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Shukla Cohomology and Triples*

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1. INTRODUCTION

The purpose of this paper is to compare the cohomology theory of associative K-algebras given by Shukla [7] with the cotriple cohomology induced by the free associative K-algebra triple on the category of sets. If the respective cohomology groups are denoted by $H^n(\Lambda, M)$ and $\tilde{H}^n(\Lambda, M)$ for a K-algebra Λ and a Λ-bimodule M, then we will show

THEOREM 1.1. *There is a natural family of isomorphisms*

$$\tilde{H}^n(\Lambda, M) \xrightarrow{\approx} \begin{cases} \mathrm{Der}(\Lambda, M), & n = 0, \\ H^{n+1}(\Lambda, M), & n > 0. \end{cases}$$

Thus the comparison ends in a manner fully as satisfactory as in [1] and [2]. The proof uses essentially the same method of acyclic models as was used in those papers. A standard (i.e., functorial) complex for each theory is constructed and the hypotheses of the theorem of acyclic models of [2] are shown to be satisfied, from which chain equivalence of these complexes follows. Here the standard complexes are not generally acyclic. Thus, chain equivalence as modules is required, in contrast to the previous situations in which it would have sufficed to show that a certain complex was merely acyclic.

The following notations will be used throughout. K denotes a commutative ring with unit, Λ is a fixed unitary K-algebra and M is a fixed Λ-bimodule. A $\otimes$ without subscript denotes $\otimes_K$ and if A is any K-module we let $A^{(n)}$ denote an nth tensor power of A over K. If B is the free R-module on the base X, for some ring R, we will let $\langle x \rangle$ denote the basis element corresponding to an $x \in X$. We let (Alg $-K, \Lambda$) denote the category whose

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objects are morphisms $\Gamma \xrightarrow{\gamma} \Lambda$ of K-algebras (always unit-preserving) and whose morphisms are commutative triangles $\Lambda \xleftarrow{\gamma} \Gamma \xrightarrow{\varphi} \Gamma' \xrightarrow{\gamma'} \Lambda$ where φ is a map of K-algebras. The elements of this category will be called algebras and morphisms *over* Λ. If $U\Lambda$ denotes the underlying set of Λ, we let $(\mathscr{S}, U\Lambda)$ denote the category of sets and functions over $U\Lambda$ in the same way. Then there is an obvious underlying functor, which we also denote by $U : (\text{Alg} -K, \Lambda) \to (\mathscr{S}, U\Lambda)$, taking $\Gamma \xrightarrow{\gamma} \Lambda$ to the underlying set of Γ mapped by the function underlying γ. U has a left adjoint,

$$F : (\mathscr{S}, U\Lambda) \to (\text{Alg} -K, \Lambda),$$

which can be described by saying that if $X \xrightarrow{\xi} U\Lambda$ is an object over $U\Lambda$, FX is the polynomial ring over K in noncommuting variables $[x]$, one for each $x \in X$. $F\xi : FX \to \Lambda$ is the unique K-algebra morphism for which $F\xi[x] = \xi x$.

2. Cohomology and Acyclic Models

If F and U are as above, let $\alpha : 1 \to UF$ and $\beta : FU \to 1$ be the adjointness morphisms. Then it is known that $\mathbf{G} = (FU, \beta, F\alpha U)$ is a cotriple on $(\text{Alg} -K, \Lambda)$ (see [*4*] and [*5*]). If $T : (\text{Alg} -K, \Lambda) \to \mathscr{A}$ is a functor to an Abelian category and $G = FU$, the left-derived functors $L_n T$ with respect to $\mathbf{G}$ are the homology groups of the complex

$$\cdots \to TG^{n+1} \to \cdots \to TG^3 \to TG^2 \to TG \to 0$$

whose boundary is $\sum (-1)^i \, TG^i\beta G^{n-i} : TG^{n+1} \to TG^n$. If T is contravariant, we get right-derived functors $R_n T$ in the analogous way. To define cohomology with coefficients in the Λ-bimodule M, we take $T_M : (\text{Alg} -K, \Lambda) \to \mathscr{A}\mathscr{b}$, the category of Abelian groups, to be the contravariant functor whose value on an object $\Gamma \xrightarrow{\gamma} \Lambda$ (which will by abuse of notation simply be denoted by Γ) is

$$\text{Der}(\Gamma, M) = \{f : \Gamma \to M \mid f(x_1 x_2) = \gamma x_1 \cdot f x_2 + f x_1 \cdot \gamma x_2\}.$$

This is a group using addition in M and is called the group of derivations (or crossed homomorphisms) of Γ to M. Then $\tilde{H}^n(\Gamma, M) = R_n T_M \Gamma$.

