6.2.8 (Completeness). *For propositional BI formulas p, q, $p \vdash q$ iff $\{p\} \subseteq \{q\}$.*

Proof:

$$\begin{array}{rcrcrl} p \vdash q & \text{iff} & [p] & \leq & [q] & \text{by definition of } BS \\ & \text{iff} & \downarrow[p] & \subseteq & \downarrow[q] & \text{since the Yoneda functor is full and faithful} \\ & \text{iff} & \{p\} & \subseteq & \{q\} & \text{by Lemma 6.2.7} \\ & \text{iff} & p & \models_{\mathrm{Sh}(BS)} & q. & \end{array}$$

□

Note that the reason why we need to consider Grothendieck sheaves as models to get completeness is the failure of the Yoneda embedding to preserve $\vee, \bot$. The multiplicative connectives do not pose any problems of this sort as Proposition 6.2.5 shows.

6.3 Kripke-Joyal semantics for BI

We are going to expand the Kripke-Joyal semantics to include the new connectives $*, \mathrel{-\!\!*}$ and I. For ease of reference we give the complete table here.

Theorem 6.3.1 (Kripke-Joyal semantics for BI). *Let $\mathcal{T}$ be a doubly closed topos (then $\mathrm{Sub}_{\mathcal{T}}(1)$ is a BI algebra) and $\mathcal{M} = (\mathcal{T}, \llbracket\cdot\rrbracket)$ a categorical model of provability in* **HBI** *as defined in 6.2.1. Given propositions p, q and $C \in \mathrm{Obj}(\mathcal{T})$, then*

(0) $C \Vdash p$ *iff* $\mathrm{char}(p)1_C = \top 1_C$,

(1) *$C \Vdash p \wedge q$ iff $C \Vdash p$ and $C \Vdash q$,*

(2) *$C \Vdash \top$ always,*

(3) *$C \Vdash \bot$ iff C is the initial object of $\mathcal{T}$,*

(4) *$C \Vdash p \vee q$ iff there exists an epi $[k, l] : D + E \to C$ such that $D \Vdash p$ and $E \Vdash q$,*

(5) *$C \Vdash p \to q$ iff, for all $h : D \to C$, if $D \Vdash p$ then $D \Vdash q$.*

(6) *$C \Vdash p * q$ iff there exists an arrow $h : D \to E \otimes F$ and an epi $k : D \twoheadrightarrow C$, where $D, E, F \in \mathrm{Obj}(\mathcal{T})$, such that $E \Vdash p$ and $F \Vdash q$.*

(7) *$C \Vdash p \mathrel{-\!\!*} q$ iff for all $D \in \mathrm{Obj}(\mathcal{T})$, $D \Vdash p$ implies $C \otimes D \Vdash q$.*

(8) *$C \Vdash \mathrm{I}$ iff there exists an arrow $h : C \to \llbracket \mathrm{I} \rrbracket$.*

Proof:

(0)-(5) are proven in Theorem 5.3.3.

(6) Suppose $C \Vdash p * q$, that is $C \to \llbracket p * q \rrbracket$, where

$$\begin{tikzcd}
\llbracket p \rrbracket \otimes \llbracket q \rrbracket \arrow[r, two heads, "e"] & \mathrm{Im}(p * q) = \llbracket p * q \rrbracket \arrow[r, tail] & 1
\end{tikzcd}$$

Now take the pullback of $C \to \mathrm{Im}(p * q)$ along e:

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, two heads, "k"'] & \llbracket p \rrbracket \otimes \llbracket q \rrbracket \arrow[d, two heads, "e"] \\
C \arrow[r] & \mathrm{Im}(p * q)
\end{tikzcd}$$

$k : D \twoheadrightarrow C$ is epi since pullbacks preserve epis, and we always have $\llbracket p \rrbracket \Vdash p$ and $\llbracket q \rrbracket \Vdash q$. On the other hand, suppose we have a diagram in $\mathcal{T}$

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, two heads, "k"'] & E \otimes F \\
C &
\end{tikzcd}$$

such that $E \Vdash p, F \Vdash q$, then

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[drr, two heads, "k"'] & E \otimes F \arrow[r] & \llbracket p \rrbracket \otimes \llbracket q \rrbracket \arrow[r, two heads, "e"] & \mathrm{Im}(p * q) \arrow[r, tail] & 1 \\
 & & C \arrow[r, two heads] & \mathrm{Im}(1_C) \arrow[u, dotted, "u"] \arrow[ur, tail] &
\end{tikzcd}$$

The arrow u exists by Proposition 2.2.1, so there is an arrow $C \to \llbracket p * q \rrbracket$ as required.

(7) Suppose $C \Vdash p \mathrel{-\!\!*} q$, i.e., there is an arrow $C \to [\![p \mathrel{-\!\!*} q]\!] = ([\![p]\!] \multimap [\![q]\!])$, by adjunction this is the same as an arrow $C \otimes [\![p]\!] \to [\![q]\!]$. Suppose $D \Vdash p$, then we have arrows

$$C \otimes D \to C \otimes [\![p]\!] \to [\![q]\!],$$

i.e., $C \otimes D \Vdash q$.

For the other implication, suppose that for all $D \in \mathrm{Obj}(\mathcal{T})$, $D \Vdash p$ implies $C \otimes D \Vdash q$. In particular, $[\![p]\!] \Vdash p$ implies $C \otimes [\![p]\!] \Vdash q$, which means that there is an arrow $C \to ([\![p]\!] \multimap [\![q]\!])$, i.e., $C \Vdash p \mathrel{-\!\!*} q$.

(8) This is the definition.

□

Proposition 6.3.2 (Kripke monotonicity). *If there exists an arrow $C \to D$ in $\mathcal{T}$ and $D \Vdash p$ for $p \in \mathrm{Prop}(\mathcal{L})$, then $C \Vdash p$.*

Proof: The proposition just states that $[\![p]\!]$ is a subfunctor of 1, which it is by definition of a categorical model. □

6.3.1 Kripke-Joyal semantics for BI in functor categories

We now expand the examples given in the previous chapter to include the multiplicative connectives.

Kripke-Joyal semantics for BI in presheaf categories

Theorem 6.3.3. *Let $(\mathcal{C}, \cdot, e)$ be a small symmetric monoidal category and $\mathcal{M} = (\widehat{\mathcal{C}}, [\![\cdot]\!])$ a categorical model of provability in* **HBI** *as defined in 6.2.1. Given propositions p, q and $C \in \mathrm{Obj}(\mathcal{C})$, then, writing C for the representable* $\mathbf{y}C$*,*

(0) *$C \Vdash p$ iff $[\![p]\!](C) \neq \emptyset$,*

(1) *$C \Vdash p \wedge q$ iff $C \Vdash p$ and $C \Vdash q$,*

(2) *$C \Vdash \top$ always,*

(3) *$C \Vdash \bot$ never,*

(4') *$C \Vdash p \vee q$ iff $C \Vdash p$ or $C \Vdash q$,*

(5) *$C \Vdash p \to q$ iff, for all $h : D \to C$ in $\mathcal{C}$, if $D \Vdash p$ then $D \Vdash q$.*

(6) *$C \Vdash p * q$ iff for some $D, D' \in \mathrm{Obj}(\mathcal{C})$ such that there is an arrow $h : C \to D \cdot D'$ in $\mathcal{C}$, $D \Vdash p$ and $D' \Vdash q$.*

(7) *$C \Vdash p \mathrel{-\!\!*} q$ iff for all $D \in \mathrm{Obj}(\mathcal{C})$, $D \Vdash p$ implies $C \cdot D \Vdash q$.*

(8) *$C \Vdash \mathrm{I}$ iff there is an arrow $h : C \to e$ in $\mathcal{C}$.*

Proof: Using Day's construction we get a symmetric monoidal closed structure on $\widehat{\mathcal{C}}$, since $\widehat{\mathcal{C}}$ is also a topos we can use Theorem 6.3.1.

(0)-(5) Follows immediately from the corresponding theorem for intuitionistic fragment 5.3.11.

(6) Suppose $h : C \to D \cdot D'$ and $D \Vdash p$ and $D' \Vdash q$, the identity $C \twoheadrightarrow C$ is an epi, so by clause (6) of Theorem 6.3.1, $C \Vdash p * q$. For the converse, we consider the concrete definition of the subobject $[\![p * q]\!]$ which is the image of the arrow $[\![p]\!] \otimes [\![q]\!] \to 1$. By assumption we have an arrow $\mathbf{y}C \to \mathrm{Im}([\![p]\!] \otimes [\![q]\!])$, which implies that $\mathrm{Im}([\![p]\!] \otimes [\![q]\!])(C) \neq \emptyset$. Now

$$\mathrm{Im}([\![p]\!] \otimes [\![q]\!])(C) = \mathrm{Im}(\int^{D,D'} [\![p]\!](D) \times [\![q]\!](D') \times \mathcal{C}(C, D \cdot D'))$$

it follows that there exists objects D, D' with $C \to D \cdot D'$, and $[\![p]\!](D) \neq \emptyset$ and $[\![q]\!](D') \neq \emptyset$, i.e., $D \Vdash p$ and $D' \Vdash q$.

(7) To see that clause (7) of Theorem 6.3.1 holds when D is restricted to be in a generating set, suppose

$$\text{For all } D \in \mathrm{Obj}(\mathcal{C}).D \Vdash p \text{ implies } C \cdot D \Vdash q. \tag{6.1}$$

It must be shown that $[\![p \twoheadrightarrow q]\!](C) \neq \emptyset$. $[\![p \twoheadrightarrow q]\!](C) = [\![p]\!] \multimap [\![q]\!](C) = \widehat{\mathcal{C}}([\![p]\!], [\![q]\!](C \cdot -))$ for all C. Assumption 6.1 gives that for all $D \in \mathcal{C}$, if $[\![p]\!](D) \neq \emptyset$ then $[\![q]\!](C \cdot D) \neq \emptyset$ so there is a natural transformation $[\![p]\!] \to [\![q]\!](C \cdot -)$, showing that $[\![p \twoheadrightarrow q]\!](C) \neq \emptyset$.

For the converse, use (7) of Theorem 6.3.1 and the identity $\mathbf{y}(C \cdot D) \cong \mathbf{y}(C) \otimes \mathbf{y}(D)$.

(8) $C \Vdash \mathrm{I}$ iff there is an arrow $\mathbf{y}C \to [\![\mathrm{I}]\!] = \mathbf{y}e$ iff there is an arrow $C \to e$ in $\mathcal{C}$.

□

Remark 6.3.4. *In [Pym02] a Kripke model is defined as a triple*

$$\langle \widehat{\mathcal{M}}, \models, [\![-]\!] \rangle$$

where $\mathcal{M}$ is a preordered commutative monoid, $[\![-]\!] : \mathrm{Prop}(\mathcal{L}) \to \mathrm{Obj}(\widehat{\mathcal{M}})$ a partial function, and $\models$ a satisfaction relation satisfying the constraints given in 6.3.3, such that Kripke monotonicity is satisfied.

Though the interpretation $[\![p]\!]$ of a proposition p is a functor in $\widehat{\mathcal{M}}$, the constraints in Theorem 6.3.3 actually refers to the image of the arrow $[\![p]\!] \to 1$ which is a subobject of 1, which means that interpretation of propositions actually is in $\mathrm{Sub}_{\widehat{\mathcal{M}}}(1)$. So the definition in [Pym02] coincides with the Kripke semantics given in Theorem 6.3.3.

