Some Constructions of Algebraic Model Categories

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

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2021

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Abstract

We show that for a large class of algebraic model categories, the *compact algebraic model categories*, the projective model structure on the functor category of any diagram exists and is an algebraic model category. For a large class of these compact algebraic model categories, the projective algebraic model structures themselves will be compact. This generalizes a result of [Rie11] for cofibrantly generated algebraic model categories. To prove our result, we fix an issue with and generalize Garner's construction of free algebraic weak factorization systems [Gar08] and more fully develop the theory of algebraic model categories. We then present an easy proof that the h-model structure on k-spaces is a compact algebraic model structure. This gives a method for computing homotopy colimits of any shape of diagram in the h-model structure.

We also define quasiaccessible categories, which both generalize locally presentable categories and include the categories of topological spaces and k-spaces. We define quasiaccessible model structures on quasiaccessible categories, prove they have associated algebraic model structures, and show how the Bousfield-Friedlander theorem can be applied to produce a Bousfield localization of a quasiaccessible category that is itself an algebraic model category. We then prove that the h-model structure on topological spaces is a quasiaccessible model structure. We conclude with a characterization of certain accessible model categories inspired by Smith's theorem for combinatorial model categories.

The results of this thesis provide general methods for dealing with large classes of non-cofibrantly generated model structures on reasonably well-behaved categories.

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Table of Contents

Chapter 1: Introduction

1.1 Neglected Model Categories on Spaces

The h-model structure on spaces, also called the Hurewicz model structure or the Strøm model structure, has an advantage over the Quillen model structure in that every object in the h-model category is both fibrant and cofibrant. Despite this fact, the Quillen model structure on **Top** and the Quillen equivalent Quillen model structure on **sSet** are far more widely used across homotopy theory. One reason for this is that the Quillen model structures on **Top** and **sSet** are cofibrantly generated whereas the h-model structure is not. Another reason, in the case of **sSet**, is that the category of simplicial sets has a lot of useful structure, much of which is captured by the fact that it is a locally presentable category.

Cofibrant generation is a smallness condition that makes many manipulations of the weak factorization systems (WFSs) in a model category possible. Among other things, cofibrantly generated model categories can be lifted along certain right adjoints. When $\mathscr{C}$ is a cofibrantly generated model category, this make it possible to define the projective model structure on the functor category $\mathscr{C}^{\mathscr{D}}$ for any small category $\mathscr{D}$. We can then compute the homotopy colimit of a diagram $\mathscr{D} \to \mathscr{C}$ by taking its cofibrant replacement in the projective model category.

Cofibrantly generated WFSs were generalized by Garner in [Gar08] to cofibrantly generated algebraic weak factorization systems. Algebraic weak factorization systems (AWFSs), introduced by Grandis and Tholen in [GT06] under the name natural weak factorization systems, replace the *property* of having a lift in a WFS with the *structure* of a lift. Moreover, the functorial factorization of an AWFS on a category $\mathscr{C}$ *naturally* factors maps in $\mathscr{C}$ into maps in the left class followed by maps in the right class. This gives us a sort of generalized version of "reflections" and "coreflections" of arrows in $\mathscr{C}$ into the right and left classes, respectively. The right class of a cofibrantly generated AWFS on a category $\mathscr{C}$ is characterized by the structure of a natural right lift with respect to a diagram of arrows in $\mathscr{C}$, rather than the property of having a right lift with respect to a set of maps in $\mathscr{C}$. Under a mild condition on the category $\mathscr{C}$, every cofibrantly generated WFS has an associated cofibrantly generated AWFS. A few distinct technical advantages of AWFSs over WFSs are the pointwise AWFSs of [Rie11, §4.2] and the lift of the (co)fibrant replacement functor to a category of (co)cofibrant objects described in [Rie11, 3.5].

Not only does the structure of an AWFS have some technical advantages over that of a WFS, but there are cofibrantly generated AWFSs which are not cofibrantly generated WFSs. Riehl showed in [Rie11, §4.4] that when $\mathscr{C}$ has the structure of a cofibrantly generated algebraic model category, the projective model structure on the functor category $\mathscr{C}^{\mathscr{D}}$ for any small category $\mathscr{D}$ exists and is a cofibrantly generated algebraic model category. This makes it possible to compute homotopy colimits for a larger class of model categories.