Der(Γ, —) is represented, as a functor on the category of Λ-bimodules, by the module $J\Gamma = \text{coker}\, \varphi$ where $\varphi : \Lambda \otimes \Gamma \otimes \Gamma \otimes \Lambda \to \Lambda \otimes \Gamma \otimes \Lambda$ is defined by

$$\varphi(l \otimes x \otimes x' \otimes l') = l\gamma(x) \otimes x' \otimes l' - l \otimes xx' \otimes l' + l \otimes x \otimes \gamma(x')\, l'$$

for $l, l' \in \Lambda$, $x, x' \in \Gamma$ (see [*3*]). Hence as a standard complex for computing the cohomology given above we may use the complex of functors $Q = \{Q_n\}$ where $Q_n \Gamma = J G^{n+1} \Gamma$ with boundary as above. In Section 3 we will construct a standard complex E for computing the modified Shukla cohomology (as appears in Theorem 1.1). To show that these are equivalent we use the theorem of acyclic models which appears in dual form in [2]. The form we shall use states

THEOREM 2.1. *Suppose that* $H_0(E) \approx H_0(Q)$ *and that there are natural transformations of* $E_n \to E_nG$ *and* $Q_n \to Q_nG$ *for* $n \geqslant 0$ *such that* $E_n \to E_nG \xrightarrow{E_n\beta} E_n$ *and* $Q_n \to Q_nG \xrightarrow{Q_n\beta} Q_n$ *are the respective identities and that both augmented complexes* $EG \to H_0(EG) \to 0$ *and* $QG \to H_0(QG) \to 0$ *are contractible by natural transformations. Then there are natural transformations* $f : E \to Q$ *and* $g : Q \to E$, *and natural homotopies* $fg \sim 1_Q$ *and* $gf \sim 1_E$.

For $Q_n \to Q_nG$ we may take $JF\alpha UG^n$ for $n \geqslant 0$. It is easily seen that $H_0(Q) \approx J$ so that we may take $s_{-1} : H_0(QG) \to Q_0G$ to be $JF\alpha U$ and $s_n : Q_nG \to Q_{n+1}G$ to be $JF\alpha UG^{n+1}$ for $n \geqslant 0$. Using the naturality of α, this is easily checked to be a contraction. To complete the proof of Theorem 1.1 it is necessary to define E and show it has similar properties (see Section 3).

3. THE COHOMOLOGY GROUPS OF SHUKLA

These are described in detail in [7], but we give here a brief description which is suitable for our purposes. Given $\Gamma \to \Lambda$ we will describe a standard complex $S\Gamma$ for computing Shukla's cohomology $H(\Gamma, M)$ with coefficients in the Λ-bimodule M. After adjusting this complex at the bottom we will be in a position to apply acyclic models and compare this with $Q\Gamma$ described above.

We begin by letting $V_{-1}\Gamma = N_{-1}\Gamma = \Gamma$. If $V_i\Gamma$ is defined to be the free K-module generated by the underlying set of $N_{i-1}\Gamma$, we have a natural epimorphism $\epsilon_i : V_i\Gamma \to N_{i-1}\Gamma$ whose kernel we define to be $N_i\Gamma$. It is shown in [7] how the terms of $V\Gamma$ of non-negative degree form a differential graded algebra in which the differential d is the composite $V_i\Gamma \xrightarrow{\epsilon_i} N_{i-1}\Gamma \xrightarrow{\subset} V_{i-1}\Gamma$ and such that $\epsilon_0 : V_0\Gamma \to \Gamma$ induces a map $\epsilon : V\Gamma \to \Gamma$ of such algebras (where Γ has trivial differential and grading). For example, the multiplication in $V_0\Gamma$ is the unique K-linear product for which $\langle x\rangle \langle y\rangle = \langle xy\rangle$, $x, y \in \Gamma$. Also, the composite $\gamma\epsilon$ makes $V\Gamma$ into an algebra over Λ, from which it follows that ϵ is a map over Λ. We now form the Λ-bimodule $S\Gamma = \sum_{n\geqslant 0} S_n\Gamma$ where $S_n\Gamma = \Lambda \otimes (V\Gamma)^{(n)} \otimes \Lambda$. If this is given suitable differential and grading, it is clear from the discussion between Theorems 2 and 3 ([7], p. 178) that $H(\mathrm{Hom}_{\Lambda-\Lambda}(S\Gamma, M))$ is the Shukla cohomology of Γ with coefficients in the Λ-bimodule M. The differential comes from the usual one for the bar resolution of a differential graded algebra as described in [6], p. 306. That is, $\partial = \partial S = \partial' + \partial''$ where, if $\exp(m) = (-1)^m$ for an integer m, $l, l' \in \Lambda$, $v_1, \ldots, v_n \in V\Gamma$, then

