

Fibrations of bicategories

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Abstract

The main purpose of this work is to describe a generalization of the Grothendieck construction for pseudofunctors which provides a biequivalence between the 2-categories of pseudofunctors and fibrations. For any bicategory \mathcal{B} , we describe a tricategory of \mathcal{B} -indexed bicategories which are the weakest possible generalization of pseudofunctors to the immediate next level. Then we introduce a notion of a fibration of bicategories, which generalizes 2-fibrations of strict 2-categories defined by Hermida. Finally we give an explicit construction of an analog of the Grothendieck construction which provides a triequivalence between the 3-category of \mathcal{B} -indexed bicategories and the 3-category of 2-fibrations over \mathcal{B} .

1 Introduction

The main purpose of this paper is to introduce the notion of a *fibration of bicategories* as a natural generalization of many existing familiar notions existing in the category theory. This theory has its historical origins in two separate lines of mathematical development: *fibration of categories* or *fibred categories*, introduced by Grothendieck [29] for purely geometric reasons, and *bicategories*, introduced by Bénabou [7], as the algebraic structures which are the weakest possible generalization of categories, to the immediate next level.

The first coming-together of these two lines of development can be traced in mostly unpublished work by Bénabou, in his investigations of logical aspects of fibred categories, and his attempt to give foundations of naive category theory [8] by using fibred categories.

It was already pointed by Street that fibrations of bicategories are quite different from *fibrations in bicategories*, in the paper [53] bearing that name. The simple observation that functors which are equivalences of categories are not necessarily fibred categories, led him to introduce a bicategorical notion of two-sided fibration, which generalized his earlier notion of two-sided fibrations in 2-categories [52]. These fibrations are certain spans in a bicategory, and by using this construction, it was possible to express fibred and cofibred categories [25] as such spans in a 2-category Cat of small categories.

The main distinction between fibrations of bicategories and fibrations in bicategories from the previous paragraph is that fibrations of bicategories should be seen as fibrations in the tricategory $Bicat$ of bicategories, their homomorphisms, pseudonatural transformations and modifications. *Tricategories* were introduced by Gordon, Power and Street [24] as the weakest possible generalization of bicategories, on the 3-dimensional level. Their motivation to introduce tricategories came from several different sources. One of these sources was within category theory and Walters's work on categories enriched in a bicategory [54], [55] which required *monoidal bicategories* which are precisely one object tricategories. Another example of such a structure is the bicategory of relations in a regular category explored by Carboni and Walters [17]. There were also another sources of motivation to introduce tricategories from outside category theory. Namely, homotopy 3-types of Joyal and Tierney [41] are special types of tricategories. Monoidal bicategories bear the same relation to the algebraic aspects of Knizhnik-Zamolodchikov equations [35], [36] analogous to the relation between monoidal categories and Yang-Baxter equation.

Gordon, Power and Street defined *trihomomorphisms* between tricategories as the weakest possible generalization of homomorphisms of bicategories, on the 3-dimensional level. Special types of trihomomorphisms, whose domain is a fixed bicategory and codomain the tricategory $Bicat$, are generalization of pseudofunctors, extensively used by Grothendieck, who showed the biequivalence between them and fibred categories. Such special trihomomorphisms will be called *indexed bicategories*, as they are natural generalization of *indexed categories* used by Janelidze, Schumacher and Street [39] in the context of categorical Galois theory. Indexed bicategories will play a fundamental role in this paper since they are objects of a tricategory which is triequivalent to the 3-category of fibrations of bicategories.

This triequivalence is a generalization of the famous Grothendieck construction [29], which for any category \mathcal{C} , provides a biequivalence

$$j: \text{Cat}^{\mathcal{C}^{op}} \rightarrow \text{Fib}(\mathcal{C}) \quad (1.1)$$

between the 2-category $\text{Cat}^{\mathcal{C}^{op}}$ whose objects are *indexed bicategories* or *pseudofunctors* $P: \mathcal{C}^{op} \rightarrow \text{Cat}$, and the 2-category $\text{Fib}(\mathcal{C})$ of fibered categories over \mathcal{C} , introduced in [29]. The fibered category $F: \mathcal{E} \rightarrow \mathcal{C}$ is the functor, which is characterized by the property which associates to any pair (f, E) , where $f: X \rightarrow Y$ is a morphism in \mathcal{C} and E is an object in \mathcal{E} such that $F(E) = X$, a cartesian morphism $\tilde{f}: F \rightarrow E$, having certain universal property, such that $F(\tilde{f}) = f$. After we choose a cartesian lifting for each such pair, their universal property gives rise to the coherence law for the natural transformations which are part of the data for the obtained pseudofunctor. Moreover, for any pseudofunctor $P: \mathcal{C}^{op} \rightarrow \text{Cat}$, this coherence laws are precisely responsible for the associativity and unit laws of the composition in the category $\int_{\mathcal{C}} P$ obtained from the Grothendieck construction.

In this article, we introduce the generalization of the Grothendieck construction, which for any bicategory \mathcal{B} , provides a triequivalence

$$j: \text{Bicat}^{\mathcal{B}^{coop}} \rightarrow 2\text{Fib}(\mathcal{B}) \quad (1.2)$$

between a tricategory $\text{Bicat}^{\mathcal{B}^{coop}}$ whose objects are indexed bicategories $P: \mathcal{B}^{coop} \rightarrow \text{Bicat}$, and the 3-category $2\text{Fib}(\mathcal{B})$ of 2-fibrations of bicategories (or fibered bicategories) over \mathcal{B} . The objects of the 3-category $2\text{Fib}(\mathcal{B})$ are strict homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$, again characterized by the existence of cartesian liftings, now for both 1-morphisms and 2-morphisms. In the strict case, when both \mathcal{E} and \mathcal{B} are strict 2-categories, the definition of such fibrations was given by Hermida in [33]. Although he did not define explicitly 2-fibrations for bicategories, in his subsequent paper [34], he proposed their definition by using the bireflection of the category Bicat of bicategories and their homomorphisms into the category 2-Cat of (strict) 2-categories and (strict) 2-functors. Here we give their explicit definition, using properties of certain biadjunctions introduced by Gray in his monumental work [27].

There are two major reasons why we restricted our attention to strict homomorphisms of bicategories. The first one is that these are the morphisms of a category Bicat_s , which is essentially algebraic in the sense of Freyd [21], and as such Bicat_s is both complete and cocomplete, and in fact locally finitely presentable [1]. More precisely, Bicat_s is a category of Eilenberg-Moore algebras for some monad K_2 on the category Cat-Gph of Cat-graphs. For any symmetrical monoidal category \mathcal{V} satisfying certain (co)completeness conditions, Wolff showed in [56] that the forgetful functor $U_{\mathcal{V}}: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Gph}$ is monadic, with a left adjoint $F_{\mathcal{V}}: \mathcal{V}\text{-Gph} \rightarrow \mathcal{V}\text{-Cat}$. This was the starting point of Batanin's work [6] who gave an explicit description of a monad K_n on $n\text{-Gph}$, defined by $U_n F_n: n\text{-Gph} \rightarrow n\text{-Gph}$ in the special case when \mathcal{V} is a symmetrical monoidal category $n\text{-Gph}$ of n -graphs. In this way, Batanin realized weak n -categories as objects in a category of Eilenberg-Moore algebras for such monads, whose morphisms are strict, preserving all their structure precisely.

The practical importance of the fact that the category $Bicat_s$ of bicategories and their strict homomorphisms is both complete and cocomplete, unlike the category $Bicat$, is that it allows us to deal with strict fibers of any strict homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$ of bicategories. *Strict fibers* of such homomorphisms are defined as their (strict) pullbacks in $Bicat_s$ by a strict homomorphism from the free bicategory on one object. If we would have been working instead with general homomorphisms of bicategories, as morphisms of the category $Bicat$, we would be forced to deal with *homotopy fibers* which would inevitably cause many (unnecessary) technical difficulties in our constructions.

This leads us to the second, even more important reason, why we restricted our attention to strict homomorphisms of bicategories. Even when we apply the Grothendieck construction (6.2) to the most general indexed bicategory $P: \mathcal{B}^{coop} \rightarrow Bicat$, whose codomain is the tricategory $Bicat$ of homomorphisms of bicategories, pseudonatural transformations and modifications, we still obtain a fibration of bicategories which is strict homomorphism of bicategories! On the other side, the same result is obtained when we apply the Grothendieck construction (6.2) to an indexed 2-category $P: \mathcal{B}^{coop} \rightarrow 2\text{-Cat}$ whose codomain is the 3-category 2-Cat of 2-categories and 2-functors, 2-transformations and modifications.

Although it may seem that strict homomorphisms of bicategories are rare in practice in category theory or its applications, they occur quite naturally in the homotopy theory, were Lack used them [44] in order to introduce a closed model structure on the category 2-Cat of 2-categories and 2-functors, which he extended to the category $Bicat_s$ of bicategories and their strict homomorphisms, in his subsequent paper [45]. These model structures are closely related to the model structure of Moerdijk and Svensson in [51], and our fibrations of bicategories naturally fit in these examples. Strict homomorphisms were also used by Hardie, Kamps and Kieboom who defined fibrations of bigroupoids in [32], generalizing the notion of fibration of 2-groupoids by Moerdijk [50]. They used Brown's construction [11] in order to derive an exact nine term sequence from such fibrations, and they applied their theory to the construction of a homotopy bigroupoid of a topological space [31].

One of the first author's motivations for this work, was to use the Grothendieck construction (6.2) as an interpretation theorem for the third nonabelian cohomology $\mathcal{H}^3(\mathcal{B}, \mathcal{K})$ of a bigroupoid \mathcal{B} with coefficients in a bundle of 2-groups \mathcal{K} . In the case of groupoids, this approach was taken in [9], where the second nonabelian cohomology of a groupoid \mathcal{G}

$$\mathcal{H}^2(\mathcal{G}, \mathcal{K}) := [\mathcal{G}, AUT(\mathcal{K})]$$

is the set $[\mathcal{G}, AUT(\mathcal{K})]$ of connected components of pseudofunctors $P: \mathcal{G}^{op} \rightarrow AUT(\mathcal{K})$ to the full sub 2-groupoid $AUT(\mathcal{K})$ of the 2-groupoid Gpd of groupoids, induced naturally by the bundle of groups \mathcal{K} over the objects G_0 of the groupoid \mathcal{G} . Then the Grothendieck construction (6.1) was used as an interpretation theorem for the classification of fibrations of groupoids, seen as short exact sequences of groupoids

$$1 \longrightarrow K \longrightarrow G \longrightarrow B \longrightarrow 1$$

over the same underlying set of objects M , by means of the higher Schreier theory.

2 Bicategories and their homomorphisms

Bicategories were originally introduced by Bénabou [7]. In this section we will not follow Bénabou's original presentation of bicategories, but rather we will use a more recent approach based on simplicial techniques.

Definition 2.1. *Skeletal simplicial category Δ consists of the following data:*

- *objects are finite nonempty ordinals $[n] = \{0 < 1 < \dots < n\}$,*
- *morphisms are monotone maps $f: [n] \rightarrow [m]$, which for all $i, j \in [n]$ such that $i \leq j$, satisfy $f(i) \leq f(j)$.*

We also call Δ the topologist's simplicial category, and this is a full subcategory of the algebraist's simplicial category $\bar{\Delta}$, which has an additional object $[-1] = \emptyset$, given by a zero ordinal, that is an empty set.

Skeletal simplicial category Δ may be presented by means of generators as in a diagram

$$[0] \begin{array}{c} \xrightarrow{\partial_1} \\ \xleftarrow{\partial_0} \end{array} [1] \begin{array}{c} \xrightarrow{\partial_2} \\ \xleftarrow{\partial_0} \end{array} [2] \begin{array}{c} \xrightarrow{\partial_3} \\ \xleftarrow{\partial_0} \end{array} [3]$$

and relations given by the maps $\partial_i: [n-1] \rightarrow [n]$ for $0 \leq i \leq n$, called coface maps, which are injective maps that omit i in the image, and the maps $\sigma_i: [n] \rightarrow [n-1]$ for $0 \leq i \leq n-1$, called codegeneracy maps, which are surjective maps which repeat i in the image. These maps satisfy following cosimplicial identities:

$$\begin{aligned} \partial_j \partial_i &= \partial_i \partial_{j-1} & (i < j) \\ \sigma_j \sigma_i &= \sigma_i \sigma_{j+1} & (i \leq j) \\ \sigma_j \partial_i &= \partial_i \sigma_{j-1} & (i < j) \\ \sigma_j \partial_i &= id & (i = j, i = j + 1) \\ \sigma_j \partial_i &= \partial_i \sigma_{j+1} & (i > j + 1). \end{aligned} \tag{2.1}$$

We will use the following factorization of monotone maps by cofaces and codegeneracies.

Lemma 2.2. *Any monotone map $f: [m] \rightarrow [n]$ has a unique factorization given by*

$$f = \partial_{i_1}^n \partial_{i_2}^{n-1} \dots \partial_{i_s}^{n-s+1} \sigma_{j_t}^{m-t} \dots \sigma_{j_2}^{m-2} \sigma_{j_1}^{m-1} \tag{2.2}$$

where $0 \leq i_s < i_{s-1} < \dots < i_1 \leq n$, $0 \leq j_t < j_{t-1} < \dots < j_1 \leq m$ and $n = m - t + s$.

Proof. The proof is a direct consequence of the injective-surjective factorization in Set and cosimplicial identities. \square

Definition 2.3. *An internal simplicial object S in a category \mathcal{C} is a presheaf $S: \Delta^{op} \rightarrow \mathcal{C}$ on the skeletal simplicial category Δ with values in \mathcal{C} .*

We can regard each ordinal $[n]$ as a category whose objects are elements of a set $\{0, 1, \dots, n\}$, and such that there exists a unique morphisms from i to j if and only if $i \leq j$. In this way, we obtain a fully faithful embedding from the skeletal category Δ

$$i: \Delta \rightarrow \text{Cat} \quad (2.3)$$

into the category Cat of small categories and functors between them, since a monotone maps between any two ordinals $[k]$ and $[m]$ are precisely functors between corresponding categories $i([k])$ and $i([m])$. We shall write $[\mathcal{C}, \mathcal{D}]$ or $\mathcal{D}^{\mathcal{C}}$ for a category whose objects are functors from \mathcal{C} to \mathcal{D} , and whose morphisms are natural transformations between them. We will denote by \bullet an image of the zeroth ordinal $[0]$ by a fully faithful embedding (2.3), but more often we will abbreviate notation and use the term $[k]$ to denote both the k -th ordinal and the category $i([k])$. We shall always identify categories $\bullet \times \mathcal{C}$ and \mathcal{C}^\bullet with \mathcal{C} , and use two natural embeddings

$$\begin{aligned} \partial_0 \times \mathcal{C}: \mathcal{C} &\rightarrow [1] \times \mathcal{C} \\ \partial_1 \times \mathcal{C}: \mathcal{C} &\rightarrow [1] \times \mathcal{C} \end{aligned} \quad (2.4)$$

For any category \mathcal{C} , we also have two functors from the category $\mathcal{C}^{[1]}$ of *morphisms* of \mathcal{C}

$$\begin{aligned} \mathcal{C}^{\partial_0}: \mathcal{C}^{[1]} &\rightarrow \mathcal{C} \\ \mathcal{C}^{\partial_1} \times \mathcal{C}^{[1]} &\rightarrow \mathcal{C} \end{aligned} \quad (2.5)$$

whose objects are morphisms of \mathcal{C} , and whose morphisms are their commutative squares. We call the two functors (2.5) the *codomain* and the *domain* functor, respectively. More generally, for any ordinal $[n]$ we have a category $\mathcal{C}^{[n]}$ whose objects are strings of n composable morphisms of \mathcal{C} , and whose morphisms are appropriate commutative diagrams. These categories $\mathcal{C}^{[n]}$ together with degeneracy functors $\mathcal{C}^{\sigma_i}: \mathcal{C}^{[n-1]} \rightarrow \mathcal{C}^{[n]}$ for all $0 \leq i \leq n-1$, and face functors $\mathcal{C}^{\partial_i}: \mathcal{C}^{[n]} \rightarrow \mathcal{C}^{[n-1]}$ for all $0 \leq i \leq n$, form an internal simplicial object in Cat .

The category $\mathcal{C}^{[2]}$ of *composable pairs of morphisms* of \mathcal{C} is a following pullback in Cat

$$\begin{array}{ccc} \mathcal{C}^{[2]} & \xrightarrow{\mathcal{C}^{\partial_2}} & \mathcal{C}^{[1]} \\ \mathcal{C}^{\partial_0} \downarrow & & \downarrow \mathcal{C}^{\partial_0} \\ \mathcal{C}^{[1]} & \xrightarrow{\mathcal{C}^{\partial_1}} & \mathcal{C} \end{array} \quad (2.6)$$

and we adopt the following convention: for any functor $P: \mathcal{D} \rightarrow \mathcal{C}$, the first of the symbols

$$\mathcal{E} \times_{\mathcal{C}} \mathcal{D} \quad \text{and} \quad \mathcal{D} \times_{\mathcal{C}} \mathcal{E}$$

will denotes the pullback of P and D_0 , and the second one is the pullback of D_1 and P .

Definition 2.4. A *Cat-graph* \mathcal{B} consists of a discrete category \mathcal{B}_0 of objects, and a category \mathcal{B}_1 of morphisms, together with two functors

$$\mathcal{B}_1 \begin{array}{c} \xrightarrow{D_1} \\ \xrightarrow{D_0} \end{array} \mathcal{B}_0 \quad (2.7)$$

where $D_1: \mathcal{B}_1 \rightarrow \mathcal{B}_0$ is called a *domain functor* and $D_0: \mathcal{B}_1 \rightarrow \mathcal{B}_0$ a *codomain functor*. There exists a category *Cat-Gph* whose objects are *Cat-graphs*, and morphisms $F: \mathcal{B} \rightarrow \mathcal{G}$ of *Cat-graphs* consists of two functors $F_0: \mathcal{B}_0 \rightarrow \mathcal{G}_0$ and $F_1: \mathcal{B}_1 \rightarrow \mathcal{G}_1$ such that diagrams

$$\begin{array}{ccc} \mathcal{B}_1 & \xrightarrow{F_1} & \mathcal{G}_1 \\ \downarrow D_0 & & \downarrow D_0 \\ \mathcal{B}_0 & \xrightarrow{F_0} & \mathcal{G}_0 \end{array} \quad \begin{array}{ccc} \mathcal{B}_1 & \xrightarrow{F_1} & \mathcal{G}_1 \\ \downarrow D_1 & & \downarrow D_1 \\ \mathcal{B}_0 & \xrightarrow{F_0} & \mathcal{G}_0 \end{array} \quad (2.8)$$

commute.

If for any two objects x and y of a *Cat-graph* \mathcal{B} , a category $\mathcal{B}(x, y)$ is defined as a pullback

$$\begin{array}{ccc} \mathcal{B}(x, y) & \xrightarrow{J_{x,y}} & \mathcal{B}_1 \\ \downarrow T & & \downarrow (D_0, D_1) \\ \bullet & \xrightarrow{(x,y)} & \mathcal{B}_0 \times \mathcal{B}_0 \end{array} \quad (2.9)$$

then and morphism $F: \mathcal{B} \rightarrow \mathcal{G}$ of *Cat-graphs* may be seen as a function $F_0: \mathcal{B}_0 \rightarrow \mathcal{G}_0$ together with a family of functors $F_{x,y}: \mathcal{B}(x, y) \rightarrow \mathcal{G}(F_0(x), F_0(y))$ for any two objects x, y in \mathcal{B}_0 . Let the two categories \mathcal{B}_2 and \mathcal{B}_3 be defined as the following two pullbacks in *Cat*

$$\begin{array}{ccc} \mathcal{B}_2 & \xrightarrow{D_0} & \mathcal{B}_1 \\ \downarrow D_2 & & \downarrow D_1 \\ \mathcal{B}_1 & \xrightarrow{D_0} & \mathcal{B}_0 \end{array} \quad \begin{array}{ccc} \mathcal{B}_3 & \xrightarrow{D_0} & \mathcal{B}_1 \\ \downarrow D_3 & & \downarrow D_2 \\ \mathcal{B}_1 & \xrightarrow{D_0} & \mathcal{B}_0 \end{array} \quad (2.10)$$

of horizontally composable pairs and triples, respectively, of 1- and 2-morphisms in \mathcal{B} .

Definition 2.5. A bicategory \mathcal{B} consists of the following data:

- two categories, a discrete category \mathcal{B}_0 of objects, and a category \mathcal{B}_1 of morphisms, whose objects f, g, h, \dots are called 1-morphisms of the bicategory \mathcal{B} and whose morphisms $\phi, \psi, \theta, \dots$ are called 2-morphisms of the bicategory \mathcal{B} , and their composition in the category \mathcal{B}_1 , which we call the vertical composition of the bicategory \mathcal{B} , is denoted by the concatenation $\psi\phi$, whenever the composition of ϕ and ψ is defined
- two functors $D_0, D_1: \mathcal{B}_1 \rightarrow \mathcal{B}_0$, called target and source functors, respectively, a unit functor $I: \mathcal{B}_0 \rightarrow \mathcal{B}_1$, any a horizontal composition functor $H: \mathcal{B}_2 \rightarrow \mathcal{B}_1$, whose value $H(g, f)$ on object (g, f) in \mathcal{B}_2 is denoted by $g \circ f$. For any two objects x and y in \mathcal{B}_0 , we denote by $\mathcal{B}(x, y)$ a hom-category whose objects are 1-morphisms $f: x \rightarrow y$ in \mathcal{B} such that $D_0(f) = y$ and $D_1(f) = x$.
- a natural isomorphism

$$\begin{array}{ccc}
 \mathcal{B}_3 & \xrightarrow{H \times Id_{\mathcal{B}_1}} & \mathcal{B}_2 \\
 \downarrow Id_{\mathcal{B}_1} \times H & \searrow \alpha & \downarrow H \\
 \mathcal{B}_2 & \xrightarrow{H} & \mathcal{B}_1
 \end{array} \tag{2.11}$$

- natural isomorphisms

$$\begin{array}{ccc}
 & \mathcal{B}_2 & \\
 S_1 \nearrow & \downarrow H & \nwarrow S_0 \\
 \mathcal{B}_1 & \xrightarrow{H} & \mathcal{B}_1
 \end{array} \tag{2.12}$$

where the functor $S_0: \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is defined by the composition

$$\mathcal{B}_1 \xrightarrow{(D_0, Id_{\mathcal{B}_1})} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_0 \xrightarrow{I \times Id_{\mathcal{B}_1}} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1,$$

and the functor $S_1: \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is defined by the composition

$$\mathcal{B}_1 \xrightarrow{(Id_{\mathcal{B}_1}, D_1)} \mathcal{B}_0 \times_{\mathcal{B}_0} \mathcal{B}_1 \xrightarrow{Id_{\mathcal{B}_1} \times I} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1,$$

so that the values of these two functors on any 1-morphism $f: x \rightarrow y$ in \mathcal{B} are $S_0(f) = (f, i_x)$ and $S_1(f) = (i_y, f)$, respectively, and similarly for 2-morphisms in \mathcal{B} .

This data are required to satisfy the following axioms:

- for any object (k, h, g, f) in \mathcal{B}_4 we have the following commutative pentagon

$$\begin{array}{ccc}
 & ((k \circ h) \circ g) \circ f & \\
 \alpha_{k, h, g \circ f} \swarrow & & \searrow \alpha_{k \circ h, g, f} \\
 (k \circ (h \circ g)) \circ f & & (k \circ h) \circ (g \circ f) \\
 \alpha_{k, h \circ g, f} \searrow & & \swarrow \alpha_{k, h, g \circ f} \\
 k \circ ((h \circ g) \circ f) & \xrightarrow{k \circ \alpha_{h, g, f}} & k \circ (h \circ (g \circ f))
 \end{array} \tag{2.13}$$

- for any object (g, f) in \mathcal{B}_2 we have the following commutative triangle

$$\begin{array}{ccc}
 (g \circ i_y) \circ f & \xrightarrow{\alpha_{g, i_y, f}} & g \circ (i_y \circ f) \\
 \rho_{g \circ f} \searrow & & \swarrow g \circ \lambda_f \\
 & g \circ f &
 \end{array} \tag{2.14}$$

Remark 2.6. Note that a definition of the horizontal composition functor $H: \mathcal{B}_2 \rightarrow \mathcal{B}_1$ implies that for any composable pair of morphisms in the category \mathcal{B}_2 , as in a diagram

$$\begin{array}{ccccc}
 & f_1 & & f_2 & \\
 & \curvearrowright & & \curvearrowright & \\
 x & \xrightarrow{\quad} & y & \xrightarrow{\quad} & z \\
 & \Downarrow \phi_1 & & \Downarrow \psi_1 & \\
 & \Downarrow \phi_2 & & \Downarrow \psi_2 & \\
 & \curvearrowleft & & \curvearrowleft & \\
 & h_1 & & h_2 &
 \end{array}$$

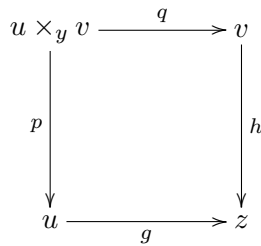
we have a following Godement interchange law

$$(\psi_2 \circ \psi_1)(\phi_2 \circ \phi_1) = (\psi_2 \psi_1) \circ (\phi_2 \phi_1). \tag{2.15}$$

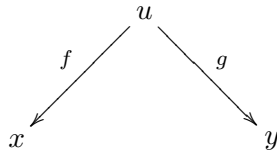
Example 2.7. (*Strict 2-categories*) A bicategory in which associativity and left and right identity natural isomorphisms are identities is called a (strict) 2-category. A basic example is the 2-category Cat of small categories, functors and natural transformations.

Example 2.8. (*Monoidal categories*) Any monoidal category \mathcal{M} may be seen as a bicategory for which the discrete category of objects is a terminal category \bullet with only one object and one morphism. Consequently, any strict monoidal category \mathcal{M} may be seen as a strict 2-category with a terminal category \bullet of objects. Then it is clear that for any object x in the bicategory \mathcal{B} , a hom-category $\mathcal{B}(x, x)$ is a (weak) monoidal category, and similarly for any object x in the strict 2-category \mathcal{C} , a hom-category $\mathcal{C}(x, x)$ is a (strict) monoidal category.

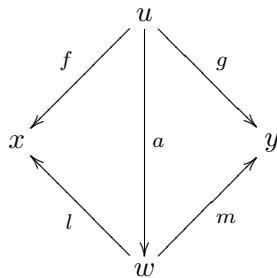
Example 2.9. (*Bicategory of spans*) Let \mathcal{C} be a cartesian category, that is a category with pullbacks. First we make a choice of the pullback



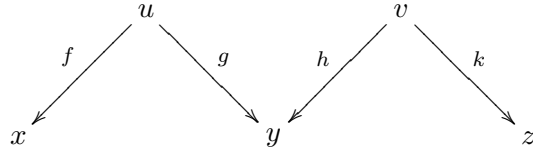
for any such diagram $x \xrightarrow{f} z \xleftarrow{g} y$ in a category \mathcal{C} . Then there exists a bicategory $\text{Span}(\mathcal{C})$ of spans in the category \mathcal{C} whose objects are the same as objects of \mathcal{C} . For any two objects x, y in $\text{Span}(\mathcal{C})$, a 1-morphism $u: x \rightrightarrows y$ is a span



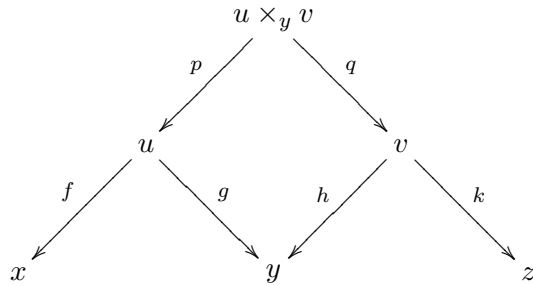
and a 2-morphism $a: u \rightrightarrows w$ between any two such spans is given by a commutative diagram



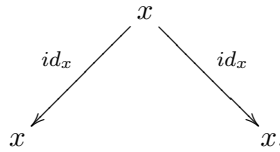
The vertical composition of 2-morphisms is given by the composition in \mathcal{C} . The horizontal composition of 1-morphisms



is given by the pullback



and the horizontal identity $i_x: x \rightrightarrows x$ for this composition is a span



It is obvious that the horizontal composition is not strictly associative, but it follows from the universal property of pullbacks that it is associative up to canonical isomorphisms which define components of the associativity coherence for given choices of pullbacks.