Kripke-Joyal semantics for BI in sheaf categories

Theorem 6.3.5 (Kripke semantics for $\mathrm{Sh}(X)$). *Let $(\mathcal{O}(X), \cdot, \{e\})$ be a topological monoid (Definition 4.2.10), and let $\mathcal{M} = (\mathrm{Sh}(X), [\![\cdot]\!])$ be a categorical model of provability as defined in 5.2.1. Given propositions p, q and $U \in \mathcal{O}(X)$, then, writing U for the representable $\mathbf{y}U$,*

(0) *$U \Vdash p$ iff $[\![p]\!](U) \neq \emptyset$,*

(1) *$U \Vdash p \wedge q$ iff $U \Vdash p$ and $U \Vdash q$,*

(2) *$U \Vdash \top$ always,*

(3) *$U \Vdash \bot$ iff $U = \emptyset$,*

(4) *$U \Vdash p \vee q$ iff $U = V \cup W$ for some $V, W \in \mathcal{O}(X)$ such that $V \Vdash p$ and $W \Vdash q$,*

(5) *$U \Vdash p \to q$ iff, for all $V \subseteq U$, if $V \Vdash p$ then $V \Vdash q$,*

(6) *$U \Vdash p * q$ iff for some $V, V' \in \mathcal{O}(X), U \subseteq V \cdot V'$ and $V \Vdash p$ and $V' \Vdash q$.*

(7) *$U \Vdash p \mathrel{-\!\!*} q$ iff for all $V \in \mathcal{O}(X)$, $V \Vdash p$ implies $U \cdot V \Vdash q$.*

(8) *$U \Vdash \mathrm{I}$ iff $U \subseteq \{e\}$.*

Proof:

(0)-(5) These are proven in Theorem 5.3.13. For the clauses (6)-(8) we would like to deduce them from the more general semantics given in Theorem 6.3.1 like we did when we gave the Kripke semantics for presheaves, however, we do not in general have a symmetric monoidal closed structure on the category of sheaves so the requirements for the Theorem 6.3.1 can not be met.

What we do have is a BI algebra on Sub(1) in the category of sheaves, and since all the action of propositional logic is taking place here, we can prove the theorem by direct calculations in Sub(1).

(6) Consider the ideals $\overline{\llbracket \mathrm{p} \rrbracket}$ and $\overline{\llbracket \mathrm{q} \rrbracket}$ corresponding to the interpretations of p and q. (That is, $U \in \overline{\llbracket \mathrm{p} \rrbracket}$ iff $\llbracket p \rrbracket(U) \neq \emptyset$.) By the calculations given in Example 4.4.4 and Corollary 4.4.9,

$$\overline{\llbracket \mathrm{p} * \mathrm{q} \rrbracket} = \downarrow \{V \cdot V' \mid V \in \overline{\llbracket \mathrm{p} \rrbracket}, V' \in \overline{\llbracket \mathrm{q} \rrbracket}\}.$$

Clearly $U \in \overline{\llbracket \mathrm{p} * \mathrm{q} \rrbracket}$ iff $U \subseteq V \cdot V', V \in \overline{\llbracket \mathrm{p} \rrbracket}, V' \in \overline{\llbracket \mathrm{q} \rrbracket}$.

(7) Again consider the ideals corresponding to the interpretations, we have

$$\overline{\llbracket \mathrm{p} \mathrel{-\!\!*} \mathrm{q} \rrbracket} = \{U \mid V \in \overline{\llbracket \mathrm{p} \rrbracket} \text{ implies } U \cdot V \in \overline{\llbracket \mathrm{q} \rrbracket}\}.$$

The result is now immediate.

(8) $U \Vdash \mathrm{I}$ iff there exists an arrow $\mathbf{y}U \to \llbracket \mathrm{I} \rrbracket = \mathbf{y}e$ iff $U \subseteq \{e\}$.

□

Since for every $p \in \mathrm{Prop}(\mathcal{L})$, $\llbracket p \rrbracket$ is a subsheaf of 1, we get:

Proposition 6.3.6 (Local character). *If $\bigcup_i U_i = U$ is a cover of U and $p \in \mathrm{Prop}(\mathcal{L})$, then if $U_i \Vdash p$ for all i, then $C \Vdash p$.*

Kripke-Joyal semantics for BI in Grothendieck sheaf categories

Theorem 6.3.7 (Kripke semantics for Grothendieck sheaves). *Let J be a Grothendieck topology over a symmetric monoidal category $(\mathcal{C}, \cdot, e)$ such that $\cdot$ is cover preserving with respect to J, and let $\mathcal{M} = (\mathrm{Sh}(\mathcal{C}, J), \llbracket \cdot \rrbracket)$ be a categorical model of provability as defined in 6.2.1. Given propositions p, q and $C \in \mathcal{C}$, then, writing C for the representable $\mathbf{ay}C$,*

(0) $C \Vdash p$ *iff there is an arrow* $\mathbf{ay}(C) \to \llbracket p \rrbracket$
iff $\llbracket p \rrbracket(C) \neq \emptyset$
iff $C \in \overline{\llbracket \mathrm{p} \rrbracket}$

(1) *$C \Vdash p \wedge q$ iff $C \Vdash p$ and $C \Vdash q$,*

(2) *$C \Vdash \top$ always,*

(3) *$C \Vdash \bot$ iff $\emptyset \in J(C)$,*

(4) *$C \Vdash p \vee q$ iff there exists $S \in J(C)$ such that for any $C_i \in S$, $C_i \Vdash p$ or $C_i \Vdash q$,*

(5) *$C \Vdash p \to q$ iff, for all $h : D \to C$ in $\mathcal{C}$, if $D \Vdash p$ then $D \Vdash q$.*

(6) *$C \Vdash p * q$ iff there exists a cover $S \in J(C)$ such that for any $f_i : C_i \to C \in S$ there exists $D_p, D_q \in \mathcal{C}$ such that there is an arrow $C_i \to D_p \cdot D_q$ and $D_p \Vdash p$ and $D_q \Vdash q$.*

(7) *$C \Vdash p \mathrel{-\!\!*} q$ iff for any $D \in \mathcal{C}$, $D \Vdash p$ implies $C \cdot D \Vdash q$.*

(8) *$C \Vdash \mathrm{I}$ iff there exists $S \in J(C)$ such that for any $f_i : C_i \to C \in S$, there is an arrow $C_i \to e$ in $\mathcal{C}$.*

Proof:

(0)-(5) Proven in Theorem 5.3.16. As for sheaves over topological spaces, the multiplicative clauses must be proven by direct calculations in Sub(1) since we do not have a symmetric monoidal closed structure on the whole category.

(6) Let $\overline{\llbracket \mathrm{p} \rrbracket}$ and $\overline{\llbracket \mathrm{q} \rrbracket}$ be the ideals corresponding to the interpretations of p, q, (as defined in Section 2.6) then $\overline{\llbracket \mathrm{p} * \mathrm{q} \rrbracket} =$
$\{C \in \mathrm{Obj}(\mathcal{C}) \mid$ there exists a cover $S \in J(C)$ such that for any $f_i : C_i \to C$ in $S, C_i \in \overline{\llbracket \mathrm{p} \rrbracket} * \overline{\llbracket \mathrm{q} \rrbracket}\}$ where

$$\overline{\llbracket \mathrm{p} \rrbracket} * \overline{\llbracket \mathrm{q} \rrbracket} = \downarrow \{C \cdot D \mid C \in \llbracket p \rrbracket, D \in \llbracket q \rrbracket\}$$

The ideal $\overline{\llbracket \mathrm{p} * \mathrm{q} \rrbracket}$ is the sheafification of the presheaf $\llbracket p \rrbracket * \llbracket q \rrbracket$ (see Proposition 4.4.8). Now suppose $C \Vdash p * q$, then $C \in \overline{\llbracket \mathrm{p} * \mathrm{q} \rrbracket}$ so there exists a cover $S \in J(C)$ such that $C_i \in \llbracket p \rrbracket * \llbracket q \rrbracket$ for any $f_i : C_i \to C$ in S. This means that for any such C_i there are D_p, D_q with $C_i \to D_p \cdot D_q$ and $D_p \in \llbracket p \rrbracket$ and $D_q \in \llbracket q \rrbracket$.

The converse also follows from the definition of $\overline{\llbracket \mathrm{p} * \mathrm{q} \rrbracket}$.

(7) The ideal corresponding to $\llbracket p \mathrel{-\!\!*} q \rrbracket$ is given by

$$\overline{\llbracket \mathrm{p} \mathrel{-\!\!*} \mathrm{q} \rrbracket} = \{C \in \mathrm{Obj}(\mathcal{C}) \mid D \in \overline{\llbracket \mathrm{p} \rrbracket} \text{ implies } C \cdot D \in \overline{\llbracket \mathrm{q} \rrbracket}\},$$

which immediately gives (7).

(8) $C \Vdash \mathrm{I}$ iff $\llbracket \mathrm{I} \rrbracket(C) \neq \emptyset$, $\llbracket \mathrm{I} \rrbracket = \mathrm{Im}(\mathbf{ay}(e))$, so by equation 4.12, $\llbracket \mathrm{I} \rrbracket(C) \neq \emptyset$ iff there exists a cover $S \in J(C)$ such that for all $f_i : C_i \to C \in S$, $\mathbf{y}(e)(C_i) \neq \emptyset$.

□

Proposition 6.3.8 (Local character). *If $\{f_i : C_i \to C\}$ is a cover of C and $p \in \mathrm{Prop}(\mathcal{L})$ such that $C_i \Vdash p$ for all i, then $C \Vdash p$.*

Proof: Again, this is just a reformulation of the fact that $[\![p]\!]$ is a subsheaf of 1, which it is by definition of a categorical model. □

In [Pym02] the Grothendieck resource model is defined as a Grothendieck topology on a preordered commutative monoid M, which is cover preserving (continuous), together with an interpretation function

$$[\![-]\!] : \mathcal{L} \to \mathcal{P}(M)$$

from the set of propositional letters to the powerset over M. The interpretation function is subject to two conditions (called (K) and (Sh)) which ensures that $[\![p]\!]$ is in fact an ideal (i.e., a subsheaf of 1). The interpretation is then extended to all propositions Prop($\mathcal{L}$) using the clauses of Theorem 6.3.7, the extended interpretation is well-defined if Kripke monotonicity (K) and local character (Sh) holds for all $p \in$ Prop($\mathcal{L}$). In other words, a Grothendieck resource model is a special case of a categorical model of provability as defined in 6.2.1.

Chapter 7

Predicate BI

Literature: [Pit02] and [Yan02].

This chapter presents new research results due to Lars Birkedal, Noah Torp-Smith and the author.

One goal of this thesis has been to clarify the relation between separation logic and BI. It turns out that separation logic is predicate BI, though not in the sense that Pym defines predicate BI in [Pym02]. (We have not been able to understand Pym's suggestion of predicate BI; see the discussion in Appendix A.) Free variables are kept in a set as usually, and not in bunches as in [Pym02], so there are no substructural constraints on the level of variables, only on the propositional level.

We first recall the notion of a first order hyperdoctrine which is the "minimal" structure needed to soundly model first order predicate logic, and state soundness and completeness for these. This definition is altered slightly to provide models for first order predicate BI, these structures are named BI-hyperdoctrines and they have soundness and completeness results similar to the ones for hyperdoctrines.

Finally we show how separation logic can be seen as predicate BI (in this sense) with a special signature (specification of types, function symbol and predicate symbols) and the pointer model then becomes a model of separation logic, that is, a BI-hyperdoctrine. We show how the forcing semantics for the pointer model can be derived from the BI-hyperdoctrine.

7.1 First order hyperdoctrines

In this section we recall the notion of first order hyperdoctrines and how first order logic can be interpreted in these.

First we define what we mean by first order intuitionistic logic. Many-sorted first order intuitionistic logic consists of a set of types $X, Y, \ldots$, countably infinite many variables of each type $x_1 : X, x_2 : X, \ldots$, a set of function symbols $f : X_1, \ldots, X_n \to X$ (constants are functions of arity 0) and relation symbols $R \subseteq X_1, \ldots, X_n$.

Well-formed terms. Let Γ, denote a context [1] of the form $\{y_1 : Y_1, \ldots, y_m : Y_m\}$, then well-formed terms are

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[1] The way we have formulated the signature, a context is the same as a set of free variables since the variables are born with a type.

- variables and constants,
- If $t_1, \ldots, t_n$ are terms of types $X_1, \ldots, X_n$ with free variables in Γ (i.e., $t_i : \Gamma \to X_i$), and $f : X_1, \ldots, X_n \to X$ a function symbol, then $f(t_1, \ldots, t_n) : \Gamma \to X$ is a term.

Atomic formulas.

- If t, t' are both terms of type X, then $t =_X t'$ is an atomic formula.
- If $t_1, \ldots, t_n$ are terms of types $X_1, \ldots, X_n$ with free variables in Γ and $R \subseteq X_1, \ldots, X_n$ a relation symbol, then $R(t_1, \ldots, t_n) \subseteq \Gamma$ is an atomic formula.

