140 If f is an isomorphism we call f an *isometry*. If $q_i = q_{B_i}$ then we define $q_1 \perp q_2 = q_{B_1 \perp B_2}$ on $P_1 \oplus P_2$, and $q_1 \otimes q_2 = q_{B_1 \otimes B_2}$ on $P_1 \otimes P_2$. It is easily checked that these definition are unambiguous. ◻

2 The hyperbolic functor

Let P be a k-module and define

$$B_0^P \in \mathrm{Bil}((P \oplus P^*) \times (P \oplus P^*)) \text{ by } B_0^P((x_1, y_1), (x_2, y_2)) = \langle y_1, x_2 \rangle_P,$$

and let $q^P = q_{B_0^P}$ be the induced quadratic form:

$$q^P(x, y) = \langle y, x \rangle_P \qquad (x \in P, y \in P^*).$$

Let $B^P = B_0^P + (B_0^P)^*$ be the associated bilinear form, $B^P = B_{q^P}$. Then

$$B^P((x_1, y_1), (x_2, y_2)) = \langle y_1, x_2 \rangle_P + \langle y_2, x_1 \rangle_P.$$

If $d_P : P \to P^{**}$ is the natural map then it is easily checked that

$$d_{B^P} : P \oplus P^* \to (P \oplus P^*)^* = P^* \oplus P^{**}$$

is represented by the matrix

$$\begin{pmatrix} 0 & 1_{P^*} \\ d_P & 0 \end{pmatrix}.$$

Consequently, B^P *is non-singular if and only if P is reflexive*. If, in this case, we identify $P = P^{**}$ then the matrix above becomes $\left(\begin{smallmatrix} 0 & 1_{P^*} \\ 1_P & 0 \end{smallmatrix}\right)$.

We will write 141

$$\mathbb{H}(P) = (P \oplus P^*, q^P)$$

and call this quadratic module the *hyperbolic form* on P.

Suppose $f : P \to Q$ is an isomorphism of k-modules. Define

$$\mathbb{H}(f) = f \oplus (f^*)^{-1} : \mathbb{H}(P) \to \mathbb{H}(Q).$$

$$q^Q(\mathbb{H}(f)(x, y)) = q^Q(fx, (f^*)^{-1}y) = \langle (f^{-1})^* y, fx \rangle_Q$$

$$= \langle y, f^{-1} f x\rangle_P = q^P(x, y), \text{ so } \mathbb{H}(f) \text{ is an isometry.}$$

If we identify $(P_1 \oplus P_2)^* = P_1^* \oplus P_2^*$ so that

$$\langle (y_1, y_2), (x_1, x_2)\rangle_{P_1 \oplus P_2} = \langle y_1, x_1\rangle_{P_1} + \langle y_2, x_2\rangle_{P_2}$$

then the natural homomorphism

$$f : \mathbb{H}(P_1) \perp \mathbb{H}(P_2) \to \mathbb{H}(P_1 \oplus P_2),$$

$f((x_1, y_1), (x_2, y_2)) = ((x_1, x_2), (y_1, y_2))$. is an isometry.

Summarizing the above remarks, $\mathbb{H}$ *is a product preserving functor* (in the sense of chapter 1) *from (modules, isomorphisms*, $\oplus$) to (*quadratic modules, isometries,* $\perp$). We now characterize non-singular hyperbolic forms.

Lemma 2.1. *A non-singular quadratic module* (P, q) *is hyperbolic if and only if* P *has a direct summand* U *such that* $q|U = 0$ *and* $U = U^\perp$. *In this case* $(P, q) \approx \mathbb{H}(U)$ *(isometry).*

142 *Suppose* P *is finitely generated and projective. If* U *is a direct summand such that* $q|U = 0$ *and* $[P : k] \leq 2[U : k]$ *then* $(P, q) \approx \mathbb{H}(U)$.