Both cofibrant generation with respect to a set and the more general cofibrant generation with respect to a diagram can be thought of as smallness conditions on (algebraic) WFSs. In this thesis, we introduce a much more general smallness condition on AWFSs and algebraic model categories, which we call *compactness*. Both accessible WFSs and the WFSs of the h-model structure on spaces can be given the structure of a compact AWFS. We promote the new perspective that compact AWFSs and algebraic model categories are a good setting to do homotopy theory. In particular, we are able to show that compact AWFSs can be lifted along certain right adjoints. We then show that when $\mathscr{C}$ has the structure of an algebraic model category and $\mathscr{D}$ is small, the projective algebraic model structure on the functor category $\mathscr{C}^{\mathscr{D}}$ exists and is algebraic. So we have a method for computing homotopy colimits in this context. When the AWFSs of the algebraic model category on $\mathscr{C}$ satisfy the stronger smallness condition of being *$\mathcal{E}$-compact*, we can show that the projective algebraic model structure is $\mathcal{E}$-compact.

The compactness of an AWFS is a very general condition. A compact AWFS is one satisfying a smallness condition about behaving well with respect to certain filtered colimits. This is weaker than having to preserve filtered colimits and even weaker than preserving certain filtered colimits of monomorphisms. As such, accessible WFSs are just one example of compact AWFSs. The most well-behaved form of compactness is $\mathcal{E}$-compactness. Rather than requiring that the endofunctors of an AWFS preserve filtered colimits, $\mathcal{E}$-compactness only requires that they preserve filtered colimits "up to epimorphisms", or at least some class of epimorphisms. Examples of $\mathcal{E}$-compact AWFSs are accessible AWFSs and the WFSs of the h-model structure on k-spaces.

The fact that the h-model structure on k-spaces is an $\mathcal{E}$-compact algebraic model category is a major result of this thesis. Many of the constructions previously reserved for cofibrantly generated categories can now be done for the h-model structure. In particular, we can lift it along certain right adjoints and we are able to show that the projective algebraic model structure exists for the h-model structure on any category of diagrams. Moreover the projective model structure is $\mathcal{E}$-compact itself. This result could open the way for h-model structures to have a more important role in modern homotopy theory.

By [Col06b], there is a mixed model structure on k-spaces whose weak equivalences are weak homotopy equivalences and whose fibrations are Hurewicz fibrations. We call this the m-model structure. We are able to use the $\mathcal{E}$-compactness of the h-model structure to show that the m-model structure is algebraic. Unfortunately, we are not able to show that the m-model structure is compact.

It has come to the author's attention that it is shown in [Gau19, §4] that the q-, h-, and m-model structures on the locally presentable category of delta-generated spaces are accessible model structures. So there is some precedent for the ideas presented here.

The category of simplicial sets has many nice properties that often make it the preferred model for a category of spaces over topological spaces themselves. Many of these properties are captured by the structure of a locally presentable category. The canonical counterexample to a locally presentable category is the category of topological spaces. The category **Top** is too large to be able to express every object as a filtered colimit of spaces in a fixed set. We present an alternative to this perspective as well. We define *quasiaccessible categories*, which are, roughly, categories whose objects can be expressed as certain filtered colimits "up to epimorphisms" of objects in a fixed set. Every locally presentable category is quasiaccessible. Moreover, we prove that the categories of topological spaces and k-spaces are both quasiaccessible categories.

We are able to recover much of the theory of accessible categories in the context of quasiaccessible categories. We define quasiaccessible WFSs and prove that every quasiaccessible WFSs has an associated algebraic weak factorization system, an analog of a result of Rosický in [Ros17] for accessible WFSs. This allows us to show that every quasiaccessible model category has an associated algebraic model category. We prove that the Bousfield-Friedlander theorem can be used to localize quasiaccessible model categories and that the localized model category we get from this process is a quasiaccessible model category. Then the localized model category has an associated algebraic model category. Finally, we prove that the h-model structure on topological spaces is quasiaccessible. So it is possible to Bousfield-localize the h-model structure and get back an algebraic model category.

In the final section of this thesis, we give a characterization of accessible model categories whose weak equivalences are accessible and accessibly embedded. This result was motivated by Smith's theorem for combinatorial model categories.

1.2 Organization of this Thesis

Chapter 2 serves the dual purpose of establishing the conventions and much of the notation that we will use throughout this thesis and of presenting G. M. Kelly's construction of free monads and monoids in [Kel80]. Most of the important, nonstandard definitions and notation are in sections 2.1.1 - 2.1.5 and section 2.4.1. We in particular want to point out that in section 2.1.5 we commit to using the term "R-algebra" for the algebras of a pointed endofunctor R. Even when R is a monad, the unqualified R-algebras will refer to the algebras for the underlying pointed endofunctor. Likewise, the unqualified term "L-coalgebra" will refer to a coalgebra for the copointed endofunctor L.

