COALGEBRAS IN A CATEGORY OF ALGEBRAS

by

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Let $\underline{A}$ be a category and $\mathbb{T} = (T,\eta,\mu)$ be a triple on $\underline{A}$. Then we may form $\underline{B} = \underline{A}^{\mathbb{T}}$, the category of $\mathbb{T}$ algebras. There is then an adjoint pair $\underline{A} \underset{U}{\overset{F}{\rightleftarrows}} \underline{B}$ and we may ask whether or not F is cotripleable. More explicitly, we may form a cotriple $\mathbb{G} = (FU,\epsilon,F\eta U)$ where $\epsilon\colon FU \longrightarrow 1$ and $\eta\colon 1 \longrightarrow UF = T$ are the adjointness morphisms. Then there is a natural functor $\Psi\colon \underline{A} \longrightarrow \underline{B}_{\mathbb{G}} = \underline{C}$ and we are asking whether Ψ is an equivalence.

For general $\underline{A}$, the problem seems very difficult. It would, of course, be possible to continue forming categories $\underline{A}^{\mathbb{T}}$, $(\underline{A}^{\mathbb{T}})_{\mathbb{G}}$, ... Myles Tierney has given an example to show that this process needn't ever terminate. Presumably it is also possible that it terminate at any finite step. Thus we have, perhaps, a concept of a dimension of a category.

Here we give a complete answer when $\underline{A}$ is the category of sets (denoted by $\underline{S}$), pointed sets (denoted by $(1,\underline{S})$), or vector spaces over the field K (denoted by $\underline{V}_K$). Ignoring those $\mathbb{T}$ for which the functor T is constant, we show that $\dim\underline{S} = 1$ while $\dim(1,\underline{S}) = \dim\underline{V}_K = 0$. Since the main difference between $\underline{S}$ and $(1,\underline{S})$ is that the former contains some monomorphisms which do not split (any $\phi \longrightarrow X$), this concept of dimension does seem to be some kind of homological measure of the category. For the meaning and statements of the various tripleableness theorems we refer to [Be] and [Li].

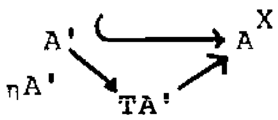
1. PRELIMINARIES

According to the dual of the tripleableness theorem we must consider the question of whether F reflects isomorphisms and whether it preserves equalizers of F-split pairs (since all of our categories are complete.) But U reflects isomorphisms and creates all limits and an F-split pair is also UF-split. Thus we need only consider these questions for T itself. First we consider the question of T reflecting isomorphisms. In any concrete category we may call a triple consistent if it has a model of cardinality at least 2. We henceforth assume that $\underline{A}$ is one of the three categories mentioned above. The following lemma is from [La].

Lemma 1.

Let $\mathbb{T} = (T, \eta, \mu)$ be a consistent triple in $\underline{A}$. Then η is 1-1.

Proof. Any object $A \in \underline{A}$ with cardinality ≥ 2 is a cogenerator. Then for all $A' \in \underline{A}$, $A' \subset A^X$ for some set X. If A is a $\mathbb{T}$-algebra, so is A^X, and the given embedding $A' \hookrightarrow A^X$ can be completed to



so that $\eta A'$ is 1-1.

From now on we assume $\mathbb{T}$ is consistent. It is easily seen that on $\underline{S}$ there are 2 inconsistent triples ($TX = 1$ for all X is one and $TX = 1$ for all $X \neq \phi$, $T\phi = \phi$ is the other), while on $(1,\underline{S})$ and $\underline{V}_K$ there is exactly one.

<u>Proposition 2.</u>

T is faithful.

<u>Proof.</u> If $f \neq g: X \longrightarrow Y$, then since η is 1-1, $Tf.\eta X = \eta Y.f \neq \eta Y.g = Tg.\eta X$ so $Tf \neq Tg$.

<u>Theorem 3.</u>

T (and hence F) reflects isomorphisms.