Formulas. Well-formed formulas ϕ are given by the following grammar

$$\phi ::= \phi \mid \top \mid \bot \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \to \psi \mid \forall x : X.\phi \mid \exists x : X.\phi$$

where ϕ, ψ ranges over atomic formulas and formulas.

Deduction. For each finite set of variables X we define a binary relation $\vdash_X$ between formulas such that their free variables are contained in X as follows.

1. Structural rules.

1.1 $p \vdash_X p$

1.2 $\dfrac{p \vdash_X q \quad q \vdash_X r}{p \vdash_X r}$

1.3 $\dfrac{p \vdash_X q}{p \vdash_{X \cup X'} q}$

1.4 $\dfrac{\phi(x) \vdash_X \psi(x)}{\phi(b) \vdash_{X \setminus \{x\}} \psi(b)}$

if $x : B$ and b is a term of type B with free variables among $X \setminus \{x\}$, and b is substitutable for x on both sides.

2. Logical rules.

2.1 $p \vdash_X \top, \qquad \bot \vdash_X p$

2.2 $\dfrac{r \vdash_X p_1 \quad r \vdash_X p_2}{\overline{r \vdash_X p_1 \wedge p_2}} \qquad \dfrac{p_1 \vdash_X r \quad p_2 \vdash_X r}{\overline{p_1 \vee p_2 \vdash_X r}}$

2.3 $\dfrac{p \wedge q \vdash_X r}{\overline{p \vdash_X q \to r}}$

2.4 $$\dfrac{p \vdash_X \forall x : X.\phi(x)}{\overline{p \vdash_{X \cup \{x\}} \phi(x)}} \qquad \dfrac{\exists x : X.\phi(x) \vdash_X p}{\overline{\phi(x) \vdash_{X \cup \{x\}} p}}$$

where the double rules $\dfrac{P}{\overline{Q}}$ means $\dfrac{Q}{P}$ and $\dfrac{P}{Q}$.

3. Axioms for equality.

3.1 $\vdash_{\{x\}} x = x$

3.2 $x_1 = x_2 \vdash_{\{x_1, x_2\}} x_2 = x_1$ where x_1, x_2 have the same type.

3.3 $x_1 = x_2 \wedge x_2 = x_3 \vdash_{\{x_1, x_2, x_3\}} x_1 = x_3$ where x_1, x_2, x_3 have the same type.

3.4 $\bigwedge \overline{\mathrm{x}} = \overline{\mathrm{y}} \vdash_{\{\overline{\mathrm{x}}, \overline{\mathrm{y}}\}} R\overline{\mathrm{x}} \leftrightarrow R\overline{\mathrm{y}}$,

for any relation symbol R and using some obvious abbreviations.

The following definition and example 7.1.3 are taken from [Pit02].

First order hyperdoctrines are categorical structures tailored to model first order predicate logic with equality. The structure has a base category $\mathcal{C}$ with finite products for modeling the types and terms of a first order theory, and a $\mathcal{C}$-indexed category $\mathcal{P}$ for modeling its formulas. Since we are only concerned with provability rather than proofs, we restrict our attention to indexed partially ordered sets rather than indexed categories. The following definition recalls the properties of $(\mathcal{C}, \mathcal{P})$ needed to soundly model first order intuitionistic predicate logic with equality.

Definition 7.1.1. *Let $\mathcal{C}$ be a category with finite products. A* first order hyperdoctrine *$\mathcal{P}$ over $\mathcal{C}$ is a contravariant functor $\mathcal{P} : \mathcal{C}^{op} \to$* Poset *from $\mathcal{C}$ into the category* Poset *of partially ordered sets and monotone functions, with the following properties.*

1. *For each $\mathcal{C} -$ object X, the partially ordered set $\mathcal{P}(X)$ is a Heyting algebra.*
2. *For each $\mathcal{C}$-morphism $f : X \to Y$, the monotone function $\mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(X)$ is a homomorphism of Heyting algebras.*
3. *For each diagonal morphism $\Delta_X : X \to X \times X$ in $\mathcal{C}$, the left adjoint to $\mathcal{P}(\Delta_X)$ at the top element $\top \in \mathcal{P}(X)$ exists. In other words, there is an element $=_X$ of $\mathcal{P}(X \times X)$ satisfying for all $A \in \mathcal{P}(X \times X)$ that*
$$\top \leq \mathcal{P}(\Delta_X)(A) \quad \textit{iff} \quad =_X \leq A.$$
4. *For each product projection $\pi : \Gamma \times X \to \Gamma$ in $\mathcal{C}$, the monotone function $\mathcal{P}(\pi) : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma \times X)$ has both a left adjoint $(\exists X)_\Gamma$ and a right adjoint $(\forall X)_\Gamma$:*
$$A \leq \mathcal{P}(\pi)(A') \quad \textit{if and only if} \quad (\exists X)_\Gamma(A) \leq A'$$
$$\mathcal{P}(\pi)(A') \leq A \quad \textit{if and only if} \quad A' \leq (\forall X)_\Gamma(A).$$

Moreover, these adjoint are natural in Γ, *i.e., given* $s : \Gamma \to \Gamma'$ *in* $\mathcal{C}$, *we have*

$$\begin{tikzcd}
\mathcal{P}(\Gamma' \times X) \arrow[r, "\mathcal{P}(s\times \mathrm{id}_X)"] \arrow[d, "(\exists X)_{\Gamma'}"'] & \mathcal{P}(\Gamma \times X) \arrow[d, "(\exists X)_{\Gamma}"] & \mathcal{P}(\Gamma' \times X) \arrow[r, "\mathcal{P}(s\times \mathrm{id}_X)"] \arrow[d, "(\forall X)_{\Gamma'}"'] & \mathcal{P}(\Gamma \times X) \arrow[d, "(\forall X)_{\Gamma}"] \\
\mathcal{P}(\Gamma') \arrow[r, "\mathcal{P}(s)"'] & \mathcal{P}(\Gamma) & \mathcal{P}(\Gamma') \arrow[r, "\mathcal{P}(s)"'] & \mathcal{P}(\Gamma).
\end{tikzcd}$$

The elements of $\mathcal{P}(X)$, *as* X *ranges over* $\mathcal{C}$*-objects, will be referred to as* $\mathcal{P}$-predicates.

Interpretation of predicate logic in a first order hyperdoctrine. Given a first order signature of types X, countably infinite many variables $x : X$ of each type, function symbols $f : X_1, \ldots, X_n \to X$, (constants are functions of arity 0) and relation symbols $R \subseteq X_1, \ldots, X_n$, an *interpretation* for the signature is a first order hyperdoctrine $(\mathcal{C}, \mathcal{P})$ that assigns a $\mathcal{C}$-object $[\![X]\!]$ to each type, a $\mathcal{C}$-morphism $[\![f]\!] : [\![X_1]\!] \times \cdots \times [\![X_n]\!] \to [\![X]\!]$ to each function symbol, and a $\mathcal{P}$-predicate $[\![R]\!] \in \mathcal{P}([\![X_1]\!] \times \cdots \times [\![X_n]\!])$ to each relation symbol.

Each term t over the signature, with variables in $\Gamma = \{y_1 : Y_1, \ldots, y_n : Y_n\}$ and of sort X say, can be interpreted as a $\mathcal{C}$-morphism $[\![t]\!] : [\![\Gamma]\!] \to [\![X]\!]$, where $[\![\Gamma]\!] = [\![Y_1]\!] \times \cdots \times [\![Y_n]\!]$, by induction on the structure of t. Interpretation of terms is given by

$$\begin{array}{lcl} [\![x : X]\!] & = & \mathrm{id}_{[\![X]\!]} \\ [\![f(t_1, \ldots, t_n)]\!] & = & [\![f]\!] \circ (\langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle), \end{array}$$

assuming $[\![t_i]\!] : [\![\Gamma]\!] \to [\![X_i]\!]$ for $i = 1, \ldots, n$. Each formula ϕ with free variables in Γ can be interpreted as a $\mathcal{P}$-predicate $[\![\phi]\!] \in \mathcal{P}([\![\Gamma]\!])$ by induction on the structure of ϕ using the properties given in Definition 7.1.1 as follows. Interpretation of atomic formulas: We have assumed interpretation of the predicate symbols $[\![R]\!] \in \mathcal{P}([\![X_1]\!] \times \cdots \times [\![X_n]\!])$. Given terms t_i, such that $[\![t_i]\!] : [\![\Gamma]\!] \to [\![X_i]\!]$,

$$[\![R(t_1, \ldots, t_n)]\!] = \mathcal{P}(\langle [\![t_1]\!], \cdots, [\![t_n]\!] \rangle)([\![R]\!]) \in \mathcal{P}([\![\Gamma]\!])$$

in particular the atomic formula $t =_X t'$ is mapped to the $\mathcal{P}$-predicate $\mathcal{P}(\langle [\![t]\!], [\![t']\!] \rangle)(=_{[\![X]\!]})$.

Interpretation of formulas: Assume ϕ, ψ are formulas with free variables in $\Gamma \cup \{x : X\}$, then

$$\begin{array}{lcl} [\![\phi \wedge \psi]\!] & = & [\![\phi]\!] \wedge_H [\![\psi]\!] \\ [\![\phi \vee \psi]\!] & = & [\![\phi]\!] \vee_H [\![\psi]\!] \\ [\![\phi \to \psi]\!] & = & [\![\phi]\!] \to_H [\![\psi]\!] \\ [\![\top]\!] & = & \top_H \\ [\![\bot]\!] & = & \bot_H \\ [\![\forall x : X.\phi]\!] & = & (\forall [\![X]\!])_{[\![\Gamma]\!]}([\![\phi]\!]) \in \mathcal{P}([\![\Gamma]\!]) \\ [\![\exists x : X.\phi]\!] & = & (\exists [\![X]\!])_{[\![\Gamma]\!]}([\![\phi]\!]) \in \mathcal{P}([\![\Gamma]\!]) \end{array}$$

where $\wedge_H, \vee_H$, etc. is the Heyting algebra structure on $\mathcal{P}([\![\Gamma]\!] \times [\![X]\!])$.

We say that a formula $\phi \in \mathcal{P}(X)$ is *satisfied* if $[\![\phi]\!]$ is the top element of $\mathcal{P}(X)$.

Remark 7.1.2. *Note that if the source category* $\mathcal{C}$ *is ccc with a* generic *object* $G \in \mathrm{Obj}(\mathcal{C})$, *i.e., an object which is an internal Heyting algebra, then we can interpret higher order logic in any (first order) hyperdoctrine over* $\mathcal{C}$, *where the "powerset-types"* PA, *for a type* A *are interpreted by* $G^{[\![A]\!]}$. *To keep it simple we have restricted our attention to the first order fragment, but the extension to higher order logic is straight forward. The examples that we give all have the structure needed to interpret higher order logic.*

Example 7.1.3 (Hyperdoctrine of a complete Heyting algebra). *Let H be a complete Heyting algebra. It determines a first order hyperdoctrine over the category* Set *as follows. For each set X we take $\mathcal{P}(X) = H^X$, the X-fold product of H in the category of Heyting algebras. The $\mathcal{P}$-predicates are then indexed families of elements of H, ordered componentwise. They can also be seen as functions from X into H, where X ranges over* Set. *Given $f : X \to Y, \mathcal{P}(f) : H^Y \to H^X$ is the Heyting algebra homomorphism given by re-indexing along f (or, thinking in terms of functions, precomposing with f). For example if $s, t \in \mathcal{P}(Y)$, i.e., $s, t : Y \to H$, then $f(s) = s \circ f : X \to H$ and $s \wedge t$ is defined pointwise as $(s \wedge t)(y) = s(y) \wedge t(y)$. Equality predicates $=_X$ in $H^{X \times X}$ are given by*

$$=_X (x, x') \overset{def}{=} \begin{cases} \top & \text{if } x = x' \\ \bot & \text{if } x \neq x' \end{cases}$$

where $\top$ and $\bot$ are respectively the greatest and least elements of H. The quantifiers use set-indexed joins ($\bigvee$) and meets ($\bigwedge$), which H possesses because it is complete: given $A \in H^{\Gamma \times X}$ one has

$$(\exists X)_\Gamma(A) \overset{def}{=} \lambda i \in \Gamma. \bigvee_{x \in X} A(i, x) \quad (\forall X)_\Gamma(A) \overset{def}{=} \lambda i \in \Gamma. \bigwedge_{x \in X} A(i, x)$$

in H^Γ.

