A COHOMOLOGY THEORY FOR COMMUTATIVE ALGEBRAS. I[1]

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1. **Introduction.** D. K. Harrison has recently developed a cohomology theory for commutative algebras over a field [2]. A few key theorems are proved and the results applied to the theory of local rings and eventually to algebraic geometry. The main problem is that both his definitions and proofs require involved calculations.

In this paper we define a cohomology theory which (a) relies on more or less straightforward techniques of homological algebra and (b) defines a cohomology theory for an algebra over any commutative ring K, whose H^2 group is the group of all singular extensions, whether K-split or not. Of course, a suitable specialization of the theory gives relative case (see §4, below). The definitions are based on an idea of M. Gerstenhaber [3].

2. **Definitions.** Throughout this paper K is a commutative ring, R a commutative K-algebra (with unit) and M is a (left) R-module (equivalently, M is a symmetric two-sided R-module). A derivation $d: R\to M$ is a K-linear mapping with $d(xy)=xd(y)+yd(x)$. An n-long singular extension of R by M is an exact sequence

$$0\longrightarrow M\xrightarrow{d_n}M_{n-1}\xrightarrow{d_{n-1}}M_{n-2}\xrightarrow{d_{n-2}}\cdots\longrightarrow M_1\xrightarrow{d_1}T\xrightarrow{\phi}R\longrightarrow 0$$

in which T is a commutative K-algebra and ϕ is a morphism of K-algebras whose kernel M_0 has square zero (and hence is an R-module) and the remainder of the sequence is an exact sequence of R-modules. In particular a 1-long singular extension will be called a short singular extension. If E and E' are two singular extensions, a morphism $f: E\to E'$ is a sequence of maps $f_i: M_i\to M_i'$ with suitable maps at the ends so that the following is commutative

$$\begin{array}{ccccccccccccc}
0\to & M & \to & M_{n-1} & \to & \cdots & \to & M_1 & \to & T & \to & R & \to 0\\
 & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle 1_R} & \\
0\to & M' & \to & M'_{n-1} & \to & \cdots & \to & M'_1 & \to & T' & \to & R & \to 0.
\end{array}$$

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In particular if f_n is an isomorphism, then f is called an equivalence of E with E' (written $f\colon E\approx E'$). The concept of an ω-long extension of R is obvious. A short singular extension is called generic if it admits a morphism to every singular extension.

PROPOSITION 1. *Short generic singular extensions exist.*

PROOF. For let $P\rightarrow R$ any K projective module mapping onto R and $S(P)$ the symmetric algebra over P. Then $S(P)$ has the property that any K-linear mapping of P to a commutative K-algebra T can be extended uniquely to a K-algebra morphism of $S(P)$ to T. In particular, we have $\alpha\colon S(P)\rightarrow R$, an epimorphism of K-algebras. Let $N=\ker\alpha$ and $F=S(P)/N^2$. Then we have a singular extension $0\rightarrow N/N^2\rightarrow F\rightarrow^{\alpha} R\rightarrow 0$. If $0\rightarrow M\rightarrow T\rightarrow R\rightarrow 0$ is any other singular extension, we have $T\rightarrow^{\phi} R\rightarrow 0$ gives a map $\beta\colon P\rightarrow T$ so that $\alpha\beta=\phi$ which induces an algebra map $S(\beta)\colon S(P)\rightarrow T$ with the same property. But then $\beta(N)\subset M$ and so $\beta(N^2)\subset M^2=0$ and thus β induces a mapping of $S(P)/N^2\rightarrow T$ whose restriction to N/N^2 maps the latter to M. Q.E.D.