The following two examples of bicategories are important and they can be both applied to the special case of bicategory $\text{Span}(\mathcal{C})$ of spans in a cartesian category \mathcal{C} from the previous example.

Example 2.10. (*Bimodules*) Let Bim denote the bicategory whose objects are rings with identity. For any two rings A and B , $\text{Bim}(A, B)$ will be a category of A - B -bimodules and their homomorphisms. Horizontal composition is given by the tensor product, and associativity and identity constraints are the usual ones for the tensor product.

Example 2.11. (*Bimodules internal in a bicategory*) Let \mathcal{B} be a bicategory, whose hom-categories have coequalizers, which are preserved by a (pre)composition with 1-morphisms in \mathcal{B} . Then a construction from a previous example can be extended to such bicategory \mathcal{B} in the following way. There exists a bicategory $\text{Bim}(\mathcal{B})$ whose objects are pairs (x, M) ,

where x is an object in the bicategory \mathcal{B} , and M is a monoid in a monoidal hom-category $\mathcal{B}(x, x)$. This means that the 1-morphism $M: x \rightarrow x$ is supplied with two 2-morphisms $\mu_M: M \circ M \Rightarrow M$ and $\epsilon_M: i_x \Rightarrow M$ in $\mathcal{B}(x, x)$, such that the following diagrams

$$\begin{array}{ccc}
(M \circ M) \circ M & \xrightarrow{\mu_M \circ \iota_M} & M \circ M \\
\Downarrow \alpha_{M, M, M} & & \Downarrow \mu_M \\
M \circ (M \circ M) & \xrightarrow{\iota_M \circ \mu_M} M \circ M \xrightarrow{\mu_M} & M
\end{array}$$

$$\begin{array}{ccccc}
i_x \circ M & \xrightarrow{\epsilon_M \circ \iota_M} & M \circ M & \xleftarrow{\iota_M \circ \epsilon_M} & M \circ i_x \\
\Downarrow \lambda_M & & \Downarrow \mu_M & & \Downarrow \rho_M \\
& & M & &
\end{array}$$

commute. We will usually refer to a monoid M in $\mathcal{B}(x, x)$ without mentioning explicitly the these two 2-morphisms. Then for any monoid M in $\mathcal{B}(x, x)$, and any monoid N in $\mathcal{B}(y, y)$, a 1-morphism $L: x \rightarrow y$ is called a N - M -bimodule if there are 2-morphisms $\phi: N \circ L \Rightarrow L$ and $\psi: L \circ M \Rightarrow L$ such that the diagrams

$$\begin{array}{ccc}
(N \circ N) \circ L & \xrightarrow{\mu_N \circ \iota_L} & N \circ L \\
\Downarrow \alpha_{N, N, L} & & \Downarrow \phi \\
N \circ (N \circ L) & \xrightarrow{\iota_N \circ \phi} N \circ L \xrightarrow{\phi} & L
\end{array}$$

$$\begin{array}{ccc}
(L \circ M) \circ M & \xrightarrow{\psi} & L \circ M \\
\Downarrow \alpha_{L, M, M} & & \Downarrow \psi \\
L \circ (M \circ M) & \xrightarrow{\iota_L \circ \mu_M} L \circ M \xrightarrow{\phi} & L
\end{array}$$

$$\begin{array}{ccc}
(N \circ L) \circ M & \xrightarrow{\phi \circ \iota_M} & L \circ M \\
\Downarrow \alpha_{N, L, M} & & \Downarrow \psi \\
N \circ (L \circ M) & \xrightarrow{\iota_M \circ \psi} M \circ L \xrightarrow{\phi} & M
\end{array}$$

$$\begin{array}{ccc}
i_y \circ L & \xrightarrow{\epsilon_{N \circ L}} & N \circ L \\
& \searrow \lambda_L & \downarrow \phi \\
& & L
\end{array}
\qquad
\begin{array}{ccc}
L \circ M & \xleftarrow{\iota_{L \circ M}} & L \circ i_x \\
& \downarrow \psi & \swarrow \rho_L \\
& & L
\end{array}$$

commute. Then a 1-morphism from (x, M) to (y, N) in $Bim(\mathcal{B})$ is an N - M -bimodule L . A 2-morphism between N - M -bimodules K and L in $Bim(\mathcal{B})$ is a 2-morphism $\nu: K \Rightarrow L$ in \mathcal{B} such that diagrams

$$\begin{array}{ccc}
N \circ K & \xrightarrow{\kappa} & K \\
\downarrow \iota_{N \circ \nu} & & \downarrow \nu \\
N \circ L & \xrightarrow{\phi} & L
\end{array}
\qquad
\begin{array}{ccc}
K \circ M & \xrightarrow{\theta} & K \\
\downarrow \nu \circ \iota_M & & \downarrow \nu \\
L \circ M & \xrightarrow{\psi} & L
\end{array}$$

commute. For any N - M -bimodule $L: x \rightarrow y$, which we denote by ${}_N L_M$, and for any P - N -bimodule $T: y \rightarrow z$, denoted by ${}_P T_N$, their horizontal composition is a tensor product

$${}_P T_N \otimes_N L_M$$

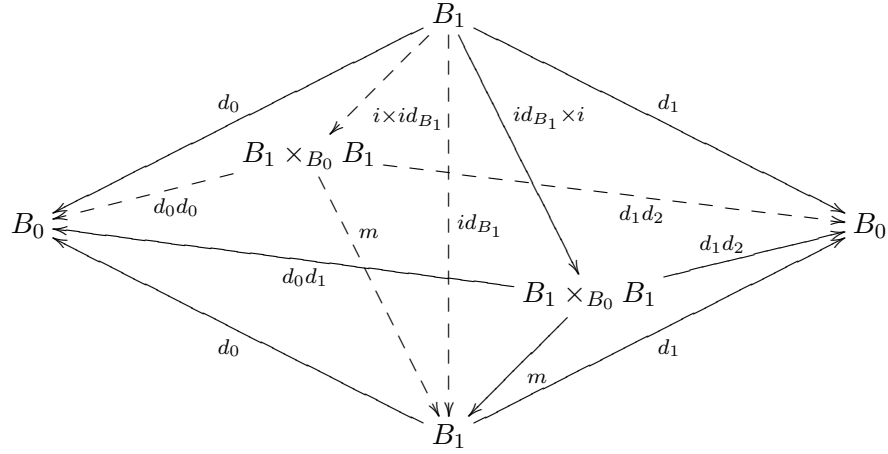
which is a P - M -bimodule $T \otimes_N L: x \rightarrow z$ obtained as a coequalizer in the category $\mathcal{B}(x, z)$

$$T \circ N \circ L \begin{array}{c} \xrightarrow{\psi \circ L} \\ \xrightarrow{\quad \quad} \\ \xrightarrow{T \circ \phi} \end{array} T \circ L \xrightarrow{e} T \otimes_N L.$$

The composition of 2-morphisms is induced by the universal property of the coequalizer, and the associativity coherence follows since these coequalizers are preserved by a composition with 1-morphisms in \mathcal{B} .

Remark 2.12. If the monoidal category Ab of abelian groups, is seen as a bicategory ΣAb with one object, with the usual tensor product of abelian groups as a horizontal composition, then the construction of a bicategory $Bim(\Sigma Ab)$ from the previous example yields a usual bicategory Bim of bimodules from Example 1.9, so that we have an identity of bicategories

$$Bim(\Sigma Ab) = Bim$$



This means that the span (2.16) is an underlying graph of an internal category \mathcal{B} in \mathcal{E} .

Therefore, the objects of the bicategory $Bim(Span(\mathcal{E}))$ are internal categories in \mathcal{E} . For any two internal categories \mathcal{B} and \mathcal{C} , a 1-morphism from \mathcal{B} to \mathcal{C} is a span

$$\begin{array}{ccc}
 & K_0 & \\
 \mu_0 \swarrow & & \searrow \mu_1 \\
 B_0 & & C_0
 \end{array} \tag{2.17}$$

together with a morphism $l: B_1 \times_{B_0} K_0 \rightarrow K_0$ which is a part of a morphism of spans

$$\begin{array}{ccccc}
 & & B_1 \times_{B_0} K_0 & & \\
 & d_0 pr_1 \swarrow & \downarrow l & \searrow \mu_1 pr_2 & \\
 B_0 & & K_0 & & C_0 \\
 & \mu_0 \swarrow & & \searrow \mu_1 & \\
 & & & &
 \end{array}$$

and a morphism $r: K_0 \times_{C_0} C_1 \rightarrow K_0$ which is a part of the following morphism of spans

$$\begin{array}{ccccc}
 & & K_0 \times_{C_0} C_1 & & \\
 & \mu_0 pr_1 \swarrow & \downarrow r & \searrow d_1 pr_2 & \\
 B_0 & & K_0 & & C_0 \\
 & \mu_0 \swarrow & & \searrow \mu_1 & \\
 & & & &
 \end{array}$$

Obviously, these two morphisms are left and right action of internal categories \mathcal{B} and \mathcal{C} respectively, and by constructing the category of elements \mathcal{K} for these two actions, one

obtains the span of internal categories which is a discrete fibration [52], [53] from \mathcal{B} to \mathcal{C}

$$\begin{array}{ccc} & \mathcal{K} & \\ D_0 \swarrow & & \searrow D_1 \\ \mathcal{B} & & \mathcal{C} \end{array} \tag{2.18}$$

Finally, 2-morphisms between such discrete fibrations are just morphisms between their underlying spans. Therefore, the bicategory $Bim(Span(\mathcal{E}))$ is isomorphic to the bicategory $Prof$ of profunctors or distributors.

The following result is a typical instance of the coherence theorem for bicategories, which roughly says that any diagram in the bicategory which is made of associativity, left and right identity coherence isomorphisms must commute [49].

Theorem 2.13. *Let \mathcal{B} be a bicategory. Then the diagrams*

$$\begin{array}{ccc}
 (i_z \circ g) \circ f & \xrightarrow{\alpha_{i_z, g, f}} & i_z \circ (g \circ f) \\
 \searrow \lambda_g \circ f & & \swarrow i_z \circ \lambda_{g \circ f} \\
 & g \circ f &
 \end{array}
 \qquad
 \begin{array}{ccc}
 (g \circ f) \circ i_x & \xrightarrow{\alpha_{g, f, i_x}} & g \circ (f \circ i_x) \\
 \searrow \rho_{g \circ f} & & \swarrow g \circ \rho_f \\
 & g \circ f &
 \end{array}
 \tag{2.19}$$

commute for any pair of 1-morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in \mathcal{B} .

Proof. For any triple of 1-morphisms $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$, we consider the diagram

$$\begin{array}{ccccc}
 & & ((h \circ i_z) \circ g) \circ f & & \\
 & \swarrow \alpha_{h, i_z, g \circ f} & \downarrow (\rho_h \circ g) \circ f & \searrow \alpha_{h \circ i_z, g, f} & \\
 (h \circ (i_z \circ g)) \circ f & \xrightarrow{(h \circ \lambda_g) \circ f} & (h \circ g) \circ f & & (h \circ i_z) \circ (g \circ f) \\
 \searrow \alpha_{h, i_z \circ g, f} & & \downarrow \alpha_{h, g, f} & \swarrow \rho_h \circ (g \circ f) & \\
 & & h \circ (g \circ f) & & \\
 \swarrow h \circ (\lambda_g \circ f) & & \downarrow h \circ \lambda_{g \circ f} & \searrow \alpha_{h, i_z, g \circ f} & \\
 h \circ ((i_z \circ g) \circ f) & \xrightarrow{h \circ \alpha_{i_z, g, f}} & h \circ (i_z \circ (g \circ f)) & &
 \end{array}$$

in which two triangles (beside the bottom one) commute because of the triangle coherence for identities, and two deformed squares commute by the naturality of associativity coherence. Since all the terms are 2-isomorphisms, then the bottom triangle also commutes. By taking $h = i_z$, we obtain the identity

$$i_z \circ (\lambda_{g \circ f} \alpha_{i_z, g, f}) = i_z \circ (\lambda_g \circ f)$$

from which it follows that the back face of the cube

$$\begin{array}{ccccc}
& & i_z \circ ((i_z \circ g) \circ f) & \xrightarrow{i_z \circ (\lambda_{g \circ f})} & i_z \circ (g \circ f) \\
& \swarrow \lambda_{(i_z \circ g) \circ f} & \downarrow & \swarrow \lambda_{g \circ f} & \parallel \\
(i_z \circ g) \circ f & \xrightarrow{\lambda_{g \circ f}} & g \circ f & & \\
\downarrow \alpha_{i_z, g, f} & & \downarrow i_z \circ \alpha_{i_z, g, f} & & \parallel \\
& & i_z \circ (i_z \circ (g \circ f)) & \xrightarrow{i_z \circ \lambda_{g \circ f}} & i_z \circ (g \circ f) \\
& \swarrow \lambda_{i_z \circ (g \circ f)} & \downarrow & \swarrow \lambda_{g \circ f} & \parallel \\
i_z \circ (g \circ f) & \xrightarrow{\lambda_{g \circ f}} & g \circ f & &
\end{array}$$

commutes. The top, bottom and right faces commute from the naturality of the left identity coherence, and the right face commutes trivially. Since all edges are 2-isomorphisms we conclude that the front face also commutes, which proves that the first triangle in lemma commutes. Similarly, we prove the commutativity of the other triangle. \square

The statement preceding the theorem, can be made more precise as an instance of a *supercoherence* introduced by Jardine in [40]. For a bicategory \mathcal{B} , and any ordinal $[n]$, we can consider the category \mathcal{B}_n of horizontally composable strings of 1-morphisms in \mathcal{B} , in an analogy with (2.10). A *string* from $[n_0]$ to $[n_k]$ is a factorization of a map $f: [n_0] \rightarrow [n_k]$

$$n_0 \xrightarrow{\theta_1} n_1 \xrightarrow{\theta_2} \dots \xrightarrow{\theta_k} n_k \quad (2.20)$$

in the category Δ , such that each θ_i is either coface or codegeneracy, but (2.20) does not necessarily have a canonical form (2.2). Then if we define *face functors* $D_i: \mathcal{B}_n \rightarrow \mathcal{B}_{n-1}$ by

$$D_i(f_n, f_{n-1}, \dots, f_2, f_1) = \begin{cases} (f_n, f_{n-1}, \dots, f_3, f_2) & i = 0 \\ (f_n, \dots, f_{i+1} \circ f_i, \dots, f_1) & 0 < i < n \\ (f_{n-1}, f_{n-2}, \dots, f_2, f_1) & i = n \end{cases} \quad (2.21)$$

by composing out i^{th} object, and if we define the *degeneracy functors* $S_i: \mathcal{B}_n \rightarrow \mathcal{B}_{n+1}$ by

$$S_i(f_n, f_{n-1}, \dots, f_2, f_1) = \begin{cases} (f_n, f_{n-1}, \dots, f_1, i_{x_0}) & i = 0 \\ (f_n, \dots, f_{i+1}, id_{x_i}, f_i, \dots, f_1) & 0 < i < n \\ (i_{x_n}, f_n, \dots, f_2, f_1) & i = n \end{cases} \quad (2.22)$$

expanding the i^{th} object by its identity morphism, the collection of categories \mathcal{B}_n and face and degeneracy functors form a *supercoherent structure*, which we will make precise by first introducing a notion of a homomorphism of bicategories.

Definition 2.14. A homomorphism $F: \mathcal{B} \rightarrow \mathcal{B}'$ of bicategories consists of the following data:

- a (discrete) functor $F_0: \mathcal{B}_0 \rightarrow \mathcal{B}'_0$, and a functor $F_1: \mathcal{B}_1 \rightarrow \mathcal{B}'_1$,
- two natural transformations

$$\begin{array}{ccc}
\mathcal{B}_2 & \xrightarrow{F_2} & \mathcal{B}'_2 \\
\downarrow H & \swarrow \mu & \downarrow H' \\
\mathcal{B}_1 & \xrightarrow{F_1} & \mathcal{B}'_1
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{B}_0 & \xrightarrow{F_0} & \mathcal{B}'_0 \\
\downarrow I & \swarrow \eta & \downarrow I' \\
\mathcal{B}_1 & \xrightarrow{F_1} & \mathcal{B}'_1
\end{array}
\tag{2.23}$$

given by components $\mu_{g,f}: F(g) \circ F(f) \Rightarrow F(g \circ f)$ and $\eta_x: i'_{F(x)} \Rightarrow F(i_x)$, respectively.

These data are required to satisfy the following axioms:

- for any object (h, g, f) in \mathcal{B}_3 there is a commutative diagram

$$\begin{array}{ccccc}
(F(h) \circ F(g)) \circ F(f) & \xrightarrow{\mu_{h,g} \circ F(f)} & F(h \circ g) \circ F(f) & \xrightarrow{\mu_{h \circ g, f}} & F((h \circ g) \circ f) \\
\Downarrow a'_{F(h), F(g), F(f)} & & & & \Downarrow F(a_{h,g,f}) \\
F(h) \circ (F(g) \circ F(f)) & \xrightarrow{F(h) \circ \mu_{g,f}} & F(h) \circ F(g \circ f) & \xrightarrow{\mu_{h, g \circ f}} & F(h \circ (g \circ f))
\end{array}
\tag{2.24}$$

- for any object f in \mathcal{B}_1 there is a commutative diagram

$$\begin{array}{ccccc}
F(f) \circ i'_{F(x)} & \xrightarrow{F(f) \circ \eta_x} & F(f) \circ F(i_x) & \xrightarrow{\mu_{f, i_x}} & F(f \circ i_x) \\
\Downarrow \rho'_{F(f)} & & & & \Downarrow F(\rho_f) \\
F(f) & \xlongequal{\quad\quad\quad} & & & F(f)
\end{array}
\tag{2.25}$$

- for any object f in \mathcal{B}_1 there is a commutative diagram

$$\begin{array}{ccc}
i'_{F(y)} \circ F(f) & \xrightarrow{\eta_y \circ F(f)} & F(i_y) \circ F(f) & \xrightarrow{\mu_{i_y, f}} & F(i_y \circ f) \\
\Downarrow \lambda'_{F(f)} & & & & \Downarrow F(\lambda_f) \\
F(f) & \xlongequal{\quad\quad\quad} & & & F(f)
\end{array} \tag{2.26}$$

If the components $\mu_{g, f}: F(g) \circ F(f) \Rightarrow F(g \circ f)$ and $\eta_x: i'_{F(x)} \Rightarrow F(i_x)$ of two natural transformations (2.23) are the identity 2-morphisms, which means that the two diagrams (2.23) commute strictly, then we say that the homomorphism $F: \mathcal{B} \rightarrow \mathcal{B}'$ is strict.

Remark 2.15. If both \mathcal{B} and \mathcal{B}' are strict 2-categories, the the coherence (2.24) becomes

$$\begin{array}{ccc}
F(h) \circ F(g) \circ F(f) & \xrightarrow{\mu_{h, g \circ F(f)}} & F(h \circ g) \circ F(f) \\
\Downarrow F(h) \circ \mu_{g, f} & & \Downarrow \mu_{h \circ g, f} \\
F(h) \circ F(g \circ f) & \xrightarrow{\mu_{h, g \circ f}} & F(h \circ g \circ f)
\end{array}$$

and the coherence for the right (2.25) and left identity (2.26) become the two diagrams

$$\begin{array}{ccc}
& F(f) \circ F(i_x) & \\
F(f) \circ \eta_x \nearrow & & \searrow \mu_{f, i_x} \\
F(f) & \xlongequal{\quad\quad\quad} & F(f)
\end{array}
\qquad
\begin{array}{ccc}
& F(i_y) \circ F(f) & \\
\eta_y \circ F(f) \nearrow & & \searrow \mu_{i_y, f} \\
F(f) & \xlongequal{\quad\quad\quad} & F(f)
\end{array}$$

The proof of the following important result can be found in Bénabou's paper [7].

Theorem 2.16. *There exist categories Bicat and Bicat_s whose objects are bicategories, and morphisms are homomorphisms and strict homomorphisms of bicategories, respectively.*

In the rest of the paper, we will be mostly concerned with strict homomorphisms of bicategories, since all the examples of fibrations of bicategories in Section 4 will be such strict homomorphisms of bicategories. Now, we will give two basic examples of (nonstrict) homomorphism of bicategories.

Example 2.17. (*Weak monoidal functors*) Since any two monoidal categories \mathcal{M} and \mathcal{M}' may be seen as bicategories $\Sigma\mathcal{M}$ and $\Sigma\mathcal{M}'$ with only one object, any weak monoidal functor

$$F: \mathcal{M} \rightarrow \mathcal{M}'$$

may be seen as a homomorphism $\Sigma F: \Sigma\mathcal{M} \rightarrow \Sigma\mathcal{M}'$ of bicategories $\Sigma\mathcal{M}$ and $\Sigma\mathcal{M}'$.

Example 2.18. (*Pseudofunctors*) For any category \mathcal{C} , any pseudofunctor $F: \mathcal{C}^{op} \rightarrow \text{Cat}$ may be seen as a homomorphism from a locally discrete bicategory \mathcal{C}^{op} .

The collection of categories \mathcal{B}_n , face functors (2.21) and degeneracy functors (2.22)

$$\mathcal{B}_0 \begin{array}{c} \xleftarrow{D_1} \\ \xrightarrow{D_0} \end{array} \mathcal{B}_1 \begin{array}{c} \xleftarrow{D_2} \\ \xrightarrow{D_0} \end{array} \mathcal{B}_2 \begin{array}{c} \xleftarrow{D_3} \\ \xrightarrow{D_0} \end{array} \mathcal{B}_3 \dots \quad (2.27)$$

may be presented as in the above diagram in Cat , in which we denoted only extremal face functors $D_0, D_n: \mathcal{B}_n \rightarrow \mathcal{B}_{n-1}$, while we omitted all degeneracy functors $S_i: \mathcal{B}_n \rightarrow \mathcal{B}_{n+1}$. These functors do not satisfy simplicial identities, since there exist natural isomorphisms

$$\begin{aligned} \alpha: D_i D_j &\Rightarrow D_{j-1} D_i && (i < j) \\ \alpha: S_i S_j &\Rightarrow S_{j+1} S_i && (i \leq j) \\ \alpha: D_i S_j &\Rightarrow S_{j-1} D_i && (i \leq j) \\ \alpha: D_i S_j &\Rightarrow Id && (i = j, i = j + 1) \\ \alpha: D_i S_j &\Rightarrow S_{j-1} D_i && (i > j + 1) \end{aligned} \quad (2.28)$$

all of which are denoted by α . The three functors $D_0, D_1, D_2: \mathcal{B}_2 \rightarrow \mathcal{B}_1$ satisfy identities

$$\begin{aligned} D_0 D_1 &= D_0 D_0 \\ D_1 D_2 &= D_1 D_1 \\ D_0 D_2 &= D_1 D_0 \end{aligned} \quad (2.29)$$

where the first and the second identity correspond to the compatibility of the horizontal composition with the target and source functors, respectively, and the third identity correspond to the second pullback in a diagram (2.10). The first nontrivial simplicial natural isomorphisms in (2.28) arise from the associativity coherence (2.11) in the bicategory \mathcal{B}

$$\alpha: D_1 D_2 \Rightarrow D_1 D_1 \quad (2.30)$$

which is the only nontrivial simplicial identity between the face functors from \mathcal{B}_3 to \mathcal{B}_1 . The left and right identity coherence (2.12) provide the following two natural isomorphisms

$$\begin{aligned} \lambda: D_1 S_1 &\Rightarrow Id_{\mathcal{B}_1} \\ \rho: D_1 S_0 &\Rightarrow Id_{\mathcal{B}_1} \end{aligned} \quad (2.31)$$

and these natural isomorphisms, together with (2.30), satisfy three commutative diagrams corresponding to the associativity, left and right identity coherence, which are the only nontrivial coherence conditions out of seventeen commutative diagrams described in [40], so that collection of categories and functors (2.27), together with natural isomorphisms (2.28) constitute a *supercoherent structure*, which we call a *supercoherent nerve* \mathcal{NB} of \mathcal{B} .

For any such supercoherent structure \mathcal{NB} , we consider the set $Str([n], [m])$ of strings (2.20) from $[n]$ to $[m]$, and to any string $(\theta_k, \dots, \theta_2, \theta_1)$ we associate the composite functor $\theta_k \dots \theta_1: \mathcal{B}_n \rightarrow \mathcal{B}_m$ defined with respect to the supercoherent structure \mathcal{NB} . A *coherence isomorphism*

$$(\theta_k, \dots, \theta_1) \rightarrow (\gamma_l, \dots, \gamma_1) \quad (2.32)$$

is defined to be a composite of natural isomorphisms of the form $\zeta_k \dots \zeta_{i+1} \alpha^\epsilon \zeta_i \dots \zeta_1$, where $\epsilon = \pm 1$ and each ζ_i is either a face or a degeneracy of \mathcal{NB} . It follows that $\theta_k \dots \theta_1 = \gamma_l \dots \gamma_1$ if there exist a coherence isomorphism (2.32), so that we can consider a category $Str(\theta)$ whose objects are all string $(\theta_k, \dots, \theta_2, \theta_1)$ such that $\theta = \theta_k \dots \theta_1$, and whose morphisms are coherence isomorphisms. The following *supercoherence theorem* is proved in [40].

Theorem 2.19. *Let \mathcal{NB} be a supercoherent nerve of a bicategory \mathcal{B} . Then the category $Str(\theta)$ is a trivial groupoid.*

Remark 2.20. *Since a trivial groupoid is a category with exactly one morphism between any two objects, the supercoherence theorem asserts that any two strings (2.20) representing the same monotone map can be connected by a path of coherence isomorphism (2.32), and moreover any two such paths have equal composites, which means that all diagrams of coherence isomorphisms necessarily commute.*

We will justify the term a *supercoherent nerve*, by the construction which associates to any supercoherent structure a *pseudosimplicial category*.

Definition 2.21. *A pseudosimplicial category \mathcal{B}_\bullet is a pseudofunctor $\mathcal{B}: \Delta^{op} \rightarrow \text{Cat}$ from the skeletal simplicial category Δ to the 2-category Cat of small 2-categories.*

Now, suppose that \mathcal{NB} is a supercoherent nerve of the bicategory \mathcal{B} . For any monotone map $\theta: [n] \rightarrow [m]$, we choose its canonical form (2.2), and we define the induced functor

$$\theta_* = S_{j_1} \dots S_{j_t} D_{i_s} \dots D_{i_1}: \mathcal{B}_n \rightarrow \mathcal{B}_m \quad (2.33)$$

with respect to the supercoherent structure \mathcal{NB} . If $\tau: [m] \rightarrow [k]$ is another monotone map, the the two functors $\tau_* \theta_*$ and $(\tau\theta)_*$ come from strings which have the same composition, and by the supercoherence theorem there exists a unique coherence natural isomorphism

$$\mu_{\tau, \theta}: \tau_* \theta_* \Rightarrow (\tau\theta)_*. \quad (2.34)$$

Then we immediately obtain the following result whose proof can be also found in [40].

Theorem 2.22. *Let \mathcal{NB} be a supercoherent nerve of a bicategory \mathcal{B} . Then the assignment (2.33), together with natural isomorphisms (2.34) defines a pseudosimplicial category.*

Proof. The proof is essentially the consequence of the supercoherence theorem since it involves diagrams of coherence natural isomorphism (2.34) which necessarily commute. \square

For any strict homomorphism of bicategories $P: \mathcal{E} \rightarrow \mathcal{B}$, and every object x in \mathcal{B} , a fiber bicategory \mathcal{E}_x is a (strict) pullback

$$\begin{array}{ccc}
 \mathcal{E}_x & \xrightarrow{\mathcal{J}_x} & \mathcal{E} \\
 \downarrow T & & \downarrow P \\
 \mathcal{J} & \xrightarrow{x} & \mathcal{B}
 \end{array} \tag{2.35}$$

where \mathcal{J} is the *free bicategory on one object*. Let us recall that a free bicategory is a functor

$$\mathcal{F} : \text{Cat-Gph} \rightarrow \text{Bicat}_s \tag{2.36}$$

from the category Cat-Gph of Cat -graphs to the category Bicat_s of bicategories and their strict homomorphisms. Objects of the category Cat-Gph are graphs \mathcal{G} consisting of the set G_0 of objects together with a category $\mathcal{G}(x, y)$ for any two objects x, y in G_0 . A morphism $F: \mathcal{G} \rightarrow \mathcal{H}$ of Cat -graphs consists of the function $F_0: G_0 \rightarrow H_0$ and functors $F_{x,y}: \mathcal{G}(x, y) \rightarrow \mathcal{H}(F_0(x), F_0(y))$ for any two objects in G_0 .