If we take H to be a complete Boolean algebra we get a model for *classical* first order logic. There are plenty of examples of complete Heyting algebras: for any Grothendieck sheaf topos $\mathcal{E}$ and object X of $\mathcal{E}$, $\mathrm{Sub}_{\mathcal{E}}(X)$ is a complete Heyting algebra (according to Corollary 2.3.8 and Theorem 2.3.11).

Example 7.1.4 (Hyperdoctrine over a topos). *Let $\mathcal{E}$ be a topos, then $X \mapsto \mathrm{Sub}_{\mathcal{E}}(X)$ defines a hyperdoctrine over the topos. For $f : X \to Y, \mathrm{Sub}_{\mathcal{E}}(f) : \mathrm{Sub}_{\mathcal{E}}(Y) \to \mathrm{Sub}_{\mathcal{E}}(X)$ is pullback along f.*

This is the first order fragment of the usual subobject semantics: terms are interpreted as morphisms in the topos, predicates as subobjects.

For each diagonal $\Delta_X : X \to X \times X$ in $\mathcal{E}$, the element $=_X$ is defined by $\langle \mathrm{id}_X, \mathrm{id}_X \rangle : X \rightarrowtail X \times X$. It is also standard that each projection $\pi : \Gamma \times X \to \Gamma$ has both left and right adjoints $(\exists X)_\Gamma, (\forall X)_\Gamma$ and that these satisfies the Beck-Chevalley condition.

Knowing that Ω is an internal Heyting algebra, we could also define a hyperdoctrine over a topos by

$$\mathcal{E}(-, \Omega) : \mathcal{E} \to \mathrm{Poset}$$

which would be isomorphic to the subobject hyperdoctrine since $\mathrm{Sub}(X) \cong \mathcal{E}(X, \Omega)$.

Soundness and completeness.

Theorem 7.1.5 (Soundness). *For any formulas ϕ, ψ if $\phi \vdash_X \psi$ then $[\![\phi]\!] \leq [\![\psi]\!]$ for any interpretation in any hyperdoctrine.*

Proof: By induction on the structure of proofs. □

Theorem 7.1.6 (Completeness). *For any formulas ϕ, ψ if $[\![\phi]\!] \leq [\![\psi]\!]$ for all interpretations in all hyperdoctrines, then $\phi \vdash_X \psi$.*

Proof: (Sketch) Consider the *classifying category* $\mathcal{CL}$ (defined in [Jac99] section 2.1). Objects are contexts $\Gamma = \{x_1 : X_1, \ldots, x_n : X_n\}$ of variables with types, and morphisms $\Gamma \to \Delta$, where $\Delta = \{y_1 : Y_1, \ldots, y_m : Y_m\}$, are m-tuples $(M_1, \ldots, M_m)$ of terms of types $M_i : Y_i$ with free variables in Γ.

The classifying category has finite products. The empty context is the terminal object, the product of two contexts $\Gamma = \{x_1 : X_1, \ldots, x_n : X_n\}, \Delta = \{y_1 : Y_1, \ldots, y_m : Y_m\}$ is their concatenation $\{x_1 : X_1, \ldots, x_n : X_n, y_1 : Y_1, \ldots, y_m : Y_m\}$, where we assume that no variable in Γ appears in Δ, this does not imply a loss of generality, since we can always rename a variable.

Define a syntactic model $S : \mathcal{CL} \to \text{Poset}$ by

$$\Gamma \mapsto \{[\phi] \mid \mathrm{FV}(\phi) \subseteq \Gamma\}$$

where $[\phi]$ is the equivalence class of provability, that is $[\phi] = \{\psi \mid \;\vdash_\Gamma \psi \leftrightarrow \phi\}$. For a morphism $\overline{\mathrm{t}} : \Gamma \to \Delta$, $S(\overline{\mathrm{t}})$ is substitution. We claim that this defines a hyperdoctrine.

For each context Γ the equivalence classes ordered by

$$[\phi] \leq_\Gamma [\psi] \text{ iff } \phi \vdash_\Gamma \psi,$$

constitutes a Heyting algebra.

The equality predicate $=_\Gamma \in S(\Gamma \times \Gamma)$ is $[\overline{\mathrm{v}} = \overline{\mathrm{v}}']$, where $\overline{\mathrm{v}}$ is the tuple of variables from Γ and $\overline{\mathrm{v}}'$ is the tuple of renamed variables.

Right and left adjoints to $\pi : \Gamma \times X \to \Gamma$ are given by

$$(\forall X)_\Gamma([\phi(x, \overline{\mathrm{y}})]) = [\forall x : X.\phi(x, \overline{\mathrm{y}})],$$

similarly for $\exists$.

Define an interpretation in $(\mathcal{CL}, S)$ by

$$[\![\phi]\!] := [\phi]$$

then $\phi \vdash_\Gamma \psi$ iff $[\![\phi]\!] \leq_\Gamma [\![\psi]\!]$. □

7.1.1 First order BI-hyperdoctrines

First order BI is first order intuitionistic logic extended with

Formulas.

- I,
- $\phi * \psi$,
- $\phi \mathrel{-\!\!*} \psi$,

satisfying the following logical rules

2.5 $(p * q) * r \vdash_X p * (q * r) \qquad p * (q * r) \vdash_X (p * q) * r$

2.6 $\vdash_X p \leftrightarrow p * \mathrm{I} \qquad \vdash_X p * \mathrm{I} \leftrightarrow \mathrm{I} * p$

2.7 $$\frac{p \vdash_X q \qquad r \vdash_X s}{p * r \vdash_X q * s}$$

2.8 $$p * q \vdash_X q * p$$

2.9 $$\frac{p * q \vdash_X r}{\overline{p \vdash_X q \mathrel{-\!\!*} r}}$$

We now define the obvious extension of a hyperdoctrine to get a structure which is rich enough to interpret first order BI. (Note that the only difference is that “Heyting” is substituted with “BI” in 1. and 2.).

Definition 7.1.7 (BI-hyperdoctrines). *Let $\mathcal{C}$ be a category with finite products. A* first order BI-hyperdoctrine *$\mathcal{P}$ over $\mathcal{C}$ is a contravariant functor $\mathcal{P} : \mathcal{C}^{op} \to$* Poset *from $\mathcal{C}$ into the category* Poset *of partially ordered sets and monotone functions, with the following properties.*

1. *For each $\mathcal{C}-$object X, the partially ordered set $\mathcal{P}(X)$ is a BI algebra.*
2. *For each $\mathcal{C}$-morphism $f : X \to Y$, the monotone function $\mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(X)$ is a homomorphism of BI algebras.*
3. *For each diagonal morphism $\Delta_X : X \to X \times X$ in $\mathcal{C}$, the left adjoint to $\mathcal{P}(\Delta_X)$ at the top element $\top \in \mathcal{P}(X)$ exists. In other words there is an element $=_X$ of $\mathcal{P}(X \times X)$ satisfying for all $A \in \mathcal{P}(X \times X)$ that*
$$\top \leq \mathcal{P}(\Delta_X)(A) \quad \textit{iff} \quad =_X \leq A.$$
4. *For each product projection $\pi : \Gamma \times X \to \Gamma$ in $\mathcal{C}$, the monotone function $\mathcal{P}(\pi) : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma \times X)$ has both a left adjoint $(\exists X)_\Gamma$ and a right adjoint $(\forall X)_\Gamma$:*
$$A \leq \mathcal{P}(\pi)(A') \quad \textit{if and only if} \quad (\exists X)_\Gamma(A) \leq A'$$
$$\mathcal{P}(\pi)(A') \leq A \quad \textit{if and only if} \quad A' \leq (\forall X)_\Gamma(A).$$
Moreover, these adjoints are natural in Γ, i.e., given $s : \Gamma \to \Gamma'$ in $\mathcal{C}$, we have
$$\begin{tikzcd}
\mathcal{P}(\Gamma' \times X) \arrow[r, "\mathcal{P}(s \times \mathrm{id}_X)"] \arrow[d, "(\exists X)_{\Gamma'}"'] & \mathcal{P}(\Gamma \times X) \arrow[d, "(\exists X)_\Gamma"] \\
\mathcal{P}(\Gamma') \arrow[r, "\mathcal{P}(s)"'] & \mathcal{P}(\Gamma)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\Gamma' \times X) \arrow[r, "\mathcal{P}(s \times \mathrm{id}_X)"] \arrow[d, "(\forall X)_{\Gamma'}"'] & \mathcal{P}(\Gamma \times X) \arrow[d, "(\forall X)_\Gamma"] \\
\mathcal{P}(\Gamma') \arrow[r, "\mathcal{P}(s)"'] & \mathcal{P}(\Gamma).
\end{tikzcd}$$

*The elements of $\mathcal{P}(X)$, as X ranges over $\mathcal{C}$-objects, will be referred to as $\mathcal{P}$-*predicates*.*

A BI algebra is a Heyting algebra with an additional residuated structure, so we can interpret first order BI in the obvious way: the additive part as described in the previous section the new connectives as $[\![\phi * \psi]\!] = [\![\phi]\!] *' [\![\psi]\!]$ where $[\![\phi]\!], [\![\psi]\!] \in \mathcal{P}(X)$ and $*'$ is the monoid composition in the BI algebra $\mathcal{P}(X)$, $\mathrel{-\!\!*}$ and I are interpreted in the same manner.

Here are two examples of BI-hyperdoctrines.

Example 7.1.8 (BI-hyperdoctrine of a complete BI algebra). *Let B be a complete BI algebra (cBIa). It determines a first order BI-hyperdoctrine over the category* Set *as follows. For each set X we take $\mathcal{P}(X) = B^X$, the X-fold product of B in the category of BI algebras. The $\mathcal{P}$-predicates are then indexed families of elements of B, ordered componentwise. Given $f : X \to Y, \mathcal{P}(f) : B^Y \to B^X$ is the BI algebra homomorphism given by re-indexing along f. For example if $s, t \in \mathcal{P}(Y)$, i.e., $s, t : Y \to B$, then $f(s) = s \circ f : X \to B$ and $s * t$ is defined pointwise as $(s * t)(y) = s(y) * t(y)$. Equality predicates $=_X$ in $B^{X \times X}$ are given by*

$$=_X (x, x') \overset{def}{=} \begin{cases} \top & \textit{if } x = x' \\ \bot & \textit{if } x \neq x' \end{cases}$$

where $\top$ and $\bot$ are respectively the greatest and least elements of B. The quantifiers use set-indexed joins ($\bigvee$) and meets ($\bigwedge$), which B possesses because it is complete: given $A \in H^{\Gamma \times X}$ one has

$$(\exists X)_\Gamma(A) \overset{def}{=} \lambda i \in \Gamma.\ \bigvee_{x \in X} A(i, x) \quad (\forall X)_\Gamma(A) \overset{def}{=} \lambda i \in \Gamma.\ \bigwedge_{x \in X} A(i, x)$$

in B^Γ.

If we take B to be a complete Boolean BI algebra, i.e., the intuitionistic (additive) part of the BI algebra is Boolean rather than Heyting, we get a model for *classical* first order BI.

There are plenty of examples of complete BI algebras presented in this thesis: for any Grothendieck topos $\mathcal{E}$ with an additional symmetric monoidal closed structure, $\mathrm{Sub}_{\mathcal{E}}(1)$ is a complete BI algebra, and for any monoidal category $\mathcal{C}$ such that the monoid is cover preserving w.r.t. the Grothendieck topology J, $\mathrm{Sub}_{\mathrm{Sh}(\mathcal{C},J)}(1)$ is a cBIa even though the category of sheaves $\mathrm{Sh}(\mathcal{C}, J)$ is not doubly closed in general (Proposition 4.4.8).