Proof. If $(P, q) \approx \mathbb{H}(U) = (U \oplus U^*, q^U)$ then the non-singularity of (P, q) implies U is reflexive, and it is easy to check that $U \subset U \oplus U^*$ satisfies $q^U|U = 0$ and $U = U^\perp$. □

Conversely, suppose given a direct summand U of P such that $q|U = 0$ and $U = U^\perp$. Write $q = q_{B_0}$, so that $B_q = B_0 + B_0^*$. According to Lemma 1.4 we can write $P = U^\perp \oplus V = U \oplus V$ and B_q induces a non-singular pairing on $U \times V$. Moreover we can arrange that $B_0(v, v) = 0$ for all $v \in V$, i.e. that $q|V = 0$. Let $d : V \to U^*$ be the isomorphism induced by B_q; $\langle dv, u\rangle_U = B_q(v, u)$ for $u \in U$, $v \in V$.

Let

$$f = 1_U \oplus d : P = U \oplus V \to U \oplus U^*.$$

This is an isomorphism, and we want to check that

$$q^U((u, dv)) = q(u, v) \text{ for } u \epsilon U, v \epsilon V. q^U((u, dv)) = \langle dv, u\rangle_U = B_q(v, u),$$

while $q(u, v) = q(u) + q(v) + B_q(u, v) = B_q(u, v)$, since $q/U = 0$ and $q/V = 0$.

The last assertion reduces to the preceding ones we show that $U = U^\perp$. Lemma 1.2 shows that $U^\perp$ is a direct summand of rank $[U^\perp : k] = [P : k] - [U : k] \leq [U : k]$, because, by assumption, $[P : k] \leq 2[U : k]$. But we also have $q/U = 0$ so $U \subset U^\perp$, and therefore $U = U^\perp$, as claimed.

Lemma 2.2. *A quadratic module (P, q) is non-singular if and only if*

$$(P, q) \perp (P, -q) \approx \mathbb{H}(P),$$

provided P is reflexive.

Proof. P reflexive implies $\mathbb{H}(P)$ is non-singular, and hence likewise for **143**
any orthogonal summand.

Suppose now that (P, q) is non-singular. Then so is $(P, q) \perp (P, -q) = (P \oplus P, q_1 = q \perp (-q))$.

Let $U = \{(x, x) \epsilon P \oplus P | x \in P\}$. Then $q_1/U = 0$, and U is a direct summand of $P \oplus P$, isomorphic to P. If $U \subsetneq U^\perp$ we can find a $(0, y) \epsilon U^\perp$, $y \neq 0$. Then, for all $x \in P$,

$$\begin{aligned} 0 &= B_{q_1}((x, x), (0, y)) = q_1(x, x + y) - q_1(x, x) - q_1(0, y) \\ &= q(x) - q(x + y) + q(y) \\ &= -B_q(x, y). \end{aligned}$$

Since B_q is non-singular this contradicts $y \neq 0$. Now the Lemma follows from Lemma 2.1. □

Lemma 2.3. *Let P be a reflexive module and let (Q, q) be a non-singular quadratic module with Q finitely generated and projective. Then*

$$\mathbb{H}(P) \otimes (Q, q) \approx \mathbb{H}(P \otimes Q).$$

Proof. The hypothesis on Q permits us to identify $(P \otimes Q)^* = P^* \otimes Q^*$, so it follows that $(W, q_1) = \mathbb{H}(P) \otimes (Q, q)$ is non-singular. We shall apply Lemma 2.1 by taking

$U = P \otimes Q \subset W = (P \otimes Q) \oplus (P^* \otimes Q)$. If $\sum x_i \otimes y_i \epsilon U$, then $q_1(\Sigma x_i \otimes y_i) = \Sigma q^P(x_i) q(y_i) + \sum_{i<j} B_{q_1}(x_i \otimes y_i, x_j \otimes y_j) = \sum_{i<j} B^P(x_i, x_j) B_q(y_i, y_j) = 0$, because $q^P/P = 0$ in $\mathbb{H}(P)$. Thus $U \subset U^\perp$, and to show equality it suffices clearly to show that $(P^* \otimes Q) \cap U^\perp = 0$. If $\Sigma x_i \otimes y_i \in U$ and $\Sigma w_j \otimes z_j \in (P^* \otimes Q) \cap U^\perp$ then $0 = B_{q_1}(\Sigma x_i \otimes y_i, \Sigma w_j \otimes z_j) = \sum_{i,j} B^P(x_i, w_j) B_q(y_i, z_j)$. □

144 Since $(P^* \otimes Q)^* = P \otimes Q^*$ (P is reflexive) the non-singularity of q guarantees that all linear functionals on $P^* \otimes Q$ have the form $\sum_i B^P(x_i, \ ) B_q(y_i, \ )$, so $\Sigma w_j \otimes z_j$ is killed by all linear functionals, hence is zero. We have now shown $U = U^\perp$ so the lemma follows from Lemma 2.1.

