Holomorphic symplectic geometry

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Why is it interesting?

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Thus holomorphic symplectic manifolds (also called hyperkähler) are building blocks for manifolds with K trivial, which are themselves building blocks in the classification of projective (or compact Kähler) manifolds.

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- dim > 2? Idea: take S^r for S K3. Many symplectic forms:

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  φ invariant under $\mathfrak{S}_r$, i.e. φ comes from $S^{(r)} := S^r/\mathfrak{S}_r =$ {subsets of r points of S, counted with multiplicities}

- $S^{(r)}$ is singular, but admits a natural desingularization $S^{[r]} :=$ {finite analytic subspaces of S of length r} (Hilbert scheme)

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Example: $V \subset \mathbb{P}^5$ cubic fourfold. $F(V) := \{\text{lines contained in } V\}$ is holomorphic symplectic, deformation of $S^{[2]}$ with S K3.

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(Beware that $\mathcal{M}_L$ is non Hausdorff in general.)

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Gives very precise information on the structure of $\mathcal{M}_L$ and the geometry of X.

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$\rightsquigarrow$ explicit solutions of the ODE X_{h_i} (e.g. in terms of θ functions): "algebraically completely integrable system". Classical examples: geodesics of the ellipsoid, Lagrange and Kovalevskaya tops, etc.

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Is there a simple characterization of Lagrangian fibration?

Conjecture

$\exists\ X \dashrightarrow \mathbb{P}^r$ Lagrangian $\iff \exists\, L$ on X, $q(c_1(L)) = 0$.

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a classical result of Kummer.

II. Contact geometry

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Again the definition makes sense in the holomorphic set-up $\rightsquigarrow$ holomorphic contact manifold. We will be looking for *projective* contact manifolds.

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$\big(\Rightarrow$ classical conjecture in Riemannian geometry: classification of compact quaternion-Kähler manifolds (LeBrun, Salamon).$\big)$

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Again this makes sense for X complex manifold, τ holomorphic.

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5. If X is Poisson, any $X \times Y$ is Poisson.

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