Example 7.1.9 (BI-hyperdoctrine over a topos with an internal BI algebra). *We do not in general have a BI structure on the subobject lattice in a (doubly closed) topos, but assuming a topos $\mathcal{E}$ has an object B which is an internal BI algebra, we can define a BI-hyperdoctrine by*

$$\mathcal{E}(-, B) : \mathcal{E} \to \mathrm{Poset}$$

for $f : X \to Y, \mathcal{E}(f, B) : \mathcal{E}(Y, B) \to \mathcal{E}(X, B)$ is precomposing with f.

To see how formulas are interpreted in this structure consider for example $\phi * \psi$, assuming $[\![\phi]\!], [\![\psi]\!] \in \mathcal{E}(X, B)$, then $[\![\phi * \psi]\!]$ is the composite

$$X \xrightarrow{\langle [\![\phi]\!], [\![\psi]\!] \rangle} B \times B \xrightarrow{*} B.$$

For $s, t \in \mathcal{E}(X, B)$ the order is defined by $s \leq t$ iff $s \wedge t = s$, where $s \wedge t : X \to B$ is the arrow $\wedge \circ \langle s, t \rangle$.

The two examples we have just given are BI-hyperdoctrines over Cartesian closed categories so they actually model higher order predicate logic in the sense that for each type A, interpreted as the object $[\![A]\!]$ in $\mathcal{E}$, the exponent $B^{[\![A]\!]}$ is an object adequate to interpret the "powerset-type" PA.

Soundness and completeness.

Theorem 7.1.10 (Soundness). *For any formulas* ϕ, ψ *if* $\phi \vdash_X \psi$ *then* $[\![\phi]\!] \leq [\![\psi]\!]$ *for any interpretation in any BI-hyperdoctrine.*

Proof: By induction on the structure of proofs. □

Theorem 7.1.11 (Completeness). *For any formulas* ϕ, ψ *if* $[\![\phi]\!] \leq [\![\psi]\!]$ *for all interpretations in all hyperdoctrines, then* $\phi \vdash_X \psi$.

Proof: To proof completeness, proceed as in the proof of Theorem 7.1.6 using BI algebras instead of Heyting algebras. □

7.2 Separation logic modelled by BI-hyperdoctrines

We give a brief presentation of separation logic (for a more thorough presentation see for instance [Rey02]), which was also discussed in the Introduction, and show how it can be modeled by first order BI-hyperdoctrines.

Separation logic consists of a single type Val of values and a unit type 1, terms t are defined by

$$t ::= x \mid n \mid t + t \mid t - t$$

where n : Val are constants.

Formulas are defined by

$$\phi ::= \top \mid \bot \mid t = t \mid t \mapsto t \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi * \phi \mid \phi \mathrel{-\!\!*} \phi \mid \mathrm{emp}$$

where t ranges over terms.

What we have defined here is really just a signature (specification of types, function symbols and predicate symbols) for first order BI with a single type Val, function symbols $+, -$: Val, Val $\to$ Val, constants n : Val, a relation symbol $\mapsto \subseteq$ Val, Val and units $\top \subseteq 1, \bot \subseteq 1, \mathrm{emp} \subseteq 1$. Thus, the general soundness and completeness results for BI hyperdoctrines apply to separation logic.

The pointer model. The (classical) pointer model is an example of a model of separation logic in a hyperdoctrine over Set. The Pointer model consists of a set $[\![\mathrm{Val}]\!]$ interpreting the type Val and a set $[\![\mathrm{Loc}]\!]$ of locations such that $[\![\mathrm{Loc}]\!] \subseteq [\![\mathrm{Val}]\!]$ and binary functions on $[\![\mathrm{Val}]\!]$ interpreting the function symbols $+, -$. Furthermore we require a set of heaps $H = [\![\mathrm{Loc}]\!] \rightharpoonup_{fin} [\![\mathrm{Val}]\!]$ of finite partial functions with the discrete order (one can also define another order on H, which gives rise to an intuitionistic model, this will be discussed briefly at the end of this section). The set of heaps has a partial operation $*$ defined by

$$h_1 * h_2 = \begin{cases} h_1 \cup h_2 & \text{if } h_1 \# h_2 \\ \text{undefined} & \text{otherwise} \end{cases}$$

where # is a binary relation on heaps defined by $h_1 \# h_2$ iff $\mathrm{dom}(h_1) \cap \mathrm{dom}(h_2) = \emptyset$. The interpretation of the relation $\mapsto \subseteq [\![\mathrm{Val}]\!] \times [\![\mathrm{Val}]\!]$ is the subset of singleton heaps, i.e., for $h \in H$, $h \in \mapsto$ iff $h = \{(v_1, v_2)\}$. To define the interpretation we assume a partial function (stack)

$s : \mathrm{Var} \to [\![\mathrm{Val}]\!]$ on the set of variables [2], the interpretation of terms depends on the stack and is defined by

$$\begin{array}{lcl} [\![x]\!]s & = & s(x) \\ [\![n]\!]s & = & [\![n]\!] \\ [\![t_1 \pm t_2]\!]s & = & [\![t_1]\!]s \pm [\![t_2]\!]s \end{array}$$

The set of heaps H is our set of possible worlds. Interpretation is usually defined by a forcing relation $s, h \Vdash \phi$ where $\mathrm{FV}(\phi) \subseteq \mathrm{dom}(s)$ as follows

$$\begin{array}{lll} s, h \Vdash t_1 = t_2 & \text{iff} & [\![t_1]\!]s = [\![t_2]\!]s \\ s, h \Vdash t_1 \mapsto t_2 & \text{iff} & \mathrm{dom}(h) = \{[\![t_1]\!]s\} \text{ and } h([\![t_1]\!]s) = [\![t_2]\!]s \\ s, h \Vdash \mathrm{emp} & \text{iff} & h = \emptyset \\ s, h \Vdash \top & & \text{always} \\ s, h \Vdash \bot & & \text{never} \\ s, h \Vdash \phi * \psi & \text{iff} & \text{there exists } h_1, h_2 \in H.\, h_1 * h_2 = h \text{ and} \\ & & s, h_1 \Vdash \phi \text{ and } s, h_2 \Vdash \psi \\ s, h \Vdash \phi \mathrel{-\!\!*} \psi & \text{iff} & \text{for all } h', h' \# h \text{ and } s, h' \Vdash \phi \text{ implies } s, h * h' \Vdash \psi \\ s, h \Vdash \phi \vee \psi & \text{iff} & s, h \Vdash \phi \text{ or } s, h \Vdash \psi \\ s, h \Vdash \phi \wedge \psi & \text{iff} & s, h \Vdash \phi \text{ and } s, h \Vdash \psi \\ s, h \Vdash \phi \to \psi & \text{iff} & s, h \Vdash \phi \text{ implies } s, h \Vdash \psi \\ s, h \Vdash \forall x.\phi & \text{iff} & \text{for all } v \in [\![\mathrm{Val}]\!].s[x \mapsto v], h \Vdash \phi \\ s, h \Vdash \exists x.\phi & \text{iff} & \text{there exists } v \in [\![\mathrm{Val}]\!].s[x \mapsto v], h \Vdash \phi \end{array}$$

This forcing semantics is trivially Kripke monotone since the order on the heaps H is discrete. Clearly, this definition resembles the Kripke semantics for propositional BI given earlier, the connection between the two settings goes via BI-hyperdoctrines.

The pointer model as a BI-hyperdoctrine. We now show how the pointer model is an instance of a BI-hyperdoctrine of a complete BI algebra (example 7.1.8).

Let $(H_\bot, *)$ be the set of heaps with a bottom element added to represent undefined, order is flat (i.e., discrete with an added bottom) and $\bot * h = \bot$ for all $h \in H_\bot$. This defines a partially ordered commutative monoid with emp (the empty heap) as the unit for $*$. The powerset of $H, \mathcal{P}(H)$ (without $\bot$) is a complete BI algebra, with inclusion as the order. This can be shown in at least two ways:

1. Define it directly using the monoid composition $* : H_\bot \times H_\bot \to H_\bot$ pointwise and then remove $\bot$ from the resulting set. The unit is $\{\mathrm{emp}\}$, the subset containing the empty heap, and the right adjoint is defined by
$$U \mathrel{-\!\!*} V := \bigcup \{W \subseteq H \mid (W * U) \subseteq V\}.$$
2. Another way to see it is by noticing that there is a one-one correspondence between $\mathcal{P}(H)$ and non-empty sieves on $H_\bot$ by respectively adding and removing $\bot$. These are precisely the ideals with respect to the "semi-trivial" Grothendieck topology on $H_\bot$ defined by
$$J(h) = \begin{cases} \{h\} & \text{if } h \neq \bot \\ \{\{\bot\}, \emptyset\} & \text{otherwise.} \end{cases}$$

---

[2] If we have more than one type, s must be a type-respecting partial function from Var to the disjoint union of the interpretation of the types. In other words, s will be the disjoint union of partial functions: $s_X : \mathrm{Var}_X \to [\![X]\!]$, where Var_X is a finite set of variables of type X, for each type X.

This means that $\mathcal{P}(H) \cong \mathrm{Sub}_{\mathrm{Sh}(H_\perp,J)}(1)$, and $\mathrm{Sub}_{\mathrm{Sh}(H_\perp,J)}(1)$ is a complete BI algebra. To see this, note that $*$ preserves covers of J, then it follows from Proposition 4.4.8.

So, $\mathcal{P}(H)$ is a complete BI algebra which is even Boolean since $\mathcal{P}(H)$ is a Boolean algebra.

We define a hyperdoctrine $S : \mathrm{Set}^{op} \to \mathcal{P}(H)$ by $X \mapsto \mathcal{P}(H)^X$, where $\mathcal{P}(H)^X \cong \mathrm{Set}(X, \mathcal{P}(H))$ as described in example 7.1.8. A term t in context $\Gamma = \{x_1 : X_1, \ldots, x_n : X_n\}$ is interpreted as a morphism between sets:

- $[\![x : \mathrm{Val}]\!] = \mathrm{id}_{[\![\mathrm{Val}]\!]}$,
- $[\![n]\!]$ is the map $[\![n]\!] : 1 \to [\![\mathrm{Val}]\!]$, that sends $*$ to $[\![n]\!]$,
- $[\![t_1 \pm t_2]\!] = [\![t_1]\!] \pm [\![t_2]\!] : [\![\Gamma]\!] \to [\![\mathrm{Val}]\!]$, where $t_i : [\![\Gamma]\!] \to [\![\mathrm{Val}]\!], i = 1, 2$.

Formulas are S-predicates: suppose $\mathrm{FV}(\phi) = \{x_1 : X_1, \ldots, x_n : X_n\}$, [3] let $[\![\Gamma]\!] = [\![X_1]\!] \times \cdots \times [\![X_n]\!]$, then we want to give an inductive definition of formulas such that $[\![\phi]\!] : [\![\Gamma]\!] \to \mathcal{P}(H)$ satisfies

$$(v_1, \ldots, v_n) \mapsto \{h \mid [x_1 \mapsto v_1, \ldots, x_n \mapsto v_n], h \Vdash \phi\}.$$