Our treatment of free monads closely follows the approach in [Kel80]. We prove some of the results there in more detail. We also place an emphasis on free monad sequences first, before the smallness condition on pointed endofunctors. If the free monad sequence on a pointed endofunctor (T, τ) converges objectwise, then the free monad on (T, τ) exists and is given on each object by the value the free monad sequence evaluated on that object converges to. This is true even if the endofunctor T does not satisfy the smallness condition of section 2.4. One departure from [Kel80] is our definition of weakly convergent free monad sequences in section 2.3.4. Weak convergence is more general than objectwise convergence, but it is still strong enough to prove that the free monad on a pointed endofunctor exists. Weak convergence plays an important role in the proof in chapter 3 that free AWFSs on certain left algebraic weak factorization systems (LAWFSs) exist. In fact, there was an issue with Garner's original proof of this result and weak convergence is what is necessary to make his proof go through.

In section 2.4, we introduce a smallness condition on endofunctors that guarantees their free monad sequences converge objectwise and therefore converge weakly. We also discuss a special case of this smallness condition that is inherited by the free monad in section 2.4.4. This is a new result, which makes many of the nice properties of $\mathcal{E}$-compact AWFS possible in later sections, including our proof in chapter 5 that the h-model structure on k-spaces is an $\mathcal{E}$-compact algebraic model category.

In section 2.5, we show that weak convergence of the free monoid sequence for pointed objects in certain strict monoidal categories implies the existence of free monoids on those objects.

In chapter 3, we present our correction of Garner's construction of free AWFSs in [Gar08] and [Gar07] alongside a few original results. We also generalize Garner's construction to make use of the full generality of Kelly's paper in the process. After introducing AWFSs and LAWSFs, we prove some results about how their categories of algebras and coalgebras behave with respect to categorical lifts in 3.1.3. In 3.2.1, we present a variant of Garner's proof that the category of LAWFSs has the structure of a strict monoidal category and that the monoids in this category are the AWFSs. This makes it possible to apply the results of section 2.5 to the construction of free AWFSs on LAWFSs. We also define compact and $\mathcal{E}$-compact endofunctors, functorial factorization systems, and model categories in section 3.2.2. The notation and definitions we introduce in section 3.2.2 are important for the rest of the chapters in this thesis. We show that the free monoid sequence on a compact LAWFSs converges weakly. So free AWFSs exist on compact comonads and compact LAWFSs.

We prove some useful results in 3.3 about $\mathcal{E}$-compact objects. Specifically, we show that free AWFSs on $\mathcal{E}$-compact LAWFSs are $\mathcal{E}$-compact and that a model category whose weak factorization systems have associated $\mathcal{E}$-compact LAWFSs can be given the structure of an $\mathcal{E}$-compact algebraic model category.

In sections 4.1.1 and 4.1.2, we define compact adjunctions, show that it is possible to transfer a compact AWFS along a compact right adjoint, and prove that when an acyclicity condition is satisfied, we can transfer an algebraic model category along a compact right adjoint. While this result is known for cofibrantly generated model categories [GS07, 3.6], cofibrantly generated algebraic model categories [Rie11, §3.3], and for accessible model categories [GKR20], it has not been shown in this level of generality before. We use this to prove in 4.1.3 that when $\mathscr{C}$ has the structure of a compact algebraic model category, the projective model structure on the functor category $\mathscr{C}^{\mathscr{D}}$ exists and is algebraic for any small category $\mathscr{D}$. Furthermore, when the algebraic model structure on $\mathscr{C}$ is $\mathcal{E}$-compact, then the projective algebraic model structure is $\mathcal{E}$-compact.

In chapter 5, we prove that the h-model structure on k-spaces has an associated $\mathcal{E}$-compact algebraic model structure and that the m-model structure on k-spaces has an associated algebraic model structure. We reduce the existence of an $\mathcal{E}$-compact algebraic h-model structure on any topologically bicomplete category to the $\mathcal{E}$-compactness of two LAWFSs.

So this provides an entirely different condition for the existence of the h-model structure on a topologically bicomplete category than the monomorphism hypothesis in [BR13]. So even just the existence of an h-model structure in this case is a new result, but we also get that the h-model structure is algebraic and $\mathcal{E}$-compact. While [BR13] needed the monomorphism hypothesis to prove the existence of the h-model structure and fix an issue with the proofs in [Col06a] and [MS06, §4], the monomorphism hypothesis does not seem to be adequate to show the h-model structure is algebraic. In [BR13] they could only show one of the factorizations of the h-model structure was algebraic. In 5.1.4, we reduce our condition that the two LAWFSs are $\mathcal{E}$-compact to conditions on the topologically bicomplete category itself. We then show in 5.1.5 that the category of k-spaces satisfies these conditions. Therefore the h-model structure on k-spaces exists, is algebraic, and is $\mathcal{E}$-compact.