The free bicategory functor (2.36) can be described on the level of objects in the following way. For any Cat -graph \mathcal{G} , the free bicategory $\mathcal{F}\mathcal{G}$ has an underlying 2-truncated globular set whose objects $\mathcal{F}G_0$ is the set G_0 . The set $\mathcal{F}G_1$ of 1-morphisms is defined inductively to include identity 1-morphism i_x as new object of $\mathcal{G}(x, x)$ for any object x in G_0 , then all objects of the category $\mathcal{G}(x, y)$ for any two objects x, y in G_0 , and for any objects f in $\mathcal{G}(x, y)$ and g in $\mathcal{G}(y, z)$ their horizontal composition $g \circ f$. The set $\mathcal{F}G_2$ of 2-morphisms is defined inductively by taking new 2-isomorphisms $\alpha_{h,g,f}: (h \circ g) \circ f \Rightarrow h \circ (g \circ f)$ for any object f in $\mathcal{G}(x, y)$, g in $\mathcal{G}(y, z)$ and h in $\mathcal{G}(z, w)$, with $\lambda_f: i_x \circ f \Rightarrow f$ and $\rho_f: f \circ i_x \Rightarrow f$ for any element f in $\mathcal{G}(x, y)$, and their horizontal compositions with all morphisms in $\mathcal{G}(x, y)$ for any two objects x, y in G_0 . Then we form vertically composable strings of the horizontal compositions of these 2-morphisms, and we quotient out by the equivalence relation generated by naturality of the associativity, left and right identity coherence 2-morphisms, the Godement interchange law, and the compatibility of the horizontal composition with the vertical composition in categories $\mathcal{G}(x, y)$ for any two objects x, y in G_0 .

Then the free bicategory $\mathcal{F}(\bullet)$ on the Cat -graph whose $G_0 = \{\bullet\}$ and $\mathcal{G}(\bullet, \bullet) = \emptyset$ is actually a bigroupoid \mathcal{J} with a unique object \bullet , 1-morphisms are horizontal compositions

$$i_\bullet, \quad i_\bullet \circ i_\bullet, \quad (i_\bullet \circ i_\bullet) \circ i_\bullet, \quad i_\bullet \circ (i_\bullet \circ i_\bullet), \quad ((i_\bullet \circ i_\bullet) \circ i_\bullet) \circ i_\bullet, \quad (i_\bullet \circ i_\bullet) \circ (i_\bullet \circ i_\bullet), \quad \dots$$

of an identity 1-morphism $i_\bullet: \bullet \rightarrow \bullet$ and whose 2-morphisms are horizontal compositions of coherence 2-isomorphisms $\alpha_{i_\bullet, i_\bullet, i_\bullet}: (i_\bullet \circ i_\bullet) \circ i_\bullet \Rightarrow i_\bullet \circ (i_\bullet \circ i_\bullet)$ and $\lambda_{i_\bullet} = \rho_{i_\bullet}: i_\bullet \circ i_\bullet \Rightarrow i_\bullet$.

The free bicategory functor (2.36) has a right adjoint, which is the forgetful functor

$$\mathcal{U}: \mathit{Bicat}_s \rightarrow \mathit{Cat}\text{-}\mathit{Gph} \quad (2.37)$$

taking a bicategory \mathcal{B} to its underlying graph \mathcal{UB} with $\mathcal{UB}_0 = B_0$ and $\mathcal{UB}(x, y) = \mathcal{B}(x, y)$. This means that the free bicategory functor (2.36) has the following universal property: for any bicategory \mathcal{B} and any morphism $G: \mathcal{G} \rightarrow \mathcal{UB}$ from a graph \mathcal{G} to an underlying graph \mathcal{UB} of \mathcal{B} , there exists a unique strict homomorphism $\tilde{G}: \mathcal{FG} \rightarrow \mathcal{B}$ of bicategories

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\eta_G} & \mathcal{UFG} \\ & \searrow G & \downarrow \mathcal{U}\tilde{G} \\ & & \mathcal{UB} \end{array} \quad (2.38)$$

such that diagram commutes, where $\eta_G: \mathcal{G} \rightarrow \mathcal{UFG}$ is component of the unit of adjunction.

A strict homomorphism $x: \mathcal{J} \rightarrow \mathcal{B}$ takes the unique object \bullet of \mathcal{J} to the object x in \mathcal{B} and the identity 1-morphism $i_\bullet: \bullet \rightarrow \bullet$ in \mathcal{J} to an identity 1-morphism $i_x: x \rightarrow x$ in \mathcal{B} . Actually, $x: \mathcal{J} \rightarrow \mathcal{B}$ is uniquely determined by (2.38) by the image of the unique object \bullet since \mathcal{J} is the free bicategory on the object \bullet . Therefore, \mathcal{E}_x is a bicategory whose objects are all those objects in \mathcal{E} mapped to the object x in \mathcal{B} and whose 1-morphisms are all those 1-morphisms $j: E \rightarrow F$ whose image by P are parenthesized strings of 1-morphisms

$$i_x, i_x \circ i_x, (i_x \circ i_x) \circ i_x, i_x \circ (i_x \circ i_x), ((i_x \circ i_x) \circ i_x) \circ i_x, (i_x \circ i_x) \circ (i_x \circ i_x), \dots \quad (2.39)$$

and whose 2-morphisms $\phi: j \rightarrow k$ are mapped by P to such strings of identity 2-morphisms. Therefore, any 1-morphism $j: F \rightarrow E$ in \mathcal{E}_x is mapped to a 1-morphism of the type (2.39) and we define its order $\epsilon(j)$ as the number of occurrences of identity morphisms i_x in $P(j)$.

We can use a supercoherent nerve \mathcal{NB} of the bicategory \mathcal{B} with faces $D_j: \mathcal{B}_n \rightarrow \mathcal{B}_{n-1}$, for any $1 \leq j \leq n$, and degeneracies $S_j: \mathcal{B}_{n-1} \rightarrow \mathcal{B}_n$, for any $1 \leq j \leq n-1$, as functors between categories \mathcal{B}_n of horizontally composable strings of n morphisms, which satisfy pseudosimplicial identities. Then any parenthesized string of the type (2.39) may be expressed as $D_{i_1} \dots D_{i_n} S_0^n(i_x)$ where $S_0^n(i_x)$ is a sequence of $n+1$ identity morphisms

$$\underbrace{(i_x, i_x, \dots, i_x, i_x)}_{n+1 \text{ times}}$$

and $D_j: \mathcal{B}_n \rightarrow \mathcal{B}_{n-1}$ factors out j^{th} object in such strings for any $1 \leq j \leq n$. The indices are such that $1 = i_1 \leq i_2 \leq \dots \leq i_n \leq n$ and $i_{n-1} < i_n$ when $i_n = n$. Therefore, we introduce a notation $\theta(j) = (i_1, i_2, \dots, i_{\epsilon(j)-1})$, and we usually write $P(j) = i_x^{\theta(j)}$ when

$$P(j) = D_{i_1} \dots D_{i_{\epsilon(j)-1}} S_0^{\epsilon(j)-1}(i_x)$$

such that $1 \leq i_1 \leq i_2 \leq \dots \leq i_{\epsilon(j)-1} \leq \epsilon(j)$ and $i_{\epsilon(j)-2} < i_{\epsilon(j)-1}$ when $i_{\epsilon(j)-1} = \epsilon(j)$.

Definition 2.23. A pseudonatural transformation $\sigma: F \Rightarrow G$ is defined by the data:

- a natural transformation $\sigma_0: F_0 \rightarrow G_0$ between (discrete) functors,
- a natural isomorphism

$$\begin{array}{ccc}
 \mathcal{B}_1 & \xrightarrow{G_1} & \mathcal{B}'_1 \\
 \downarrow F_1 & \Downarrow \sigma_1 & \downarrow \sigma_0^* \\
 \mathcal{B}'_1 & \xrightarrow{\sigma_{0*}} & \mathcal{B}'_1
 \end{array}$$

whose component indexed by an object $f: x \rightarrow y$ in \mathcal{B}_1 is given by a 2-morphism $\sigma_f: G(f) \circ \sigma_x \Rightarrow \sigma_y \circ F(f)$ as in a diagram

$$\begin{array}{ccc}
 F(x) & \xrightarrow{\sigma_x} & G(x) \\
 \downarrow F(f) & \Downarrow \sigma_f & \downarrow G(f) \\
 F(y) & \xrightarrow{\sigma_y} & G(y)
 \end{array}$$

These data are required to satisfy the following axioms:

- for any object (g, f) in \mathcal{B}_2 , there is a commutative diagram

$$\begin{array}{ccc}
 (G(g) \circ G(f)) \circ \sigma_x & \xrightarrow{\alpha'_{G(g), G(f), \sigma_x}} & G(g) \circ (G(f) \circ \sigma_x) \xrightarrow{G(g) \circ \sigma_f} G(g) \circ (\sigma_y \circ F(f)) \\
 \downarrow \mu_{g, f}^G \circ \sigma_x & & \downarrow \alpha'^{-1}_{G(g), \sigma_y, F(f)} \\
 G(g \circ f) \circ \sigma_x & & (G(g) \circ \sigma_y) \circ F(f) \\
 \downarrow \sigma_{g \circ f} & & \downarrow \sigma_g \circ F(f) \\
 \sigma_z \circ F(g \circ f) & \xleftarrow{\sigma_z \circ \mu_{g, f}^F} \sigma_z \circ (F(g) \circ F(f)) & \xleftarrow{\alpha'_{\sigma_z, F(g), F(f)}} (\sigma_z \circ F(g)) \circ F(f)
 \end{array} \quad (2.40)$$

- for any object x in \mathcal{B} there is a commutative diagram

$$\begin{array}{ccc}
 i'_{G(x)} \circ \sigma_x & \xrightarrow{\lambda'_{\sigma_x}} & \sigma_x \xrightarrow{\rho'^{-1}_{\sigma_x}} & \sigma_x \circ i'_{F(x)} \\
 \eta_x^G \circ \sigma_x \Downarrow & & & \Downarrow \sigma_x \circ \eta_x^F \\
 G(i_x) \circ \sigma_x & \xrightarrow{\sigma_{i_x}} & & \sigma_x \circ F(i_x)
 \end{array} \tag{2.41}$$

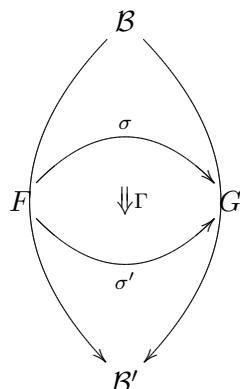
Remark 2.24. If both \mathcal{B} and \mathcal{B}' are strict 2-categories, then the coherence (2.40) becomes a commutative diagram

$$\begin{array}{ccc}
 G(g) \circ G(f) \circ \sigma_x & \xrightarrow{G(g) \circ \sigma_f} & G(g) \circ \sigma_y \circ F(f) \\
 \mu_{g,f}^G \circ \sigma_x \Downarrow & & \Downarrow \sigma_g \circ F(f) \\
 G(g \circ f) \circ \sigma_x & & \\
 \sigma_{g \circ f} \Downarrow & & \\
 \sigma_z \circ F(g \circ f) & \xleftarrow{\sigma_z \circ \mu_{g,f}^F} & \sigma_z \circ F(g) \circ F(f)
 \end{array}$$

The second coherence (2.41) becomes the commutative diagram

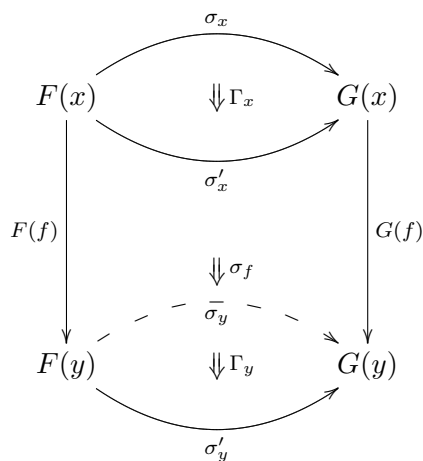
$$\begin{array}{ccc}
 & \sigma_x & \\
 \eta_x^G \circ \sigma_x \swarrow & & \searrow \sigma_x \circ \eta_x^F \\
 G(i_x) \circ \sigma_x & \xrightarrow{\sigma_{i_x}} & \sigma_x \circ F(i_x)
 \end{array}$$

Definition 2.25. For any two pseudonatural transformations, a modification $\Gamma: \sigma \Rightarrow \sigma'$



consists of the following data:

- for each object x in \mathcal{B} , a 2-morphism $\Gamma_x: \sigma_x \Rightarrow \sigma'_x$ in the bicategory \mathcal{B}' such that the following diagram



commutes, which means that a diagram of 2-morphisms in the bicategory \mathcal{B}'

$$\begin{array}{ccc}
 G(f) \circ \sigma_x & \xrightarrow{G(f) \circ \Gamma_x} & G(f) \circ \sigma'_x \\
 \sigma_f \Downarrow & & \Downarrow \sigma'_f \\
 \sigma_y \circ F(f) & \xrightarrow{\Gamma_y \circ F(f)} & \sigma'_y \circ F(f)
 \end{array}$$

commutes.

Theorem 2.26. *Homomorphisms from a bicategory \mathcal{B} to \mathcal{E} are objects of a bicategory*

$$\mathcal{H}om(\mathcal{B}, \mathcal{E}) \quad (2.42)$$

where 1-morphisms are pseudonatural transformations and 2-morphisms are modifications.

Proof. The horizontal composition $\xi \circ \sigma$ of two pseudonatural transformations $\sigma: P \Rightarrow R$ and $\xi: R \Rightarrow S$ has a component indexed by an object B in \mathcal{B} , defined by an identity

$$(\xi \circ \sigma)_B := \xi_B \circ \sigma_B \quad (2.43)$$

and for any 1-morphism $f: A \rightarrow B$ in the bicategory \mathcal{B} , the pasting composite of a diagram

$$\begin{array}{ccccc} P(A) & \xrightarrow{\sigma_A} & R(A) & \xrightarrow{\xi_A} & S(A) \\ \downarrow P(f) & \Downarrow \sigma_f & \downarrow R(f) & \Downarrow \xi_f & \downarrow S(f) \\ P(B) & \xrightarrow{\sigma_B} & R(B) & \xrightarrow{\xi_B} & S(B) \end{array}$$

defines the component $(\xi \circ \sigma)_f$ of the pseudonatural transformation $\xi \circ \sigma$ by an identity

$$(\xi \circ \sigma)_f := (\xi_B \circ \sigma_f)(\xi_f \circ \sigma_A). \quad (2.44)$$

The associativity coherence for this composition is given by an invertible modification

$$\tilde{\alpha}_{\omega, \xi, \sigma}: (\omega \circ \xi) \circ \sigma \Rightarrow \omega \circ (\xi \circ \sigma) \quad (2.45)$$

whose components for any three pseudonatural transformations $P \xrightarrow{\sigma} R \xrightarrow{\xi} S \xrightarrow{\omega} T$

$$\begin{array}{ccc} & \begin{array}{c} \xrightarrow{[(\omega \circ \xi) \circ \sigma]_A} \\ \downarrow \alpha_{\omega_A, \xi_A, \sigma_A} \\ \xrightarrow{[\omega \circ (\xi \circ \sigma)]_A} \end{array} & \\ P(A) & & T(A) \\ \downarrow P(f) & & \downarrow T(f) \\ & \begin{array}{c} \downarrow [\omega \circ (\xi \circ \sigma)]_f \\ \xrightarrow{[(\omega \circ \xi) \circ \sigma]_B} \\ \downarrow \alpha_{\omega_B, \xi_B, \sigma_B} \\ \xrightarrow{[\omega \circ (\xi \circ \sigma)]_B} \end{array} & \\ P(B) & & T(B) \end{array}$$

are given by associativity coherence 2-morphisms $\alpha_{\omega_A, \xi_A, \sigma_A} : (\omega_A \circ \xi_A) \circ \sigma_A \Rightarrow \omega_A \circ (\xi_A \circ \sigma_A)$. For a homomorphism $R: \mathcal{B} \rightarrow \mathcal{E}$ of bicategories, an identity pseudonatural transformation

$$\iota_R: R \Rightarrow R \quad (2.46)$$

has an identity 1-morphism $i_{R(B)}: R(B) \rightarrow R(B)$ as a component $(\iota_R)_B$ indexed by an object B in \mathcal{B} , and its component $(\iota_R)_f: R(f) \circ i_{R(B)} \Rightarrow i_{R(A)} \circ R(f)$ indexed by any 1-morphism $f: A \rightarrow B$ in \mathcal{B} , is as in a diagram

$$\begin{array}{ccc} R(A) & \xrightarrow{i_{R(A)}} & R(A) \\ \downarrow R(f) & \searrow R(f) & \downarrow R(f) \\ & \swarrow \rho_{R(f)}^{-1} & \\ R(B) & \xrightarrow{i_{R(B)}} & R(B) \end{array}$$

(Note: The diagram also includes a diagonal arrow from $R(A)$ to $R(B)$ labeled $\lambda_{R(f)}$ and another diagonal arrow from $R(A)$ to $R(B)$ labeled $\rho_{R(f)}^{-1}$.)

and is equal to $\rho_{R(f)}^{-1} \lambda_{R(f)}$. Then for any pseudonatural transformations $\sigma: P \Rightarrow R$ and $\xi: R \Rightarrow S$, a left and right identity coherence modifications

$$\tilde{\lambda}_\sigma: \iota_R \circ \sigma \Rightarrow \sigma \quad (2.47)$$

$$\tilde{\rho}_\xi: \xi \circ \iota_R \Rightarrow \xi \quad (2.48)$$

have components for any object B in \mathcal{B} given by the left and right identity coherence

$$\lambda_{\sigma_B}: i_{R(B)} \circ \sigma_B \Rightarrow \sigma_B \quad (2.49)$$

$$\rho_{\xi_B}: \xi_B \circ i_{R(B)} \Rightarrow \xi_B \quad (2.50)$$

respectively. The horizontal composition of any two modifications in $\mathcal{H}om(\mathcal{B}, \mathcal{E})$

$$\begin{array}{ccccc} & \sigma & & \xi & \\ & \curvearrowright & & \curvearrowright & \\ P & & R & & S \\ & \curvearrowleft & & \curvearrowleft & \\ & \tau & & \zeta & \end{array}$$

(Note: The diagram also includes vertical double arrows from P to R labeled Γ and from R to S labeled Ω .)

is defined by the horizontal composition of its components indexed by objects B in \mathcal{B}

$$(\Omega \circ \Gamma)_B := \Omega_B \circ \Gamma_B \quad (2.51)$$

Associativity and identity coherence for the horizontal composition follows immediately from associativity and identity coherence of the horizontal composition in the bicategory \mathcal{E} . Vertical composition of modifications $\sigma \xRightarrow{\Gamma} \tau \xRightarrow{\Pi} \pi$ in $\mathcal{H}om(\mathcal{B}, \mathcal{E})$ is defined by

$$(\Pi\Gamma)_B := \Pi_B \Gamma_B \tag{2.52}$$

and the two compositions (2.51) and (2.52) satisfy the Godement interchange law. \square

3 Indexed bicategories and tricategories

All definitions in this section are taken from [24], and $Bicat$ denote a tricategory of bicategories, their homomorphisms, pseudonatural transformations and modifications.

Definition 3.1. A tricategory \mathcal{T} consists of the following data:

- a set \mathcal{T}_0 whose elements are called objects of \mathcal{T}
- for any two objects x and y in \mathcal{T} , a bicategory $\mathcal{T}(x, y)$ which we call a hom-bicategory, whose objects f, g, h, \dots are called 1-morphisms of \mathcal{T} , whose 1-morphisms $\phi, \psi, \theta, \dots$ are called 2-morphisms of \mathcal{T} , and whose 2-morphisms Π, Θ, \dots are called 3-morphisms of \mathcal{T} . The horizontal composition of 1-morphisms ϕ and ψ in $\mathcal{T}(x, y)$ is denoted by $\psi \circ \phi$, and the vertical composition of 2-morphisms Π and Θ in $\mathcal{T}(x, y)$ is denoted by the concatenation $\Theta\Pi$, whenever these compositions are defined

- for any three objects x, y and z in \mathcal{T} , a homomorphism of bicategories

$$\otimes: \mathcal{T}(y, z) \times \mathcal{T}(x, y) \rightarrow \mathcal{T}(x, z) \quad (3.1)$$

called a composition, whose coherence conditions are not named

- for any object x in \mathcal{T} , a homomorphism of bicategories

$$I_x: \mathcal{I} \rightarrow \mathcal{T}(x, x) \quad (3.2)$$

where \mathcal{I} is the free 2-category on one object,

- for any three objects x, y and z in \mathcal{T} , a pseudonatural equivalence

$$\begin{array}{ccc} \mathcal{T}(z, w) \times \mathcal{T}(y, z) \times \mathcal{T}(x, y) & \xrightarrow{\otimes \times Id_{\mathcal{T}(x, y)}} & \mathcal{T}(y, w) \times \mathcal{T}(x, y) \\ \downarrow Id_{\mathcal{T}(z, w)} \times \otimes & \Downarrow \alpha & \downarrow \otimes \\ \mathcal{T}(z, w) \times \mathcal{T}(x, z) & \xrightarrow{\otimes} & \mathcal{T}(x, w) \end{array} \quad (3.3)$$

- for any two objects x and y in \mathcal{T} , pseudonatural equivalences

$$\begin{array}{ccc} & \mathcal{T}(x, y) & \\ & \parallel & \\ I_y \times Id_{\mathcal{T}(x, y)} & \swarrow & \searrow Id_{\mathcal{T}(x, y)} \times I_x \\ & \Downarrow \tau & \Downarrow \rho \\ \mathcal{T}(y, y) \times \mathcal{T}(x, y) & \xrightarrow{\otimes} & \mathcal{T}(x, y) & \xleftarrow{\otimes} & \mathcal{T}(x, y) \times \mathcal{T}(x, x) \end{array} \quad (3.4)$$

- for any three objects x, y and z in \mathcal{T} , two isomodifications λ and ρ inside diagrams

$$\begin{array}{ccc}
 \begin{array}{c}
 \mathcal{T}_3 \\
 \begin{array}{ccc}
 \nearrow^{I \times 1 \times 1} & \downarrow \lambda \times 1 & \searrow^{\otimes \times 1} \\
 \mathcal{T}_2 & \text{---} & \mathcal{T}_2 \\
 \downarrow \otimes & \downarrow 1 \times \otimes & \downarrow \otimes \\
 \mathcal{T} & \mathcal{T}_2 & \mathcal{T} \\
 \downarrow \otimes & \nearrow^{I \times 1} & \searrow^{\otimes} \\
 \mathcal{T} & & \mathcal{T}
 \end{array}
 \end{array}
 &
 \begin{array}{c}
 \mathcal{T}_3 \\
 \begin{array}{ccc}
 \nearrow^{1 \times 1 \times I} & \downarrow 1 \times \rho & \searrow^{1 \times \otimes} \\
 \mathcal{T}_2 & \text{---} & \mathcal{T}_2 \\
 \downarrow \otimes & \downarrow 1 \times \otimes & \downarrow \otimes \\
 \mathcal{T} & \mathcal{T}_2 & \mathcal{T} \\
 \downarrow \otimes & \nearrow^{1 \times I} & \searrow^{\otimes} \\
 \mathcal{T} & & \mathcal{T}
 \end{array}
 \end{array}
 &
 (3.7)
 \end{array}$$

Example 3.2. (Tricategory *Bicat*) Bicategories are objects of the tricategory *Bicat*, and for any two bicategories \mathcal{B} and \mathcal{E} a hom-bicategory $\text{Bicat}(\mathcal{B}, \mathcal{E})$ is the one (2.42) given in Theorem 2.16. A composition homomorphism

$$\otimes : \text{Bicat}(\mathcal{E}, \mathcal{P}) \times \text{Bicat}(\mathcal{B}, \mathcal{E}) \rightarrow \text{Bicat}(\mathcal{B}, \mathcal{P}) \quad (3.8)$$

is defined on the level of objects for any two homomorphisms $F : \mathcal{B} \rightarrow \mathcal{E}$ and $G : \mathcal{E} \rightarrow \mathcal{P}$ by

$$G \otimes F = GF. \quad (3.9)$$

The composition of homomorphisms of bicategories is strictly associative, by Theorem 2.16, so that the tricategory *Bicat* has an underlying category which we denote by the same name *Bicat*. For any two pseudonatural transformations

$$\begin{array}{ccccc}
 & & F & & G \\
 & & \curvearrowright & & \curvearrowright \\
 \mathcal{B} & & & \mathcal{E} & & \mathcal{P} \\
 & & \Downarrow \phi & & \Downarrow \psi \\
 & & \curvearrowleft & & \curvearrowleft \\
 & & L & & M
 \end{array}$$

the composition homomorphism (3.8) is defined in the following way. First, we define a pseudonatural transformation $\psi \otimes F : G \otimes F \Rightarrow M \otimes F$ whose component indexed by an object

B in \mathcal{B} is $(\psi \otimes F)_B = \psi_{F(B)}$, and whose component indexed by a 1-morphism $f: B \rightarrow C$ in \mathcal{B} is $(\psi \otimes F)_f = \psi_{F(f)}$ as in a diagram

$$\begin{array}{ccc} GF(B) & \xrightarrow{\psi_{F(B)}} & MF(B) \\ \downarrow GF(f) & \not\cong \psi_{F(f)} & \downarrow MF(f) \\ GF(C) & \xrightarrow{\psi_{F(C)}} & MF(C) \end{array}$$

Second, we define a pseudonatural transformation $M \otimes \phi: M \otimes F \Rightarrow M \otimes G$ whose component indexed by an object B in \mathcal{B} is $(M \otimes \phi)_B = M(\phi_B)$, and whose component $(M \otimes \phi)_f$

$$\begin{array}{ccc} MF(B) & \xrightarrow{M(\phi_B)} & MG(B) \\ \downarrow MF(f) & \not\cong (M \otimes \phi)_f & \downarrow MG(f) \\ MF(C) & \xrightarrow{M(\phi_C)} & MG(C) \end{array}$$

indexed by a 1-morphism $f: B \rightarrow C$ in \mathcal{B} , is defined by a composition of 2-morphisms

$$MG(f) \circ M(\phi_B) \xrightarrow{\mu_{G(f), \phi_B}} M(G(f) \circ \phi_B) \xrightarrow{M(\phi_f)} M(\phi_C \circ F(f)) \xrightarrow{\mu_{\phi_C, F(f)}^{-1}} M(\phi_C) \circ MF(f).$$

It is easy to see that both $\psi \otimes F$ and $M \otimes \phi$ are pseudonatural transformations, and these are 1-morphisms in the hom-bicategory $\text{Bicat}(\mathcal{B}, \mathcal{P})$ which will also be denoted by ψF and $M\phi$, respectively. Then we use horizontal composition (2.43) and (2.44) in the bicategory $\text{Bicat}(\mathcal{B}, \mathcal{P})$ to define the composition (3.8) of pseudonatural transformations ψ and ϕ by

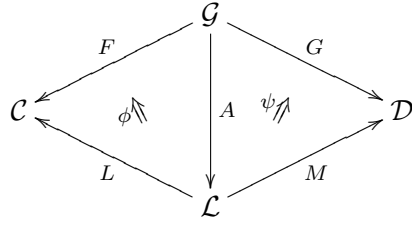
$$\psi \otimes \phi = (\psi \otimes F) \circ (M \otimes \phi) \quad (3.10)$$

which we will also write in abbreviated form $\psi \otimes \phi = \psi F \circ M\phi$.