If $\overline{\mathrm{v}} = (v_1, \ldots, v_n)$, the definition is as follows (using the hyperdoctrine structure as described in example 7.1.8):

$$\begin{array}{lcl}
[\![t_1 \mapsto t_2]\!](\overline{\mathrm{v}}) & = & \{h \mid \mathrm{dom}(h) = \{[\![t_1]\!](\overline{\mathrm{v}})\} \text{ and } h([\![t_1]\!](\overline{\mathrm{v}})) = [\![t_2]\!](\overline{\mathrm{v}})\} \\
[\![t_1 = t_2]\!](\overline{\mathrm{v}}) & = & H \text{ if } [\![t_1]\!](\overline{\mathrm{v}}) = [\![t_2]\!](\overline{\mathrm{v}}) \\
 & & \emptyset \text{ otherwise} \\
[\![\top]\!](*) & = & H \\
[\![\bot]\!](*) & = & \emptyset \\
[\![\mathrm{emp}]\!](*) & = & \{h \mid \mathrm{dom}(h) = \emptyset\} \\
[\![\phi \wedge \psi]\!](\overline{\mathrm{v}}) & = & [\![\phi]\!](\overline{\mathrm{v}}) \cap [\![\psi]\!](\overline{\mathrm{v}}) \\
[\![\phi \vee \psi]\!](\overline{\mathrm{v}}) & = & [\![\phi]\!](\overline{\mathrm{v}}) \cup [\![\psi]\!](\overline{\mathrm{v}}) \\
[\![\phi \to \psi]\!](\overline{\mathrm{v}}) & = & \{h \mid h \in [\![\phi]\!](\overline{\mathrm{v}}) \text{ implies } h \in [\![\psi]\!](\overline{\mathrm{v}})\} \\
[\![\phi * \psi]\!](\overline{\mathrm{v}}) & = & [\![\phi]\!](\overline{\mathrm{v}}) * [\![\psi]\!](\overline{\mathrm{v}}) \\
 & = & \{h_1 * h_2 \mid h_1 \in [\![\phi]\!](\overline{\mathrm{v}}) \text{ and } h_2 \in [\![\psi]\!](\overline{\mathrm{v}})\} \setminus \{\bot\} \\
[\![\phi \mathrel{-\!\!*} \psi]\!](\overline{\mathrm{v}}) & = & [\![\phi]\!](\overline{\mathrm{v}}) \mathrel{-\!\!*} [\![\psi]\!](\overline{\mathrm{v}}) \\
 & = & \{h \mid [\![\phi]\!](\overline{\mathrm{v}}) * \{h\} \subseteq [\![\psi]\!](\overline{\mathrm{v}})\} \\
[\![\forall x : X.\phi]\!](\overline{\mathrm{v}}) & = & \bigcap_{v_x \in [\![X]\!]}([\![\phi]\!](v_x, \overline{\mathrm{v}})) \\
[\![\exists x : X.\phi]\!](\overline{\mathrm{v}}) & = & \bigcup_{v_x \in [\![X]\!]}([\![\phi]\!](v_x, \overline{\mathrm{v}}))
\end{array}$$

We can easily derive the forcing semantics of the pointer model:

Proposition 7.2.1. *$h \in [\![\phi]\!](v_1, \ldots, v_n)$ iff $[x_1 \mapsto v_1, \ldots, x_n \mapsto v_n], h \Vdash \phi$.*

Proof: This is immediate by structural induction on formulas ϕ. □

Now, soundness of separation logic follows directly from the general soundness result of hyperdoctrines. Also, Kripke monotonicity of the forcing relation $\Vdash$ follows from the fact that $\mathcal{P}(H) \cong \mathrm{Sub}_{\mathrm{Sh}(H_\perp,J)}(1)$ so that $[\![\phi]\!](\overline{\mathrm{v}})$ is a subobject of 1, in particular $[\![\phi]\!](\overline{\mathrm{v}})$ is a presheaf which is a property that corresponds exactly to monotonicity. This may not be very interesting in the present case since we already noted that Kripke monotonicity is trivial, but for other models it is less trivial.

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[3] In fact, since we only have one type, Val, apart from the unit type, all the X_i's are Val.

An intuitionistic model. Consider again the set of heaps $(H_\perp, *)$ with an added bottom $\perp$, satisfying $h * \perp = \perp$ for all $h \in H$. The order is now defined by

$$h_1 \sqsupseteq h_2 \qquad \text{iff} \qquad \mathrm{dom}(h_1) \subseteq \mathrm{dom}(h_2) \text{ and for all } x \in \mathrm{dom}(h_1).h_1(x) = h_2(x).$$

or equivalently

$$h_1 \sqsupseteq h_2 \qquad \text{iff} \qquad \exists h_3 \in H.h_1 * h_3 = h_2.$$

The Grothendieck topology on $H_\perp$ is again the "semi-trivial" one. From Proposition 4.4.8 it follows that $\mathrm{Sub}_{Sh(H_\perp,J)}(1)$ is a complete BI algebra.

Now the cBIa $\mathrm{Sub}_{Sh(H_\perp,J)}(1)$ corresponds to the ideals on $H_\perp$, an ideal in this case is a non-empty, downwards closed subset of $H_\perp$. Again we remove the added bottom and get $\mathrm{Sub}_{Sh(H_\perp,J)}(1)$ is in one-one correspondence with sieves on H (downwards closed subsets of H) which in turn corresponds to $\mathrm{Sub}_{\widehat{H}}(1)$. This shows that $\mathrm{Sub}_{\widehat{H}}(1)$ is a cBIa with inclusion as order.

Now define a hyperdoctrine $T : \mathrm{Set} \to \mathrm{Sub}_{\widehat{H}}(1)$ as before except that the predicates should now be functions from a set to $\mathrm{Sub}_{\widehat{H}}(1)$. Terms are defined as before. The definition of $\mapsto$ is

$$[\![t_1 \mapsto t_2]\!](\overline{v}) = \{h \mid h([\![t_1]\!](\overline{v})) = [\![t_2]\!](\overline{v})\},$$

which defines a downwards closed subset of H. Now use the BI-hyperdoctrine structure to interpret the formulas inductively (which essentially means to use the BI structure of $\mathrm{Sub}_{Sh(H_\perp,K)}(1)$ pointwise and then remove the $\perp$ to get an element of $\mathrm{Sub}_{\widehat{H}}(1)$). This will yield a semantics which is slightly different, for example,

$$[\![\phi * \psi]\!](\overline{v}) = [\![\phi]\!](\overline{v}) * [\![\psi]\!](\overline{v}) = (\downarrow\{h_1 * h_2 \mid h_1 \in [\![\phi]\!](\overline{v}), h_2 \in [\![\psi]\!](\overline{v})\}) \setminus \{\perp\}$$

using the definition of $*$ in $\mathrm{Sub}_{\mathrm{Sh}(H_\perp,K)}(1)$ as described in Proposition 4.4.8 and using the fact that Day's tensor restricts to sheaves for this topology, Proposition 4.3.4.

Conclusion: There is nothing revolutionary in the idea of interpreting predicate BI in BI-hyperdoctrines, but the link to separation logic makes it relevant for at least two reasons:

It shows that the stack $s : \mathrm{Var} \to [\![\mathrm{Val}]\!]$, which usually maps free variables to a *set* of values $[\![\mathrm{Val}]\!]$ (one can think of it as a substitution operator or an environment) can be a map to something more general than a set, if we model separation logic in a hyperdoctrine over some other category than Set.

It also shows how separation logic, being a hyperdoctrine over Set, possesses all the expressional power of the internal language of the topos Set.[4] In particular all the relations introduced in [Rey04] to prove correctness for a garbage collector using separation logic can be seen as abbreviations or names of predicates of the internal language of Set, since any set is a predicate of the internal language of Set. All the rules or axioms introduced in [Rey04] for these relations are well-known properties of sets, so they automatically become valid when we note that we are using the internal logic of Set. One thing should be noted, though; we are actually working with two different kinds of predicates: the usual predicates of the topos

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[4] The *internal language* of a topos $\mathcal{T}$ has a type for each object of $\mathcal{T}$, a variable of type X for each map $1 \to X$, a function symbol for each morphism, predicate symbols are function symbols $f : X \to \Omega$ of type Ω. The terms and formulas over this signature form the internal language of $\mathcal{T}$. The *internal logic* is defined by $\phi \vdash_X \psi$ iff for all objects $C \in \mathcal{T}$ and all arrows $h : C \to X$, if $\phi h = \top 1_C$ then $\psi h = \top 1_C$, where $\phi : X \to \Omega$ (see [LS86]).

set, i.e., terms $\phi : X \to \Omega$ of type Ω and predicates that involve the heap (those are the ones defined by the forcing relation $s, h \Vdash \phi$). When we interpret the logic in a hyperdoctrine $S : \mathrm{Set}^{op} \to \mathrm{Poset}$, all predicates must be interpreted as functions from the set X of their free variables to the powerset of heaps $\mathcal{P}(H)$. To see why this is not a problem, i.e., that this interpretation is sound for the internal logic of Set, note that $\Omega = \{0, 1\}$ in Set so Ω factors through $\mathcal{P}(H)$ by sending 0 to the least element, $\emptyset$ and 1 to the top element, H. This implies that any set-theoretic predicate $\phi : X \to \Omega$ can be interpreted as a predicate $\{\phi\}$ in the hyperdoctrine by the composite

$$\begin{tikzcd} \llbracket X \rrbracket \arrow[r, "\mathrm{char}\llbracket \phi \rrbracket"] & \Omega \arrow[r] & \mathcal{P}(H), \end{tikzcd}$$

where $\mathrm{char}\llbracket \phi \rrbracket$ is the usual interpretation of ϕ in Set. In the usual subobject interpretation we have $\vdash \phi$ implies $\llbracket \phi \rrbracket = \llbracket X \rrbracket$ (the maximal subobject), which is equivalent (in Set) to $\mathrm{char}\llbracket \phi \rrbracket : \llbracket X \rrbracket \to \Omega$ satisfying $\mathrm{char}\llbracket \phi \rrbracket(x) = 1$ for all $x \in \llbracket X \rrbracket$. The hyperdoctrine interpretation of ϕ then satisfies

$$\{\phi\}(x) = H = \top_{\mathcal{P}(H)}, \text{ for all } x \in \llbracket X \rrbracket.$$

Thus, for any predicate $\phi : X \to \Omega$ of the internal language of Set we have

$$\begin{array}{lll} & \vdash \phi & \text{iff } \phi \text{ is provable in the internal logic of Set} \\ \text{iff} & \llbracket \phi \rrbracket = \llbracket X \rrbracket & \text{in the subobject interpretation in Set} \\ \text{iff} & \mathrm{char}\llbracket \phi \rrbracket(x) = 1 & \text{for all } x \in \llbracket X \rrbracket \\ \text{iff} & \{\phi\}(x) = H = \top_{\mathcal{P}(H)} & \text{for all } x \in \llbracket X \rrbracket \\ \text{iff} & s, h \Vdash \phi & \text{always.} \end{array}$$

We believe that what Yang does in his thesis [Yan96] can also be simplified by using this framework.

Appendix A

On Pym's notion of predicate BI

Literature: [Pym02] and [Amb91].

This chapter contains mainly comments on the book [Pym02], part II, concerning predicate BI and it will probably not make much sense to someone who has not had a look in this book. We have not been able to understand Pym's suggestion of predicate BI. In this chapter we discuss some problems we have encountered. The chapter does not present predicate BI, but we point out some important issues that should be considered by anyone who attempts to define and model predicate BI (and other substructural predicate logics). The main point is that for a logic that does not allow weakening for variables, some sort of variable balancing (on each side of the $\vdash$) is needed for the syntax, if there shall be any hope to give a subobject semantics for such a logic.

A.1 The axiom relation

In [Pym02] David Pym defines a syntax of predicate BI, which involves a symmetric, transitive, binary relation on bunches of variables. In this section we shall try to give a precise definition of this relation and use this to point out some problems of the syntax.

The Axiom relation is a finite equivalence relation on bunches. In the beginning of a proof the Axiom relation $\mathcal{A} \subset \text{Bunch} \times \text{Bunch}$ is empty, we have a rule for introducing elements to $\mathcal{A}$:

$$\frac{X_1 \vdash \phi(X_1) : \text{Prop} \quad X_2 \vdash \phi(X_2) : \text{Prop} \qquad \mathcal{A} = A}{(X_1, X_2)\phi(X_1) \vdash \phi(X_2) \qquad \mathcal{A} = (A \cup \{(X_1, X_2)\})}\ \mathit{Axiom}$$

where $(A \cup \{(X_1, X_2)\})$ is the symmetric, reflexive, transitive closure of $A \cup \{(X_1, X_2)\}$. There is also a rule that removes elements

$$\frac{(Y(X, X'))\Gamma \vdash \phi \qquad \mathcal{A} = (A \cup \{(X, X')\})}{Y(X)\Gamma[X/X'] \vdash \phi[X/X'] \qquad \mathcal{A} = A}\ C$$

again we take the symmetric, reflexive, transitive closure.

The substitution rule and the rules for quantifiers also involves the Axiom relation in their side conditions, but we shall not consider these here.

Because we are mixing the multiplicative end the additive parts of the logic and keeping variables in bunches, we need something like the Axiom relation to make sure that we get an extension of intuitionistic predicate logic and not something completely different. For example we want to have a rule

$$\frac{X \vdash \phi(X) : \text{Prop} \qquad X \vdash \psi(X) : \text{Prop}}{X \vdash \phi(X) \wedge \psi(X)\, \text{Prop}} .$$

This can easily be derived from the rule

$$\frac{X \vdash \phi(X) : \text{Prop} \qquad Y \vdash \psi(Y) : \text{Prop}}{X; Y \vdash \phi(X) \wedge \psi(Y)\, \text{Prop}} ,$$

using the Axiom rule and the cut rule.