We show in section 5.2.1 that under certain conditions a mixed model structure inherits the structure of an algebraic model category. Under stricter conditions this mixed algebraic model category will be $\mathcal{E}$-compact. We show in section 5.2.2 that the m-model structure on k-spaces has an associated algebraic model structure.

In chapter 6, we define quasiaccessible categories. These categories generalize locally presentable categories and include the categories of topological spaces and of k-spaces. We are able to prove an analog of a surprising number of results for accessible categories in this more general context. We introduce quasiaccessible and weakly quasiaccessible functors in 6.2.1. Through some highly technical proofs in sections 6.2.2 and 6.2.3, we are able to show in 6.2.4 that the forgetful functor $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathscr{C}$ for a quasiaccessible copointed endofunctor L on $\mathscr{C}$ is a weakly quasiaccessible functor. This, along with our slight generalization of the special adjoint functor theorem in A.3, is enough to show that U_{L} has a right adjoint. It then follows that the cofree comonad on L exists and is a compact endofunctor.

We use our results on quasiaccessible copointed endofunctors to show that every quasiaccessible WFS has an associated compact LAWFS and an associated AWFS in section 6.3.1. Therefore, every quasiaccessible model category has an associated algebraic model category. Under some mild hypotheses, we show that the localized model category produced by the Bousfield-Friedlander theorem is a quasiaccessible category when the original category is quasiaccessible in 6.3.2. Therefore, in this case, the localized category has an associated algebraic model category. We show in 6.3.3 that the h-model structure on **Top** is quasiaccessible. The author has not yet been able to prove this for the h-model structure on k-spaces.

In chapter 7, we discuss a characterization of accessible model categories whose weak equivalences are accessible and accessibly embedded. Unlike in combinatorial model categories, it does not seem to be the case that the weak equivalences in an accessible category must be accessible and accessibly embedded. Without this assumption on the weak equivalences, the problem of characterizing accessible model categories seems intractable to the author. Even giving ourselves this assumption, the characterization we do obtain is of questionable utility. It certainly is nowhere near as easy to apply as Smith's theorem for combinatorial categories.

1.3 Contributions

Most of chapter 2 is expository. Our treatment of the weak convergence of free monad sequences and free monoid sequences in sections 2.3.4 and 2.5.1 is original. Also the results in 2.4.4 are new.

The definitions of the categorical lift operations $(-)^{\boxslash}$ and ${}^{\boxslash}(-)$ in section 3.1.3 are due to Garner [Gar08]. While the author is aware of [BG16a, §2.7], the results about categorical lifts in the remainder of section 3.1.3 are more general, complete, and organized than what currently exists in the literature. Our definitions of compact and $\mathcal{E}$-compact endofunctors are generalizations of ideas in [Gar08] and [BR13]. In section 3.2.3, we use weak convergence to fix a mistake in Garner's proof of the existence of free AWFSs on LAWFSs. In the process we generalize his results. Our generalization is a straightforward application of the results in [Kel80] and was no doubt known by Garner, although it is not present in his papers. The result 3.2.19 (3.1.4) seems to be new and is enabled by our work on categorical lifts in section 3.1.3. The results in section 3.3 are all new.

The results in chapter 4 are original in the algebraic context. Special cases of 4.1.8 and 4.1.12 are known for cofibrantly generated algebraic model categories [Rie11, §3.3, §4.4] and for accessible model categories [GKR20].

While the factorization system $(\mathrm{L}_{t1}, \mathrm{R}_1)$ in section 5.1.3 was constructed in [Col06a] and its properties were shown in [BR13], as far as the author is aware, the factorization system $(\mathrm{L}_1, \mathrm{R}_{t1})$ is new. Theorem 5.1.9 is original. The results in sections 5.1.4 and 5.1.5 are all new. The algebraic parts of section 5.2 are new.

The material in chapter 6 closely mirrors results in [AR94], but is all entirely new in this context.

The results in section 7.2 are the author's, except where otherwise indicated.

The propositions in A.3 are slight variations on more well-known results. Proposition A.5.1 in A.5 may have been folklore, but we actually work out the proof. This result is not used anywhere else in this thesis, but seemed interesting enough to include.

1.4 Future Work

It is likely that many of the important concepts in this thesis are not in their correct final form. Since it seems unlikely that the free AWFS on an $(\mathcal{E}, \mathcal{M}')$-compact LAWFS will satisfies a version of compactness itself, it would probably be better to define compact AMCs to be the ones freely generated by $(\mathcal{E}, \mathcal{M}')$-compact LAWFSs $(\mathrm{C}_t, \mathrm{F})$ and $(\mathrm{C}, \mathrm{F}_t)$ with $|\mathbf{Alg}_{\mathrm{F}}| = \mathcal{F}$ and $|\mathbf{Alg}_{\mathrm{F}_t}| = \mathcal{F} \cap \mathcal{W}$. It would then likely be possible to transport these model structures along adjunctions and show that the projective model structure on any diagram category $\mathscr{C}^{\mathscr{D}}$ exists by transporting the compact LAWFSs. This perspective already seems to be implicit in our proofs of 5.2.1 and 6.3.4.