$$\begin{aligned}
&\partial'(l \otimes v_1 \otimes \cdots \otimes v_n \otimes l') \\
&\quad = l\gamma\epsilon(v_1) \otimes v_2 \otimes \cdots \otimes v_n \otimes l' \\
&\qquad + \sum_{i=1}^{n-1} \exp(i + \deg v_1 + \cdots + \deg v_i)\, l \otimes v_1 \otimes \cdots \\
&\qquad \otimes v_i v_{i+1} \otimes \cdots \otimes v_n \otimes l' \\
&\qquad + \exp(n + \deg v_1 + \cdots + \deg v_n)\, l \otimes v_1 \otimes \cdots \\
&\qquad \otimes v_{n-1} \otimes \gamma\epsilon(v_n)\, l' \qquad \text{and} \qquad \partial''(l \otimes v_1 \otimes \cdots \otimes v_n \otimes l') \\
&\quad = \sum_{i=1}^{n} \exp(i - 1 + \deg v_1 + \cdots + \deg v_{i-1})\, l \otimes v_1 \otimes \cdots \\
&\qquad \otimes dv_i \otimes \cdots \otimes v_n \otimes l'.
\end{aligned}$$

Similarly, we grade it by setting

$$\deg(l \otimes v_1 \otimes \cdots \otimes v_n \otimes l') = n + \deg v_1 + \cdots + \deg v_n .$$

We let $E\Gamma$ denote the complex consisting of those terms of $S\Gamma$ of strictly positive degree, grading reduced by 1 and differential $\partial E = -\partial S$, except, of course, in (new) degree zero where the differential now is zero. It is clear that, for $n > 0$, $H^n(\mathrm{Hom}_{\Lambda-\Lambda}(E\Gamma, M))$ is just the $(n + 1)$ st Shukla cohomology group of Γ with coefficients in the Λ-bimodule M. We can now extend E in the obvious way on morphisms over Λ so that it becomes a complex of functors $E = \{E_n\}$, $E_n : (\mathrm{Alg}\ -K, \Lambda) \to \Lambda$-bimodules. It is this complex that we will show is naturally equivalent to Q.

Proposition 3.1. *$H_0(E\Gamma) \approx J\Gamma$ so that*

$$H^0(\mathrm{Hom}_{\Lambda-\Lambda}(E\Gamma, M)) \approx \mathrm{Der}(\Gamma, M).$$

Proof. We know that $E_0\Gamma = \Lambda \otimes V_0\Gamma \otimes \Lambda$. If

$$\pi : \Lambda \otimes \Gamma \otimes \Lambda \to \mathrm{coker}\, \varphi = J\Gamma$$

is the projection, then clearly $\pi(1 \otimes \epsilon_0 \otimes 1) : E_0\Gamma \to J\Gamma$ is an epimorphism. It is immediate that $(1 \otimes \epsilon_0 \otimes 1)\, \partial'' = 0$, while it is a direct computation that $(1 \otimes \epsilon_0 \otimes 1)\, \partial'(l \otimes v \otimes v' \otimes l') = \varphi(l \otimes \epsilon_0 v \otimes \epsilon_0 v' \otimes l')$ so that $\pi(1 \otimes \epsilon_0 \otimes 1)\, \partial' = \pi(1 \otimes \epsilon_0 \otimes 1)\, \partial = 0$. Thus $\mathrm{Im}\, \partial \subset \ker \pi(1 \otimes \epsilon_0 \otimes 1)$. The proposition is proved if we show the reverse inclusion. So suppose $w \in \ker \pi(1 \otimes \epsilon_0 \otimes 1)$. Then $(1 \otimes \epsilon_0 \otimes 1)\, w \in \ker \pi = \mathrm{Im}\, \varphi$ and so

$$(1 \otimes \epsilon_0 \otimes 1)\, w = \varphi\left(\sum l_i \otimes x_i \otimes x_i' \otimes l_i'\right), \qquad l_i, l_i' \in \Lambda,\ x_i, x_i' \in \Gamma.$$

If $u = \sum l_i \otimes \langle x_i \rangle \otimes \langle x_i' \rangle \otimes l_i'$ then it can be directly calculated that $(1 \otimes \epsilon_0 \otimes 1)\, \partial u = (1 \otimes \epsilon_0 \otimes 1)\, \partial' u = (1 \otimes \epsilon_0 \otimes 1)\, w$ so that

$$(1 \otimes \epsilon_0 \otimes 1)(w - \partial u) = 0.$$

But the exactness of $V_1\Gamma \xrightarrow{d} V_0\Gamma \xrightarrow{\epsilon_0} \Gamma \to 0$ implies the same about $\Lambda \otimes V_1\Gamma \otimes \Lambda \xrightarrow{1 \otimes d \otimes 1} \Lambda \otimes V_0\Gamma \otimes \Lambda \xrightarrow{1 \otimes \epsilon_0 \otimes 1} \Lambda \otimes \Gamma \otimes \Lambda \to 0$ so there is a $t \in \Lambda \otimes V_1\Gamma \otimes \Lambda$ such that $(1 \otimes d \otimes 1)\, t = \partial'' t = \partial t = w - \partial u$ which completes the proof.