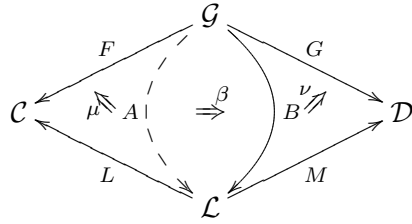
Example 3.3. (Tricategory of spans in Cat) There is tricategory $\text{Span}(\text{Cat})$ of spans in the 2-category Cat , whose objects are small categories. For any two small categories \mathcal{C} and \mathcal{D} , a 1-morphism $(F, \mathcal{G}, G): \mathcal{C} \rightarrow \mathcal{D}$ is a span

$$\begin{array}{ccc} & \mathcal{G} & \\ F \swarrow & & \searrow G \\ \mathcal{C} & & \mathcal{D} \end{array}$$

and a 2-morphism $(\phi, A, \psi): (F, \mathcal{G}, G) \Rightarrow (L, \mathcal{L}, M)$ in $\text{Span}(\text{Cat})$ is a diagram



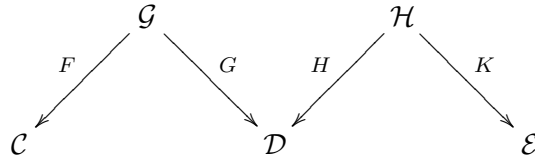
consisting of a functor $A: \mathcal{G} \Rightarrow \mathcal{L}$ together with natural transformations $\phi: L \circ A \Rightarrow F$ and $\psi: M \circ A \Rightarrow G$. A 3-morphism $\beta: (\phi, A, \psi) \Rightarrow (\mu, B, \nu)$ in $\text{Span}(\text{Cat})$ is given by a diagram



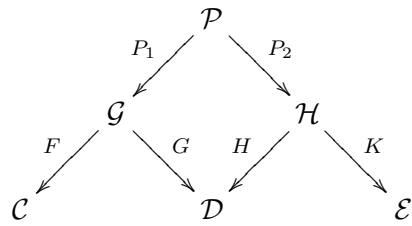
in which $\beta: A \Rightarrow B$ is a natural transformation, such that $\mu(L \circ \beta) = \phi$ and $\nu(M \circ \beta) = \psi$. It is obvious that for any two small categories \mathcal{C} and \mathcal{D} , a hom-bicategory $\text{Span}(\text{Cat})(\mathcal{C}, \mathcal{D})$ is actually a strict 2-category. The composition homomorphism

$$\otimes: \text{Span}(\text{Cat})(\mathcal{D}, \mathcal{E}) \times \text{Span}(\text{Cat})(\mathcal{C}, \mathcal{D}) \rightarrow \text{Span}(\text{Cat})(\mathcal{C}, \mathcal{E}) \quad (3.11)$$

is defined on the level of spans of categories



by the composite left and right legs of a diagram where a square is a pullback in Cat



It is obvious that an identity $I_C: C \rightarrow C$ for the horizontal composition (3.11) is a span

$$\begin{array}{ccc} & C & \\ Id_C \swarrow & & \searrow Id_C \\ C & & C. \end{array}$$

and that the horizontal composition (3.11) is not strictly associative or unital, but it follows from a universal property of pullbacks that it is associative and unital up to canonical isomorphisms of categories.

From the previous example, it follows that for any small category \mathcal{B} , the following span

$$\begin{array}{ccc} & \mathcal{B}^I & \\ D_1 \swarrow & & \searrow D_0 \\ \mathcal{B} & & \mathcal{B} \end{array} \quad (3.12)$$

is a monoid in a monoidal 2-category $Span(Cat)(\mathcal{B}, \mathcal{B})$, where \mathcal{B}^I is a category of morphisms. By identifying any functor $P: \mathcal{E} \rightarrow \mathcal{B}$ with a span $(I_{\mathcal{E}}, \mathcal{E}, P)$, we have a composition

$$\begin{array}{ccccc} & & (\mathcal{B}, P) & & \\ & & P_1 \swarrow & & \searrow P_2 \\ & & \mathcal{B}^I & & \mathcal{E} \\ D_1 \swarrow & & D_0 \searrow & & P \swarrow & & \searrow I_{\mathcal{E}} \\ \mathcal{B} & & \mathcal{B} & & \mathcal{E} & & \mathcal{E} \end{array} \quad (3.13)$$

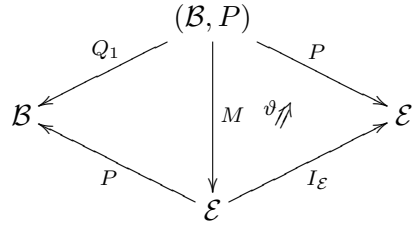
which means that for a slice 2-category (Cat, \mathcal{B}) of functors over \mathcal{B} , we have a 2-functor

$$(\mathcal{B}, -): (Cat, \mathcal{B}) \rightarrow (Cat, \mathcal{B}) \quad (3.14)$$

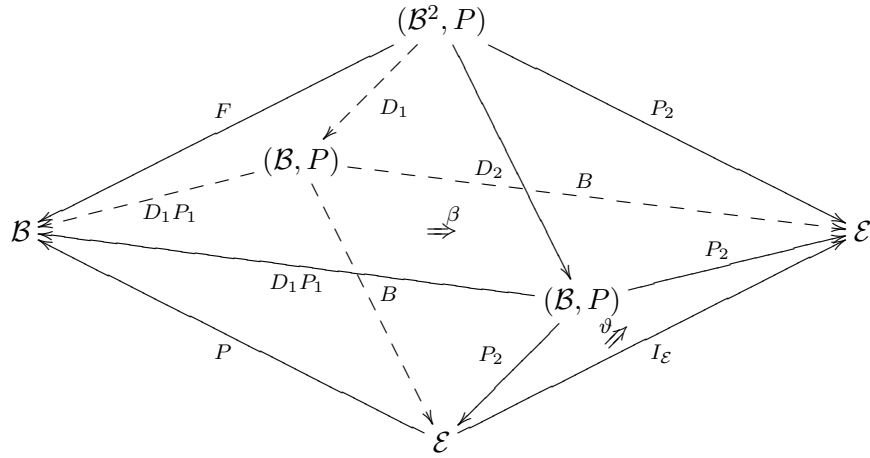
which is actually a 2-monad on (Cat, \mathcal{B}) by the commutativity of the following diagram

$$\begin{array}{ccccc} & & (\mathcal{B}^3, P) & & \\ & & F \swarrow & & \searrow P_2 \\ & & (\mathcal{B}^2, P) & & \mathcal{E} \\ D_1 \swarrow & & D_1 \searrow & & D_2 \searrow & & \searrow B \\ \mathcal{B} & & \mathcal{B}^2, P & & \mathcal{B}^2, P & & \mathcal{E} \\ D_1 P_1 \swarrow & & D_1 P_1 \searrow & & P_2 \swarrow & & \searrow I_{\mathcal{E}} \\ & & (\mathcal{B}, P) & & \mathcal{B} & & \mathcal{E} \\ P \swarrow & & P_2 \swarrow & & P_2 \swarrow & & \searrow I_{\mathcal{E}} \\ & & (\mathcal{B}, P) & & \mathcal{B} & & \mathcal{E} \end{array}$$

Then a pseudoalgebra structure for the 2-monad (3.14) is given by a functor $P: \mathcal{E} \rightarrow \mathcal{B}$ and a functor $M: (\mathcal{B}, P) \rightarrow \mathcal{E}$ which is a part of a morphism of spans



such that the following diagram in $Span(\mathcal{B}, \mathcal{E})$



commutes.

The left leg of a span (4.17) is a functor from the comma category (\mathcal{B}, P)

$$D_1P_1: (\mathcal{B}, P) \rightarrow \mathcal{B}. \tag{3.15}$$

and this functor is actually a fibration which we prove by the following theorem.

Theorem 3.4. *For any functor $P: \mathcal{E} \rightarrow \mathcal{B}$, the functor (3.15) is a fibration of categories, called a canonical fibration associated to P .*

Definition 3.5. Let \mathcal{B} be a small bicategory. A \mathcal{B} -indexed bicategory is a trihomomorphism $\mathcal{F}: \mathcal{B}^{\text{coop}} \rightarrow \text{Bicat}$ of tricategories from $\mathcal{B}^{\text{coop}}$ to the tricategory Bicat which consists of:

- for every object x in \mathcal{B} , a bicategory $\mathcal{F}(x)$, which we will also denote by \mathcal{F}_x ,
- for every two objects x and y in \mathcal{B} , a homomorphism of bicategories

$$\mathcal{F}_{x,y}: \mathcal{B}(x,y)^{\text{op}} \rightarrow \text{Hom}(\mathcal{F}_y, \mathcal{F}_x) \quad (3.16)$$

from the opposite $\mathcal{B}(x,y)^{\text{op}}$ of the category $\mathcal{B}(x,y)$ of 1-morphisms and 2-morphisms with x and y as a 0-source and 0-target, respectively, into the bicategory $\text{Hom}(\mathcal{F}_y, \mathcal{F}_x)$. For any 1-morphism $f: x \rightarrow y$ in \mathcal{B} , its image $\mathcal{F}_{x,y}(f)$ by (3.16) is a homomorphism

$$x \xrightarrow{f} y \quad \xrightarrow{\mathcal{F}_{x,y}} \quad \mathcal{F}_x \xleftarrow{f^*} \mathcal{F}_y \quad (3.17)$$

on the right side, and for any 2-morphism $\beta: f \Rightarrow g$ in \mathcal{B} , its image $\mathcal{F}_{x,y}(\beta)$ by (3.16)

$$\begin{array}{ccc} \begin{array}{c} x \xrightarrow{f} y \\ \Downarrow \beta \\ x \xrightarrow{g} y \end{array} & \xrightarrow{\mathcal{F}_{x,y}} & \begin{array}{c} \mathcal{F}_x \xleftarrow{f^*} \mathcal{F}_y \\ \beta^* \Uparrow \\ \mathcal{F}_x \xleftarrow{g^*} \mathcal{F}_y \end{array} \end{array} \quad (3.18)$$

is a pseudonatural transformation $\beta^*: g^* \Rightarrow f^*$ on the right side of (3.18).

- for any composable pair $f \xrightarrow{\beta} g \xrightarrow{\gamma} h$ of 2-morphisms in \mathcal{B} , an isomodification

$$\mu_{\gamma,\beta}: \beta^* \circ \gamma^* \Rightarrow (\gamma\beta)^* \quad (3.19)$$

which satisfy coherence with respect to the vertical composition of 2-morphisms in \mathcal{B}

$$\begin{array}{ccccc} (\beta^* \circ \gamma^*) \circ \delta^* & \xrightarrow{\mu_{\gamma,\beta \circ \delta^*}} & (\gamma\beta)^* \circ \delta^* & \xrightarrow{\mu_{\delta,\gamma\beta}} & [\delta(\gamma\beta)]^* \\ \Downarrow \tilde{\alpha}_{\beta^*,\gamma^*,\delta^*} & & & & \Downarrow \text{id} \\ \beta^* \circ (\gamma^* \circ \delta^*) & \xrightarrow{\beta^* \circ \mu_{\delta,\gamma}} & \beta^* \circ (\delta\gamma)^* & \xrightarrow{\mu_{\delta\gamma,\beta}} & [(\delta\gamma)\beta]^* \end{array} \quad (3.20)$$

and such that the following normalization conditions for modifications are satisfied

$$\mu_{\iota_g,\beta} = \tilde{\lambda}_{\beta^*} \quad (3.21)$$

$$\mu_{\beta, \iota_f} = \tilde{\rho}_{\beta^*} \quad (3.22)$$

where modifications on the right side are the components of the left and right identity coherence modifications (2.47) and (2.48) in the bicategory $\mathcal{H}om(\mathcal{F}_x, \mathcal{F}_x)$, respectively.

- for any 1-morphisms $f: x \rightarrow y$ and $g: y \rightarrow z$ in \mathcal{B} , two pseudonatural equivalences

$$\chi_{g,f}: f^*g^* \Rightarrow (g \circ f)^* \quad (3.23)$$

$$\chi_{g,f}^*: (g \circ f)^* \Rightarrow f^*g^* \quad (3.24)$$

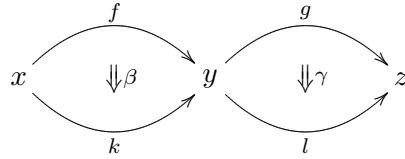
in the bicategory $\mathcal{H}om(\mathcal{F}_x, \mathcal{F}_x)$, together with isomodifications

$$\Theta_{g,f}: \chi_{g,f}^* \otimes \chi_{g,f} \Rightarrow \iota_{f^* \circ g^*} \quad (3.25)$$

$$\Lambda_{g,f}: \chi_{g,f} \otimes \chi_{g,f}^* \Rightarrow \iota_{(g \circ f)^*} \quad (3.26)$$

such that (3.23) and (3.24) are adjoint pseudonatural equivalences.

- for any horizontally composable pair of 2-morphisms in \mathcal{B}



an isomodification $\chi_{\gamma, \beta}: (\gamma \circ \beta)^* \chi_{l,k} \Rightarrow \chi_{g,f}(\beta^* \otimes \gamma^*)$, as in a following diagram

$$\begin{array}{ccc}
 k^*l^* & \xrightarrow{\chi_{l,k}} & (l \circ k)^* \\
 \beta^* \otimes \gamma^* \downarrow & \swarrow \chi_{\gamma, \beta} & \downarrow (\gamma \circ \beta)^* \\
 f^*g^* & \xrightarrow{\chi_{g,f}} & (g \circ f)^*
 \end{array} \quad (3.27)$$

- for any three $x \xrightarrow{f} y \xrightarrow{g} w \xrightarrow{h} z$ 1-morphisms in \mathcal{B} , an isomodification

$$\begin{array}{ccc}
f^* g^* h^* & \xrightarrow{\chi_{g,f} h^*} & (g \circ f)^* h^* \xrightarrow{\chi_{h,g \circ f}} [(h \circ (g \circ f))^* \\
\parallel & & \parallel \\
f^* \chi_{h,g} & & \Downarrow \omega_{h,g,f} \\
f^* (h \circ g)^* & \xrightarrow{\chi_{h \circ g, f}} & [(h \circ g) \circ f]^*
\end{array} \quad (3.28)$$

This data are required to satisfy the following coherence conditions:

- commutative cube

$$\begin{array}{ccccc}
& & f^* g^* h^* k^* & \xrightarrow{f^* \chi_{h,g} k^*} & f^* (h \circ g)^* k^* \\
& & \parallel & & \parallel \\
& & \chi_{g,f} h^* k^* & \Downarrow \theta_x \omega_{h,g,f} & f^* \chi_{k,h \circ g} \\
& & \parallel & & \parallel \\
& & (g \circ f)^* h^* k^* & \xrightarrow{f^* \alpha_{h,g,f}^*} & f^* [k \circ (h \circ g)]^* \\
& & \parallel & & \parallel \\
& & \chi_{h,g \circ f} k^* & & \parallel \\
& & \parallel & & \parallel \\
& & [h \circ (g \circ f)]^* k^* & \xrightarrow{\alpha_{h,g,f}^* k^*} & [(h \circ g) \circ f]^* k^* \\
& & \parallel & & \parallel \\
& & \chi_{k,h \circ (g \circ f)} & & \parallel \\
& & \parallel & & \parallel \\
& & [k \circ (h \circ (g \circ f))]^* & \xrightarrow{(k \circ \alpha_{h,g,f})^*} & [k \circ ((h \circ g) \circ f)]^* \\
& & \parallel & & \parallel \\
& & \omega_{h,g,f} \theta_z & & \parallel \\
& & \parallel & & \parallel \\
& & [(k \circ (h \circ g)) \circ f]^* & \xrightarrow{(\alpha_{k,h,g \circ f})^*} & [(k \circ (h \circ g)) \circ f]^* \\
& & \parallel & & \parallel \\
& & \alpha_{k \circ h, g, f}^* & & \parallel \\
& & \parallel & & \parallel \\
& & [(k \circ h) \circ (g \circ f)]^* & \xrightarrow{\alpha_{k \circ h, g, f}^*} & [((k \circ h) \circ g) \circ f]^*
\end{array}$$

- for any object x in \mathcal{B} , the image by (3.16) of an identity 1-morphism $i_x: x \rightarrow x$ satisfies

$$i_x^* = I_{\mathcal{F}_x} \quad (3.29)$$

where $I_{\mathcal{F}_x}: \mathcal{F}_x \rightarrow \mathcal{F}_x$ is an identity homomorphism in the hom-bicategory $\mathcal{H}om(\mathcal{F}_x, \mathcal{F}_x)$.

- for any 1-morphism $f: x \Rightarrow y$ in \mathcal{B} , the image by (3.16) of an identity 2-morphism $\iota_f: f \Rightarrow f$ in \mathcal{B} , satisfies the identity

$$\iota_f^* = \iota_{f^*} \quad (3.30)$$

where $\iota_{f^*}: f^* \Rightarrow f^*$ is an identity pseudonatural transformation defined in (2.46).

- for any 1-morphism $f: x \rightarrow y$ in \mathcal{B} , two identities of pseudonatural transformations

$$\chi_{i_y, f} = \lambda_f^* \quad (3.31)$$

$$\chi_{f, i_x} = \rho_f^* \quad (3.32)$$

- for any 2-morphism $\beta: f \Rightarrow g$ in the bicategory \mathcal{B} , the following two modifications

$$\begin{array}{ccc} g^* & \xrightarrow{\chi_{i_y, g} = \lambda_g^*} & (i_y \circ g)^* \\ \beta^* \downarrow & \Downarrow \chi_{\iota_y, \beta} & \downarrow (\iota_y \circ \beta)^* \\ f^* & \xrightarrow{\chi_{\iota_y, f} = \lambda_f^*} & (i_y \circ f)^* \end{array} \quad \begin{array}{ccc} g^* & \xrightarrow{\chi_{g, i_x} = \rho_g^*} & (g \circ i_x)^* \\ \beta^* \downarrow & \Downarrow \chi_{\beta, \iota_x} & \downarrow (\beta \circ \iota_x)^* \\ f^* & \xrightarrow{\chi_{f, i_x} = \rho_f^*} & (f \circ i_x)^* \end{array}$$

are equal to the composition of modifications

$$(\iota_y \circ \beta)^* \circ \lambda_g^* \xrightarrow{\mu_{\lambda_g, \iota_y \circ \beta}} [\lambda_g(\iota_y \circ \beta)]^* \xrightarrow{id} [\beta \lambda_f]^* \xrightarrow{\mu_{\beta, \lambda_f}^{-1}} \lambda_f^* \circ \beta^* \quad (3.33)$$

$$(\beta \circ \iota_x)^* \circ \rho_g^* \xrightarrow{\mu_{\rho_g, \beta \circ \iota_x}} [\rho_g(\beta \circ \iota_x)]^* \xrightarrow{id} [\beta \rho_f]^* \xrightarrow{\mu_{\beta, \rho_f}^{-1}} \rho_f^* \circ \beta^* \quad (3.34)$$

respectively, which follows from the naturality of the left and right identity coherence

$$\begin{array}{ccc} i_y \circ f & \xrightarrow{\lambda_f} & g \\ \downarrow \iota_y \circ \beta & & \downarrow \beta \\ i_y \circ g & \xrightarrow{\lambda_g} & g \end{array} \quad \begin{array}{ccc} f \circ i_x & \xrightarrow{\rho_f} & f \\ \downarrow \beta \circ \iota_x & & \downarrow \beta \\ g \circ i_x & \xrightarrow{\rho_g} & g \end{array}$$

Remark 3.6. When we replace the tricategory $Bicat$ by the Gray-category 2-Cat of strict 2-categories, then we obtain a notion of an \mathcal{B} -indexed 2-category. We will later use this notion in order to show that when our fundamental construction, the Grothendieck construction for bicategories, is applied to a \mathcal{B} -indexed 2-category, we still obtain a bicategory as in the more general case of \mathcal{B} -indexed bicategories.

A notion of an \mathcal{B} -indexed bicategory is a natural generalization of a pseudofunctor, or an indexed category to the immediate next level of dimension. Since a natural notion of a map between indexed categories is a cartesian functor, we also define its generalization for \mathcal{B} -indexed bicategories.

Definition 3.7. Let \mathcal{B} be a small bicategory. A cartesian homomorphism $\theta: \mathcal{F} \Rightarrow \mathcal{G}$ between \mathcal{B} -indexed bicategories $\mathcal{F}, \mathcal{G}: \mathcal{B}^{coop} \rightarrow Bicat$ consists of the following data:

- a homomorphism $\theta_x: \mathcal{F}_x \rightarrow \mathcal{G}_x$ of bicategories, for any object x in \mathcal{B} ,
- a pseudonatural transformation $\theta_f: \theta_x f^* \Rightarrow f^\dagger \theta_y$

$$\begin{array}{ccc}
 \mathcal{F}_y & \xrightarrow{f^*} & \mathcal{F}_x \\
 \theta_y \downarrow & \Downarrow \theta_f & \downarrow \theta_x \\
 \mathcal{G}_y & \xrightarrow{f^\dagger} & \mathcal{G}_x
 \end{array}$$

for any 1-morphism $f: x \rightarrow y$ in \mathcal{B} , where $f^*: \mathcal{F}_y \rightarrow \mathcal{F}_x$ and $f^\dagger: \mathcal{G}_y \rightarrow \mathcal{G}_x$ are homomorphisms between fibers of \mathcal{B} -indexed bicategories $\mathcal{F}, \mathcal{G}: \mathcal{B}^{coop} \rightarrow Bicat$ respectively,

- for any composable pair $x \xrightarrow{f} y \xrightarrow{g} z$ of 1-morphisms in \mathcal{B} , an isomodification

$$\begin{array}{ccccc}
 \theta_x f^* g^* & \xrightarrow{\theta_f g^*} & f^\dagger \theta_y g^* & \xrightarrow{f^\dagger \theta_g} & f^\dagger g^\dagger \theta_z \\
 \theta_x \chi_{g,f} \Downarrow & & \Downarrow \Pi_{g,f} & & \Downarrow \chi_{g,f} \theta_z \\
 \theta_x (g \circ f)^* & \xrightarrow{\theta_{g \circ f}} & & & (g \circ f)^\dagger \theta_z
 \end{array}$$

4 Fibrations of bicategories

In this section, we introduce fibrations of bicategories. Before we do that, we recall definitions of fibrations of categories and cartesian morphisms introduced by Grothendieck [29].

Definition 4.1. Let $F: \mathcal{D} \rightarrow \mathcal{C}$ be a functor. A morphism $f: x_1 \rightarrow x_2$ in the category \mathcal{D} is cartesian if the diagram

$$\begin{array}{ccc}
 \mathcal{D}(x_0, x_1) & \xrightarrow{\mathcal{D}(x_0, f)} & \mathcal{D}(x_0, x_2) \\
 \downarrow F & & \downarrow F \\
 \mathcal{C}(F(x_0), F(x_1)) & \xrightarrow{\mathcal{C}(F(x_0), F(f))} & \mathcal{C}(F(x_0), F(x_2))
 \end{array} \tag{4.1}$$

is a pullback. This means that $f: x_1 \rightarrow x_2$ satisfies the following property: for any morphism $g: x_0 \rightarrow x_2$ in \mathcal{D} and any morphism $u: F(x_0) \rightarrow F(x_1)$ in \mathcal{B} , such that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 x_0 & & \\
 \downarrow \tilde{u} & \searrow g & \\
 x_1 & \xrightarrow{f} & x_2
 \end{array} & \xrightarrow{F} & \begin{array}{ccc}
 F(x_0) & & \\
 \downarrow u & \searrow F(g) & \\
 F(x_1) & \xrightarrow{F(f)} & F(x_2)
 \end{array}
 \end{array} \tag{4.2}$$

$F(g) = F(f) \circ u$ there exists a unique morphism $\tilde{u}: x_0 \rightarrow x_1$ in \mathcal{D} , such that $P(\tilde{u}) = u$ and $\tilde{u}f = g$. We say that the morphism $f: x_1 \rightarrow x_2$ is a **cartesian morphism**, and we call a morphism $\tilde{u}: x_0 \rightarrow x_1$ a **lifting** of $u: F(x_0) \rightarrow F(x_1)$ by f along g .

Definition 4.2. We say that a functor $F: \mathcal{D} \rightarrow \mathcal{C}$ has enough cartesian morphisms if for any morphism $f: y \rightarrow x$ in \mathcal{B} and any object E in the fiber \mathcal{D}_x , i.e. such that $F(E) = x$, there exists a cartesian morphism $\tilde{f}_E: F \rightarrow E$ in \mathcal{E} such that $F(\tilde{f}_E) = f$.

Definition 4.3. A functor $F: \mathcal{D} \rightarrow \mathcal{C}$ is called a **fibration of categories** (or a **fibered category**) if it has enough cartesian morphisms.

Definition 4.4. Let $P: \mathcal{E} \rightarrow \mathcal{B}$ be a strict homomorphism of bicategories. A 1-morphism $f: x_1 \rightarrow x_2$ in \mathcal{E} is cartesian if the diagram

$$\begin{array}{ccc}
 \mathcal{E}(x_0, x_1) & \xrightarrow{\mathcal{E}(x_0, f)} & \mathcal{E}(x_0, x_2) \\
 \downarrow P & & \downarrow P \\
 \mathcal{B}(P(x_0), P(x_1)) & \xrightarrow{\mathcal{B}(P(x_0), P(f))} & \mathcal{B}(P(x_0), P(x_2))
 \end{array} \tag{4.3}$$

is a **bipullback** in Cat . This means that $f: x_1 \rightarrow x_2$ satisfies the following two properties:

- 1) for any 1-morphism $g: x_0 \rightarrow x_2$ in \mathcal{E} and a 1-morphism $u: P(x_0) \rightarrow P(x_1)$ in \mathcal{B} , and a 2-isomorphism $\beta: P(g) \Rightarrow P(f) \circ u$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 x_0 & & \\
 \downarrow \tilde{u} & \searrow g & \\
 x_1 & \xrightarrow{f} & x_2
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 P(x_0) & & \\
 \downarrow u & \searrow P(g) & \\
 P(x_1) & \xrightarrow{P(f)} & P(x_2)
 \end{array}
 \end{array} \tag{4.4}$$

$\Downarrow_{\tilde{\beta}}$ \Downarrow_{β}

there exists a 1-morphism $\tilde{u}: x_0 \rightarrow x_1$ and a 2-isomorphism $\tilde{\beta}: g \Rightarrow f \circ \tilde{u}$ in \mathcal{E} , such that $P(\tilde{u}) = u$ and $P(\tilde{\beta}) = \beta$. We say that the 1-morphism $f: x_1 \rightarrow x_2$ is a **1-cartesian 1-morphism**, and we call a pair $(\tilde{u}, \tilde{\beta})$ a **lifting** of (u, β) by f along g .

- 2) for any 2-morphism $\phi: g \Rightarrow h$ in \mathcal{E} , and any 2-morphism $\psi: u \Rightarrow v$ in \mathcal{B} , such that $(P(f) \circ \psi)\beta = \gamma P(\phi)$ for some 2-isomorphism $\gamma: P(h) \Rightarrow P(f) \circ v$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 x_0 & & \\
 \downarrow \tilde{u} & \searrow h & \\
 x_1 & \xrightarrow{f} & x_2
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 P(x_0) & & \\
 \downarrow u & \searrow P(h) & \\
 P(x_1) & \xrightarrow{P(f)} & P(x_2)
 \end{array}
 \end{array} \tag{4.5}$$

$\Downarrow_{\tilde{\psi}}$ \Downarrow_{ψ}

there exists a unique 2-morphism $\tilde{\psi}: \tilde{u} \Rightarrow \tilde{v}$, such that $P(\tilde{\psi}) = \psi$ and $(f \circ \tilde{\psi})\tilde{\beta} = \tilde{\gamma}\phi$. We say that the 1-morphism $f: x_1 \rightarrow x_2$ is a **2-cartesian 1-morphism**, and we call a 2-morphism $\tilde{\psi}$ a **lifting** of a 2-morphism ψ by f along ϕ .