Another concept that would likely benefit from reformulation is the notation of quasiaccessible functor. It seems that our current definition is too strict, since it does not apply to accessible functors or to the h-model structure on k-spaces. It is the belief of the author that all of the results will still go through if we only require that quasiaccessible functors preserve $\mathcal{E}$-tightness of $(\mathcal{M}, \lambda)$-cocones and send objects in some sufficiently dense subcollection $\mathcal{M}' \subseteq \mathcal{M}$ to objects in $\mathcal{M}$. This subcollection should be the λ-pure morphisms in the case of accessible functors and the closed subspace inclusions in the case of the factorizations of the h-model structure on k-spaces. It may be easier to also include this weakening in the definition of quasiaccessible categories.

It may be possible to prove that quasiaccessible model categories lift along certain left adjoints. We would then be able to prove that the injective model structure exists on any category of diagrams in a quasiaccessible model category. This would provide a method for computing homotopy limits of arbitrary diagrams in the h-model structure.

There is likely much more that can be said about Bousfield localization of compact algebraic model categories beyond the Bousfield-Friedlander theorem for quasiaccessible model categories.

Chapter 2: Free Monads and Monoids

The problem of the construction and existence of free monads on pointed endofunctors has its origin in the orthogonal subcategory problem. Kelly's paper [Kel80] is the culmination of the work of many mathematicians in the 70's on these problems. Kelly's paper unified these results, generalized them, and corrected some mistakes in the literature. The introduction to [Kel80] provides a nice summary.

Our goals in this chapter are to establish the notation we will use throughout this thesis, to give an exposition of the relevant results in [Kel80] about the existence of free monads on pointed endofunctors, and to present a couple additions to Kelly's work that will be useful in later sections. Most of the definitions and notation that we need in later chapters that is nonstandard or not well-known is concentrated in sections 2.1.1 - 2.1.5 and some in section 2.4.1. Section 2.2 is important for understanding what a free objects is and our approach to the construction of free monads on pointed endofunctors. The reader who is not interested in the intricate details of the free monad construction may skip much of the material in sections 2.3 and 2.4. The important theorems from those sections are listed below. Section 2.5 shows how to apply our results on the existence of free monads to the existence of free monoids.

Theorem 2.3.22. *If the free monad sequence for* (T,τ) *converges weakly, then the free monad on* (T,τ) *exists.*

Theorem 2.3.23. *If the free monad sequence for* (T,τ) *converges objectwise, then it converges weakly.*

Theorem 2.4.23. *Let* $\mathscr{C}$ *be a cocomplete category equipped with well-copowered, left proper, orthogonal factorization systems* $(\mathcal{E},\mathcal{M})$ *and* $(\mathcal{E}',\mathcal{M}')$. *If* (T,τ) *is a pointed endofunctor on* $\mathscr{C}$ *and* $\mathrm{T}:\mathscr{C}\to\mathscr{C}$ *preserves* $\mathcal{E}$*-tightness of* $(\mathcal{M}',\lambda)$*-cocones for a regular cardinal* λ*, then the free monad sequence on* (T,τ) *converges objectwise.*

Proposition 2.4.25. *Let* $\mathscr{C}$ *be a cocomplete category equipped with a well-copowered, left proper, orthogonal factorization system* $(\mathcal{E},\mathcal{M})$. *If* (T,τ) *is a pointed endofunctor on* $\mathscr{C}$ *and* $\mathrm{T}:\mathscr{C}\to\mathscr{C}$ *preserves* $\mathcal{E}$*-tightness of* λ*-filtered cocones for a regular cardinal* λ*, then the free monad on* (T,τ) *preserves* $\mathcal{E}$*-tightness of* λ*-filtered cocones.*

2.1 Category Theory

2.1.1 Preliminaries

We will work in a Grothendieck universe. In a few places we will need a second Grothendieck universe that contains the first, so that we can speak of *metacategories*, like **CAT** and $\mathbf{CAT}/\mathscr{C}^{\mathbf{2}}$ in section 3.2.5. The terms *small collection* and *large collection* will refer to sets and proper classes, respectively. A set $\mathcal{X}$ is λ-*small* for a cardinal λ if there are fewer than λ elements in $\mathcal{X}$.