A *quadratic space* is a non-singular quadratic module (P, q) with P finitely generated and projective, i.e. $P \epsilon \operatorname{obj} \underline{\underline{P}}$, the category of such modules. We define the category

$$\underline{\underline{\text{Quad}}} = \underline{\underline{\text{Quad}}}(k)$$

with

objects : quadratic spaces
morphisms : isometries
product : $\perp$

The discussion at the beginning of this section shows that

$$\mathbb{H} : \underline{\underline{P}} \to \underline{\underline{\text{Quad}}}$$

is a product preserving functor of categories with product (in the sense of chapter 1), and Lemma 2.1 shows that $\mathbb{H}$ is cofinal. We thus obtain an exact sequence from Theorem 4.6 of chapter 1. We summarize this:

Proposition 2.4. *The hyperbolic functor*

$$\mathbb{H} : \underline{\underline{P}} \to \underline{\underline{\text{Quad}}}$$

is a cofinal functor of categories with product. It therefore induces (Theorem 4.6 of chapter 1) an exact sequence

$$K_1 \underline{\underline{P}} \to K_1 \underline{\underline{\text{Quad}}} \to K_0 \Phi \mathbb{H} \to K_0 \underline{\underline{P}} \to K_0 \underline{\underline{\text{Quad}}} \to \text{Witt}(k) \to 0,$$

where we define Witt (k) = *coker* $(K_0\mathbb{H})$.

We close this section with some remarks about the multiplicative 145
structures. Tensor products endow K_0Quad with a commutative multiplication, and Lemma 2.3 shows that the image of $K_0\mathbb{H}$ is an ideal, so Witt (k) also inherits a multiplication. The difficulty is that, if 2 is not invertible in k, then these are rings without identity elements. For the identity should be represented by the form $q(x) = x^2$ on k. But then $B_q(x, y) = 2xy$ is not non-singular unless 2 is invertible.

Here is one natural remedy. Let Symbil denote the category of non-singular symmetric bilinear forms, (P, B) with $P \epsilon$ obj P. If $(P, B) \in$ Symbil and $(Q, q) \epsilon$ Quad define

$$(P, B) \otimes (Q, q) = (P \otimes Q, B \otimes q), \tag{2.5}$$

where $B \otimes q$ is the quadratic form $q_{B \otimes B_0}$, for some $B_0 \epsilon\, \mathrm{Bil}(Q \times Q)$ such that $q = q_{B_0}$. It is easy to see that $B \otimes q$ does not depend on the choice of B_0. Moreover, the bilinear form associated to $B \otimes q$ is $(B \otimes B_0) + (B \otimes B_0)^* = (B \otimes B_0) + (B^* \otimes B_0^*) = B \otimes (B_0 \otimes B_0^*) = B \otimes B_q$, because $B = B^*$. Since B and B_q are non-singular so is $B \otimes B_q$ so $(P \otimes Q, B \otimes q) \in$ Quad.

If $a \epsilon k$ write $\langle a \rangle$ for the bilinear module (k, B) with $B(x, y) = axy$ for $x, y \epsilon k$. If a is a unit then $\langle a \rangle \epsilon$Symbil.

Tensor products in Symbil make K_0Symbil a commutative ring, with 146
identity $\langle 1 \rangle$, and (2.5) makes K_0Quad a K_0Symbil−module. The "forgetful" functor Quad → Symbil, $(P, q) \longmapsto (P, B_q)$, induces a K_0 Symbil-homomorphism K_0Quad → K_0Symbil, so its image is an ideal. The hyperbolic forms generate a K_0Symbil submodule, image $K_0\mathbb{H}$), of K_0Quad, so Witt (k) is a K_0Symbil-module. This follows from an analogue of Lemma 2.3 for the operation (2.5)

Similarly, the hyperbolic forms, $(P \oplus P^*, B^P)$, generate an ideal in K_0Symbil which annihilates $Witt(k)$. Lemma 2.2 says that $\langle 1 \rangle \perp \langle -1 \rangle$ also annihilates Witt (k).