<u>Proof.</u> For T, being faithful, reflects both epimorphisms and monomorphisms, and $\underline{A}$ has the property that a map which is both is an isomorphism.

2. SPLIT EQUALIZERS

In order to apply the cotripleableness theorem, it is necessary to have a combinatorial description of split equalizers. It turns out to be the same (almost) in all three categories.

Theorem 4.

Two maps $X \overset{d^\circ}{\underset{d^1}{\rightrightarrows}} Y$ can be put into a split equalizer diagram

$$E \underset{t}{\overset{d}{\rightleftarrows}} X \underset{s}{\overset{}{\rightrightarrows}} Y$$

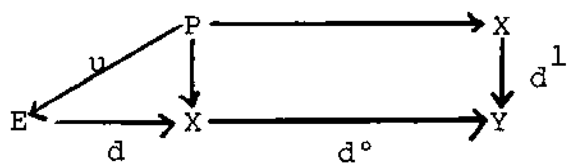
satisfying $td = E$, $sd^\circ = X$, $sd^1 = dt$, $d^\circ d = d^1 d$ if the following conditions are satisfied.

(i) Equalizer $(d^\circ, d^1) \neq \phi$ (or $X = Y = \phi$).

(ii) d° is 1-1.

(iii) If $d^\circ x = d^1 x'$, then $d^\circ x = d^1 x$.

Condition (iii) may be described by saying that there is a map $u\colon P \longrightarrow E$ making the following diagram commute



where P is the pullback (kernel pair) and E the equalizer.

Proof. The necessity of (i) and (ii) is clear (condition (i) refers only to $\underset{\sim}{S}$ anyway.) As for (iii), if $d^\circ x = d^1 x'$, then $d^1 x = d^1 s d^\circ x = d^1 s d^1 x' = d^1 d t x' = d^\circ d t x' = d^\circ s d^1 x' = d^\circ s d^\circ x = d^\circ x$.

It is seen that these conditions will be necessary in any concrete category. Their sufficiency depends very heavily on the explicit categories at hand. We must consider cases. It is only necessary to find $s\colon Y \longrightarrow X$ with $sd^\circ = X$

and $d^\circ sd^1 = d^1sd^1$, for then $t: Y \longrightarrow E$ can be chosen so that $dt = sd^1$ by the nature of equalizers.

Case $\underline{S}$. Write $Y = Y_o + Y_1$ where $Y_o = \mathrm{Im}X$. Then $d^\circ: X \xrightarrow{\cong} Y_o$ (abusing notation). Choose $x_o \in X$ with $d^\circ x_o = d^1 x_o$. Then define $s: Y \longrightarrow X$ by $s \mid Y_o = (d^\circ)^{-1}$ and $s \mid Y_1$ is constant x_o. Clearly $sd^\circ = X$. If $d^1 x \notin \mathrm{Im}d^\circ$, then $sd^1x = x_o$ and $d^1sd^1x = d^\circ sd^1x$. If $d^1x = d^\circ x'$, then $d^\circ x' = d^1x'$, and then $d^1sd^1x = d^1sd^\circ x' = d^1x' = d^\circ x' = d^\circ sd^\circ x' = d^\circ sd^1x$.

Case $(1,\underline{S})$. This is exactly the same except that x_o should be taken to be the basepoint.

Case $\underline{V}_K$. Let E be the equalizer of d° and d^1 and write $X = X_o + X_1$ where $d: E \xrightarrow{\cong} X_o$. We may assume $E = X_o$. Now since d° is 1-1, we can assume that $Y = X_o + X_1 + Y_2$ and d° is the inclusion. Moreover, $y \in \mathrm{Im}\, d^\circ \cap \mathrm{Im}\, d^1 \Longrightarrow y = d^\circ x = d^1x'$. Then $y = d^\circ x = d^1x$ and so $y \in \mathrm{Im}\, d^\circ d = X_o$. Hence we may choose Y_2 so that $\mathrm{Im}\, d^1 \subset X_o + Y_2$. In terms of this decomposition the maps d, d°, d^1 have matrix representations as follows.

$$d = \begin{pmatrix} X_o \\ \\ 0 \end{pmatrix}, \qquad d^\circ = \begin{pmatrix} X_o & 0 \\ 0 & X_1 \\ 0 & 0 \end{pmatrix}, \qquad d^1 = \begin{pmatrix} X_o & \alpha \\ 0 & 0 \\ 0 & \beta \end{pmatrix}$$

where $\alpha: X_1 \longrightarrow X_o$ and $\beta: X_1 \longrightarrow Y_2$ are arbitrary.