A long singular extension (possibly ω-long) is called a generic resolution if it admits a morphism to any long singular extension. Using standard techniques and Proposition 1 it is easily proved that,

PROPOSITION 2. *Suppose* $0\rightarrow N\rightarrow F\rightarrow^{\alpha} R\rightarrow 0$ *is a generic singular extension and* $\cdots\rightarrow X_i\rightarrow^{\epsilon_i} X_{i-1}\rightarrow^{\epsilon_{i-1}}\cdots\rightarrow X_1\rightarrow^{\epsilon_1} N\rightarrow 0$ *is an R-projective resolution of N, then* $\cdots\rightarrow X_i\rightarrow^{\epsilon_i} X_{i-1}\rightarrow^{\epsilon_{i-1}}\cdots\rightarrow X_1\rightarrow^{\epsilon_1} F\rightarrow^{\alpha} R\rightarrow 0$ *is a generic resolution of R. This will be abbreviated by* $X\rightarrow^{\epsilon} F\rightarrow^{\alpha} R\rightarrow 0$.

In [4], Baer addition of equivalence classes of short singular extensions of algebras is defined. When the extensions are commutative, so is their Baer sum. In an obvious way this can be extended to addition of equivalence classes of long singular extensions. The monoid thus formed of equivalence classes of $(n+1)$-long singular extensions is denoted by $H^n(R, M)$, for $n>1$ and we let $H^1(R, M)=\mathrm{Der}(R, M)$ the group of derivations of R to M. If $X\rightarrow^{\epsilon} F\rightarrow^{\alpha} R\rightarrow 0$ is a generic resolution we let $\overline{H}^n(R, M)$ denote the $(n-1)$st cohomology group of the complex, $0\rightarrow\mathrm{Der}(F, M)\rightarrow\mathrm{Hom}_R(X, M)$ where the first map is defined to be composition with ϵ_1 which, strangely enough, maps derivations to R-homomorphisms. The facts that $H^n(R, M)$ is a group, and in particular, a set and that $\overline{H}^n(R, M)$ does not depend on the particular choice of a generic resolution are clear from

THEOREM 3. *There is a natural isomorphism of $H^n(R, M)$ with $\overline{H}^n(R, M)$.*

Here "natural" is used somewhat loosely for it has not been shown that either H^n or $\overline{H}^n$ is a functor. However, $\overline{H}^n$ clearly is and H^n can be easily shown to be, and the isomorphism we construct is a natural equivalence. For $n=1$, the result is obvious given that $\mathrm{Der}(-, M)$ is a left exact contravariant functor (see [4]). For $n>1$ we construct a mapping of $H^n(R, M)$ to $\overline{H}^n(R, M)$ and leave the details of the proof to the reader. It is well defined and gives an isomorphism. So let $X \to^{\epsilon} F \to^{\alpha} R \to 0$ be a generic singular resolution of R and $N = \ker \alpha$. Suppose

$$0 \to M \to M_{n-1} \to \cdots \to M_1 \to T \xrightarrow{\beta} R \to 0$$

is a long singular extension and $M_0 = \ker \beta$. Then $0 \to N \to F \to R \to 0$ is a short generic singular extension so it maps to $0 \to M_0 \to T \to R \to 0$ which induces $f_0: N \to M_0$. But then $\cdots \to X_n \to \cdots \to X_1 \to N \to 0$ is a projective resolution of N and we can map it to $0 \to M \to M_{n-1} \to \cdots \to M_1 \to M_0 \to 0$ so that it commutes with f_0. Thus we have defined (not uniquely; it must be shown to be unique up to homotopy) a map $f_n: X_n \to M$. Then the isomorphism is the function which associates the above extensions to the class of f_n in $\mathrm{Hom}_R(X_n, M)$.

Henceforth, using this isomorphism, we do not distinguish between H^n and $\overline{H}^n$.

Proposition 4. *H^n is a connected functor, i.e. if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, then there are connecting homomorphisms such that*

$$\begin{aligned} &0 \to \mathrm{Der}(R, M') \to \mathrm{Der}(R, M) \to \mathrm{Der}(R, M'') \\ &\to H^2(R, M') \to H^2(R, M) \to H^2(R, M'') \\ &\to \cdots\cdots\cdots\cdots \\ &\to H^n(R, M') \to H^n(R, M) \to H^n(R, M'') \to \cdots \end{aligned}$$

is exact.

Proof. This follows from the fact that when F is a generic extension of R,

$$0 \to \mathrm{Der}(F, M') \to \mathrm{Der}(F, M) \to \mathrm{Der}(F, M'') \to 0$$

is exact (see [4]), together with well-known results.