Remark 4.5. *The extent of the uniqueness of vertical 1-morphisms $\tilde{u}: x_0 \rightarrow x_1$ in (4.4) is implied by the universal property (4.5) for 2-cartesian 1-morphisms. More precisely, for any two lifts $(\tilde{u}, \tilde{\beta})$ and $(\tilde{u}', \tilde{\beta})$ of (u, β) by a cartesian 1-morphism $f: x_1 \rightarrow x_2$ along $g: x_0 \rightarrow x_1$, we have a unique 2-isomorphism $\psi: \tilde{u} \Rightarrow \tilde{u}'$ which we see by putting $\beta = \gamma$ and $\phi = i_g$ in (4.5). Therefore, the induced vertical factorizations are unique up to a unique 2-isomorphism. Instead, we could have insisted on the uniqueness of vertical 1-morphisms $\tilde{u}: x_0 \rightarrow x_1$ in (4.4), which would correspond to the diagram (4.3) being a **pseudopullback** [43] which means that its universal property is determined up to an isomorphism of categories, and not up to an equivalence of categories, as in a bipullback. Later, we will emphasize once more this distinction in the construction of indexed bicategories by fibers, when we will construct homomorphisms between fibers of a fibration of bicategories.*

In the case when \mathcal{E} and \mathcal{B} are (strict) 2-categories and the diagram (4.4) is a (strict) pullback in Cat for a (strict) 2-functor $P: \mathcal{E} \rightarrow \mathcal{B}$, the resulting universal properties of a 1-morphism $f: x_1 \rightarrow x_2$ would correspond to those of Hermida, who use them to explicitly describe 2-fibrations of 2-categories in [33] and [34]. He proposed a definition of fibration of bicategories by using coherence for the bireflection of bicategories and their homomorphisms into (strict) 2-categories and (strict) 2-functors. Although that would be sensible approach to define fibrations of bicategories, we take here a more direct approach instead.

Definition 4.6. *We say that a strict homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$ of bicategories has enough cartesian 1-morphisms if for any 1-morphism $f: y \rightarrow x$ in \mathcal{B} and any object E in the fiber \mathcal{E}_x , there exists a cartesian 1-morphism $\tilde{f}_E: F \rightarrow E$ in \mathcal{E} such that $P(\tilde{f}_E) = f$.*

Definition 4.7. *A strict homomorphism of bicategories $P: \mathcal{E} \rightarrow \mathcal{B}$ is called a 2-fibration (or just a fibration) if the following conditions are satisfied:*

- *there are enough cartesian 1-morphisms*
- *a homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$ is locally a fibration, which means that for any two objects y, z in \mathcal{E} , the induced functor $P_{y,z}: \mathcal{E}(y, z) \rightarrow \mathcal{B}(y, z)$ between the Hom-categories is the usual (Grothendieck) fibration*
- *for any 1-morphism $f: x \rightarrow y$ in \mathcal{E} the induced precomposition functor*

$$\begin{array}{ccc}
 \mathcal{E}(y, z) & \xrightarrow{\mathcal{E}(f, z)} & \mathcal{E}(x, z) \\
 \downarrow P_{y,z} & & \downarrow P_{x,z} \\
 \mathcal{B}(P(y), P(z)) & \xrightarrow{\mathcal{B}(P(f), P(z))} & \mathcal{B}(P(x), P(z))
 \end{array} \tag{4.6}$$

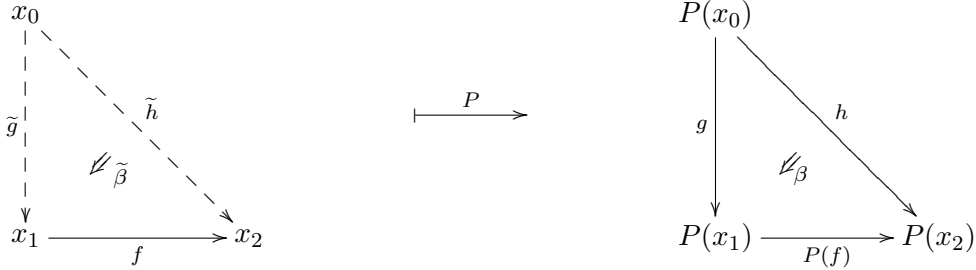
between fibrations is a Cartesian functor, i.e. it preserves cartesian 2-morphisms.

Now we will show that many familiar examples of fibrations in the existing literature are the special cases of fibrations of bicategories.

Example 4.8. (*Grothendieck fibrations*) Any functor $P: \mathcal{E} \rightarrow \mathcal{B}$ between categories may be seen as a strict homomorphism of locally discrete bicategories. Then it follows from the conjunction of the two defining properties of cartesian 1-morphisms that the functor P is a 2-fibration of (locally discrete) bicategories if and only if it is a Grothendieck fibration [29] of categories.

Example 4.9. (*Fibrations of groupoids*) Any fibration $P: \mathcal{H} \rightarrow \mathcal{G}$ of groupoids, introduced by Brown in [11], is a special case of a Grothendieck fibration of categories, and therefore it may be seen as a 2-fibration of (locally discrete) bicategories as in the previous example. From such fibrations, Brown derived a family of exact sequences familiar in homotopy theory, including a six term exact sequences familiar in nonabelian cohomology, which naturally led to the definition of a nonabelian cohomology of groupoids with coefficients in groupoids.

Example 4.10. (*Fibrations of 2-groupoids*) The category 2-Gpd_{str} of 2-groupoids and their strict homomorphisms have a closed model structure, described by Moerdijk and Svensson in [51], and it was shown earlier by Moerdijk in [50] that its homotopy category is equivalent to the homotopy category of a closed model structure on the category 2-Gpd of 2-groupoids and their homomorphisms. Fibrations in these model categories are (not necessarily) strict homomorphisms $P: \mathcal{A} \rightarrow \mathcal{B}$ of 2-groupoids such that for any 1-morphism $f: x_1 \rightarrow x_2$ in \mathcal{A} and any 1-morphisms $g: P(x_0) \rightarrow P(x_1)$ and $h: P(x_0) \rightarrow P(x_2)$, together with a 2-isomorphism $\beta: h \Rightarrow P(f) \circ g$ in \mathcal{B}



there exists 1-morphisms $\tilde{g}: x_0 \rightarrow x_1$ and $\tilde{h}: x_0 \rightarrow x_2$, together with a 2-isomorphism $\tilde{\beta}: \tilde{h} \Rightarrow f \circ \tilde{g}$ in \mathcal{A} , such that $P(\tilde{g}) = g$, $P(\tilde{h}) = h$ and $P(\tilde{\beta}) = \beta$. A strict homomorphism $P: \mathcal{A} \rightarrow \mathcal{B}$ of 2-groupoids is such fibration if and only if it is a 2-fibration of bicategories.

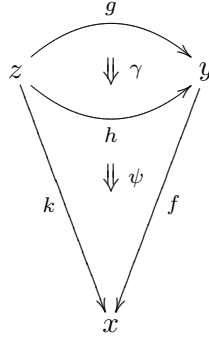
Example 4.11. (*Fibrations of bigroupoids*) The concept of a fibration of bigroupoids, introduced by Hardie, Kamps and Kieboom in [32], generalized the notion of fibration of 2-groupoids by Moerdijk from the previous example. They used Brown's construction in order to derive an exact nine term sequence from such fibrations, and they applied their theory to the construction of a homotopy bigroupoid of a topological space [31].

Example 4.12. (Fibrations in closed model structures on categories 2-Cat and Bicat_s) The category 2-Cat of 2-categories and 2-functors has a closed model structure, introduced by Lack in [44], which he extended to the category Bicat_s of bicategories and their strict homomorphisms, in his subsequent paper [44]. These model structures are closely related to the model structure of Moerdijk and Svensson in [51]. Fibrations in these model categories are strict homomorphisms $F: \mathcal{A} \rightarrow \mathcal{B}$ having the equivalence lifting property:

- a) for any equivalence $f: x \rightarrow y$ in \mathcal{B} and for any object B in \mathcal{A} such that $F(B) = y$, there exist an equivalence $\tilde{f}: A \rightarrow B$ in \mathcal{A} such that $F(\tilde{f}) = f$.
- b) for any 1-morphism $v: A \rightarrow B$ in \mathcal{A} and any 2-isomorphism $\beta: g \Rightarrow h$ in \mathcal{B} such that $F(v) = h$, there exist a 2-isomorphism $\tilde{\beta}: u \Rightarrow v$ in \mathcal{A} such that $F(u) = g$ and $F(\tilde{\beta}) = \beta$.

Any equivalence in \mathcal{A} is a cartesian 1-morphism since it satisfies properties (4.4) and (4.5), in analogy with the fact that for any functor $F: \mathcal{E} \rightarrow \mathcal{B}$, any isomorphism $f: x \rightarrow y$ in \mathcal{E} is cartesian. Therefore, 2-fibrations of bicategories are special cases of fibrations in closed model structures on categories 2-Cat and Bicat_s , as those strict homomorphisms $F: \mathcal{A} \rightarrow \mathcal{B}$ having the lifting property for all 1-morphisms in \mathcal{A} , and not just for equivalences.

Example 4.13. (A domain fibration of the homotopy fiber of a bicategory) For any bicategory \mathcal{B} , and any object x in \mathcal{B} , a homotopy fiber $\mathcal{B}_{\tilde{x}}$ is a bicategory whose objects are 1-morphisms $f: y \rightarrow x$ of \mathcal{B} whose 0-target is x . For any other object $k: z \rightarrow x$ in $\mathcal{B}_{\tilde{x}}$, a 1-morphism $(g, \phi): k \rightarrow f$ in $\mathcal{B}_{\tilde{x}}$ consists of a 1-morphism $g: z \rightarrow y$ together with a 2-morphism $\phi: k \Rightarrow g \circ f$ in \mathcal{B} . Finally, a 2-morphism $\gamma: (g, \phi) \Rightarrow (h, \psi)$ in $\mathcal{B}_{\tilde{x}}$



is a 2-morphism $\gamma: k \Rightarrow g \circ f$ in \mathcal{B} such that $\psi = (f \circ \gamma)\phi$. A domain fibration of the homotopy fiber $\mathcal{B}_{\tilde{x}}$ is a strict homomorphism $J_x: \mathcal{B}_{\tilde{x}} \rightarrow \mathcal{B}$ of bicategories defined by $J_x(f) = y$ for any object $f: y \rightarrow x$ in $\mathcal{B}_{\tilde{x}}$, $J_x(g, \phi) = g$ for any 1-morphism $(g, \phi): k \rightarrow f$ in $\mathcal{B}_{\tilde{x}}$ and $J_x(\gamma) = \gamma$ for any 2-morphism $\gamma: (g, \phi) \Rightarrow (h, \psi)$ in $\mathcal{B}_{\tilde{x}}$. It is obvious that a strict homomorphism $J_x: \mathcal{B}_{\tilde{x}} \rightarrow \mathcal{B}$ is a fibration since for any 1-morphism $g: z \rightarrow y$ in \mathcal{B} and any object $f: y \rightarrow x$ in $\mathcal{B}_{\tilde{x}}$ over y , i.e. such that $J_x(f) = y$, there exists a canonical 1-morphism $(g, i_{f \circ g}): f \circ g \rightarrow f$ in $\mathcal{B}_{\tilde{x}}$, which is clearly cartesian.

There is a 2-dimensional analog of the characterization of those fibrations $F: \mathcal{D} \rightarrow \mathcal{C}$ of categories which are *fibred in groupoids*, i.e. for which all the fibers \mathcal{D}_x are groupoids. It is well known that this holds if and only if all the morphisms in the category \mathcal{D} are cartesian. First we give an analogous definition in a 2-dimensional case.

Definition 4.14. *A strict homomorphism of bicategories $P: \mathcal{E} \rightarrow \mathcal{B}$ is called a fibration in bigroupoids (or a bigroupoid fibration) if all the fibers \mathcal{E}_x are bigroupoids.*

Now we prove the mentioned 2-dimensional characterization of fibrations in bigroupoids.

Theorem 4.15. *A strict homomorphism of bicategories $P: \mathcal{E} \rightarrow \mathcal{B}$ is a fibration in bigroupoids if and only if all 1-morphisms and 2-morphisms in the bicategory \mathcal{E} are cartesian.*

Proof. Suppose that all morphisms in \mathcal{E} are cartesian and let $u: E \rightarrow G$ and $v: F \rightarrow G$ be two lifts of a 1-morphism $f: x_1 \rightarrow x_2$ in the bicategory \mathcal{B} . Then for any 1-morphism

$$\begin{array}{ccc}
 \begin{array}{ccc}
 F & & \\
 \downarrow j & \searrow v & \\
 E & \xrightarrow{u} & G
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 x_1 & & \\
 \downarrow i_{x_1}^{\theta(j)} & \searrow f & \\
 x_1 & \xrightarrow{f} & x_2
 \end{array}
 \end{array}$$

$\Downarrow \beta$ $\Downarrow \rho_f^{-\theta(j)}$

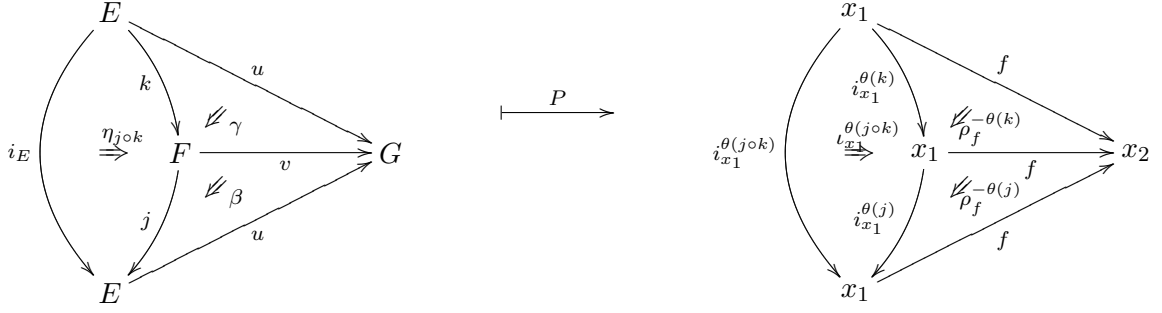
$j: E \rightarrow F$ in the fiber bicategory \mathcal{E}_{x_1} such that $P(j) = i_{x_1}^{\theta(j)}$, there exists a 2-isomorphism $\beta: v \Rightarrow u \circ j$ such that $P(\beta) = \rho_f^{-\theta(j)}$ by (4.4) since we assumed that $u: F \rightarrow G$ is cartesian. By the same argument, since $v: F \rightarrow G$ is cartesian, we find a 1-morphism $k: F \rightarrow E$ in the fiber bicategory \mathcal{E}_{x_1} together with a 2-isomorphism $\gamma: u \Rightarrow v \circ k$ such that $P(\gamma) = \rho_f^{-\theta(k)}$.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 F & & \\
 \downarrow k & \searrow u & \\
 E & \xrightarrow{v} & G
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 x_1 & & \\
 \downarrow i_{x_1}^{\theta(k)} & \searrow f & \\
 x_1 & \xrightarrow{f} & x_2
 \end{array}
 \end{array}$$

$\Downarrow \gamma$ $\Downarrow \rho_f^{-\theta(k)}$

From an identity $\rho_f^{-\theta(j \circ k)} = (\rho_f^{-\theta(j)} \circ i_{x_1}^{\theta(k)}) \rho_f^{-\theta(k)}$ which follows from the coherence theorem

for bicategories [49], we use (4.5) in order to obtain a unique 2-isomorphism $\eta_{j \circ k}: i_E \Rightarrow j \circ k$



such that $P(\eta_{j \circ k}) = i_{x_1}^{\theta(j \circ k)}$. The same argument in other direction would give a unique 2-isomorphism $\eta_{k \circ j}: i_F \Rightarrow k \circ j$ such that $P(\eta_{k \circ j}) = i_{x_1}^{\theta(k \circ j)}$ which shows that $j: E \rightarrow F$ is an equivalence with a weak inverse $k: F \rightarrow E$. The fact that any 2-morphism in the fiber bicategory \mathcal{E}_{x_1} is a 2-isomorphism follows by the same argument as in the ordinary 1-dimensional case since $P: \mathcal{E} \rightarrow \mathcal{B}$ is locally a Grothendieck fibration. \square

Example 4.16. (*2-gerbes*) The corresponding 2-dimensional sheaf-theoretical notions which may be built from bicategories are introduced by Breen in [10] under a name of 2-gerbes. These geometric objects are 2-dimensional analogs of gerbes, introduced by Giraud in [23], which are locally nonempty and locally connected stacks in groupoids. In order to provide a complete set of cohomological invariants for 2-dimensional nonabelian cohomology, Breen defined 2-gerbes as locally nonempty and locally connected 2-stacks fibred in bigroupoids. Although he used a more restrictive notion of 2-stacks over a topological space X as trihomomorphisms from the opposite of a locally discrete tricategory $\mathcal{O}(X)$ of open subsets of X

$$\mathbb{G}: \mathcal{O}(X)^{op} \rightarrow \text{Bicat}$$

which satisfy effective descent conditions for objects, 1-morphisms and 2-morphisms, one can use a more general notion of 2-gerbes over any site \mathcal{C} , that is a category \mathcal{C} equipped with the Grothendieck topology. In this way, a 2-gerbe over a site \mathcal{C} is a fibration in bigroupoids

$$\mathbb{G}: \mathcal{E} \rightarrow \mathcal{C}$$

which satisfy effective descent conditions for objects, 1-morphisms and 2-morphisms in the bicategory \mathcal{E} , where the category \mathcal{C} is again regarded as a locally discrete bicategory.

Definition 4.17. Let $P: \mathcal{E} \rightarrow \mathcal{B}$ be a strict homomorphism of bicategories. A 1-morphism $f: x_0 \rightarrow x_1$ in \mathcal{E} is cocartesian if the diagram

$$\begin{array}{ccc}
 \mathcal{E}(x_1, x_2) & \xrightarrow{\mathcal{E}(f, x_2)} & \mathcal{E}(x_0, x_2) \\
 \downarrow F & & \downarrow P \\
 \mathcal{B}(P(x_1), P(x_2)) & \xrightarrow{\mathcal{B}(P(f), P(x_2))} & \mathcal{B}(P(x_0), P(x_2))
 \end{array} \tag{4.7}$$

is a **bipushout** in Cat . This means that $f: x_1 \rightarrow x_2$ satisfies the following two properties:

- 1) for any 1-morphism $g: x_0 \rightarrow x_2$ in \mathcal{E} and a 1-morphism $u: P(x_0) \rightarrow P(x_1)$ in \mathcal{B} , and a 2-isomorphism $\beta: u \circ P(f) \Rightarrow P(g)$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 x_0 & \xrightarrow{f} & x_1 \\
 & \searrow g & \downarrow \tilde{u} \\
 & & x_2
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 P(x_0) & \xrightarrow{P(f)} & P(x_1) \\
 & \searrow P(g) & \downarrow u \\
 & & P(x_2)
 \end{array}
 \end{array} \tag{4.8}$$

there exists a 1-morphism $\tilde{u}: x_1 \rightarrow x_2$ and a 2-isomorphism $\tilde{\beta}: \tilde{u} \circ f \Rightarrow g$ in \mathcal{E} , such that $P(\tilde{u}) = u$ and $P(\tilde{\beta}) = \beta$. We say that the 1-morphism $f: x_0 \rightarrow x_1$ is a **1-cocartesian 1-morphism**, and a pair $(\tilde{u}, \tilde{\beta})$ is a **lifting** of (u, β) by f along g .

- 2) for any 2-morphism $\phi: g \Rightarrow h$ in \mathcal{E} , and any 2-morphism $\psi: u \Rightarrow v$ in \mathcal{B} , such that $P(\phi)\beta = \gamma(\psi \circ P(f))$ for some 2-isomorphism $\gamma: v \circ P(f) \Rightarrow P(h)$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 x_0 & \xrightarrow{f} & x_1 \\
 & \searrow g & \downarrow \tilde{u} \\
 & & x_2
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 P(x_0) & \xrightarrow{P(f)} & P(x_1) \\
 & \searrow P(g) & \downarrow u \\
 & & P(x_2)
 \end{array}
 \end{array} \tag{4.9}$$

there exists a unique 2-morphism $\tilde{\psi}: \tilde{u} \Rightarrow \tilde{v}$, such that $P(\tilde{\psi}) = \psi$ and $\phi\tilde{\beta} = \tilde{\gamma}(\tilde{\psi} \circ f)$. We say that the 1-morphism $f: x_1 \rightarrow x_2$ is a **2-cocartesian 1-morphism**, and we call a 2-morphism $\tilde{\psi}$ a **lifting** of a 2-morphism ψ by f along ϕ .

Remark 4.18. *The extent of the uniqueness of vertical 1-morphisms $\tilde{u}: x_1 \rightarrow x_2$ in (4.4) is implied by the universal property (4.9) for 2-cocartesian 1-morphisms. More precisely, for any two lifts $(\tilde{u}, \tilde{\beta})$ and $(\tilde{u}', \tilde{\beta})$ of (u, β) by a cocartesian 1-morphism $f: x_0 \rightarrow x_1$ along $g: x_0 \rightarrow x_2$, we have a unique 2-isomorphism $\tilde{\psi}: \tilde{u} \Rightarrow \tilde{u}'$ which we see by putting $\beta = \gamma$ and $\phi = i_g$ in (4.9). Therefore, the induced vertical factorizations are unique up to a unique 2-isomorphism. Instead, we could have insisted on the uniqueness of vertical 1-morphisms $\tilde{u}: x_1 \rightarrow x_2$ in (4.8), which would correspond to the diagram (4.7) being a **pseudopushout** which means that its universal property is determined up to an isomorphism of categories, and not up to an equivalence of categories, as in a bipushout.*

In the case when \mathcal{E} and \mathcal{B} are (strict) 2-categories and the diagram (4.8) is a (strict) pushout in Cat for a (strict) 2-functor $P: \mathcal{E} \rightarrow \mathcal{B}$, the resulting universal properties of a 1-morphism $f: x_1 \rightarrow x_2$ would correspond to 2-cofibrations of 2-categories, in a complete analogy with Hermida's notion of a 2-fibration of (strict) 2-categories [33],[34].

Definition 4.19. *We say that a strict homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$ of bicategories has enough cocartesian 1-morphisms if for any 1-morphism $f: x \rightarrow y$ in \mathcal{B} and any object E in the fiber \mathcal{E}_x , there exists a cartesian 1-morphism $f_E: E \rightarrow F$ in \mathcal{E} such that $P(f_E) = f$.*

Definition 4.20. *A strict homomorphism of bicategories $P: \mathcal{E} \rightarrow \mathcal{B}$ is called a 2-cofibration (or just a cofibration) if the following conditions are satisfied:*

- *it has enough cocartesian 1-morphisms*
- *a homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$ is locally a cofibration, which means that for any two objects y, z in \mathcal{E} , the induced functor $P_{y,z}: \mathcal{E}(y, z) \rightarrow \mathcal{B}(y, z)$ between the Hom-categories is the usual (Grothendieck) cofibration*
- *for any 1-morphism $g: y \rightarrow z$ in \mathcal{E} the induced composition functor*

$$\begin{array}{ccc}
 \mathcal{E}(x, y) & \xrightarrow{\mathcal{E}(x, g)} & \mathcal{E}(x, z) \\
 \downarrow P_{x, y} & & \downarrow P_{x, z} \\
 \mathcal{B}(P(x), P(y)) & \xrightarrow{\mathcal{B}(P(x), P(g))} & \mathcal{B}(P(x), P(z))
 \end{array} \tag{4.10}$$

between fibrations is a Cocartesian functor, i.e. it preserves cocartesian 2-morphisms.

Remark 4.21. *Note that a strict homomorphism of bicategories $P: \mathcal{E} \rightarrow \mathcal{B}$ can satisfy a combination of conditions from the previous definition and the definition of a fibration of bicategories. More precisely, we call a strict homomorphism of bicategories $P: \mathcal{E} \rightarrow \mathcal{B}$ a locally cofibred fibration of bicategories if it has enough cartesian 1-morphisms and if it is locally a cofibration, such that induced composition functors are Cocartesian. Similarly, we define locally fibred cofibrations of bicategories.*

One genuine example of a locally discretely cofibred 2-fibration of bicategories comes from actions of bicategories, introduced by the author in [5].

Example 4.22. (*Actions of bicategories*) For any action $A: \mathcal{P} \times_{\mathcal{B}_0} \mathcal{B}_1 \rightarrow \mathcal{P}$ of the bicategory \mathcal{B} on the category $\Lambda: \mathcal{P} \rightarrow \mathcal{B}_0$ over the discrete category \mathcal{B}_0 of objects of \mathcal{B} , as given in [5],

$$\begin{array}{ccc}
 P_1 & & B_2 \\
 \downarrow t & \parallel s & \downarrow t_1 \parallel s_1 \\
 P_0 & & B_1 \\
 & \searrow \Lambda_0 & \downarrow t_0 \parallel s_0 \\
 & & B_0
 \end{array} \tag{4.11}$$

there exists an action bicategory $\mathcal{P} \triangleleft \mathcal{B}$ whose underlying 2-graph consists of:

- objects $\mathcal{P} \triangleleft \mathcal{B}_0$ of an action bicategory $\mathcal{P} \triangleleft \mathcal{B}$ are objects P_0 of the category \mathcal{P}
- 1-morphisms $\mathcal{P} \triangleleft \mathcal{B}_1$ of an action bicategory $\mathcal{P} \triangleleft \mathcal{B}$ consists of pairs $(\phi, f): q \rightarrow p$

$$q \xrightarrow{(\phi, f)} p$$

where $f: \Lambda_0(q) \rightarrow \Lambda_0(p)$ is a 1-morphism in \mathcal{B} and $\phi: q \rightarrow p \triangleleft f$ is a morphism in \mathcal{P}

- 2-morphisms $\mathcal{P} \triangleleft \mathcal{B}_2$ of an action bicategory $\mathcal{P} \triangleleft \mathcal{B}$ are 2-morphisms $\gamma: (\phi, f) \Rightarrow (\psi, g)$

$$\begin{array}{ccc}
 & (\phi, f) & \\
 q & \curvearrowright & p \\
 & \Downarrow \gamma & \\
 & (\psi, g) &
 \end{array}$$

where $\gamma: f \Rightarrow g$ is a 2-morphism in \mathcal{B} , such that the diagram of morphisms in \mathcal{P}

$$\begin{array}{ccc}
 q & \xrightarrow{\phi} & p \triangleleft f \\
 & \searrow \psi & \downarrow p \triangleleft \gamma \\
 & & p \triangleleft g
 \end{array}$$

commutes.

The horizontal composition of any two horizontally composable pair of 1-morphisms in $\mathcal{P} \triangleleft \mathcal{B}$

$$r \xrightarrow{(\psi, h)} q \xrightarrow{(\phi, f)} p$$

is a 1-morphism $(\phi \circ \psi, f \circ h): r \rightarrow p$ in $\mathcal{P} \triangleleft \mathcal{B}$, where $\phi \circ \psi: r \rightarrow p \triangleleft (f \circ h)$ is a composition

$$r \xrightarrow{\psi} q \triangleleft h \xrightarrow{\phi \triangleleft h} (p \triangleleft f) \triangleleft h \xrightarrow{\kappa_{p, f, h}} p \triangleleft (f \circ h)$$

of morphisms in \mathcal{P} , and $\kappa_{p, f, h}: (p \triangleleft f) \triangleleft h \rightarrow p \triangleleft (f \circ h)$ is a component of the coherence isomorphism for an action. The vertical composition of 2-morphisms $\gamma: (\phi, f) \Rightarrow (\psi, g)$ and $\delta: (\psi, g) \Rightarrow (\varphi, k)$ is a 2-morphism $\delta\gamma: (\phi, f) \Rightarrow (\varphi, k)$ in $\mathcal{P} \triangleleft \mathcal{B}$ represented by the vertical composition $\delta\gamma: f \Rightarrow k$ of 2-morphisms in \mathcal{B} .