Let

$$s = \begin{pmatrix} X_o & 0 & 0 \\ 0 & X_1 & 0 \end{pmatrix} \quad , \quad t = \begin{pmatrix} X_o , \alpha \end{pmatrix}$$

and then the required equations are clear.

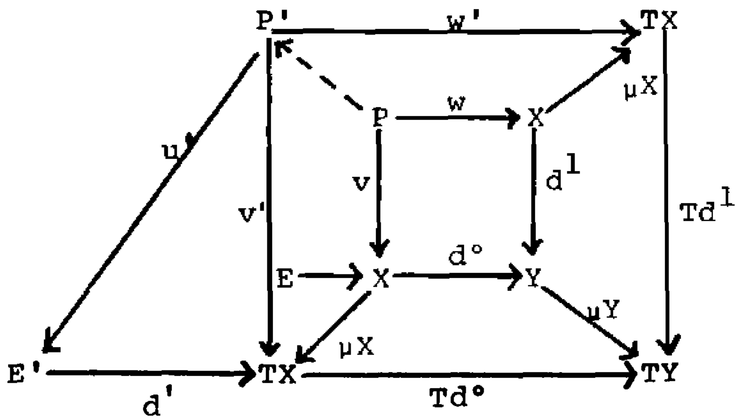
3. PRESERVATION OF T-SPLIT EQUALIZERS

In this we show that T-split equalizers are preserved in cases (1,$\underline{S}$) and $\underline{V}_K$ and examine the sole failure of this in $\underline{S}$. The methods are really quite vulgar, since they prove that T (and hence F) is co-VTT.

Proposition 5

T (and hence F) reflects conditions (ii) and (iii) of theorem 4.

Proof. If $d^\circ : X \longrightarrow Y$ and Td° is 1-1, then $\mu Y . d^\circ = Td^\circ . \mu X$ and μX is 1-1, so $\mu Y . d^\circ$ and hence d° is 1-1. For the other part we use the diagram.



It is only necessary to show that $d^{\circ}v = d^{1}v$. The universal property of P' allows a map $\pi: P \longrightarrow P'$ as indicated. Now $\mu Y.d^{\circ}.v = Td^{\circ}.\mu X.v = Td^{\circ}.v'.\pi = Td^{\circ}.d'.u'.\pi = Td^{1}.d'.u'.\pi = Td^{1}.v'.\pi = Td^{1}.\mu X.v = \mu Y.d^{1}.v$ and by lemma 1, μY is 1-1, so $d^{\circ}.v = d^{1}.v$.

<u>Theorem 6.</u>

In cases $(1,\underline{S})$ and $\underline{V}_{K}$, F is co-VTT.

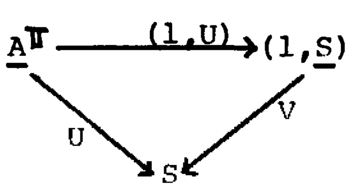
<u>Proof.</u> There is nothing left to prove.

<u>4. $\underline{\underline{S}}$</u>

There are easy examples which will become clear later which show that T does not necessarily reflect condition (i) of theorem 4.

<u>Theorem 7.</u>

If $T\phi \neq \phi$, then there is a factorization $(1,U): \underline{A}^{T} \longrightarrow (1,\underline{S})$ such that



commutes, where V is the usual underlying functor. Moreover, there is a left adjoint $(1,F) \longmapsto (1,U)$ and $(1,F)$ is co-VTT.