We could not have chosen the first rule as the definition and then used weakening to obtain the second one because this would violate the linearity restriction: a variable can occur at most once in a bunch. For example, consider $(y, x) \vdash \phi$: Prop and $(z, x) \vdash \psi$: Prop we do not have weakening for the comma so because of the linearity restriction we have to rename one of the x's, we then get $(y, x); (z, x') \vdash \phi \wedge \psi$: Prop where $(x, x') \in \mathcal{A}$. If at some point y and z are substituted for some closed terms, the bunch will have the form $(x; x')$ and then we can use the cut rule and the axiom relation to restore x from x' in ϕ.

We now give some examples to show that Lemma 12.2 in [Pym02, p. 167] (and the corrected version which appears in the Errata [Pym04] of David Pym's monograph) does not hold. Lemma 12.2 (the version in the Errata [Pym04]) states:

$$\begin{gathered}\text{If } (X)\Gamma \vdash \phi \text{ in NBI, then } X_1 \vdash \Gamma : \text{Prop and } X_2 \vdash \phi : \text{Prop,} \\ \text{where } X = X_1, X_2 \text{ or } X = X_1; X_2.\end{gathered} \tag{A.1}$$

Assume we have $X_1 \vdash \phi(X_1)$: Prop and $Y_1 \vdash \psi(Y_1)$: Prop. If X_2 is an α conversion of X_1, then we must also have $X_2 \vdash \phi(X_2)$, similarly $Y_2 \vdash \psi(Y_2)$. This gives the following proof tree (where $\mathcal{A} = \emptyset$ if nothing else is indicated)

$$\dfrac{\dfrac{\dfrac{\dfrac{X_1 \vdash \phi(X_1) : \text{Prop} \quad X_2 \vdash \phi(X_2) : \text{Prop}}{(X_1, X_2)\phi(X_1) \vdash \phi(X_2) \quad \mathcal{A} = \{(X_1, X_2)\}}}{(X_1)\phi(X_1) \vdash \phi(X_1)}\, C \qquad \dfrac{\dfrac{Y_1 \vdash \psi(Y_1) : \text{Prop} \quad Y_2 \vdash \psi(Y_2) : \text{Prop}}{(Y_1, Y_2)\psi(Y_1) \vdash \psi(Y_2) \quad \mathcal{A} = \{(Y_1, Y_2)\}}}{(Y_1)\psi(Y_1) \vdash \psi(Y_1)}\, C}{(X_1, Y_1)\phi(X_1) * \psi(Y_1) \vdash \phi(X_1) * \psi(Y_1)}}{(X_1, Y_1)\phi(X_1) \vdash \psi(Y_1) \mathrel{-\!\!*} \phi(X_1) * \psi(Y_1)} \mathrel{-\!\!*}$$

where we have $X_1 \vdash \phi(X_1)$: Prop, but not $Y_1 \vdash \psi(Y_1) \mathrel{-\!\!*} \phi(X_1) * \psi(Y_1)$: Prop, since the formation rule for $\mathrel{-\!\!*}$ has the form

$$\frac{X \vdash \phi(X) : \text{Prop} \quad Y \vdash \psi(Y) : \text{Prop}}{X, Y \vdash \phi(X) \mathrel{-\!\!*} \psi(Y)} \mathrel{-\!\!*} .$$

There is even an example that does not involve the axiom condition: The proof of A.1 would be by induction on the structure of proofs, so consider the rule

$$\frac{(X)\Gamma, \phi \vdash \psi}{(X)\Gamma \vdash \phi \mathrel{-\!\!*} \psi} \mathrel{-\!\!*} I$$

Now, induction hypothesis says that $X_1 \vdash \Gamma, \phi$: Prop and $X_2 \vdash \psi$: Prop where either $X = X_1, X_2$ or $X = X_1; X_2$. Suppose we have $X = X_1; X_2$. By the formation rules for propositions we must then have $Y \vdash \Gamma$: Prop and $Z \vdash \phi$ where $X_1 = Y, Z$ and therefore $Z, X_2 \vdash \phi \mathrel{-\!\!*} \psi$. However, it is not the case that $((Y, Z); X_2) = (Y, (Z, X_2))$ and neither that $((Y, Z); X_2) = (Y; (Z, X_2))$ so we can not split X the way we want to. Clearly A.1 can not be true.

Conclusion: The natural deduction system for predicate BI (NBI) given in [Pym02, p. 166] does not preserve well-formed propositions as defined in [Pym02, p. 161], nor does it preserve well-formed sequents $(X)\Gamma \vdash \phi$.

A.2 Semantics for predicate BI

One of the aims of this thesis has been to give subobject semantics of predicate BI corresponding to (and generalizing) the Kripke semantics of predicate BI given in [Pym02, p. 183]. This has turned out to be quite problematic for at least two reasons: the failure of A.1 and the fact that Day's tensor does not preserve pullbacks (this is needed to proof soundness for substitution).

It is worth noticing that Simon Ambler makes some of the same observations in [Amb91] where he gives a proof theory and semantics for first order linear logic.

The main point of this section is that in order to give a subobject semantics for a sequent calculus, we need all interpretations to satisfy: if $(X)\phi \vdash \psi$ then ϕ and ψ are interpreted as subobjects of the *same* object A and $[\![\phi]\!] \leq [\![\psi]\!]$ in $\mathrm{Sub}(A)$.[1] We argue that the calculus for predicate BI given in [Pym02] does not have any interpretations satisfying this necessary condition. In particular the interpretation described in [Pym02] is not sound. In the previous section we gave a proof of

$$(X_1, Y_1)\phi(X_1) \vdash \psi(Y_1) \mathrel{-\!\!*} \phi(X_1) * \psi(Y_1)$$

how can we possibly interpret the premise and conclusion in the same subobject lattice?

It is possible, though, that a small part of the calculus (without $\mathrel{-\!\!*}, \forall_{new}, \exists_{new}$) can be given a meaningful interpretation. We now give a brief description of how part of the predicate Kripke semantics given in [Pym02] relate to subobject semantics. Atomic types are interpreted as objects of the functor category $\widehat{\mathcal{M}}$, and higher types are interpreted using the doubly closed structure on $\widehat{\mathcal{M}}$, this corresponds to Definition 13.2 in [Pym02]. This defines a function $[\![-]\!]$ from bunches to $\mathrm{Obj}(\widehat{\mathcal{M}})$.

As usually in subobject semantics, a term $X \vdash t : A$ of type A and with free variables a bunch X is interpreted as a morphism $[\![t]\!] : [\![X]\!] \to [\![A]\!]$ in $\widehat{\mathcal{M}}$, going from the interpretation of the types of the free variables to the interpretation of the type of t. Predicate symbols $p \subseteq A$ are interpreted as subobjects of $[\![A]\!]$, and formulas $X \vdash \phi : \mathrm{Prop}$ are interpreted as subobjects of $[\![X]\!]$, which we can assume have the form $[\![\phi]\!] \xhookrightarrow{\iota} [\![X]\!]$ where ι is the inclusion. Interpretation of bunches of variables deserves some attention. David Pym claims ([Pym02, p. 185]) that his Kripke semantics is consistent with the subobject semantics of intuitionistic logic (as in [LS86]). In particular this presumes that if $(X)\phi \vdash \psi$ is purely intuitionistic, then the bunch of free variables, which in this case has the form $X = x_1!X_1; x_2!X_2; \ldots ; x_n!X_n$, is interpreted as the product $[\![X_1]\!] \times [\![X_2]\!] \times \cdots \times [\![X_n]\!]$ of the types of the free variables in the bunch, that is, the interpretation of the bunch of variables is the interpretation of the bunch of their types. However, Pym defines it slightly different for the basic case (see Definition 13.3 [Pym02, p. 180]): He assumes one fixed functor (object of the category) D, and defines $[\![x : A]\!] = [\![y!B]\!] = D$ for any variable x of multiplicative type A and any variable y of additive type B. This can only match the traditional subobject semantics if the language has one single type!

---

[1] In [Amb91] this is ensured by a balancing property: each variable must occur exactly once in ϕ and once in ψ.

In the same definition, Pym requires that for bunches such that $Axiom(X, Y)$ the interpretation satisfies $\llbracket X \rrbracket = \llbracket Y \rrbracket$.

The Kripke semantics defined in [Pym02] uses clauses of the form

$$(X)u \,|\, m \models \phi$$

where $m \in \mathcal{M}$ is a world, $X \vdash \phi : \mathrm{Prop}$ and $u \in \llbracket X \rrbracket(m)$. Kripke and subobject semantics in functor categories are inter-definable in the sense that $(X)u \,|\, m \models \phi$ iff there is a commuting diagram

$$\begin{tikzcd}
\mathbf{y}(m) \arrow[d, dotted] \arrow[dr, "\hat{u}"] & \\
\llbracket \phi \rrbracket \arrow[r, hook, "\iota"'] & \llbracket X \rrbracket
\end{tikzcd}$$

which is equivalent to saying that $u \in \llbracket \phi \rrbracket(m)$.

In subobject semantics the definition of $\llbracket p(t(X)) \rrbracket$ is the pullback:

$$\begin{tikzcd}
\llbracket p(t(X)) \rrbracket \arrow[r, hook] \arrow[d] & X \arrow[d, "\llbracket t \rrbracket"] \\
\llbracket p \rrbracket \arrow[r, hook] & A.
\end{tikzcd}$$

This is all standard. Now $(X)u \mid m \models p(t(X))$ iff there exists an arrow from $\mathbf{y}m$ to $\llbracket p(t(X)) \rrbracket$ making the triangle

$$\begin{tikzcd}
\mathbf{y}m \arrow[d, dotted] \arrow[dr, "u"] & \\
\llbracket p(t(X)) \rrbracket \arrow[r, hook] & X
\end{tikzcd}$$

commute. This is true iff (using the pullback property) there exists an arrow from $\mathbf{y}m$ to $\llbracket p \rrbracket$ making

$$\begin{tikzcd}
\mathbf{y}m \arrow[r, "u"] \arrow[d] & X \arrow[d, "\llbracket t \rrbracket"] \\
\llbracket p \rrbracket \arrow[r, hook] & A
\end{tikzcd}$$

commute, which again is the same as saying $\llbracket t \rrbracket_m(\check{u}) \in \llbracket p \rrbracket(m)$. The latter is Pym's definition of $(X)u \mid m \models p(t(X))$ (Pym p.183), so *assuming* interpretation of terms as morphism as described above, and *assuming* that interpretation of the bunch of free variables X is the interpretation of the type of X, Pym's semantics correspond to the standard (for the intuitionistic part, that is).

Definition A.2.1. *For $\phi \hookrightarrow X$ and $\psi \hookrightarrow Y$, $\phi * \psi \hookrightarrow X \otimes Y$ is defined as the image of ϕ and ψ:*

$$\begin{tikzcd}
\phi \otimes \psi \arrow[rr, "\iota\otimes\iota"] \arrow[dr, two heads, "\iota\otimes\iota"'] & & X \otimes Y \\
& \phi * \psi \arrow[ur, hook] &
\end{tikzcd}$$

Lemma A.2.2. *For* $\phi \hookrightarrow X$ *and* $\psi \hookrightarrow Y$*,* $(X,Y)u \mid m \models \phi * \psi$ *iff for* some (u_x, u_y, n, n') *such that* $[\check{u}_x, \check{u}_y, m \leq n \cdot n'] = \check{u}$ *in* $X \otimes Y$*, we have* $(X)u_x \mid n \models \phi$ *and* $(Y)u_y \mid n' \models \psi$*.*

Proof: Note that the equivalence relation used on $\phi * \psi$ is that of $X \otimes Y$ while it is possible that $s \neq t$ in $\phi \otimes \psi$ even if $s = t$ in $X \otimes Y$ (see Proposition 4.2.13). $\iota \otimes \iota$ works as a point-wise inclusion, i.e., $(\iota \otimes \iota)[u_x, u_y, m \leq l \cdot l'] = [\iota(u_x), \iota(u_y), m \leq l \cdot l']$. Suppose $(X,Y)u \mid m \models \phi * \psi$ that is $\check{u} \in \phi * \psi$. Then there exists some $s = [s_1, s_2, m \leq n \cdot n'] \in \phi \otimes \psi(m)$ such that $(\iota \otimes \iota)(s) = u$, i.e., $s = u$ in $\phi * \psi(m)$ and in $X \otimes Y(m)$ (but not necessarily in $\phi \otimes \psi(m)$, we do not know if $u_x \in \phi$ and $u_y \in \psi$). Since by definition $s_1 \in \phi(n)$ and $s_2 \in \psi(n')$ we can use Day's pairing to get an element $s' = [s_1, s_2, n \cdot n' = n \cdot n'] \in \phi \otimes \psi(n \cdot n')$, so at stage or world n we have an element $s_1 \in \phi(n) \subseteq X(n)$, i.e., $(X)s_1 \mid n \models \phi$ and also $(Y)s_2 \mid n' \models \psi$.