It was shown in [5] that for any such action of the bicategory \mathcal{B} on $\Lambda: \mathcal{P} \rightarrow \mathcal{B}_0$, there exists a canonical projection

$$\Lambda: \mathcal{P} \triangleleft \mathcal{B} \rightarrow \mathcal{B} \quad (4.12)$$

which is a strict homomorphism of bicategories. A homomorphism $\Lambda: \mathcal{P} \triangleleft \mathcal{B} \rightarrow \mathcal{B}$ is defined by (the object component of) the momentum functor $\Lambda(p) = \Lambda_0(p)$, for any object p in $\mathcal{P} \triangleleft \mathcal{B}$. For any 1-morphism (ϕ, f) it is defined by $\Lambda(\phi, f) = f$, and for any 2-morphism $\gamma: (\phi, f) \Rightarrow (\psi, g)$ in $\mathcal{P} \triangleleft \mathcal{B}$, it is given simply by $\Lambda(\gamma) = \gamma$.

For any 1-morphism $f: y \rightarrow x$ in \mathcal{B} and any object p in \mathcal{P} such that $\Lambda_0(p) = x$, it follows from the axioms of an action that $\Lambda_0(p \triangleleft f) = x$. Then a 1-morphism $(i_{p \triangleleft f}, f): p \triangleleft f \rightarrow p$ in $\mathcal{P} \triangleleft \mathcal{B}$ is a cartesian, since for any 1-morphism $(\psi, g): r \rightarrow p$ in $\mathcal{P} \triangleleft \mathcal{B}$ and any 1-morphism $u: z \rightarrow y$ together with a 2-morphism $\beta: g \rightarrow f \circ u$ in \mathcal{B}

$$\begin{array}{ccc}
 \begin{array}{ccc}
 r & & \\
 \downarrow (\xi, u) & \searrow (\psi, g) & \\
 p \triangleleft f & \xrightarrow{(i_{p \triangleleft f}, f)} & p
 \end{array}
 & \xrightarrow{\Lambda} &
 \begin{array}{ccc}
 z & & \\
 \downarrow u & \searrow g & \\
 y & \xrightarrow{f} & x
 \end{array}
 \end{array}$$

$\swarrow \tilde{\beta}$

 $\swarrow \beta$

there exists a 1-morphism $(\xi, u): r \rightarrow p \triangleleft f$ in $\mathcal{P} \triangleleft \mathcal{B}$, where $\xi: r \rightarrow (p \triangleleft f) \triangleleft u$ is defined by

$$r \xrightarrow{\psi} p \triangleleft g \xrightarrow{p \triangleleft \beta} p \triangleleft (f \circ u) \xrightarrow{\kappa_{p, f, u}^{-1}} (p \triangleleft f) \triangleleft u$$

for which $\beta: (\psi, g) \Rightarrow (i_{p \triangleleft f}, f) \circ (\xi, u)$ is obviously a 2-morphism in the bicategory $\mathcal{P} \triangleleft \mathcal{B}$

$$\begin{array}{c}
 r \xrightarrow{\psi} p \triangleleft g \\
 \parallel \\
 r \xrightarrow{\psi} p \triangleleft g \xrightarrow{p \triangleleft \beta} p \triangleleft (f \circ u) \xrightarrow{\kappa_{p,f,u}^{-1}} (p \triangleleft f) \triangleleft u \xrightarrow{i_{p \triangleleft f} \triangleleft u} (p \triangleleft f) \triangleleft u \xrightarrow{\kappa_{p,f,u}} p \triangleleft (f \circ u) \\
 \downarrow p \triangleleft \beta \\
 p \triangleleft g
 \end{array}$$

by the commutativity of the above diagram which follows from the identity $i_{p \triangleleft f} \triangleleft u = i_{(p \triangleleft f) \triangleleft u}$.
 For any 2-morphism $\phi: (\phi, f) \Rightarrow (\psi, g)$ in $\mathcal{P} \triangleleft \mathcal{B}$, and any 2-morphism $\omega: u \Rightarrow v$ in \mathcal{B}

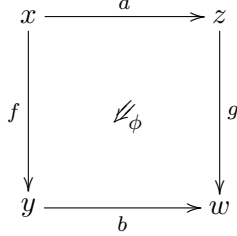
$$\begin{array}{ccc}
 \begin{array}{c}
 r \\
 \downarrow (\xi, u) \\
 p \triangleleft f \\
 \downarrow (i_{p \triangleleft f}, f) \\
 p
 \end{array}
 & \xrightarrow{P} &
 \begin{array}{c}
 z \\
 \downarrow u \\
 y \\
 \downarrow f \\
 x
 \end{array} \\
 \begin{array}{c}
 \downarrow (\zeta, v) \\
 \downarrow (\psi, g) \\
 \downarrow \tilde{\gamma}
 \end{array}
 & &
 \begin{array}{c}
 \downarrow v \\
 \downarrow g \\
 \downarrow \gamma
 \end{array}
 \end{array}$$

such that $(f \circ \omega)\beta = \gamma\phi$ for a 2-isomorphism $\gamma: h \Rightarrow f \circ v$ in \mathcal{B} , the following diagram

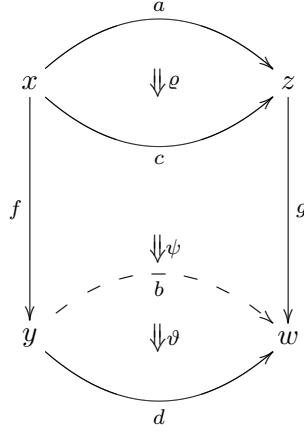
$$\begin{array}{ccccc}
 r & \xrightarrow{\psi} & p \triangleleft g & \xrightarrow{p \triangleleft \beta} & p \triangleleft (f \triangleleft u) \\
 \parallel & & \downarrow \xi & & \downarrow \kappa_{p,f,u}^{-1} \\
 r & \xrightarrow{\xi} & (p \triangleleft f) \triangleleft u & & \\
 \parallel & & \downarrow p \triangleleft \phi & & \downarrow p \triangleleft (f \triangleleft \omega) \\
 r & \xrightarrow{\theta} & p \triangleleft h & \xrightarrow{p \triangleleft \gamma} & p \triangleleft (f \triangleleft v) \\
 \parallel & & \downarrow \kappa_{p,f,v}^{-1} & & \downarrow (p \triangleleft f) \triangleleft \omega \\
 r & \xrightarrow{\zeta} & (p \triangleleft f) \triangleleft v & &
 \end{array}$$

commutes, in which the front face represents a 2-morphism $\omega: (\xi, u) \Rightarrow (\zeta, v)$ in $\mathcal{P} \triangleleft \mathcal{B}$.
 Therefore, a 1-morphism $(i_{p \triangleleft f}, f): p \triangleleft f \rightarrow p$ in $\mathcal{P} \triangleleft \mathcal{B}$ is a cartesian, and it is clear that any 2-morphism $\beta: f \Rightarrow g$ in \mathcal{B} represents a unique 2-morphism $\beta: (\phi, f) \Rightarrow (\psi, g)$ in $\mathcal{P} \triangleleft \mathcal{B}$ from any 1-morphism $(\phi, f): q \rightarrow p$ to a 1-morphism $(\psi, g): q \rightarrow p$ such that $\psi = (p \triangleleft \beta)\phi$.
 We conclude that the strict homomorphism (4.12) is locally discrete cofibration.

Example 4.23. (A fundamental fibration and a fundamental cofibration of a bicategory)
Let \mathcal{B} be a bicategory and \mathcal{I} a locally discrete bicategory which consists of two objects and one nontrivial morphism between them. The bicategory $\mathcal{B}^{\mathcal{I}}$ of 1-morphisms of \mathcal{B} , has 1-morphisms of \mathcal{B} for objects, and therefore $\mathcal{B}_0^{\mathcal{I}} = \mathcal{B}_1$. A 1-morphism $(a, \phi, b): f \rightarrow g$ in $\mathcal{B}^{\mathcal{I}}$ from $f: x \rightarrow y$ to $g: z \rightarrow w$ consists of 1-morphisms $a: x \rightarrow z$ and $b: y \rightarrow w$ in \mathcal{B} together with a 2-morphism $\phi: g \circ a \Rightarrow b \circ f$ as in the diagram



and a 2-morphism $(\varrho, \vartheta): (a, \phi, b) \Rightarrow (c, \psi, d)$ in $\mathcal{B}^{\mathcal{I}}$ consists of 2-morphisms $\varrho: a \Rightarrow c$ and $\vartheta: b \Rightarrow d$ in \mathcal{B} such that $\psi(g \circ \varrho) = (\vartheta \circ f)\phi$, which means that the following diagram in \mathcal{B}



commutes. There exists a canonical strict homomorphism of bicategories

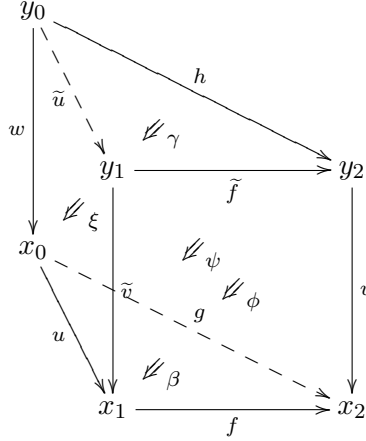
$$\partial_0: \mathcal{B}^{\mathcal{I}} \rightarrow \mathcal{B} \tag{4.13}$$

which takes any $f: x \rightarrow y$ in $\mathcal{B}^{\mathcal{I}}$ to an object y in \mathcal{B} , any 1-morphism $(a, \phi, b): f \rightarrow g$ in $\mathcal{B}^{\mathcal{I}}$ to a 1-morphism $g: z \rightarrow w$ in \mathcal{B} and any 2-morphism $(\varrho, \vartheta): (a, \phi, b) \Rightarrow (c, \psi, d)$ in $\mathcal{B}^{\mathcal{I}}$ to a 2-morphism $\vartheta: b \Rightarrow d$ in \mathcal{B} . Another canonical strict homomorphism of bicategories

$$\partial_1: \mathcal{B}^{\mathcal{I}} \rightarrow \mathcal{B} \tag{4.14}$$

is obtained in a similar way by taking sources, instead of targets of respective objects, 1-morphisms and 2-morphisms in the bicategory $\mathcal{B}^{\mathcal{I}}$.

If \mathcal{B} is a bicategory with bipullbacks then a strict homomorphism of bicategories (4.13) is a fibration of a bicategories which we call a *fundamental fibration of the bicategory \mathcal{B}* . We note that for any 1-morphism $f: x_1 \rightarrow x_2$ in \mathcal{B} and any $u: y_2 \rightarrow x_2$ in \mathcal{B}^I such that $\partial_0(u) = x_2$, their bipullback $\phi: v \circ \tilde{f} \Rightarrow f \circ \tilde{v}$ in \mathcal{B} represents a cartesian 1-morphism $(f, \phi, f): \tilde{v} \rightarrow v$ in \mathcal{B}^I . To verify this, for any 1-morphism $(h, \psi, g): w \rightarrow v$ in \mathcal{B}^I and any 1-morphism $u: x_0 \rightarrow x_1$ with a 2-morphism $\beta: g \Rightarrow f \circ u$ in \mathcal{B}



we take a 2-morphism $\alpha_{f,u,w}\beta\psi: v \circ h \Rightarrow f \circ (u \circ w)$ and we use the universal property of the bipullback in order to obtain a 1-morphism $\tilde{u}: y_0 \rightarrow y_1$ in \mathcal{B} together with 2-morphisms $\xi: \tilde{v} \circ \tilde{u} \Rightarrow u \circ w$ and $\gamma: h \Rightarrow \tilde{f} \circ \tilde{u}$ such that $(f \circ \xi)\alpha_{f,\tilde{v},\tilde{u}}(\phi \circ \tilde{u})\alpha_{v,\tilde{f},\tilde{u}}^{-1}(v \circ \gamma) = \alpha_{f,u,w}\beta\psi$.

If \mathcal{B} is a bicategory with bipushouts then a strict homomorphism of bicategories (4.14) is a *cofibration of a bicategories* which we call a *fundamental cofibration of the bicategory \mathcal{B}* .

The two strict homomorphisms (4.13) and (4.14) form a span in the tricategory $Bicat_s$

$$\begin{array}{ccc}
 & \mathcal{B}^I & \\
 \partial_1 \swarrow & & \searrow \partial_0 \\
 \mathcal{B} & & \mathcal{B}
 \end{array} \tag{4.15}$$

which is a pseudomonoid [18] in the monoidal bicategory $Span(Bicat_s)(\mathcal{B}, \mathcal{B})$, obtained in the same way as we define a monoid object in any monoidal category. This is an instance of the *microcosm principle* [2] which says that certain algebraic structures can be defined in any category equipped with a categorified version of the same structure. Consequently, $Span(Bicat_s)(\mathcal{B}, \mathcal{B})$ is a hom-bicategory of a tricategory $Span(Bicat_s)$ of spans of bicategories which can be constructed in an analogy with a construction of a bicategory $Span(\text{Cat})$ of spans of categories. Any strict homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$ of

bicategories may be regarded as a span

$$\begin{array}{ccc} & \mathcal{E} & \\ P \swarrow & & \searrow I_{\mathcal{E}} \\ \mathcal{B} & & \mathcal{E} \end{array} \tag{4.16}$$

in $\text{Span}(\text{Bicat}_s)$, and formally, we can regard an action of a span (4.15) on a span (4.16). Any such action corresponds to a so-called 2-cleavage of a strict homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$.

In order to relate a 2-cleavage of a strict homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$ of bicategories with an action of a span (4.15) on (4.16), we use a comma bicategory (\mathcal{B}, P) which is a (strict) pullback

$$\begin{array}{ccc}
 (\mathcal{B}, P) & \xrightarrow{P} & \mathcal{B}^{\mathcal{I}} \\
 \downarrow I & & \downarrow \partial_0 \\
 \mathcal{E} & \xrightarrow{P} & \mathcal{B}
 \end{array}$$

More explicitly, objects of the bicategory (\mathcal{B}, P) are triples (x, f, E) where x is an object in \mathcal{B} , E is an object in \mathcal{E} and $f: x \rightarrow P(E)$ is a 1-morphism in \mathcal{B} . A 1-morphism from (x, f, E) to (y, g, F) is a triple $(u, \phi, j): (x, f, E) \rightarrow (y, g, F)$, where $u: x \rightarrow y$ is a 1-morphism in \mathcal{B} , $j: E \rightarrow F$ is a 1-morphism in \mathcal{E} , and $\phi: P(j) \circ f \Rightarrow g \circ u$ is a 2-morphism

$$\begin{array}{ccc}
 x & \xrightarrow{f} & P(E) \\
 \downarrow u & \swarrow \phi & \downarrow P(j) \\
 y & \xrightarrow{g} & P(F)
 \end{array}$$

and a 2-morphism is a pair $(\delta, \theta): (u, \phi, j) \Rightarrow (v, \psi, k)$ where $\delta: u \Rightarrow v$ is a 2-morphism in \mathcal{B} and $\theta: j \Rightarrow k$ is a 2-morphism in \mathcal{E} as in a commutative diagram

$$\begin{array}{ccc}
 x & \xrightarrow{f} & P(E) \\
 \downarrow u & \swarrow \delta & \downarrow P(j) \\
 y & \xrightarrow{g} & P(F)
 \end{array}
 \quad
 \begin{array}{ccc}
 x & \xrightarrow{f} & P(E) \\
 \downarrow v & \swarrow \psi & \downarrow P(k) \\
 y & \xrightarrow{g} & P(F)
 \end{array}$$

(Note: The diagram above shows two separate commutative diagrams. The left one has a solid arrow δ from x to v and a solid arrow ψ from $P(E)$ to $P(F)$. The right one has a dashed arrow θ from x to v and a dashed arrow ψ from $P(E)$ to $P(F)$. The text indicates that ϕ is omitted for brevity.)

(in which we have omitted ϕ in order to avoid too many labels) commutes, which is the

same as the commutativity of the diagram of 2-morphisms

$$\begin{array}{ccc}
 P(j) \circ f & \xrightarrow{P(\theta) \circ f} & P(k) \circ f \\
 \Downarrow \phi & & \Downarrow \psi \\
 g \circ u & \xrightarrow{g \circ \delta} & g \circ v
 \end{array}$$

in \mathcal{B} . The horizontal composition $(v, \psi, k) \circ (u, \phi, j)$ of 1-morphisms in (\mathcal{B}, P)

$$\begin{array}{ccc}
 \begin{array}{ccc}
 x & \xrightarrow{f} & P(E) \\
 \downarrow u & \swarrow \phi & \downarrow P(j) \\
 y & \xrightarrow{g} & P(F) \\
 \downarrow v & \swarrow \psi & \downarrow P(k) \\
 z & \xrightarrow{h} & P(G)
 \end{array} & \xrightarrow{H} & \begin{array}{ccc}
 x & \xrightarrow{f} & P(E) \\
 \downarrow v \circ u & \swarrow \psi \boxminus \phi & \downarrow P(k \circ j) \\
 z & \xrightarrow{h} & P(G)
 \end{array}
 \end{array}$$

is a 1-morphism $(v \circ u, \psi \boxminus \phi, k \circ j): (x, f, E) \rightarrow (z, h, G)$ in (\mathcal{B}, P) , where the 2-morphism $\psi \boxminus \phi: P(k \circ j) \circ f \rightarrow h \circ (v \circ u)$ in \mathcal{B} is the left vertical 2-morphism in a diagram

$$\begin{array}{ccccc}
 (P(k) \circ P(j)) \circ f & \xrightarrow{\alpha_{P(k), P(j), f}} & P(k) \circ (P(j) \circ f) & \xrightarrow{P(k) \circ \phi} & P(k) \circ (g \circ u) \\
 \Downarrow \psi \boxminus \phi & & & & \Downarrow \alpha_{P(k), g, u}^{-1} \\
 h \circ (v \circ u) & \xleftarrow{\alpha_{h, v, u}} & (h \circ v) \circ u & \xleftarrow{\psi \circ u} & (P(k) \circ g) \circ u
 \end{array}$$

defined by the requirement that the diagram commutes.

For any strict homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$ of bicategories, we have a composition of spans

$$\begin{array}{ccccc}
 & & (\mathcal{B}, P) & & \\
 & P_1 \swarrow & & \searrow P_2 & \\
 & \mathcal{B}^{\mathcal{I}} & & \mathcal{E} & \\
 D_1 \swarrow & & D_0 \searrow & P \swarrow & \searrow I_{\mathcal{E}} \\
 \mathcal{B} & & \mathcal{B} & & \mathcal{E}
 \end{array} \tag{4.17}$$

and the left leg of a span (4.17) is a strict homomorphism from the bicategory (\mathcal{B}, P)

$$D_1 P_1: (\mathcal{B}, P) \rightarrow \mathcal{B}. \tag{4.18}$$

Theorem 4.24. *For any strict homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$ of bicategories, the homomorphism (4.18) is a fibration of bicategories, called a canonical fibration associated to P .*

Proof. For any 1-morphism $f: x \rightarrow y$ in \mathcal{B} , and any object $g: y \rightarrow P(F)$ in (\mathcal{B}, P) such that $D_1 P_1(g) = y$, the following square

$$\begin{array}{ccc}
 x & \xrightarrow{f} & y \\
 g \circ f \downarrow & \Downarrow \lambda_{g \circ f}^{-1} & \downarrow g \\
 P(F) & \xrightarrow{P(i_F)} & P(F)
 \end{array} \tag{4.19}$$

represents a cartesian 1-morphism $(f, \lambda_{g \circ f}^{-1}, i_F): (x, g \circ f, F) \rightarrow (y, g, F)$ in (\mathcal{B}, P) , since for any 1-morphism $(h, \psi, l): (z, v, F) \rightarrow (y, g, G)$ in (\mathcal{B}, P) , and a 1-morphism $u: z \rightarrow x$ in \mathcal{B}

$$\begin{array}{ccccc}
 z & & & & \\
 \downarrow v & \searrow u & & \searrow h & \\
 & x & \xrightarrow{f} & y & \\
 & \Downarrow \gamma & & & \\
 P(G) & \downarrow & & \downarrow g & \\
 & \Downarrow \xi & & \Downarrow \psi & \\
 & P(F) & \xrightarrow{P(i_F)} & P(F) & \\
 & \downarrow P(l) & \dashrightarrow & \downarrow P(l) & \\
 & \Downarrow P(\lambda_i^{-1}) & & \Downarrow \lambda_{g \circ f}^{-1} &
 \end{array}$$

with a 2-isomorphism $\gamma: h \Rightarrow f \circ u$, there exists a 2-morphism $\xi: (g \circ f) \circ u \Rightarrow P(l) \circ v$ such that the following identity

$$(P(i_F) \circ \xi) \alpha_{P(i_F), g \circ f, u} (\lambda_{g \circ f}^{-1} \circ u) \alpha_{g, f, u}^{-1} (g \circ \gamma) = \alpha_{P(i_F), P(l), v} P(\lambda_l^{-1}) \psi$$

holds, which means that a diagram of 2-morphisms in \mathcal{B}

$$\begin{array}{ccccc}
g \circ h & \xrightarrow{g \circ \gamma} & g \circ (f \circ u) & \xrightarrow{\alpha_{g, f, u}^{-1}} & (g \circ f) \circ u \\
\downarrow \psi & & & & \downarrow \lambda_{g \circ f}^{-1} \circ u \\
P(l) \circ v & & & & [i_{P(F)} \circ (g \circ f)] \circ u \\
\downarrow \lambda_{P(l) \circ v}^{-1} & & & & \downarrow \alpha_{i_{P(F)}, (g \circ f), u} \\
(i_{P(F)} \circ P(l)) \circ v & \xrightarrow{\alpha_{i_{P(F)}, P(l), v}} & i_{P(F)} \circ (P(l) \circ v) & \xleftarrow{i_{P(F)} \circ \xi} & i_{P(F)} \circ [(g \circ f) \circ u]
\end{array}$$

commutes. Then if we define a 2-morphism $\xi: (g \circ f) \circ u \Rightarrow P(l) \circ v$ by $\xi = \psi(g \circ \gamma^{-1}) \alpha_{g, f, u}$, it follows from the coherence and naturality of associativity and left identity that a diagram

$$\begin{array}{ccccc}
& & g \circ h & \xrightarrow{g \circ \gamma} & g \circ (f \circ u) & & \\
& & \downarrow \psi & & \downarrow \alpha_{g, f, u}^{-1} & & \\
& & P(l) \circ v & & (g \circ f) \circ u & & \\
& \swarrow \lambda_{P(l) \circ v}^{-1} & & \searrow g \circ \gamma & & \swarrow (\lambda_g^{-1} \circ f) \circ u & \searrow \lambda_{g \circ f}^{-1} \circ u \\
& (i_{P(F)} \circ P(l)) \circ v & & g \circ (f \circ u) & & [(i_{P(F)} \circ g) \circ f] \circ u & \xrightarrow{\alpha_{i_{P(F)}, g, f \circ u}} & [i_{P(F)} \circ (g \circ f)] \circ u \\
& \swarrow \alpha_{i_{P(F)}, P(l), v} & \downarrow \lambda_{P(l) \circ v}^{-1} & \swarrow \lambda_g^{-1} \circ (f \circ u) & \searrow \alpha_{i_{P(F)} \circ g, f, u} & & & \\
& i_{P(F)} \circ (P(l) \circ v) & & (i_{P(F)} \circ g) \circ (f \circ u) & & & & \\
& \swarrow i_{P(F)} \circ \psi & \downarrow \lambda_{g \circ h}^{-1} & \downarrow \lambda_{g \circ (f \circ u)}^{-1} & \swarrow \alpha_{i_{P(F)}, g, f \circ u} & \searrow \alpha_{i_{P(F)}, (g \circ f), u} & & \\
& i_{P(F)} \circ (g \circ h) & \xleftarrow{i_{P(F)} \circ (g \circ \gamma^{-1})} & i_{P(F)} \circ [g \circ (f \circ u)] & \xleftarrow{i_{P(F)} \circ \alpha_{g, f, u}} & i_{P(F)} \circ [(g \circ f) \circ u] & &
\end{array}$$

commutes, which proves that a square (4.19) is a 1-cartesian 1-morphism in (\mathcal{B}, P) . It also follows directly that a square (4.19) is a 2-cartesian 1-morphism in (\mathcal{B}, P) and that the homomorphism (4.18) is locally a fibration of categories. \square

For any bicategory \mathcal{B} with finite bilimits, we construct a bicategory \mathcal{E} in a following way. The objects of the bicategory \mathcal{E} are triples (p, ϵ, j) where $p: X \rightarrow I$ and $j: I \rightarrow X$ are 1-morphisms in \mathcal{B} , together with a 2-isomorphism $\epsilon: p \circ j \Rightarrow i_I$, so that j is a right weak inverse to p . For any such object (p, ϵ, j) and another one (q, θ, k) for which $\theta: q \circ k \Rightarrow i_J$, a general 1-morphism $(h, \delta, \omega, f): (p, \epsilon, j) \rightarrow (q, \theta, k)$ in \mathcal{E} is represented by a diagram

$$\begin{array}{ccc}
 P & \xrightarrow{h} & Q \\
 \downarrow p & & \downarrow q \\
 I & \xrightarrow{f} & J \\
 \uparrow j & & \uparrow k
 \end{array}
 \quad \delta$$

where $\delta: q \circ h \Rightarrow f \circ p$ and $\omega: h \circ j \Rightarrow k \circ f$ are 2-isomorphisms in \mathcal{B} such that a diagram

$$\begin{array}{ccc}
 (q \circ h) \circ j \xrightarrow{\delta \circ j} (f \circ p) \circ j \xrightarrow{\alpha_{f,p,j}} f \circ (p \circ j) & & \\
 \alpha_{q,h,k} \downarrow & & \downarrow f \circ \epsilon \\
 q \circ (h \circ j) & & f \circ i_I \\
 q \circ \omega \downarrow & & \downarrow \rho_f \\
 q \circ (k \circ f) & & \\
 \alpha_{q,k,f}^{-1} \downarrow & & \\
 (q \circ k) \circ f \xrightarrow{\theta \circ f} i_J \circ f \xrightarrow{\lambda_f} f & &
 \end{array}
 \tag{4.20}$$

commutes. A general 2-morphism $(\gamma, \beta): (h, \delta, \omega, f) \Rightarrow (l, \xi, \zeta, g)$ in \mathcal{E} is given by a diagram

$$\begin{array}{ccc}
 P & \xrightarrow{h} & Q \\
 \downarrow p & & \downarrow q \\
 I & \xrightarrow{f} & J \\
 \uparrow j & & \uparrow k
 \end{array}
 \quad \begin{array}{c}
 \Downarrow \gamma \\
 l \\
 \Downarrow \xi \\
 \overline{f} \\
 \Downarrow \beta \\
 g
 \end{array}$$

where $\gamma: h \Rightarrow l$ and $\beta: f \Rightarrow g$ are 2-morphisms in \mathcal{B} such that the diagrams of 2-morphisms

$$\begin{array}{ccc}
 q \circ h & \xrightarrow{q \circ \gamma} & q \circ l \\
 \Downarrow \delta & & \Downarrow \xi \\
 f \circ p & \xrightarrow{\beta \circ p} & g \circ p
 \end{array}
 \qquad
 \begin{array}{ccc}
 h \circ j & \xrightarrow{\gamma \circ j} & l \circ j \\
 \Downarrow \omega & & \Downarrow \zeta \\
 k \circ f & \xrightarrow{k \circ \beta} & k \circ g
 \end{array}$$

commute. The horizontal and the vertical composition of 1-morphisms and 2-morphisms in \mathcal{E} are defined in an obvious way and there exists a canonical strict homomorphism of bicategories

$$P: \mathcal{E} \rightarrow \mathcal{B} \tag{4.21}$$

defined by $P(p, \epsilon, j) = I$ for any object (p, ϵ, j) in \mathcal{E} , and $P(h, \delta, \omega, f) = f$ and $P(\gamma, \beta) = \beta$, on 1-morphisms and 2-morphisms in \mathcal{E} , respectively.