Proof. Pick a point $a\colon 1 \longrightarrow T\phi$. Define $(1,U)A$ to be $1 \xrightarrow{a} T\phi = UF\phi \xrightarrow{U*} UA$ where $*\colon F\phi \longrightarrow A$ is the unique map in $(F\phi,A) = (\phi,UA)$. Clearly $V.(1,U) = U$. Define $(1,F)$ on objects by letting $(1,F)(1 \longrightarrow X) = F(X-\{1\})$. Then $(F(X-\{1\}),A) \approx (X-\{1\},UA) \approx (1,\underline{S})(1 \longrightarrow X, (1,U)A)$, the last isomorphism being obvious. Then $(1,F)$ has a unique extension to a functor left adjoint to $(1,U)$ and hence, by theorem 6, is co-VTT. Note that $1 \longrightarrow X \rightsquigarrow X-\{1\}$ cannot be extended to a functor, which is the reason for the indirect definition of $(1,F)$ first as an object function.

Theorem 8.

If $T\phi \neq \phi$, then F is cotripleable.

Proof. If $W\colon \underline{S} \longrightarrow (1,\underline{S})$ is defined by $WX = 1 \longrightarrow 1 + X$, then $W \dashv V$. But then $F = (1,F).W$, and since $(1,F)$ is co-VTT, it is only necessary to show that W is tripleable. It is, in fact, co-CTT, as may readily be checked. That is,

$$E \longrightarrow X \rightrightarrows Y$$

is an equalizer if and only if

$$WE \longrightarrow WX \rightrightarrows WY \qquad \text{is.}$$

Theorem 9.

Suppose e°, e^{1} are the two distinct maps of $1 \longrightarrow 2$ (in $\underline{S}$). If $\phi \longrightarrow T1 \rightrightarrows T2$ is an equalizer, then F is co-tripleable.

<u>Proof</u>. It is sufficient, evidently, to show that if $E' \xrightarrow{d'} TX \overset{Td^\circ}{\underset{Td^1}{\rightrightarrows}} TY$ is a split equalizer, then the equalizer

$E \xrightarrow{d} X \overset{d^\circ}{\underset{d^1}{\rightrightarrows}} Y$ is non-empty.

At any rate condition (iii) is reflected, which means there will be a map $P \longrightarrow E$ so that if $E = \phi$, so is P. But if P is empty, this means we can find a map $2 \longrightarrow Y$ so that

$$\begin{array}{ccc} X & \overset{d^\circ}{\underset{d^1}{\rightrightarrows}} & Y \\ \downarrow & & \downarrow \\ 1 & \overset{e^\circ}{\underset{e^1}{\rightrightarrows}} & 2 \end{array}$$

commutes. This means each of the diagrams

$$\begin{array}{ccc} X & \xrightarrow{d^\circ} & Y \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[e^\circ]{} & 2 \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{d^1} & Y \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[e^1]{} & 2 \end{array}$$

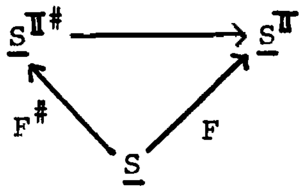
commutes. This provides us with a commutative diagram

$$\begin{array}{ccccc} E' & \longrightarrow & TX & \rightrightarrows & TY \\ \downarrow & & \downarrow & & \downarrow \\ \phi & \longrightarrow & T1 & \rightrightarrows & T2 \end{array}$$

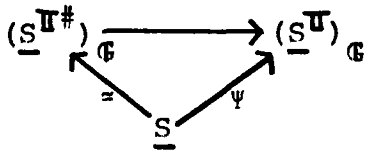
so that $E' = \phi$. This completes the proof.