3. **Main results.** In this section we obtain the main results analogous to Harrison's.

Theorem 5. *Let $R = k[X]$, the algebra of polynomials in a set X of variables and M be an R-module. Then $H^1(R, M) = M^X$ and $H^i(R, M) = 0$ for $i>1$.*

PROOF. The first assertion follows from the fact that every map of X to M can be uniquely extended to a derivation and the second from the fact that a polynomial algebra is a generic extension of itself (see [4] for details).

In what follows, $\otimes$, Tor, "flat," "projective" will mean $\otimes_K$, Tor^K, "K-flat," and "K-projective" respectively. Before going any further we require a lemma proved in [1, p. 165] which, for commutative K-algebras, may be stated as follows:

LEMMA 6. *Let L, L', L'' be commutative K-algebras, M an $L'-L''$ bimodule, M' an $L-L''$ bimodule and M'' an $L-L'$ bimodule. There is an isomorphism*

$$\mathrm{Hom}_{L\otimes L'}\left(M \underset{L''}{\otimes} M', M''\right) \approx \mathrm{Hom}_{L'\otimes L''}(M, \mathrm{Hom}_L(M', M''))$$

which establishes a natural equivalence of functors.

THEOREM 7. *Let R and R' be flat algebras and M an $R\otimes R'$-module (this is equivalent to an $R-R'$ bimodule), then*

$$H^n(R \otimes R', M) \approx H^n(R, M) \oplus H^n(R', M).$$

PROOF. Let $X \to F \to^{\alpha} R \to 0$ and $X' \to F' \to^{\alpha'} R' \to 0$ be generic singular resolutions with $N = \ker \alpha$, $N' = \ker \alpha'$.

$$F \otimes F' \xrightarrow{\alpha \otimes \alpha'} R \otimes R'$$

would be a generic singular extension if it were singular, a defect remediable by factoring out the square of the kernel. From the flatness of R and R' we get the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N \otimes N' & \to & N \otimes F' & \to & N \otimes R' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F \otimes N' & \to & F \otimes F' & \to & F \otimes R' & \to & 0 \\
 & & \downarrow & & \downarrow & \searrow & \downarrow & & \\
0 & \to & R \otimes N' & \to & R \otimes F' & \to & R \otimes R' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the diagonal arrow is $\alpha\otimes\alpha'$. Then by diagram chasing we see that $\ker(\alpha\otimes\alpha')=F\otimes N'+N\otimes F'$ and that $(F\otimes N')\cap(N\otimes F')=N\otimes N'$. But then

$$\begin{aligned}(\ker(\alpha\otimes\alpha'))^2 &= F\otimes N'^2 + N\otimes N' + N^2\otimes F' \\ &= N\otimes N'\end{aligned}$$

and the quotient is

$$(F\otimes N'/N\otimes N')\oplus(N\otimes F'/N\otimes N') = R\otimes N'\oplus N\otimes R'.$$

Thus

$$0\to R\otimes N'\oplus N\otimes R'\to F\otimes F'/N\otimes N'\to R\otimes R'\to 0$$

is a singular extension of $R\otimes R'$ which is easily seen to be generic. Now both R and R' are flat, hence R and R' projectives are also flat. This means that $X\to N\to 0$ and $X'\to N'\to 0$ are flat resolutions and that the homology of $X\otimes R'$ and $R\otimes X'$ are $\operatorname{Tor}(N, R')=N\otimes R'$ and $\operatorname{Tor}(R, N')=R\otimes N'$ respectively. But this means that $X\otimes R'\to N\otimes R'\to 0$ and $R\otimes X'\to R\otimes N'\to 0$ are acyclic. Now for any $R\otimes R'$ module M we apply Lemma 6 to obtain

$$\begin{aligned}&\operatorname{Hom}_{R\otimes R'}(R\otimes X'\oplus X\otimes R', M)\\ &\quad\approx \operatorname{Hom}_{R\otimes R'}(R\otimes X', M)\oplus\operatorname{Hom}_{R\otimes R'}(X\otimes R', M)\\ &\quad\approx \operatorname{Hom}_{R'}(X', \operatorname{Hom}_R(R, M))\oplus\operatorname{Hom}_R(X, \operatorname{Hom}_{R'}(R', M))\\ &\quad\approx \operatorname{Hom}_{R'}(X', M)\oplus\operatorname{Hom}_R(X, M).\end{aligned}$$