THEOREM 3.2. *There are natural transformations* $\theta_n : E_n \to E_n G$ *for* $n \geqslant 0$ *such that* $E_n\beta \cdot \theta_n$ *is the identity transformation of* E_n.

The proof of Theorem 3.2 is based on

PROPOSITION 3.3. *Let* Ω *and* Ω' *be in* (Alg $-K, \Lambda$) *and* $f : U\Omega \to U\Omega'$. *Then we can define functions* $N_i{}^*f : UN_i\Omega \to UN_i\Omega'$ *for* $i \geqslant -1$, *where we use* U *to denote the underlying set functor for* K*-modules as well. This construction is not functorial, but if* $V^*f = \{V_i{}^*f\}$ *where* $V_i{}^*f : V_i\Omega \to V_i\Omega'$ *denotes the unique* K*-linear map extending* N_{i-1}^*f, *then three conditions are satisfied:* (i) $N_{-1}^*f = f$; (ii) $V^*U\varphi = V\varphi$ *for* φ *a morphism of* (Alg $-K, \Lambda$); *and* (iii) *if* $f : U\Omega \to U\Omega'$ *and* $f' : U\Omega' \to U\Omega''$ *are such that* either $f = U\varphi$ or $f' = U\varphi'$, *then* $N_i{}^*f' \cdot N_i{}^*f = N_i{}^*(f' \cdot f)$.

Proof of Proposition 3.3. We let $N_{-1}^*f = f$ and having defined $N_i{}^*f$ for $i < n$, let $x_i \in N_{n-1}\Omega$, $a_i \in K$ be such that $\sum a_i\langle x_i \rangle \in N_n\Omega$ which is the same as saying that $\sum a_i x_i = 0$. We define

$$N_n{}^*f\left(\sum a_i\langle x_i\rangle\right) = \sum a_i\langle N_{n-1}^*fx_i\rangle - \left\langle \sum a_i N_{n-1}^*fx_i \right\rangle + \langle 0 \rangle,$$

which is immediately seen to be in $N_n{}^*\Omega'$. When N_{n-1}^*f is linear, the last two terms cancel so that $N_n{}^*f$ is also linear. It is clear by induction that if $f = U\varphi$ then the K-linear extension of N_{n-1}^*f is $V_n\varphi$. Also if we assume that $N_{n-1}^*f' \cdot N_{n-1}^*U\varphi = N_{n-1}^*(f' \cdot U\varphi)$ then

$$\begin{aligned}
&N_n{}^*f' \cdot N_n{}^*U\varphi\left(\sum a_i\langle x_i\rangle\right)\\
&\quad = N_n{}^*f'\left(\sum a_i\langle N_{n-1}^*U\varphi x_i\rangle\right)\\
&\quad = \sum a_i\langle N_{n-1}^*f' \cdot N_{-1}^*U\varphi x_i\rangle - \left\langle \sum a_i N_{n-1}^*f' \cdot N_{n-1}^*U\varphi x_i \right\rangle + \langle 0 \rangle\\
&\quad = \sum a_i\langle N_{n-1}^*(f' \cdot U\varphi)\, x_i\rangle - \left\langle \sum a_i N_{n-1}^*(f' \cdot U\varphi)\, x_i \right\rangle + \langle 0 \rangle\\
&\quad = N_n{}^*(f' \cdot U\varphi)\left(\sum a_i\langle x_i\rangle\right).
\end{aligned}$$

On the other hand if $f' = U\varphi'$, then

$$
\begin{aligned}
&N_n{}^*U\varphi' \cdot N_n{}^*f\left(\sum a_i\langle x_i\rangle\right)\\
&= N_n{}^*U\varphi'\left(\sum a_i\langle N_{n-1}^* f x_i\rangle - \left\langle \sum a_i N_{n-1}^* f x_i\right\rangle + \langle 0\rangle\right)\\
&= \sum a_i\langle N_{n-1}^* U\varphi' \cdot N_{n-1}^* f x_i\rangle - \left\langle N_{n-1}^* U\varphi'\left(\sum a_i N_{n-1}^* f x_i\right)\right\rangle + \langle N_{n-1}^* U\varphi' 0\rangle\\
&= \sum a_i\langle N_{n-1}^*(U\varphi' \cdot f)\, x_i\rangle - \left\langle \sum a_i N_{n-1}^* U\varphi' \cdot N_{n-1}^* f x_i\right\rangle + \langle 0\rangle\\
&= \sum a_i\langle N_{n-1}^*(U\varphi' \cdot f)\rangle - \left\langle \sum a_i N_{n-1}^*(U\varphi' \cdot f)\, x_i\right\rangle + \langle 0\rangle\\
&= N_n{}^*(U\varphi' \cdot f)\left(\sum a_i\langle x_i\rangle\right).
\end{aligned}
$$