Theorem 4.25. *For any bicategory \mathcal{B} with finite bilimits, the homomorphism (4.21) is a fibration of bicategories, called a fibration of points of \mathcal{B} .*

Proof. For any 1-morphism $f: x \rightarrow y$ in \mathcal{B} and object (q, θ, k) in \mathcal{E} such that $P(q, \theta, k) = J$, cartesian lift $(h, \delta, f): (p, \epsilon, j) \rightarrow (q, \theta, k)$ of the 1-morphism f is represented by a bipullback

$$\begin{array}{ccc}
 P & \xrightarrow{h} & Q \\
 p \downarrow & \Downarrow \delta & \downarrow q \\
 I & \xrightarrow{f} & J
 \end{array}
 \tag{4.22}$$

and by using its universal property

$$\begin{array}{ccccc}
 & & f & \rightarrow & J \\
 & & \searrow & & \searrow k \\
 I & \xrightarrow{j} & P & \xrightarrow{h} & Q \\
 & \searrow \epsilon & \Downarrow \omega & & \downarrow q \\
 & & I & \xrightarrow{f} & J \\
 & \swarrow i_I & & & \\
 & & & &
 \end{array}
 \tag{4.23}$$

applied to a 2-morphism $\vartheta: q \circ (k \circ f) \Rightarrow f \circ i_I$ defined by a composition of 2-morphisms

$$q \circ (k \circ f) \xrightarrow{\alpha_{q,k,f}^{-1}} (q \circ k) \circ f \xrightarrow{\theta \circ f} i_J \circ f \xrightarrow{\lambda_f} f \xrightarrow{\rho_f^{-1}} f \circ i_I$$

we obtain a 1-morphism $j: I \rightarrow P$ and 2-isomorphisms $\epsilon: p \circ j \Rightarrow i_I$ and $\omega: h \circ j \Rightarrow k \circ f$, such that a pasting composition of 2-morphisms in the diagram (4.23) is equal to the 2-morphism $\vartheta: q \circ (k \circ f) \Rightarrow f \circ i_I$, by commutativity of the diagram (4.20). Taking another object (r, ε, m) in \mathcal{E} with $\varepsilon: r \circ m \Rightarrow i_K$, and a 1-morphism $(l, \xi, \zeta, g): (r, \varepsilon, m) \rightarrow (q, \theta, k)$

$$\begin{array}{ccc} R & \xrightarrow{l} & Q \\ \downarrow r & & \downarrow q \\ K & \xrightarrow{g} & J \end{array} \quad \begin{array}{c} \uparrow m \\ \swarrow \xi \\ \uparrow k \end{array}$$

where $\xi: q \circ l \Rightarrow g \circ r$ and $\zeta: l \circ m \Rightarrow k \circ g$, together with a 2-isomorphism $\beta: g \Rightarrow f \circ u$

$$\begin{array}{ccccc} R & & & & Q \\ \downarrow r & \searrow v & & \searrow l & \\ & P & \xrightarrow{h} & & Q \\ \downarrow \nu & \swarrow \nu & & \swarrow \xi & \downarrow q \\ K & & & & J \\ \downarrow u & \swarrow p & \searrow g & \swarrow \delta & \\ & I & \xrightarrow{f} & & J \end{array} \quad \begin{array}{c} \swarrow \gamma \\ \swarrow \beta \end{array} \quad (4.24)$$

by using again the universal property of the pullback (4.22) applied to the composition

$$q \circ l \xrightarrow{\xi} g \circ r \xrightarrow{\beta \circ r} (f \circ u) \circ r \xrightarrow{\alpha_{f,u,r}} f \circ (u \circ r)$$

we obtain a 1-morphism $v: R \rightarrow P$ and 2-isomorphisms $\gamma: l \Rightarrow h \circ v$ and $\nu: p \circ v \Rightarrow u \circ r$, and the tedious calculation shows that the diagram (4.24) represents a 2-morphism in \mathcal{E}

$$(\gamma, \beta): (l, \xi, \zeta, g) \Rightarrow (h, \delta, \omega, f) \circ (v, \nu, \mu, u).$$

It is straightforward to show that the 1-morphism represented by the bipullback (4.22) has a universal property with respect to 2-morphisms, which makes it a 2-cartesian 1-morphism in \mathcal{E} , and one can show the strict homomorphism (4.21) is locally a fibration of categories. \square

Definition 4.26. Let $P: \mathcal{E} \rightarrow \mathcal{B}$ be a strict homomorphism between bicategories, and let $\mathcal{J}_x: \mathcal{E}_x \rightarrow \mathcal{E}$ be an inclusion of a fiber bicategory \mathcal{E}_x in \mathcal{E} . A 2-cleavage consists of the following data:

- for any 1-morphism $f: y \rightarrow x$ in \mathcal{B} , a homomorphism

$$f^*: \mathcal{E}_x \rightarrow \mathcal{E}_y \quad (4.25)$$

between the fibers, and a pseudonatural transformation

$$\vartheta^f: \mathcal{J}_y f^* \Rightarrow \mathcal{J}_x \quad (4.26)$$

- for any 2-morphism $\beta: f \Rightarrow g$ in \mathcal{B} , a pseudonatural transformation

$$\beta^*: g^* \Rightarrow f^* \quad (4.27)$$

and a modification

$$\Omega^\beta: \vartheta^f \circ \beta^* \Longrightarrow \vartheta^g \quad (4.28)$$

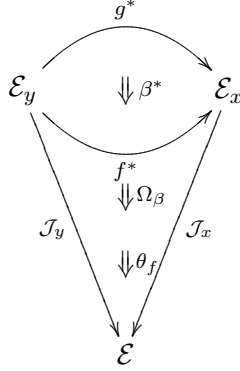
such that the following axioms hold:

- each component $\vartheta_E^f: f^* E \rightarrow E$ of the pseudonatural transformation (4.26) is cartesian lift of a 1-morphism $f: y \rightarrow x$, i.e. $P(\vartheta_E^f) = f$
- each component $\Omega_E^\beta: \vartheta_E^f \circ \beta_E^* \Rightarrow \vartheta_E^g$ of the modification (4.28) is cartesian 2-morphism

$$\begin{array}{ccc}
 \begin{array}{ccc}
 g^* E & & \\
 \downarrow \beta_E^* & \searrow \vartheta_E^g & \\
 f^* E & \xrightarrow{\vartheta_E^f} & E
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 y & & \\
 \downarrow i_y & \searrow \beta & \\
 y & \xrightarrow{f} & x
 \end{array}
 \end{array}$$

such that $P(\Omega_E^\beta) = \beta \rho_f$.

Remark 4.27. *The above data for the 2-cleavage correspond to a universal weak 3-cocone*



which represents \mathcal{E} as a tricolimit of an indexed bicategory corresponding to the 2-cleavage.

The following definition of a biadjunction between homomorphisms of bicategories is from [37].

Definition 4.28. *A homomorphism $F: \mathcal{B} \rightarrow \mathcal{C}$ of bicategories is a left biadjoint to a homomorphism $G: \mathcal{C} \rightarrow \mathcal{B}$ if there is an equivalence*

$$\mathcal{C}(F(?), -) \simeq \mathcal{B}(?, G(-))$$

in a 2-category $\text{Hom}[\mathcal{B}^{op}, \text{Hom}[\mathcal{C}, \text{Cat}]]$.

In the above definition, $\mathcal{C}(F(?), -): \mathcal{B}^{op} \rightarrow \text{Hom}(\mathcal{C}, \text{Cat})$ is a homomorphism sending each object x in \mathcal{B} to the representable 2-functor $\mathcal{C}(F(x), -): \mathcal{C} \rightarrow \text{Cat}$, and similarly $\mathcal{B}(?, G(-)): \mathcal{B}^{op} \rightarrow \text{Hom}(\mathcal{C}, \text{Cat})$ is a homomorphism sending each object x in \mathcal{B} to the homomorphism $\mathcal{B}(x, G(-)): \mathcal{C} \rightarrow \text{Cat}$.

Theorem 4.29. *A strict homomorphism $P: \mathcal{E} \rightarrow \mathcal{B}$ of bicategories is a fibration if there exist a homomorphism $L: (\mathcal{B}, P) \rightarrow \mathcal{E}^{\mathcal{I}}$ which is right biadjoint right inverse to the homomorphism $S := (F^{[1]}, \mathcal{E}^{\delta_1}): \mathcal{E}^{\mathcal{I}} \rightarrow (\mathcal{B}, P)$.*

The homomorphism $S: \mathcal{E}^{\mathcal{I}} \rightarrow (\mathcal{B}, P)$ is then defined on any object $k: G \rightarrow F$ of $\mathcal{E}^{\mathcal{I}}$, by $S(k) = (P(G), P(j), F)$. The existence of the right biadjoint right inverse $L: (\mathcal{B}, P) \rightarrow \mathcal{E}^{\mathcal{I}}$ means that for any objects (y, f, E) in (\mathcal{B}, P) and $k: G \rightarrow F$ in $\mathcal{E}^{\mathcal{I}}$, there is an equivalence

$$\text{Hom}_{(\mathcal{B}, P)}(S(k), (y, f, E)) \simeq \text{Hom}_{\mathcal{E}^{\mathcal{I}}}(k, L(y, f, E))$$

of categories, which is pseudonatural in both variables.

$$\begin{array}{ccc}
P(G) & \xrightarrow{P(k)} & P(F) \\
\downarrow u & \searrow \tilde{\beta} & \downarrow P(j) \\
y & \xrightarrow{f} & P(E)
\end{array}
\qquad
\begin{array}{ccc}
G & \xrightarrow{k} & F \\
\downarrow \tilde{u} & \searrow \tilde{\beta} & \downarrow j \\
f^*(E) & \xrightarrow{\tilde{f}_E} & E
\end{array}$$

Thus, for any object (y, f, E) in (\mathcal{B}, P) where $f: y \rightarrow P(E)$ is a 1-morphism in \mathcal{B} , we have $L(y, f, E) = \tilde{f}_E$, where $\tilde{f}_E: f^*(E) \rightarrow E$ is a biuniversal 1-morphism in \mathcal{E} .

5 The construction of indexed bicategories by fibers

Theorem 5.1. *Let $P: \mathcal{E} \rightarrow \mathcal{B}$ be strict homomorphism of bicategories and let $f: y \rightarrow x$ be a 1-morphism in \mathcal{B} . A choice of a cartesian 1-morphism $\tilde{f}_E: f^*E \rightarrow E$ for any object E in \mathcal{E}_x , such that $P(\tilde{f}_E) = f$, defines a homomorphism $f^*: \mathcal{E}_x \rightarrow \mathcal{E}_y$ of fiber bicategories.*

Proof. First, for any object E in the fiber \mathcal{E}_x , the value of a homomorphism $f^*: \mathcal{E}_x \rightarrow \mathcal{E}_y$ on E is an object f^*E in \mathcal{E}_y which is a domain of chosen cartesian 1-morphism $\tilde{f}_E: f^*E \rightarrow E$. Then we define the value of a homomorphism $f^*: \mathcal{E}_x \rightarrow \mathcal{E}_y$ on any 1-morphism $j: F \rightarrow E$ in the fiber \mathcal{E}_x such that $P(j) = i_x$. For such 1-morphism $j: F \rightarrow E$ we choose by (4.4)

$$\begin{array}{ccc}
 \begin{array}{ccc}
 f^*(F) & \xrightarrow{\tilde{f}_F} & F \\
 \downarrow f^*(j) & \swarrow \tilde{f}_j & \downarrow j \\
 f^*(E) & \xrightarrow{\tilde{f}_E} & E
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 y & \xrightarrow{f} & x \\
 \downarrow i_y & \searrow f & \downarrow i_x \\
 y & \xrightarrow{f} & x
 \end{array}
 \end{array}$$

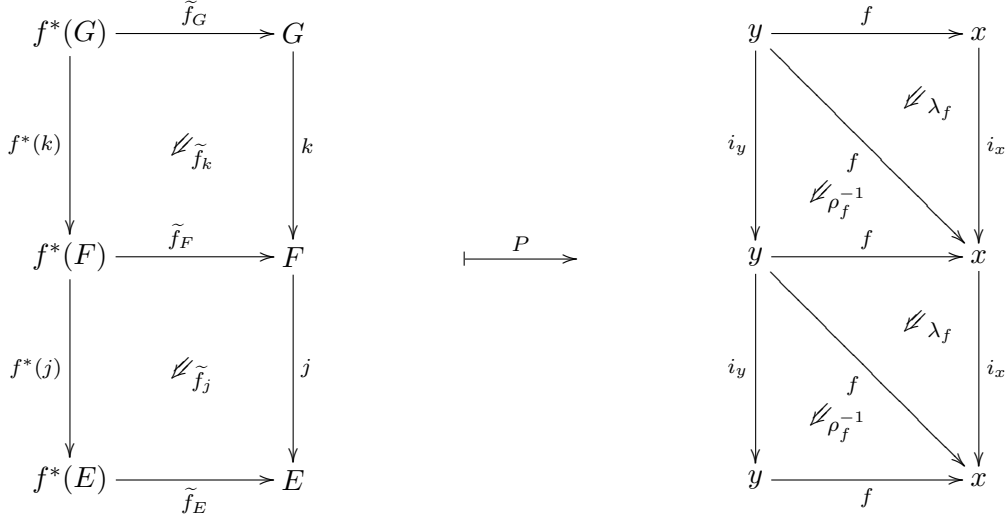
a 1-morphism $f^*(j): f^*(F) \rightarrow f^*(E)$ in \mathcal{E}_y and a 2-isomorphism $\tilde{f}_j: j \circ \tilde{f}_F \Rightarrow \tilde{f}_E \circ f^*(j)$ such that $P(\tilde{f}_j) = \rho_f^{-1} \lambda_f$. Then, for any 2-morphism $\phi: j \Rightarrow j'$ in \mathcal{E}_x such that $P(\phi) = \iota_x$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 f^*(F) & \xrightarrow{\tilde{f}_F} & F \\
 \downarrow f^*(j) & \swarrow \tilde{f}_j & \downarrow j \\
 f^*(E) & \xrightarrow{\tilde{f}_E} & E
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 y & \xrightarrow{f} & x \\
 \downarrow i_y & \searrow f & \downarrow i_x \\
 y & \xrightarrow{f} & x
 \end{array}
 \end{array}$$

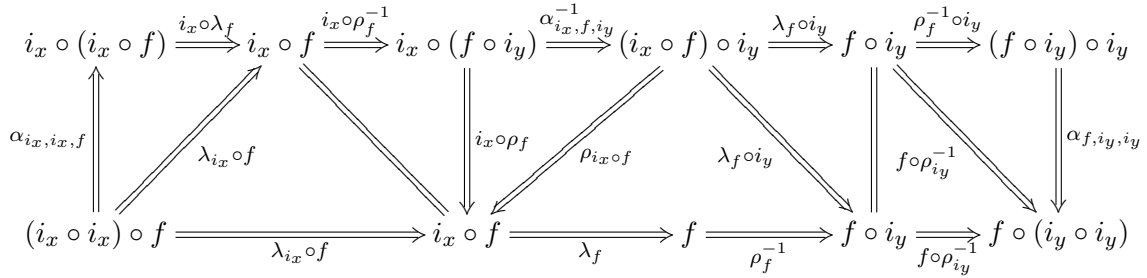
there exists a unique 2-morphism $f^*(\phi): f^*(j) \Rightarrow f^*(j')$ such that $P(f^*(\phi)) = \iota_y$ by (4.5). For a general 1-morphism $l: L \rightarrow E$ in the fiber \mathcal{E}_x such that $P(l) = i_x^{\theta(l)}$, we choose

$$\begin{array}{ccc}
 \begin{array}{ccc}
 f^*(L) & \xrightarrow{\tilde{f}_L} & L \\
 \downarrow f^*(l) & \swarrow \tilde{f}_l & \downarrow l \\
 f^*(E) & \xrightarrow{\tilde{f}_E} & E
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 y & \xrightarrow{f} & x \\
 \downarrow i_y^{\theta(l)} & \searrow f & \downarrow i_x^{\theta(l)} \\
 y & \xrightarrow{f} & x
 \end{array}
 \end{array}$$

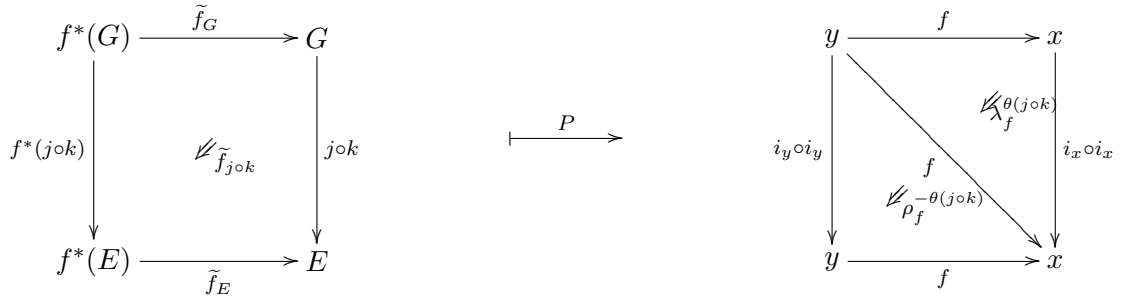
by (4.4) a 2-isomorphism $\tilde{f}_l: l \circ \tilde{f}_F \Rightarrow \tilde{f}_E \circ f^*(l)$ such that its image is $P(\tilde{f}_l) = (\rho_f^{\theta(l)})^{-1} \lambda_f^{\theta(l)}$ where $\lambda_f^{\theta(l)}: i_x^{\theta(l)} \circ f \Rightarrow f$ is the unique 2-isomorphism given by the consecutive compositions of left identity coherence $\lambda_f: i_x \circ f \Rightarrow f$, and similarly for 2-isomorphisms $\rho_f^{\theta(l)}: f \circ i_x^{\theta(l)} \Rightarrow f$. To see how this works, we take 1-morphisms $k: G \rightarrow F$ and $j: F \rightarrow E$ in a bicategory \mathcal{E}_x



and a pasting composite on the right side is the composition of vertical and top edges



of the above commutative diagram. The composition of bottom edges correspond to



and from the left vertical 2-morphism $\tilde{f}_j \boxminus \tilde{f}_k: (j \circ k) \circ \tilde{f}_G \Rightarrow \tilde{f}_E \circ (f^*(j) \circ f^*(k))$ defined by

$$\begin{array}{ccccc}
 (j \circ k) \circ \tilde{f}_G & \xrightarrow{\alpha_{j,k,\tilde{f}_G}} & j \circ (k \circ \tilde{f}_G) & \xrightarrow{j \circ \tilde{f}_k} & j \circ (\tilde{f}_E \circ f^*(k)) \\
 \Downarrow \tilde{f}_j \boxminus \tilde{f}_k & & & & \Downarrow \alpha_{j,\tilde{f}_E,f^*(k)}^{-1} \\
 \tilde{f}_E \circ (f^*(j) \circ f^*(k)) & \xleftarrow{\alpha_{\tilde{f}_E,f^*(j),f^*(k)}} & (\tilde{f}_E \circ f^*(j)) \circ f^*(k) & \xleftarrow{\tilde{f}_j \circ f^*(k)} & (j \circ \tilde{f}_E) \circ f^*(k)
 \end{array}$$

the requirement that above diagram is commutative, we see that its image by P

$$\begin{array}{ccc}
 \begin{array}{ccc}
 f^*(G) & \xrightarrow{\tilde{f}_G} & G \\
 \downarrow f^*(j) \circ f^*(k) & \swarrow \tilde{f}_j \boxminus \tilde{f}_k & \downarrow j \circ k \\
 f^*(E) & \xrightarrow{\tilde{f}_E} & E
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 y & \xrightarrow{f} & x \\
 \downarrow i_y \circ i_y & \searrow f & \downarrow i_x \circ i_x \\
 y & \xrightarrow{f} & x
 \end{array}
 \end{array}$$

is the same as for the 2-morphism $f^*(j \circ k)$ by the equality of compositions of vertical and top edges on one side and the composition of bottom edges on other side in the above commutative diagram. Therefore, following the Remark 3.2 we conclude that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 f^*(G) & \xrightarrow{\tilde{f}_G} & G \\
 \downarrow f^*(j) \circ f^*(k) & \swarrow \mu_{j,k} & \downarrow j \circ k \\
 f^*(E) & \xrightarrow{\tilde{f}_E} & E
 \end{array} & \xrightarrow{P} & \begin{array}{ccc}
 y & \xrightarrow{f} & x \\
 \downarrow i_y \circ i_y & \swarrow f & \downarrow i_x \circ i_x \\
 x & \xrightarrow{f} & x
 \end{array}
 \end{array}$$

there exists a unique 2-isomorphism $\mu_{j,k}: f^*(j) \circ f^*(k) \Rightarrow f^*(j \circ k)$ such that $P(\mu_{j,k}) = \iota_y \circ \iota_y$ by (4.5). It is straightforward to check that these 2-isomorphisms are components of the coherence for a homomorphism $f^*: \mathcal{E}_x \rightarrow \mathcal{E}_y$. \square

6 The Grothendieck construction for bicategories

The famous Grothendieck construction is a 2-functor

$$\int: \text{Cat}^{\mathcal{C}^{op}} \rightarrow \text{Fib}(\mathcal{C}) \quad (6.1)$$

from the 2-category $\text{Cat}^{\mathcal{C}^{op}}$ of pseudofunctors on \mathcal{C} , pseudonatural transformations and modifications, to the 2-category $\text{Fib}(\mathcal{C})$ of Grothendieck fibrations over \mathcal{C} . The Grothendieck construction takes any pseudofunctor $P: \mathcal{C}^{op} \rightarrow \text{Cat}$ to a canonical Grothendieck fibration

$$F: \int_{\mathcal{C}} P \rightarrow \mathcal{C}$$

which is constructed as a lax colimit, which is appropriate kind of a bicategorical limit.

In this section, we introduce a bicategorical analog of the Grothendieck construction (6.1), which for any bicategory \mathcal{B} , provides a triequivalence

$$\int: \text{Bicat}^{\mathcal{B}^{coop}} \rightarrow 2\text{Fib}(\mathcal{B}) \quad (6.2)$$

between a tricategory $\text{Bicat}^{\mathcal{B}^{coop}}$ whose objects are indexed bicategories $P: \mathcal{B}^{coop} \rightarrow \text{Bicat}$, and the 3-category $2\text{Fib}(\mathcal{B})$ of 2-fibrations of bicategories (or fibered bicategories) over \mathcal{B} .

Although we will not give a full description of the triequivalence (6.2), we do explicitly describe a construction which to any small bicategory \mathcal{B} , and any \mathcal{B} -indexed bicategory $\mathcal{F}: \mathcal{B}^{coop} \rightarrow \text{Bicat}$, associates a canonical 2-fibration of bicategories

$$F_{\mathcal{F}}: \int_{\mathcal{B}} \mathcal{F} \rightarrow \mathcal{B}$$

over \mathcal{B} , from a bicategory $\int_{\mathcal{B}} \mathcal{F}$ whose underlying Cat-graph we shall now begin to describe.

The objects of $\int_{\mathcal{B}} \mathcal{F}$ are pairs (x, E) where x is an object of \mathcal{B} , and E is an object of the bicategory $\mathcal{F}(x)$. A 1-morphism $(f, a): (x, E) \rightarrow (y, F)$ in $\int_{\mathcal{B}} \mathcal{F}$ consists of a 1-morphism $f: x \rightarrow y$ in \mathcal{B} , and a 1-morphism $a: E \rightarrow f^*(F)$ in the bicategory $\mathcal{F}(x)$. A 2-morphism $(\beta, \phi): (f, a) \Rightarrow (g, b)$ in $\int_{\mathcal{B}} \mathcal{F}$ consists of a 2-morphism $\beta: f \Rightarrow g$ in \mathcal{B} , and a 2-morphism $\phi: \beta_F^* \circ b \Rightarrow a$ in $\mathcal{F}(x)$

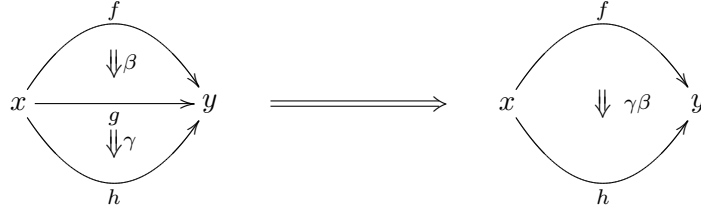
$$\begin{array}{ccc} & & g^*(F) \\ & \nearrow b & \downarrow \beta_F^* \\ E & & f^*(F) \\ & \xrightarrow{a} & \end{array}$$

$\Downarrow \phi$

where a 1-morphism $\beta_F^*: g^*(F) \Rightarrow f^*(F)$ is a component of a pseudonatural transformation $\mathcal{F}(\beta): \mathcal{F}(g) \Rightarrow \mathcal{F}(f)$ indexed by the object F in the bicategory $\mathcal{F}(y)$.

Theorem 6.1. For any object x in \mathcal{B} and E in $\mathcal{F}(x)$, the collection of all pairs (x, E) are objects $\int_{\mathcal{B}} \mathcal{F}_0$ of a Cat-graph which we denote by $\int_{\mathcal{B}} \mathcal{F}$. For any other object (y, F) in $\int_{\mathcal{B}} \mathcal{F}_0$, the category $\int_{\mathcal{B}} \mathcal{F}((x, E), (y, F))$ has for objects all 1-morphisms $(f, a): (x, E) \rightarrow (y, F)$ and for morphisms all 2-morphisms $(\beta, \psi): (f, a) \Rightarrow (g, b)$.