And we have a commuting diagram

$$\begin{tikzcd}
\mathbf{y}m \arrow[rr, "\hat{u}=\hat{s}"] \arrow[dr] & & X \otimes Y \\
& \mathbf{y}(n \cdot n') \arrow[ur, "\hat{s'}=s_1\otimes s_2"'] &
\end{tikzcd}$$

which gives the other direction.

□

This shows that the subobject $[\![\phi * \psi]\!]$ defined above matches Pym's Kripke semantics for the $*$.

To make any statements about soundness we need to be able to interpret sequences or relative truth, $(X)\phi \vdash \psi$, which is defined in [Pym02, p.184] as

$$(X)\phi \models \psi \quad \text{iff for all } m \in \mathcal{M} \text{ and all } u \in [\![X_1]\!](m),$$

$$(X_1)u \mid m \models \phi \qquad \text{implies} \qquad (X_2)u \mid m \models \psi,$$

where $X = X_1, X_2$ or $X = X_1; X_2$ and $Axiom(X_1, X_2)$. Assuming that $Axiom(X_1, X_2)$ ensures that the interpretation of the bunches X_1, X_2 satisfies $[\![X_1]\!] = [\![X_2]\!]$, the definition makes sense. It corresponds to the subobject semantics where we have to make sure that $[\![\phi]\!]$ and $[\![\psi]\!]$ are interpreted in the same subobject lattice (if we have weakening this is never a problem since we can always add dummy variables).

In particular, to be able to interpret a sequence $(X)\phi \vdash \psi$, we must have $X = X_1, X_2$ or $X = X_1; X_2$ and $Axiom(X_1, X_2)$ and $(X_1) \vdash \phi : \mathrm{Prop}$ and $(X_2) \vdash \psi : \mathrm{Prop}$, but we have already shown that A.1 can not hold, so in many cases we will not be able to interpret sequences. In fact, whenever we apply the rule $-\!\!* \, I$ the resulting sequence will fail to satisfy the above condition.

This ought to be convincing evidence that the system "predicate NBI" in [Pym02, p.166 table 12.2], and the predicate Kripke semantics p.183, Table 13.1, proposed by Pym is not sound (or, if the partially definedness of the interpretation function is to be taken literally, it is almost nowhere defined).

A.2.1 Substitution and soundness

We now make some observations assuming that the definition of $\phi[t/x]$ in the subobject semantics is the pullback

$$\begin{tikzcd}
\llbracket \phi[t/x] \rrbracket \arrow[r, tail] \arrow[d] & X \otimes Y \arrow[d, "\llbracket t \rrbracket \otimes Y"] \\
\llbracket \phi \rrbracket \arrow[r, tail] & A \otimes Y
\end{tikzcd}$$

where $X \vdash t : A$, $(x : A, Y) \vdash \phi : \mathrm{Prop}$ and t is substitutable for x in ϕ.

Since a variable can only occur once in a bunch, we have, syntactically, $(\phi * \psi)[t/x] = \phi[t/x] * \psi$ (assuming of course it occurs in ϕ), so for the above definition of $\llbracket \phi[t/x] \rrbracket$ to be well-defined we require (among other things) that $\llbracket (\phi * \psi)[t/x] \rrbracket = \llbracket \phi[t/x] * \psi \rrbracket$, i.e., that the following square is pullback

$$\begin{tikzcd}
\llbracket \phi[t/x] * \psi \rrbracket \arrow[r, tail] \arrow[d] & X \otimes Y \arrow[d, "\llbracket t \rrbracket \otimes Y"] \\
\llbracket \phi * \psi \rrbracket \arrow[r, tail] & A \otimes Y
\end{tikzcd}$$

where we can assume (by induction) that

$$\begin{tikzcd}
\llbracket \phi[t/x] \rrbracket \arrow[r, tail] \arrow[d] & X \arrow[d, "\llbracket t \rrbracket"] \\
\llbracket \phi \rrbracket \arrow[r, tail] & A
\end{tikzcd}$$

is pullback. Conclusion: since $\otimes$ does not preserve pullback by Prop. 4.2.13, it is hard to see how we can proof soundness.

Affine models. Recall that an affine model is a model where the unit of $\times$ (the terminal object) coincides with the unit of $\otimes$ such that weakening is allowed for the multiplicative part as well as for the additive part. For these models we are able to interpret all sequences since we can add dummy variables on each side as usually.

However, the failure of pullback preservation by $\otimes$ implies that even for affine models we can not prove soundness (in the usual inductive way, at least).

Remark A.2.3. *The clause*

$$\begin{array}{lll} (X)u \,|\, m \models \exists_{new} x : A.\phi & \textit{iff} & \textit{for some } n \textit{ and some } v : \mathbf{y}(n) \to \llbracket A \rrbracket \\ & & (X, x : A)u \otimes v \,|\, m \cdot n \models \phi \end{array}$$

of Pym's Kripke semantics corresponds to a subobject in the following way: Suppose $\llbracket \phi \rrbracket \rightarrowtail \llbracket X \rrbracket \otimes \llbracket A \rrbracket$. *Consider the arrows* $\pi, \sigma : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, *where* $\sigma(a, b) = a \cdot b$. *Both* π *and* σ *defines functors* $\widehat{\mathcal{C}} \to \widehat{\mathcal{C} \times \mathcal{C}}$ *by precomposing and each of these has a left adjoint* Σ_π *and* Σ_σ *and a right adjoint. Moreover, for* $X, A \in \widehat{\mathcal{C}}$,

$$\Sigma_\sigma X\pi \times A\pi' = X \otimes A,$$

Day's tensor (this is already noted in Remark 4.2.14), and we have the unit (Day's pairing)

$$X\pi \times A\pi' \to (X \otimes A)\sigma.$$

Finally, we have the identity (using density)

$$\begin{aligned}\Sigma_\pi X\pi \times A\pi' &= \int^{n,n'} Xn \times An' \times \mathcal{C}(-,n)\\ &= X \times \int^{n'} An',\end{aligned}$$

showing that there is a projection

$$\Pi : \Sigma_\pi X\pi \times A\pi' \to X$$

in $\widehat{\mathcal{C}}$. *As usual this defines a functor* $\Pi^* : \mathrm{Sub}(X) \to \mathrm{Sub}(\Sigma_\pi X\pi \times A\pi')$ *by pullback, and* Π^* *has a left adjoint* $\exists_\Pi$ *and a right adjoint* $\forall_\Pi$.

Take the pullback of $\phi\sigma$ *along the unit (Day's pairing) to get the pullback diagram (omitting the* $[\![-]\!]$*)*

$$\begin{tikzcd}
\hat{\phi} \arrow[r] \arrow[d] & X\pi \times A\pi' \arrow[d] \\
\phi\sigma \arrow[r] & (X \otimes A)\sigma
\end{tikzcd}$$

then use the functor Σ_π *on this diagram to get*

$$\begin{tikzcd}
 & \overline{\phi} \arrow[d] & \\
\Sigma_\pi(\hat{\phi}) \arrow[ur, two heads] \arrow[r] \arrow[d] & \Sigma_\pi(X\pi \times A\pi') \arrow[r, "\Pi"] \arrow[d] & X \\
\Sigma_\pi(\phi)\sigma \arrow[r] & \Sigma_\pi(X \otimes A)\sigma &
\end{tikzcd}$$

where $\overline{\phi}$ *is the image factorization. Then it can be shown that* $[\![\exists_{new} x : A.\phi]\!] = \exists_\Pi \overline{\phi}$, *which is the image factorization of the arrow* $\Sigma_\pi(\hat{\phi}) \to \Sigma_\pi(X\pi \times A\pi') \to X$.

Appendix B

Notation

- Arrows: $\rightarrowtail$: mono; $\twoheadrightarrow$: epi; $\hookrightarrow$: inclusion.
- 1_C is the unique arrow from the object C to the terminal object 1.
- If F is a bifunctor, $F(f, V)$ means $F(f, \mathrm{id}_V)$.
- $\alpha : F \Rightarrow G$ reads: α is a natural transformation from F to G.
- $\alpha : F \overset{..}{\rightarrow} G$ reads: α is a dinatural transformation from F to G.
- $f_{a \to b}$ just indicates that f is an arrow from a to b.
- $\widehat{\mathcal{C}}$ is the category $\mathrm{Set}^{\mathcal{C}^{op}}$.
- For a presheaf $F : \mathcal{M}^{op} \to \mathrm{Set}$ over a preorder $\mathcal{M}$, and $n \leq m \in \mathcal{M}$, $F_{nm} = F(n \leq m) : F(m) \to F(n)$.
- If $u \in \widehat{C}(\mathbf{y}C, F)$ then $\check{u}$ is the corresponding element in $F(C)$, via the Yoneda Lemma.
- If $u \in F(C)$, for a presheaf $F \in \mathrm{Obj}(\widehat{C})$, then $\hat{u} : \mathbf{y}C \Rightarrow F$ denotes the corresponding natural transformation.
- If $f : X \to Y$ is an arrow, $\mathrm{Im}(f)$ denotes the object which the epi-mono factorization of f factors through.
- For a functor $F : \mathcal{C} \to \mathrm{Set}$, an arrow $f : D \to C$, and an element $x \in FC$, $x \upharpoonright f$ denotes $F(f)(x)$.

Abbreviations

- ccc: Cartesian closed category.
- smcc: symmetric monoidal closed category (see 4.1.3).
- DCC: doubly closed category (see 4.1.4).
- CDCC: Cartesian doubly closed category (see 4.1.4).
- bi-CDCC: bi-Cartesian doubly closed category (see 4.1.4).
- cBIa: complete BI algebra (see 4.4.2).

Bibliography

[Amb91] Simon John Ambler. First order linear logic. 1991.

[Cac03] Mario J. Cáccamo. A formal calculus for categories. 2003.

[Jac99] Bart Jacobs. *Categorical logic and type theory*, volume 141 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[LS86] J. Lambek and P. J. Scott. *Introduction to higher order categorical logic*, volume 7 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[MLM94] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.

[O'H03] Peter O'Hearn. On bunched typing. *J. Funct. Programming*, 13(4):747–796, 2003.

[Oos] J. V. Oosten. Basic category theory.

[Pit02] Andrew M. Pitts. Tripos theory in retrospect. *Math. Structures Comput. Sci.*, 12(3):265–279, 2002. Realizability (Trento, 1999).

[Pym02] David J. Pym. *The Semantics and Proof Theory of the Logic of Bunched Implications*. Kluwer Academic Publishers, 2002.

[Pym04] David J. Pym. Errata and remarks for the semantics and proof theory of the logic of bunched implications. 2004.

[Rey02] John C. Reynolds. Separation logic: A logic for shared mutable data structures. 2002.

[Rey04] L. Birkedal & N. Torp-Smith & J. C. Reynolds. Local reasoning about a copying garbage collector. *Proceedings of the 31-st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL)*, pages 220–231, 2004.

[Win01] M. Cáccamo & J. M. E. Hyland & G. Winskel. Lecture notes in category theory. 2001.

[Yan96] Hongseok Yang. Local reasoning for stateful programs (thesis). 1996.

[Yan02] David J. Pym & Peter W. OHearn & Hongseok Yang. Possible worlds and resources. 2002.