Proof of Theorem 3.2. Let $f_\Gamma = \alpha U\Gamma : U\Gamma \to UG\Gamma$. This is just the map $x \to [x]$ mentioned in Section 1. Then $V^*f_\Gamma : V\Gamma \to VG\Gamma$ is a map of graded K-modules. Moreover,

$$V^*U\beta\Gamma \cdot V^*f_\Gamma = V^*(U\beta\Gamma \cdot \alpha U\Gamma) = V^*(1_{U\Gamma}) = V^*U(1_\Gamma) = V(1_\Gamma) = 1_{V\Gamma}.$$

Also V^*f_Γ is natural in Γ, for if $\varphi : \Gamma \to \Omega$ is a map in (Alg $-K, \Lambda$),

$$
\begin{aligned}
VG\varphi \cdot V^*f_\Gamma = V^*UG\varphi \cdot V^*f_\Gamma &= V^*(UFU\varphi \cdot \alpha U\Gamma)\\
&= V^*(\alpha U\Omega \cdot U\varphi) = V^*f_\Omega \cdot V\varphi.
\end{aligned}
$$

Then we have $(V^*f_\Gamma)^{(n)} : (V\Gamma)^{(n)} \to (VG\Gamma)^{(n)}$ which is natural in Γ and K-linear and so we can take $1 \otimes (V^*f_\Gamma)^{(n)} \otimes 1 : S_n\Gamma \to S_nG\Gamma$ which is now Λ-bilinear. Since these maps preserve degrees they define $\theta_n\Gamma : E_n\Gamma \to E_nG\Gamma$ which is a value of a well defined $\theta_n : E_n \to E_nG$ which is readily seen to have the property that $E_n\beta \cdot \theta_n = 1$.

THEOREM 3.4. *The complex $EG \xrightarrow{\varphi} JG \to 0$ has a natural contraction. That is, there are maps $s_{-1} : JG \to E_0G$ and $s_i : E_iG \to E_{i+1}G$ for $i \geqslant 0$ such that $\varphi s_{-1} = 1$, $s_{-1}\varphi + \partial s_0 = 1$ and $s_{i-1}\,\partial + \partial s_i = 1$ for $i > 0$.*

Proof of Theorem 3.4

The proof essentially consists of two parts. The first consists in showing that if $D\Gamma$ is the standard Hochschild complex and $\nu : S \to D$ is the natural transformation induced by $\epsilon : V \to 1$ then there is a $\rho : DG \to SG$ such that $\nu G \cdot \rho = 1$ and $\rho \cdot \nu G \sim 1$ by a natural homotopy h. The second is that if C is the complex of functors related to D in the same way that E is related to S, then $CG \to JG \to 0$ has a natural contraction t. Then it is easily seen that $h + \nu G \cdot t \cdot \rho$ is a contraction of $EG \to JG \to 0$.

Define the standard Hochschild complex $D\Gamma = \{D_n\Gamma\}$ by setting $D_n\Gamma = \Lambda \otimes \Gamma^{(n)} \otimes \Lambda$ for $n \geqslant 0$ with differential

$$\begin{aligned}\partial(l \otimes x_1 \otimes \cdots \otimes x_n \otimes l') \\ &= l\gamma x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes l' \\ &\quad + \sum (-1)^i\, l \otimes x_1 \otimes \cdots \otimes x_{i-1}x_i \otimes \cdots \otimes x_n \otimes l' \\ &\quad + (-1)^n\, l \otimes x_1 \otimes \cdots \otimes x_{n-1} \otimes \gamma x_n l' \qquad \text{for} \qquad l, l' \in \Lambda,\ x_i \in \Gamma.\end{aligned}$$

Then $\epsilon\Gamma : V\Gamma \to \Gamma$ induces $1 \otimes (\epsilon\Gamma)^{(n)} \otimes 1 : S_n\Gamma \to D_n\Gamma$ which defines $\nu\Gamma : S\Gamma \to \Gamma$ such that $\nu\Gamma \cdot \partial' = \partial \cdot \nu\Gamma$ and $\nu\Gamma \cdot \partial'' = 0$.