Proof. The vertical composition of any two 2-morphisms $f \xrightarrow{\beta} g \xrightarrow{\gamma} h$ in \mathcal{B}



induces a composition of morphisms $(f, a) \xrightarrow{(\beta, \phi)} (g, b) \xrightarrow{(\gamma, \psi)} (h, c)$ in $\int_{\mathcal{B}} \mathcal{F}$ as in a diagram

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & h^*(F) & \\
 c \nearrow & & \downarrow \gamma_F^* \\
 E & \xrightarrow{b} & g^*(F) \\
 \downarrow \psi & & \downarrow \beta_F^* \\
 & & f^*(F) \\
 a \searrow & & \\
 & &
 \end{array}
 & \xrightarrow{\quad} &
 \begin{array}{ccc}
 & h^*(F) & \\
 c \nearrow & & \downarrow (\gamma\beta)_F^* \\
 E & & \\
 \downarrow \psi\phi & & \\
 & & f^*(F) \\
 a \searrow & & \\
 & &
 \end{array}
 \end{array} \tag{6.3}$$

and the pasting composition of the left diagram defines a composition in $\int_{\mathcal{B}} \mathcal{F}((x, E), (y, F))$

$$(\gamma, \psi)(\beta, \phi) = (\gamma\beta, \psi\phi) \tag{6.4}$$

where the second component is a 2-morphism $\psi\phi: (\gamma\beta)_F^* \circ c \Rightarrow a$, filling a triangle on the right side of (6.3), which is defined by the vertical composition of 2-morphisms in $\mathcal{F}(x)$

$$(\gamma\beta)_F^* \circ c \xrightarrow{(\mu_{\gamma, \beta}^{-1})_F \circ c} (\beta_F^* \circ \gamma_F^*) \circ c \xrightarrow{\alpha_{\beta_F^*, \gamma_F^*, c}} \beta_F^* \circ (\gamma_F^* \circ c) \xrightarrow{\beta_F^* \circ \psi} \beta_F^* \circ b \xrightarrow{\phi} a \tag{6.5}$$

where $(\mu_{\gamma,\beta}^{-1})_F: (\gamma\beta)_F^* \Rightarrow \beta_F^* \circ \gamma_F^*$ is a component of the inverse of an isomodification (3.19) indexed by an object F in the bicategory \mathcal{F}_y . An identity 2-morphism $(g, \lambda_b): (g, b) \Rightarrow (g, b)$

$$\begin{array}{ccc}
\begin{array}{ccc}
& g^*(F) & \\
b \nearrow & & \downarrow i_{g^*(F)} \\
E & \xrightarrow{\quad} & g^*(F) \\
& \downarrow \lambda_b & \\
& b \downarrow \phi & \\
& a \searrow & \downarrow \beta_F^* \\
& & f^*(F)
\end{array}
& \Longrightarrow &
\begin{array}{ccc}
& g^*(F) & \\
b \nearrow & & \downarrow (\beta_F^*) \\
E & \xrightarrow{\quad} & g^*(F) \\
& \downarrow \lambda_b \phi = \phi & \\
& a \searrow & \downarrow \beta_F^* \\
& & f^*(F)
\end{array}
\end{array} \quad (6.6)$$

for the composition (6.4) is represented by upper triangle on the left side of a diagram (6.6) and it follows from (6.5) that a composite 2-morphism $\lambda_b \phi: \beta_F^* \circ b \Rightarrow a$ in \mathcal{F}_x is defined by

$$\beta_F^* \circ b \xrightarrow{(\mu_{\beta,\iota_g}^{-1})_F \circ b} (\beta_F^* \circ \iota_g)_F \circ b = (\beta_F^* \circ i_{g^*(F)}) \circ b \xrightarrow{\alpha_{\beta_F^*, i_{g^*(F)}, b}} \beta_F^* \circ (i_{g^*(F)} \circ b) \xrightarrow{\beta_F^* \circ \lambda_b} \beta_F^* \circ b \xrightarrow{\phi} a.$$

Then it follows from (3.22) and (2.50) that $(\mu_{\beta,\iota_g}^{-1})_F \stackrel{(3.22)}{=} (\widetilde{\rho}^{-1}_{\beta^*})_F \stackrel{(2.50)}{=} \rho_{\beta_F^*}^{-1}$ and a diagram

$$\begin{array}{ccc}
\beta_F^* \circ b & \xrightarrow{(\mu_{\beta,\iota_g}^{-1})_F \circ b} & (\beta_F^* \circ i_{g^*(F)}) \circ b \\
\parallel & \swarrow \rho_{\beta_F^* \circ b} & \downarrow \alpha_{\beta_F^*, i_{g^*(F)}, b} \\
\beta_F^* \circ b & \xleftarrow{\beta_F^* \circ \lambda_b} & \beta_F^* \circ (i_{g^*(F)} \circ b)
\end{array}$$

commutes, in which the upper right triangle commutes trivially and the lower right triangle commutes due to an identity coherence (2.14) in the bicategory \mathcal{F}_x . Therefore we conclude that a 2-morphism $(g, \lambda_b): (g, b) \Rightarrow (g, b)$ is a strict left identity for the composition (6.4), and similarly it follows from (3.21) and (2.49) that it is also a strict right identity for (6.4). In order to prove that the composition (6.4) is strictly associative, for any three morphisms

$$(f, a) \xrightarrow{(\beta,\phi)} (g, b) \xrightarrow{(\gamma,\psi)} (h, c) \xrightarrow{(\delta,\theta)} (k, d)$$

in $\int_{\mathcal{B}} \mathcal{F}((x, E), (y, F))$, we compare two composite 2-morphisms $(\theta\psi)\phi$ and $\theta(\psi\phi)$.

$$\begin{array}{ccccccc}
[\delta(\gamma\beta)]_F^* \circ d & \xrightarrow{\mu_{\delta, \gamma\beta}^F \circ d} & [(\gamma\beta)_F^* \circ \delta_F^*] \circ d & \xrightarrow{\alpha_{(\gamma\beta)_F^*, \delta_F^*, d}} & (\gamma\beta)_F^* \circ (\delta_F^* \circ d) & \xrightarrow{(\gamma\beta)_F^* \circ \theta} & (\gamma\beta)_F^* \circ c \\
\parallel & & \parallel & & \parallel & & \parallel \\
[(\delta\gamma)\beta]_F^* \circ d & & [(\beta_F^* \circ \gamma_F^*) \circ \delta_F^*] \circ d & \xrightarrow{\alpha_{\beta_F^* \circ \gamma_F^*, \delta_F^*, d}} & (\beta_F^* \circ \gamma_F^*) \circ (\delta_F^* \circ d) & \xrightarrow{(\beta_F^* \circ \gamma_F^*) \circ \theta} & (\beta_F^* \circ \gamma_F^*) \circ c \\
\parallel & & \parallel & & \parallel & & \parallel \\
\mu_{\delta, \gamma, \beta}^F \circ d & & \mu_{\gamma, \beta}^F \circ (\delta_F^* \circ d) & & \mu_{\gamma, \beta}^F \circ (\delta_F^* \circ d) & & \mu_{\gamma, \beta}^F \circ c \\
\parallel & & \parallel & & \parallel & & \parallel \\
[\beta_F^* \circ (\delta\gamma)_F^*] \circ d & \xrightarrow{(\beta_F^* \circ \mu_{\delta, \gamma}^F) \circ d} & [\beta_F^* \circ (\gamma_F^* \circ \delta_F^*)] \circ d & & & & \\
\parallel & & \parallel & & \parallel & & \parallel \\
\alpha_{\beta_F^*, (\delta\gamma)_F^*, d} & & \alpha_{\beta_F^*, \gamma_F^*, \delta_F^*, d} & & \alpha_{\beta_F^*, \gamma_F^*, \delta_F^*, d} & & \alpha_{\beta_F^*, \gamma_F^*, c} \\
\parallel & & \parallel & & \parallel & & \parallel \\
[\beta_F^* \circ (\delta\gamma)_F^*] \circ d & \xrightarrow{\beta_F^* \circ (\mu_{\delta, \gamma}^F \circ d)} & \beta_F^* \circ [(\gamma_F^* \circ \delta_F^*) \circ d] & \xrightarrow{\beta_F^* \circ \alpha_{\gamma_F^*, \delta_F^*, d}} & \beta_F^* \circ [\gamma_F^* \circ (\delta_F^* \circ d)] & \xrightarrow{\beta_F^* \circ (\gamma_F^* \circ \theta)} & \beta_F^* \circ (\gamma_F^* \circ c)
\end{array}$$

The first morphism $(\theta\psi)\phi$ is the composition of left vertical and bottom edges with

$$\beta_F^* \circ (\gamma_F^* \circ c) \xrightarrow{\beta_F^* \circ \psi} \beta_F^* \circ b \xrightarrow{\phi} a$$

and similarly $\theta(\psi\phi)$ is the composition of top and right vertical edges with that morphism. Since the above diagram commutes by the associativity coherence (2.13), naturality of the associativity, the composition coherence (3.20) of an isomodification (3.19) and its naturality, we conclude that

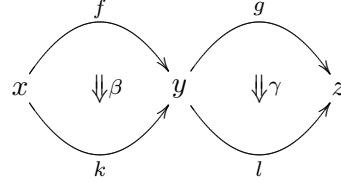
$$[(\delta, \theta)(\gamma, \psi)](\beta, \phi) = (\delta, \theta)[(\gamma\beta, \psi\phi)]$$

which proves that the composition (6.4) is strictly associative. \square

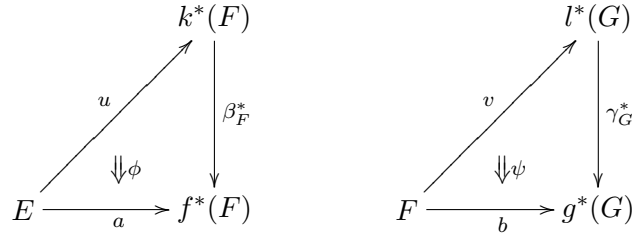
Now we define the horizontal composition on a Cat-graph $\int_{\mathcal{B}} \mathcal{F}$ by a functor

$$H: \int_{\mathcal{B}} \mathcal{F}((y, F), (z, G)) \times \int_{\mathcal{B}} \mathcal{F}((x, E), (y, F)) \rightarrow \int_{\mathcal{B}} \mathcal{F}((x, E), (z, G)) \quad (6.7)$$

which associates to any horizontally composable pair of 2-morphisms in the bicategory \mathcal{B}



and to any pair of 2-morphisms $(\beta, \phi): (x, E) \Rightarrow (y, F)$ and $(\gamma, \psi): (y, F) \Rightarrow (z, G)$ in $\int_{\mathcal{B}} \mathcal{F}$,



a 2-morphism $(\gamma \circ \beta, \psi \circ \phi)$ in $\int_{\mathcal{B}} \mathcal{F}$ where $\psi \circ \phi: v \circ u \Rightarrow b \circ a$ is a pasting of a diagram

$$\begin{array}{ccccc}
 & & k^*l^*(G) & \xrightarrow{(\chi_{l,k})_G} & (l \circ k)^*(G) \\
 & & \downarrow (k^* \otimes \gamma^*)_G & & \downarrow (\gamma \circ \beta)^*_G \\
 & k^*(F) & \xrightarrow{k^*(b)} & k^*g^*(G) & \xleftarrow{(\chi_{\gamma,\beta})_G} & k^*l^*(G) \\
 & \downarrow \beta^*_F & \swarrow k^*(v) & \downarrow k^*(\psi) & & \downarrow (k^* \otimes \gamma^*)_G \\
 E & \xrightarrow{a} & f^*(F) & \xrightarrow{f^*(b)} & f^*g^*(G) & \xrightarrow{(\chi_{g,f})_G} & (g \circ f)^*(G) \\
 & \downarrow \phi & & \swarrow (\beta^*)_b & \downarrow (\beta^* \otimes g^*)_G & & \downarrow (\gamma \circ \beta)^*_G
 \end{array} \quad (6.8)$$

The 2-morphism $(\gamma \circ \beta, \psi \circ \phi): (g \circ f, b \circ a) \Rightarrow (l \circ k, v \circ u)$ in $\int_{\mathcal{B}} \mathcal{F}$ is represented by

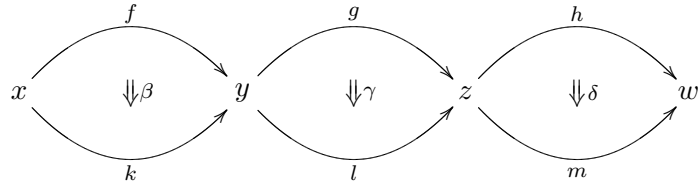
$$\begin{array}{ccc}
 & (l \circ k)^*(G) & \\
 \nearrow^{v \circ u} & \downarrow (\gamma \circ \beta)_G^* & \\
 E & \xrightarrow{b \circ a} (g \circ f)^*(G) & \\
 \downarrow \psi \circ \phi & &
 \end{array}$$

$$\begin{array}{ccccccc}
 & & k^*l^*(G) & \xrightarrow{(\chi_{l,k})_G} & (l \circ k)^*(G) & & \\
 & & \downarrow (k^* \otimes \gamma^*)_G \not\cong_{(\chi_{\gamma,k})_G} & & \downarrow (\gamma \circ k)_G^* & & \\
 & k^*(v) & & & & & \\
 & \downarrow k^*(\psi) & & & & & \\
 & k^*(F) & \xrightarrow{k^*(b)} & k^*g^*(G) & \xrightarrow{(\chi_{g,k})_G} & (g \circ k)^*(G) & \xleftarrow{\mu_{\gamma \circ k, g \circ \beta}} (g \circ \beta)_G^* & (6.9) \\
 & \downarrow \beta_F^* & \not\cong_{(\beta^*)_b} & \downarrow (\beta^* \otimes g^*)_G \not\cong_{(\chi_{g,\beta})_G} & \downarrow (g \circ \beta)_G^* & & \\
 & u & & & & & \\
 & \downarrow \phi & & & & & \\
 E & \xrightarrow{a} & f^*(F) & \xrightarrow{f^*(b)} & f^*g^*(G) & \xrightarrow{(\chi_{g,f})_G} & (g \circ f)^*(G)
 \end{array}$$

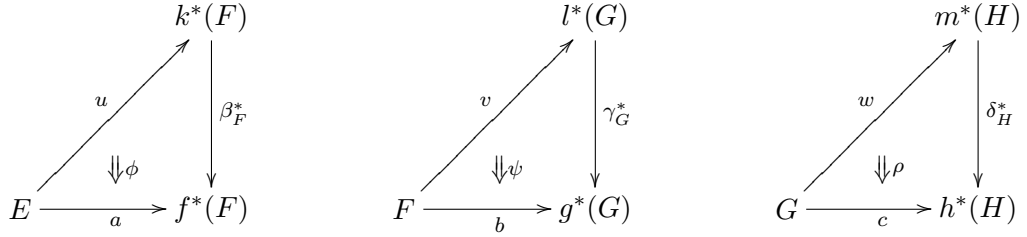
The pasting composite of the diagram (6.8) is equal to the pasting composite of the

coherence, whose square which shares with the cube is an identity 2-morphism of the 1-morphism $(\beta^* \circ \gamma^*)_G: k^*l^*(G) \rightarrow f^*g^*(G)$, which is a component of the pseudo natural transformation $\beta^* \circ \gamma^*: k^*l^* \rightarrow f^*g^*$, indexed by the object G in the bicategory F_z . The cube is just the coherence for the modification χ which is a component indexed by the horizontal composition $\beta \circ \alpha$ of the pseudo natural transformation χ , so the vertical edge of the above horizontal composition is equal to the component $(\gamma \circ \beta)_G^*: (l \circ k)^*(G) \rightarrow (g \circ f)^*(G)$ indexed by the same horizontal composite. In order to prove that \mathcal{E} is really a bicategory, we need to show that the Godement interchange law holds, and that the above horizontal composition is associative up to the coherent associativity isomorphism given by $\alpha_{c,b,a}: ((h \circ g) \circ f, (c \circ b) \circ a) \Rightarrow (h \circ (g \circ f), c \circ (b \circ a))$, for any three composable 1-morphisms.

For any three horizontally composable 2-morphisms in \mathcal{B} ,



and for any three horizontally composable 2-morphisms in $\int_{\mathcal{B}} \mathcal{F}$ which covers them



we need to show that $\beta_{w,v,u} \circ ((\rho \circ \psi) \circ \phi) = (\rho \circ (\psi \circ \phi)) \circ \beta_{c,b,a}$. The 2-morphism $\rho \circ (\psi \circ \phi)$

and the horizontal composition $(\rho \circ \psi) \circ \phi$ on the left hand side of the equation is given by

$$\begin{array}{ccccccc}
& & & & k^*l^*m^*(H) & \xrightarrow{k^*((\chi_{m,l})_H)} & k^*(ml)^*(H) & \xrightarrow{(\chi_{ml,k})_H} & (mlk)^*(H) \\
& & & & \downarrow & & \downarrow & & \downarrow \\
& & & & (k^*l^*\gamma^*)_H & \swarrow k^*((\chi_{\gamma,l})_H) & (k^*(\gamma l)^*)_H & \swarrow (\chi_{\gamma l,k})_H & (\gamma lk)^*_H \\
& & & k^*l^*(G) & \xrightarrow{k^*l^*(c)} & k^*l^*h^*(H) & \xrightarrow{k^*((\chi_{h,l})_H)} & k^*(hl)^*(H) & \xrightarrow{(\chi_{hl,k})_H} & (hlk)^*(H) \\
& & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & k^*(\beta^*_G) & \swarrow k^*((\beta^*)_c) & (k^*\beta^*h^*)_G & \swarrow k^*((\chi_{h,\beta})_H) & (k^*(h\beta)^*)_H & \swarrow (\chi_{h\beta,k})_H & (h\beta k)^*_H \\
& & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & \Downarrow k^*(\psi) & & \Downarrow k^*(\psi) & & \Downarrow k^*(\psi) & & \Downarrow k^*(\psi) \\
& & & k^*(F) & \xrightarrow{k^*(b)} & k^*g^*(G) & \xrightarrow{k^*g^*(c)} & k^*g^*h^*(H) & \xrightarrow{k^*((\chi_{h,g})_H)} & k^*(hg)^*(H) & \xrightarrow{(\chi_{hg,k})_H} & (hgf)^*(H) \\
& & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & \alpha^*_F & \swarrow \alpha^*_c & (\alpha^*g^*)_G & \swarrow (\alpha^*g^*)_u & (\alpha^*g^*h^*)_H & \swarrow \alpha^*((\chi_{hg,i})_H) & (\alpha^*(hg)^*)_H & \swarrow (\chi_{hg,\alpha})_H & (hg\alpha)_H \\
& & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
E & \xrightarrow{a} & f^*(F) & \xrightarrow{f^*(b)} & f^*g^*(G) & \xrightarrow{f^*g^*(c)} & f^*g^*h^*(H) & \xrightarrow{f^*((\chi_{h,g})_H)} & f^*(hg)^*(H) & \xrightarrow{(\chi_{hg,f})_H} & (hgf)^*(H)
\end{array}$$

the pasting of the above diagram, in which all compositions are denoted by concatenation. By the above construction, we have the following main theorem.

Theorem 6.2. *For any trihomomorphism $\mathcal{F}: \mathcal{B}^{coop} \rightarrow \text{Bicat}$ of bicategories, a canonical*

$$F_{\mathcal{B}}: \int_{\mathcal{B}} \mathcal{F} \rightarrow \mathcal{B} \quad (6.10)$$

strict homomorphism from the Grothendieck construction $\int_{\mathcal{B}} \mathcal{F}$ is a fibration of bicategories.

Proof. The strict homomorphism $F_{\mathcal{B}}: \int_{\mathcal{B}} \mathcal{F} \rightarrow \mathcal{B}$ is defined on any object (x, E) of $\int_{\mathcal{B}} \mathcal{F}$ by $F_{\mathcal{B}}(x, E) := x$. Also for any 1-morphism $(f, a): (x, E) \rightarrow (y, F)$ in $\int_{\mathcal{B}} \mathcal{F}$, we define $F_{\mathcal{B}}(f, a) := f$, and for any 2-morphism $(\alpha, \phi): (f, a) \rightarrow (g, b)$ we define $F_{\mathcal{B}}(\alpha, \phi) := \alpha$.

In order to prove that the homomorphism $F_{\mathcal{B}}: \int_{\mathcal{B}} \mathcal{F} \rightarrow \mathcal{B}$ is a fibration of bicategories, we show that for any 1-morphism $f: x_1 \rightarrow x_2$ in \mathcal{B} , and any object (x_2, F) in the fibre $\int_{\mathcal{B}} \mathcal{F}_{x_2}$, a 1-morphism $(f, i_{f^*F}): (x_1, f^*F) \rightarrow (x_2, F)$ is a 2-cartesian 1-morphism in $\int_{\mathcal{B}} \mathcal{F}$.

For any 1-morphism $(g, c): (x_0, G) \rightarrow (x_2, F)$ in $\int_{\mathcal{B}} \mathcal{F}$, and any 2-isomorphism $\beta: f \circ u \Rightarrow g$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 (x_0, G) & & \\
 \downarrow (u, b) & \searrow (g, c) & \\
 (x_1, f^*F) & \xrightarrow{(f, i_{f^*F})} & (x_2, F) \\
 & \nearrow (\beta, \epsilon) & \\
 & &
 \end{array}
 & \xrightarrow{F_{\mathcal{B}}} &
 \begin{array}{ccc}
 x_0 & & \\
 \downarrow u & \searrow g & \\
 x_1 & \xrightarrow{f} & x_2 \\
 & \nearrow \beta &
 \end{array}
 \end{array}$$

in \mathcal{B} , we define a 1-morphism $(u, b): (x_0, G) \rightarrow (x_1, f^*F)$ in $\int_{\mathcal{B}} \mathcal{F}$, where $b: G \rightarrow u^*f^*F$ is a composition $(\chi_{f,u}^*)_F \circ (\beta_F^* \circ c)$ of the string of 1-morphisms in $\int_{\mathcal{B}} \mathcal{F}_{x_0}$

$$G \xrightarrow{c} g^*(F) \xrightarrow{\beta_F^*} (f \circ u)^*(F) \xrightarrow{(\chi_{f,u}^*)_F} u^*f^*(F).$$

We have the horizontal composition $(f \circ u, i_{f^*F} \circ b) = (f, i_{f^*F}) \circ (u, b)$ where the second component $i_{f^*F} \circ b$ is defined by the composition $(\chi_{f,u}^*)_F \circ \{u^*(i_{f^*F}) \circ [(\chi_{f,u}^*)_F \circ (\beta_F^* \circ c)]\}$

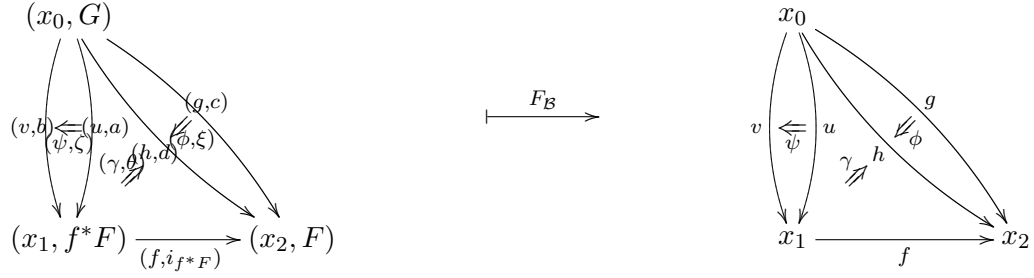
$$G \xrightarrow{c} g^*(F) \xrightarrow{\beta_F^*} (f \circ u)^*(F) \xrightarrow{(\chi_{f,u}^*)_F} u^*f^*(F) \xrightarrow{u^*(i_{f^*F})} u^*f^*(F) \xrightarrow{(\chi_{f,u}^*)_F} (f \circ u)^*(F)$$

and it is obvious that we have a (canonical) 2-isomorphism $\phi: \beta_F^* \circ c \Rightarrow i_{f^*F} \circ b$ in $\int_{\mathcal{B}} \mathcal{F}_{x_0}$

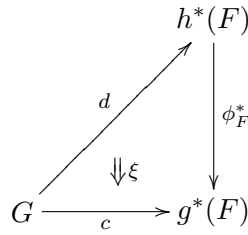
$$\begin{array}{ccc}
 & g^*(F) & \\
 & \nearrow c & \downarrow \beta_F^* \\
 G & \xrightarrow{i_{f^*F} \circ b} & (f \circ u)^*(F) \\
 & \Downarrow \phi & \\
 & &
 \end{array}$$

obtained from the vertical composition of coherence isomorphisms which are components of the modification $\Lambda_{f,u}: \chi_{f,u} \circ \chi_{f,u}^* \Rightarrow \iota_{(f \circ u)^*}$ and the homomorphism $u^*: \int_{\mathcal{B}} \mathcal{F}_{x_1} \rightarrow \int_{\mathcal{B}} \mathcal{F}_{x_0}$.

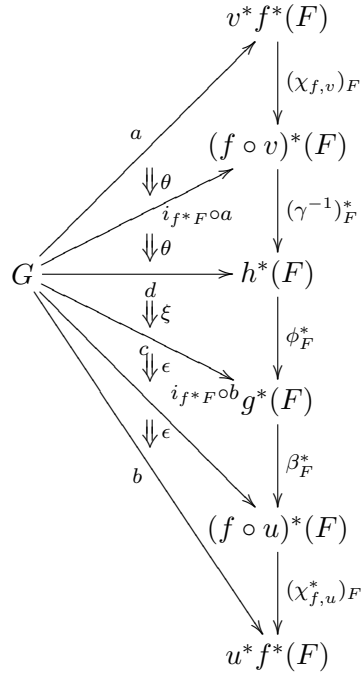
In order to prove that $(f, i_{f^*F}): (x_1, f^*F) \rightarrow (x_2, F)$ is a 2-cartesian 1-morphism in $\int_{\mathcal{B}} \mathcal{F}$, we consider a diagram



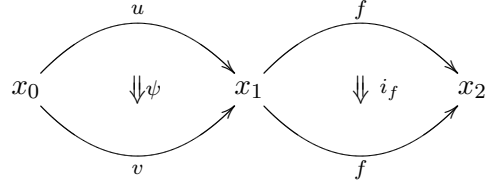
where $(\phi, \xi): (g, c) \Rightarrow (h, d)$ is a 2-morphism in $\int_{\mathcal{B}} \mathcal{F}$ for which $\xi: \phi_F^* \circ d \Rightarrow c$ is a 2-morphism



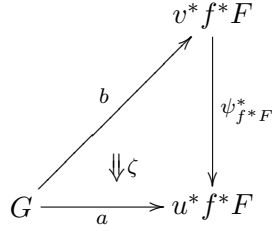
and $\phi: g \Rightarrow h$ is a 2-morphism in \mathcal{B} such that $\phi\beta = \gamma(f \circ \psi)$. Then there exists a 2-morphism



by $\phi' \phi := \phi'(\alpha_F'^* \circ \phi)$. For a horizontally composable pair of 2-morphisms in \mathcal{B} ,

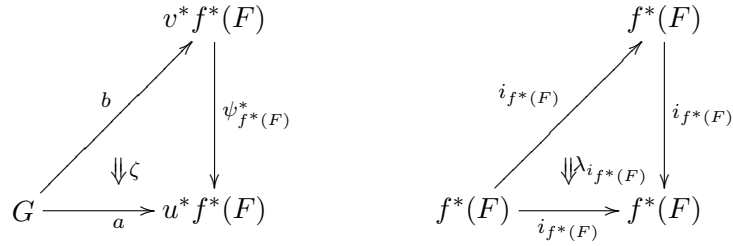


$(\psi, \zeta): (u, a) \Rightarrow (v, b)$ where $\zeta = \theta^{-1} \xi \epsilon$.

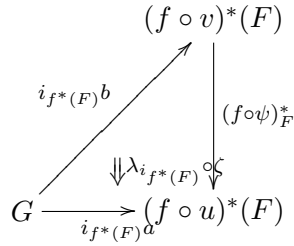


□

and for a horizontally composable pair of 2-morphisms in $\int_{\mathcal{B}} \mathcal{F}$ which covers them



their horizontal composition $(i_f \circ \psi, \lambda_{i_{f^*(F)}} \circ \zeta)$ is represented by the 2-morphism in the triangle



This horizontal composition is defined by the pasting composition of the diagram

$$\begin{array}{ccccc}
 & & v^* f^*(F) & \xrightarrow{(\chi_{f,v})_F} & (f \circ v)^*(F) \\
 & & \downarrow i_{v^* f^*(F)} & \not\approx (\chi_{i_f, i_v})_F & \downarrow i_{(f \circ v)^*(F)} \\
 & & v^* f^*(F) & \xrightarrow{v^*(i_{f^*(F)})} & v^* f^*(F) & \xrightarrow{(\chi_{f,v})_F} & (f \circ v)^*(F) \\
 & & \downarrow \psi_{f^*(F)} & \not\approx (\psi^*)_{i_{f^*(F)}} & \downarrow (\psi^* \circ f^*)_F & \not\approx (\chi_{i_f, \psi})_F & \downarrow (f \circ \psi)_F \\
 & & v^* f^*(F) & \xrightarrow{v^*(i_{f^*(F)})} & v^* f^*(F) & \xrightarrow{(\chi_{f,v})_F} & (f \circ v)^*(F) \\
 & & \downarrow \zeta & & \downarrow & & \downarrow \\
 G & \xrightarrow{a} & u^* f^*(F) & \xrightarrow{u^*(i_{f^*(F)})} & u^* f^*(F) & \xrightarrow{(\chi_{f,u})_F} & (f \circ u)^*(F)
 \end{array}$$

which is equal to the pasting composition of the diagram

$$\begin{array}{ccccccc}
 E & \xrightarrow{u} & k^*(F) & \xrightarrow{k^*(v)} & k^* l^*(G) & \xrightarrow{(\chi_{l,k})_G} & (lk)^*(G) \\
 & \searrow & \downarrow \phi & & \downarrow & & \downarrow \\
 & & k^*(F) & \xrightarrow{\alpha_F^*} & f^*(F) & \xrightarrow{f^*(v)} & f^* l^*(G) & \xrightarrow{(\chi_{l,f})_G} & (lf)^*(G) \\
 & & \downarrow \alpha_F^* & \not\approx (\alpha^*)_v & \downarrow (\alpha^* \circ l^*)_G & \not\approx (\chi_{l,\alpha})_G & \downarrow (l \circ \alpha)_G^* \\
 & & f^*(F) & \xrightarrow{f^*(v)} & f^* l^*(G) & \xrightarrow{(\chi_{l,f})_G} & (lf)^*(G) \\
 & & \downarrow f^*(\psi) & & \downarrow (f^* \circ \beta^*)_G & \not\approx (\chi_{\beta,f})_G & \downarrow (\beta \circ f)_G^* \\
 & & f^*(F) & \xrightarrow{f^*(v)} & f^* l^*(G) & \xrightarrow{(\chi_{l,f})_G} & (lf)^*(G) \\
 & & \downarrow f^*(b) & & \downarrow & & \downarrow \\
 & & f^* g^*(G) & \xrightarrow{(\chi_{g,f})_G} & (gf)^*(G)
 \end{array}$$

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