This shows both that $R\otimes X'\oplus X\otimes R'$ is projective and that the complex $\operatorname{Hom}_{R\otimes R'}(R\otimes X'\oplus X\otimes R', M)$ is the direct sum of $\operatorname{Hom}_R(X, M)$ and $\operatorname{Hom}_{R'}(X', M)$. The easily proved assertions that $\operatorname{Der}(F\otimes F', M)\approx\operatorname{Der}(F, M)\oplus\operatorname{Der}(F', M)$ and that these are all natural equivalences complete the proof.

THEOREM 8. *Let S be a multiplicatively closed subset of R with $0\notin S$, and suppose that there are no zero divisors of R in S. Suppose M is an R_S-module, then $H^n(R, M)\approx H^n(R_S, M)$.*

PROOF. Let $0\to N\to F\to^{\alpha} R\to 0$ be a generic singular extension of R and $C=\alpha^{-1}(S)$. Then by direct computation $0\to N\otimes_R R_S\to F_C\to^{\alpha_S} R_S\to 0$ is exact. I claim it is a generic singular extension. For let $0\to M\to T\to^{\beta} R_S\to 0$ be a singular extension. Then $0\to M\to\beta^{-1}(R)\to R\to 0$ is a singular extension of R and we can map ϕ: $F\to\beta^{-1}(R)$ with $\beta\phi=c$. Now given $c\in C$ we can find a unique $\theta(c)\in T$ with $\theta(c)\phi(c)=1$. Then extending ϕ to ϕ_S: $F_C\to T$ by $\phi_S(x/c)=\phi(x)\beta(c)$ the claim is proved. Now since no zero divisors of R are in S,

$$R_S = \text{inj}\lim\{(1/s)R \mid s \in S\}$$

is R-flat being a direct limit of R-flat modules. Then if $X \to N \to 0$ is an R-projective resolution of N, then homology of $X \otimes_R R_S$ is $\text{Tor}^R(N, R_S) = N \otimes_R R_S$ so that $X \otimes_R R_S \to N \otimes_R R_S \to 0$ is acyclic. Moreover, for any R_S-module M, we have by Lemma 6 $\text{Hom}_{R_S}(X \otimes_R R_S, M) \approx \text{Hom}_R(X, \text{Hom}_{R_S}(R_S, M)) \approx \text{Hom}_R(X, M)$ showing that $X \otimes_R R_S$ is R_S-projective and that the complexes $\text{Hom}_R(X, M)$ and $\text{Hom}_{R_S}(X \otimes_R R_S, M)$ are isomorphic. The fact that $\text{Der}(F, M) \approx \text{Der}(F_C, M)$ is easily seen and completes the proof.

The assumption above that S contains no zero divisors may be omitted; Robert M. Fossum has pointed out that R_S is always R-flat.

4. **Generalizations, specializations, etc.** In a subsequent paper it will appear how to specialize the theory to Z-split or K-split extensions (or, any ring in between) on the one hand and to generalize to a functor H_ψ, for any singular epimorphism of K-algebras $\psi: S' \to S$ in such a way that if $N = \ker \psi$ and M is an S-module, then we have an exact sequence

$$\begin{aligned}
0 &\to \text{Der}(S, M) \to \text{Der}(S', M) \to \text{Hom}_S(N, M) \\
&\to H^2(S, M) \to H^2_\psi(S', M) \to \text{Ext}^1_S(N, M) \\
&\to \cdots\cdots\cdots\cdots\cdots\cdots \\
&\to H^n(S, M) \to H^n_\psi(S', M) \to \text{Ext}^{n-1}_S(N, M) \to \cdots .
\end{aligned}$$

Thus we have a connected sequence in the first variable.

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