Note that $(G\Gamma, V_0G\Gamma) \approx (U\Gamma, UV_0G\Gamma)$, the first being Hom taken in (Alg $-K, \Lambda$), the second in $(\mathscr{S}, U\Lambda)$ so that there is a unique morphism $G\Gamma \to V_0G\Gamma$ in (Alg $-K, \Lambda$) such that $[x] \to \langle[x]\rangle$, $x \in \Gamma$. This map followed by the inclusion of $V_0G\Gamma$ into $VG\Gamma$ is called $\sigma\Gamma$ and clearly σ can be extended on morphisms to be a natural transformation of G to VG which is a morphism over Λ of differential graded K-algebras. In order to construct h we need

Lemma 3.5. *Suppose $P''X$ is the free K-module on a basis X, PX is the free K-module on the set underlying $P''X$ and $P'X$ is the free K-module on the set $\{p'' \in P''X \mid p'' \neq \langle x\rangle$ for any $x \in X\}$. If $fX : PX \to P''X$ is the K-morphism such that $fX\langle p''\rangle = p''$ for $p'' \in P''X$ and $eX : P'X \to PX$ is the K-morphism such that*

$$eX\left\langle \sum a_i\langle x_i\rangle\right\rangle = \left\langle \sum a_i\langle x_i\rangle\right\rangle - \sum a_i\langle\!\langle x_i\rangle\!\rangle,$$

then

$$0 \longrightarrow P'X \xrightarrow{eX} PX \xrightarrow{fX} P''X \longrightarrow 0$$

is exact.

Remark 3.6. The point of this lemma is that there is a functorial choice for $\ker f$. That is, if $g : X \to Y$ is a function then there are obvious vertical maps making the following diagram commute,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'X & \xrightarrow{eX} & PX & \xrightarrow{fX} & P''X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle P'f} & & \downarrow{\scriptstyle Pf} & & \downarrow{\scriptstyle P''f} & & \\ 0 & \longrightarrow & P'Y & \xrightarrow{eY} & PY & \xrightarrow{fY} & P''Y & \longrightarrow & 0. \end{array}$$

In fact, take $P''f\langle x\rangle = \langle fx\rangle$, $Pf\langle p''\rangle = \langle P''f(p'')\rangle$ and $P'f\langle p''\rangle = \langle P''f(p'')\rangle$.

Proof of Lemma 3.5. Clearly $eX \cdot fX = 0$, eX is $1-1$ and fX is onto. Suppose, therefore, that $\sum a_i\langle p_i''\rangle + \sum b_j\langle\!\langle x_j\rangle\!\rangle \in \ker fX$ where $a_i, b_j \in K$, for each i, $p_i'' \in P''$ but not of the form $\langle x \rangle$ for $x \in X$, and $x_j \in X$. Also suppose that among the x_j are all of the variables appearing in any of the p_i''. Then if $p_i'' = \sum c_{ij}\langle x_j \rangle$, $c_{ij} \in K$, it follows that $b_j = -\sum a_i c_{ij}$. Then we have

$$\begin{aligned} eX\Big(\sum_i a_i\langle p_i''\rangle\Big) &= \sum_i a_i\langle p_i''\rangle - \sum_i a_i \sum_j c_{ij}\langle\!\langle x_j\rangle\!\rangle \\ &= \sum_i a_i\langle p_i''\rangle - \sum_j \Big(\sum_i a_i c_{ij}\Big)\langle\!\langle x_j\rangle\!\rangle \\ &= \sum_i a_i\langle p_i''\rangle - \sum_j b_j\langle\!\langle x_j\rangle\!\rangle. \end{aligned}$$

PROPOSITION 3.7. *There is a natural homotopy* $k : 1 \sim \sigma \cdot \epsilon G$ *such that* $k \cdot \sigma = 0$ *and* $k^2 = 0$.

Proof. This is equivalent to the assertion that there is a natural contracting homotopy k in the complex $VG \xrightarrow{\epsilon_0 G} G \to 0$ such that $k^2 = 0$ and $k_{-1} = \sigma$. If $X_{-1}\Gamma$ is the set of monomials $[x_1] \cdots [x_k]$ in $G\Gamma$, for $x_1, \ldots, x_k \in \Gamma$ then it is clear that $X_{-1}\Gamma$ is a free K-basis for $G\Gamma$. Also, as K-modules $PX_{-1}\Gamma \approx V_0G\Gamma$ and $P'X_{-1}\Gamma \approx N_0G\Gamma$. Hence there is a natural basis $X_0\Gamma$ for $N_0G\Gamma$ as constructed above. We take $k_0\Gamma : V_0G\Gamma \to V_1G\Gamma$ to be the composite of the map of $V_0G\Gamma \to N_0G\Gamma$ such that $V_0G\Gamma \to N_0G\Gamma \xrightarrow{\subset} V_0G\Gamma$ is $1 - \sigma\Gamma \cdot \epsilon_0 G\Gamma$ and the unique K-linear map $N_0G\Gamma \to V_1G\Gamma$ which takes $x \to \langle x \rangle$ when x is an element of the canonical basis of $N_0G\Gamma$ constructed in (3.5). Now continue this way inductively to construct $k_i\Gamma$. It is clear that $k_0k_{-1} = k_0\sigma = 0$ and it will be similarly true that $k_ik_{i-1} = 0$ as claimed.