Now we suppose that $T\phi = \phi$, but that

$\phi \neq E \longrightarrow T1 \overset{Te^\circ}{\underset{Te^1}{\rightrightarrows}} T2$ is an equalizer. Since the underlying functor creates equalizers, it is clear that E has the structure of a $\mathbb{T}$ algebra, so that there is a map $TE \xrightarrow{\mu^{\#}\phi} E$ defining this structure. Also we let $\eta^{\#}\phi: \phi \longrightarrow E$ denote the unique map. Let $T^{\#}$ be defined by $T^{\#}X = \begin{cases} TX & \text{if } X \neq \phi \\ E & \text{if } X = \phi \end{cases}$. It is trivial to check that with $\eta^{\#}X = \eta X$ and $\mu^{\#}X = \mu X$ for $X \neq \phi$, $\mathbb{T}^{\#} = (T^{\#}, \eta^{\#}, \mu^{\#})$ becomes a triple on $\underline{S}$. Moreover, $T^{\#}\phi \neq \phi$. Clearly every $\mathbb{T}^{\#}$-algebra is also a $\mathbb{T}$-algebra, since E is the only new free algebra and it was already an algebra. This defines a functor $\underline{S}^{\mathbb{T}^{\#}} \longrightarrow \underline{S}^{\mathbb{T}}$ which is easily checked to be full, faithful and almost onto. The only difference is that ϕ is a model for $\mathbb{T}$ but not for $\mathbb{T}^{\#}$. $\underline{S}^{\mathbb{T}^{\#}}$ has an initial object E, while in $\underline{S}^{\mathbb{T}}$, E is not initial but has a single predecessor ϕ. Also the induced cotriples on each category commute (on the nose!) with the inclusion $\underline{S}^{\mathbb{T}^{\#}} \longrightarrow \underline{S}^{\mathbb{T}}$. From this it follows that $(\underline{S}^{\mathbb{T}^{\#}})_{\mathbb{G}} \longrightarrow (\underline{S}^{\mathbb{T}})_{\mathbb{G}}$ is full, faithful and almost onto, the only difference, again, being that ϕ is a coalgebra in $(\underline{S}^{\mathbb{T}})_{\mathbb{G}}$. The failure of the diagram



to commute exactly — $F^{\#}\phi = E$, $F\phi = \phi$ — also makes



not quite commute. In fact the comparison Ψ is such that $\Psi\phi = \phi$. Thus $(\underline{S}^{\mathbb{T}})_{\mathbb{G}}$ is as in the following.

Theorem 10.

Suppose $T\phi = \phi$, but the equalizer $E \longrightarrow T1 \rightrightarrows T2$ is non-empty. Then $(\underline{S}^{\mathbb{T}})_{\mathbb{G}}$ may be described as $\underline{S} \cup \{\gamma\}$ where $(\phi,\gamma) = (\gamma,\gamma) = 1$ and $(X,\gamma) = \phi$ for any other $X \in \underline{S}$. The comparison $\underline{S} \xrightarrow{\Psi} (\underline{S}^{\mathbb{T}})_{\mathbb{G}}$ is just the inclusion functor. Moreover, this induced triple on $\underline{S} \cup \{\gamma\}$ has $\underline{S}^{\mathbb{T}}$ as its algebras.

Note that if $\mathbb{T}^{\#}$ is a triple with $T^{\#}\phi \neq \phi$, we may define $\mathbb{T}$ by $T\phi = \phi$ and get a 1-1 correspondence between triples for which $T\phi = \phi$, $E \neq \phi$ and these for which $T\phi \neq \phi$ (in which case it is easily seen-- owing to the lack of empty models-- that $T\phi = E$). On the other hand, if $E = \phi$ (as when $\mathbb{T} = \mathbb{P}$), this can't happen. E is called the set of pseudo-constants. This is justified by the fact that it is the set of constant natural transformations of $U \longrightarrow U$ (whereas $T\phi$ is the set of natural transformations of $U^{\phi} \longrightarrow U$ -- the indeterminancy of $0°$ rears its ugly head again). Then we may restate our results.

Theorem 11.

If the set of pseudo-constants of $\mathbb{T}$ is equal to the set of constants, then $(\underline{S}^{\mathbb{T}})_{\mathbb{G}} = \underline{S}$. Otherwise it is $\underline{S} \cup \{\gamma\}$ as above.

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