In general, we will not require our categories to be locally small. When X and Y are two objects in a category $\mathscr{C}$, we will use the notation $\mathscr{C}(X,Y)$ for the collection of homomorphisms in $\mathscr{C}$ from X to Y. The notation *ob*$(\mathscr{C})$ will be used for the collection of objects in $\mathscr{C}$.

A category $\mathscr{C}$ is λ-*small* for a cardinal λ if there are fewer than λ morphisms in $\mathscr{C}$. A category $\mathscr{C}$ is *small* if there is a cardinal λ such that $\mathscr{C}$ is λ-small. Equivalently, a category is small if it is locally small and the collection of its objects is a set. A category is *large* if it is not small. We reserve the term *diagram* in a category $\mathscr{C}$ for a functor $D : \mathscr{D} \to \mathscr{C}$ on a small category $\mathscr{D}$. We will specify that the functor $D : \mathscr{D} \to \mathscr{C}$ is a *large diagram* in $\mathscr{C}$ when $\mathscr{D}$ is a large category. A λ-*small diagram* is a diagram $D : \mathscr{D} \to \mathscr{C}$ on a λ-small category $\mathscr{D}$.

Let X be an object in a category $\mathscr{C}$ and let $\Delta_X^{\mathscr{D}} : \mathscr{D} \to \mathscr{C}$ be the constant diagram on $\mathscr{D}$ valued at X. A *cocone* for a diagram $D : \mathscr{D} \to \mathscr{C}$ is an object X and a natural transformation $\alpha : D \to \Delta_X^{\mathscr{D}}$. We will use the notation $\alpha : D \overset{\cdot}{\longrightarrow} X$ for this natural transformation. Dually, a *cone* $\alpha : X \overset{\cdot}{\longrightarrow} D$ for a diagram $D : \mathscr{D} \to \mathscr{C}$ consists of an object X and a natural transformation $\alpha : \Delta_X^{\mathscr{D}} \to D$.

If $\mathcal{X}$ is a collection of objects in a category $\mathscr{A}$, then $\mathbf{Full}(\mathcal{X})$ and $\mathbf{Disc}(\mathcal{X})$ will denote the full and discrete subcategories of $\mathscr{A}$ on the objects in $\mathcal{X}$, respectively. If $F : \mathscr{A} \to \mathscr{B}$ is a functor, we will use the notation $F(\mathcal{X})$ for the collection of objects $\{FX \mid X \in \mathcal{X}\}$ in $\mathscr{B}$. If $\mathcal{Y}$ is a collection of objects in $\mathscr{B}$, then $F^{-1}(\mathcal{Y})$ will denote the collection of objects $\{X \mid FX \in \mathcal{Y}\}$ in $\mathscr{A}$. We extend this notation to subcategories $\mathscr{X}$ of $\mathscr{A}$ and $\mathscr{Y}$ of $\mathscr{B}$ as well. So $F(\mathscr{X}) = F(ob(\mathscr{X}))$ and $F^{-1}(\mathscr{Y}) = F^{-1}(ob(\mathscr{Y}))$.

We will use the following notation.

- **2** for the category $\cdot \to \cdot$ freely generated by a single map with distinct domain and codomain

- **3** for the category $\cdot \to \cdot \to \cdot$ freely generated by two composable maps between 3 distinct objects

- **out** for the category $\cdot \leftarrow \cdot \to \cdot$ freely generated by two maps with the same domain and distinct codomains

- **in** for the category $\textbf{.} \to \textbf{.} \leftarrow \textbf{.}$ freely generated by two maps with the same codomain and distinct domains

In general, when $\mathscr{A}$ and $\mathscr{C}$ are categories, then $\mathscr{C}^{\mathscr{A}}$ will denote the category whose objects are functors $\mathscr{A} \to \mathscr{C}$ and whose morphisms are natural transformations.

The most important functor category in this thesis is $\mathscr{C}^{\mathbf{2}}$. We will represent a map

$$\begin{tikzcd}
{} \arrow[r, "u"] \arrow[d, "f"'] & {} \arrow[d, "g"] \\
{} \arrow[r, "v"] & {}
\end{tikzcd}$$

in $\mathscr{C}^{\mathbf{2}}$ from f to g by the pair (u, v). There is a functor $\mathrm{dom} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ that sends objects $f : X \to Y$ to X and morphisms $(u, v) : f \to g$ to u. We also have a functor $\mathrm{cod} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ that sends objects $f : X \to Y$ to Y and morphisms $(u, v) : f \to g$ to v.

We will use the notation $\mathcal{E}^{\twoheaddownarrow}$, $\mathcal{E}^{s\twoheaddownarrow}$, $\mathcal{M}^{\downarrowtail}$, and $\mathcal{M}^{s\downarrowtail}$ for the collections of epimorphisms, strong epimorphisms, monomorphisms, and strong monomorphisms, respectively, in a given category.