Next observe that the complex $S_nG\Gamma$ with differential ∂'' is also the tensor product $\Lambda \otimes (VG\Gamma)^{(n)} \otimes \Lambda$ as complexes (thinking of Λ as having trivial differential and grading). Thus a repeated application of [6, Theorem V, 9.1, p. 164] gives a natural homotopy

$$h_n\Gamma : 1 \sim (1 \otimes (\sigma\Gamma)^{(n)} \otimes 1) \cdot (1 \otimes (\epsilon_0 G\Gamma)^{(n)} \otimes 1).$$

Now filter $SG\Gamma$ and $DG\Gamma$ by letting $F^nSG\Gamma$ denote $\sum_{m \leqslant n} S_mG\Gamma$ and $F^nDG\Gamma$ denote $\sum_{m \leqslant n} D_mG\Gamma$. Suppose we let $\rho\Gamma : DG\Gamma \to SG\Gamma$ be the map such that $\rho\Gamma|_{D_nG\Gamma}$ is $1 \otimes (\sigma\Gamma)^{(n)} \otimes 1$ followed by $S_nG\Gamma \xrightarrow{\subset} SG\Gamma$ and $\nu G\Gamma : SG\Gamma \to DG\Gamma$ the map such that $\nu G\Gamma|_{S_nG\Gamma}$ is $1 \otimes (\epsilon G\Gamma)^{(n)} \otimes 1$ followed by $D_nG\Gamma \xrightarrow{\subset} DG\Gamma$. It is clear that $\nu G\Gamma \cdot \rho\Gamma = 1$. Let

$$F^0h\Gamma = 1 : 1 \sim F^0\rho\Gamma \cdot F^0\nu G\Gamma$$

and suppose that natural maps $F^0h\Gamma, \ldots, F^{n-1}h\Gamma$ have been constructed so that $F^ih\Gamma : 1 \sim F^i\rho\Gamma \cdot F^i\nu G\Gamma$ and that $F^ih\Gamma|_{F^{i-1}SG\Gamma} = F^{i-1}h\Gamma$. Also suppose that $F^ih\Gamma \cdot F^i\rho\Gamma = 0$ for $i < n$. Since $\epsilon G \cdot k = 0$ it can easily be seen that

$F^n\nu G\Gamma \cdot h_n\Gamma = 0$. Similarly since $k \cdot \sigma = 0$ it follows that $h_n\Gamma \cdot F^n\rho\Gamma = 0$. Now take $x \in F^nSG\Gamma$ and define $F^nh\Gamma(x)$ by writing $x = y + z$ where $y \in F^{n-1}SG\Gamma$ and $z \in S_nG\Gamma$. Then let

$$F^nh\Gamma(x) = F^{n-1}h\Gamma(y) + h_n\Gamma(z) - F^{n-1}h\Gamma \cdot \partial' \cdot h_n\Gamma(z).$$

Then

$$\begin{aligned}
&\partial \cdot F^nh\Gamma(x) + F^nh\Gamma \cdot \partial(x)\\
&\quad = \partial \cdot F^{n-1}h\Gamma(y) + \partial \cdot h_n\Gamma(z) - \partial \cdot F^{n-1}h\Gamma \cdot \partial' \cdot h_n\Gamma(z)\\
&\qquad + F^nh\Gamma \cdot \partial(y) + F^nh\Gamma \cdot \partial'(z) + F^nh\Gamma \cdot \partial''(z)\\
&\quad = \partial \cdot F^{n-1}h\Gamma(y) + F^{n-1}h\Gamma \cdot \partial(y) + \partial \cdot h_n\Gamma(z)\\
&\qquad - (1 - F^{n-1}h \cdot \partial - F^{n-1}\rho\Gamma \cdot F^{n-1}\nu G\Gamma) \cdot \partial' \cdot h_n\Gamma(z)\\
&\qquad + F^{n-1}h\Gamma \cdot \partial'(z) + h_n\Gamma \cdot \partial''(z) - F^{n-1}h\Gamma \cdot \partial' \cdot h_n\Gamma \cdot \partial''(z)\\
&\quad = (\partial \cdot F^{n-1}h\Gamma + F^{n-1}h\Gamma \cdot \partial)(y) + (\partial \cdot h_n\Gamma + h_n\Gamma \cdot \partial')(z)\\
&\qquad - F^{n-1}h\Gamma \cdot \partial' \cdot \partial'' \cdot h_n\Gamma(z)\\
&\qquad - F^{n-1}\rho\Gamma \cdot F^{n-1}\nu G\Gamma \cdot \partial' \cdot h_n\Gamma(z) + F^{n-1}h\Gamma \cdot \partial'(z)\\
&\qquad - F^{n-1}h\Gamma \cdot \partial' \cdot (1 - \partial'' \cdot h_n\Gamma - (1 \otimes (\sigma\Gamma)^{(n)} \cdot (\epsilon G\Gamma)^{(n)} \otimes 1)(z)\\
&\quad = (1 - F^n\rho\Gamma \cdot F^n\nu G\Gamma)(y) + (1 - 1 \otimes (\sigma\Gamma)^{(n)} \cdot (\epsilon G\Gamma)^{(n)} \otimes 1)(z)\\
&\qquad - F^{n-1}h\Gamma \cdot \partial' \cdot \partial'' \cdot h_n\Gamma(z) - F^{n-1}\rho\Gamma \cdot \partial \cdot F^n\nu G\Gamma \cdot h_n\Gamma(z)\\
&\qquad + F^{n-1}h\Gamma \cdot \partial'(z) - F^{n-1}\Gamma \cdot \partial'(z) + F^{n-1}h\Gamma \cdot \partial' \cdot \partial'' \cdot h_n\Gamma(z)\\
&\qquad + F^{n-1}h\Gamma \cdot \partial' \cdot F^n\rho\Gamma \cdot F^n\nu G\Gamma(z)\\
&\quad = (1 - F^n\rho\Gamma \cdot F^n\nu G\Gamma)(y + z) + F^{n-1}h\Gamma \cdot F^{n-1}\rho\Gamma \cdot \partial \cdot F^n\nu G\Gamma(z)\\
&\quad = (1 - F^n\rho\Gamma \cdot F^n\nu G\Gamma)(x).
\end{aligned}$$

If $x \in F^nDG\Gamma$, $x = y + z$ where $y \in F^{n-1}DG\Gamma$ and $z \in D_nG\Gamma$,

$$\begin{aligned}
F^nh\Gamma \cdot F^n\rho\Gamma(x) &= F^nh\Gamma \cdot F^n\rho\Gamma(y) + F^nh\Gamma \cdot F^n\rho\Gamma(y)\\
&= F^{n-1}h\Gamma \cdot F^{n-1}\rho\Gamma(y) + h_n\Gamma \cdot F^n\rho\Gamma(z)\\
&\quad - F^{n-1}h\Gamma \cdot \partial' \cdot h_n\Gamma \cdot F^n\rho\Gamma(z) = 0.
\end{aligned}$$

Evidently $F^nh|_{F^{n-1}SG\Gamma} = F^{n-1}h$. From the latter fact it follows that $\{F^nh\}$ converges to a homotopy $h : 1 \sim \rho\Gamma \cdot \nu G\Gamma$ which is natural in Γ. This completes the first part of the proof of Theorem 3.4.

Remark 3.8. This proof is essentially just a modification of the proof of Theorem 1 [7] with everything made natural.

Now let C_n be the complex such that $C_n\Gamma = D_{n+1}\Gamma$ for $n \geqslant 0$ and $\partial C = -\partial D$. It is clear that νG, ρ and h have restrictions, which for convenience we will still denote by νG, ρ and h such that $\nu G : EG \to CG$,

$\rho : CG \to EG$, $\nu G \cdot \rho = 1$ and $h : 1 \sim \rho \cdot \nu G$. Now, as noted above, a contraction of $CG \to JG \to 0$ will imply a contraction of $EG \to JG \to 0$. This is done in [*1*] abstractly but we repeat the proof here for completeness. First note that $JG\Gamma$ must be naturally isomorphic to the free Λ-bimodule on the underlying set of Γ since both represent Der($G\Gamma$, —) on the category of Λ-bimodules (see [*3*]). Hence we take $t_{-1} : JG\Gamma \to C_0G\Gamma$ to be the composite of that isomorphism with the Λ-bilinear map such that $\langle x \rangle \to 1 \otimes \langle x \rangle \otimes 1$. For $n \geqslant 0$ define $t_n(1 \otimes u_0 \otimes \cdots \otimes u_n \otimes 1)$ for monomials $u_0, \ldots, u_n \in G\Gamma$ by induction on the length of the monomial u_0 by

$$t_n(1 \otimes 1 \otimes u_1 \otimes \cdots \otimes u_n \otimes 1) = 1 \otimes 1 \otimes 1 \otimes u_1 \otimes \cdots \otimes u_n \otimes 1$$

and for $u_0 = [x]\, u_0'$,

$$t_n(1 \otimes u_0 \otimes u_1 \otimes \cdots \otimes u_n \otimes 1) = \gamma(x)\, t_n(1 \otimes u_0' \otimes u_1 \otimes \cdots \otimes u_n \otimes 1) - 1 \otimes [x] \otimes u_0' \otimes u_1 \otimes \cdots \otimes u_n \otimes 1$$

and extending this to be Λ-bilinear. Then it is left as an exercise to show that this is a contracting homotopy which completes the proof of Theorem 3.4 and of Theorem 1.1 as well.

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