2.1.2 Cardinals and Ordinals

We will identify small categories $\mathscr{D}$ satisfying the following conditions with partially ordered sets.

- For any two objects X and Y in $\mathscr{D}$, the cardinality of $\mathscr{D}(X, Y)$ is at most 1.
- If there are maps $X \to Y$ and $Y \to X$ in $\mathscr{D}$, then $X = Y$.

A small category $\mathscr{D}$ is equivalent to a partially ordered set when it satisfies the first condition and $X \cong Y$ in the second condition.

Let **Ord** be the category of ordinal numbers. This is a large, locally small category whose objects consist of all ordinal numbers. There is a unique morphism $\alpha \to \beta$ in **Ord** from an ordinal α to an ordinal β exactly when $\alpha \leq \beta$. The ordinal 0 is the initial object of **Ord**. For each ordinal α, let $\mathbf{Ord}_{<\alpha}$ be the full subcategory of **Ord** on the ordinal numbers smaller than α. Then $\mathbf{Ord}_{<\alpha}$ is a small category which can be identified with the partially ordered set α.

Definition 2.1.1. We will take a *transfinite sequence* in a category $\mathscr{C}$ to be a functor $F : \mathbf{Ord} \to \mathscr{C}$. For each ordinal α, an *α-sequence* in $\mathscr{C}$ is a functor $F : \mathbf{Ord}_{<\alpha} \to \mathscr{C}$.

Definition 2.1.2. A transfinite sequence $F : \mathbf{Ord} \to \mathscr{C}$ is *cocontinuous* if $F(\gamma) \cong \operatorname{colim}_{\alpha \leq \gamma} F(\alpha)$ for every limit ordinal γ. A transfinite sequence of ordinals $F : \mathbf{Ord} \to \mathbf{Ord}$ is *strictly increasing* if $F(\alpha) < F(\beta)$ whenever $\alpha < \beta$.

Let α be an ordinal. We can extend the above definitions to α-sequences $F : \mathbf{Ord}_{<\alpha} \to \mathscr{C}$ and $F : \mathbf{Ord}_{<\alpha} \to \mathbf{Ord}$ by restricting to ordinals less than α. Every strictly increasing transfinite sequence (or α-sequence) in **Ord** is an injective functor on objects.

We can define a cocontinuous transfinite sequence $F : \mathbf{Ord} \to \mathscr{C}$ inductively by specifying a value for $F(0)$ and a rule that defines $F(\alpha+1)$ and $F(\alpha \to \alpha+1)$ whenever $F(\alpha)$ is defined. The transfinite sequence is completely determined by these specifications.

In a category $\mathscr{C}$, the term *transfinite composition* will refer to the constituent map $\alpha_0 : F(0) \to \operatorname{colim} F$ of the colimiting cocone $\alpha : F \overset{\centerdot}{\longrightarrow} \operatorname{colim} F$ of a cocontinuous α-sequence $F : \mathbf{Ord}_{<\alpha} \to \mathscr{C}$.

Definition 2.1.3. An infinite ordinal α is *regular* if every final full subcategory of $\mathbf{Ord}_{<\alpha}$ is isomorphic to $\mathbf{Ord}_{<\alpha}$.

We will use the notation $|\alpha|$ for the cardinality of an ordinal α.

Definition 2.1.4. An ordinal α is *initial* if $|\beta| < |\alpha|$ for every ordinal $\beta < \alpha$.

For each cardinal λ, there is a unique initial ordinal α with $|\alpha| = \lambda$. We will say α is the *initial ordinal of* λ in this case.

Every regular ordinal is an initial ordinal, but not every infinite initial ordinal is regular. Every infinite initial ordinal is a limit ordinal, but not every limit ordinal is initial.

Definition 2.1.5. An infinite cardinal λ is *regular* if

$$\sum_{i \in \mathcal{I}} \lambda_i < \lambda$$

for every λ-small set $\{\lambda_i\}_{i \in \mathcal{I}}$ of cardinals $\lambda_i < \lambda$.

Every infinite successor cardinal is a regular cardinal.

Proposition 2.1.6. *An ordinal is regular if and only if it is the initial ordinal of a regular cardinal.*

Proof. Let α be a regular ordinal. Then it is an initial ordinal. Suppose there is an $|\alpha|$-small set $\{\lambda_i\}_{i \in \mathcal{I}}$ of cardinals $\lambda_i < |\alpha|$ such that $\sum_{i \in \mathcal{I}} \lambda_i = |\alpha|$. Let β be the initial ordinal of the cardinality of $\mathcal{I}$. Every element $i \in \mathcal{I}$ corresponds to a unique object β_i in $\mathbf{Ord}_{<\beta}$ We can then construct a strictly increasing cocontinuous β-sequence of ordinals $F : \mathbf{Ord}_{<\beta} \to \mathbf{Ord}_{<\alpha}$ such that $|F(\beta_i)| \geq \lambda_i$ for each $i \in \mathcal{I}$: If we run out of ordinals less than α at an ordinal $\beta' < \beta$, the first of the following contradictions holds with β' in place of β. If the colimit of F in $\mathbf{Ord}$ is equal to α, then $\mathbf{Ord}_{<\beta} \cong \mathbf{Ord}_{<\alpha}$ and $|\alpha| = |\beta|$, which is a contradiction. If the colimit of F in $\mathbf{Ord}$ is an ordinal $\zeta < \alpha$, then $|\zeta| \geq \sum_{i \in \mathcal{I}} \lambda_i = |\alpha|$, which means α is not initial which is another contradiction. So $|\alpha|$ must be a regular cardinal.

Conversely, suppose λ is a regular cardinal. Let α be the initial ordinal of λ. Suppose $\mathscr{D}$ is a final full subcategory of $\mathbf{Ord}_{<\alpha}$. Since the underlying set of $\mathscr{D}$ is well-ordered, there is an isomorphism $\mathscr{D} \cong \mathbf{Ord}_{<\beta}$ for some $\beta \leq \alpha$. If $|\beta| < |\alpha|$, then the equality $\sum_{\zeta \in ob(\mathscr{D})} |\zeta| =$

$|\operatorname{colim}\mathscr{D}| = \lambda$ means λ is not regular, a contradiction. If $\beta < \alpha$ and $|\beta| = |\alpha|$, then α is not initial, a contradiction. So $\beta = \alpha$ and α is a regular ordinal. $\square$

Example 2.1.7. The ordinal ω is the initial ordinal of $\aleph_0$ and it is a regular ordinal. The ordinal $\omega \cdot 2$ is a limit ordinal, but it is not initial, since $|\omega \cdot 2| = \aleph_0$. Therefore it is also not regular. The cardinal $\aleph_0$ and its successors $\aleph_1, \aleph_2, \cdots$ are regular cardinals. The limit cardinal $\aleph_\omega$ is not a regular cardinal. The initial ordinal of $\aleph_\omega$ is not a regular ordinal.

Let λ be a regular cardinal. A λ-*directed set* is a partially ordered set $\mathscr{X}$ such that every λ-small subset of $\mathscr{X}$ has an upper bound. A *directed set* is an $\aleph_0$-directed set.

Definition 2.1.8. A regular cardinal κ is *sharply smaller* than a regular cardinal λ if for every λ-directed set $\mathcal{D}$, every κ-small subset of $\mathcal{D}$ is a subset of a κ-small λ-directed subset of $\mathcal{D}$.

We will use the notation $\kappa \lhd \lambda$ when κ is sharply smaller than λ. The relation $\lhd$ is transitive on the regular cardinals. For every regular cardinal λ, $\lambda \lhd \lambda^+$ and $\lambda \lhd (2^\lambda)^+$, where λ^+ is the successor cardinal of λ.

2.1.3 Cocones and Colimits

Let λ be a regular cardinal. A category is λ-*directed* if it is a λ-directed partially ordered set. Some authors define λ-directed categories as λ-directed preordered sets rather than partially ordered sets. As a consequence of proposition 2.1.14, this distinction is rarely important. A category $\mathscr{C}$ is λ-*filtered* if every λ-small diagram in $\mathscr{C}$ has a cocone. Every λ-directed category is a λ-filtered category. A category $\mathscr{D}$ is λ-*sequential* if $\mathscr{D}$ is equivalent to $\mathbf{Ord}_{<\alpha}$ for some regular ordinal α with $|\alpha| \geq \lambda$. By the following proposition, every λ-sequential category is λ-directed.

Proposition 2.1.9. *Let λ be a regular cardinal. For every regular ordinal α with $|\alpha| \geq \lambda$, $\mathbf{Ord}_{<\alpha}$ is a λ-directed category.*

Proof. Let $D : \mathscr{D} \hookrightarrow \mathbf{Ord}_{<\alpha}$ be the inclusion of a λ-small subcategory. Since the cardinality of $ob(\mathscr{D})$ is less than $|\alpha|$, $\mathscr{D}$ cannot be isomorphic to $\mathbf{Ord}_{<\alpha}$. So $\mathscr{D}$ is not a final subcategory of $\mathbf{Ord}_{<\alpha}$. This means there is an object β in $\mathbf{Ord}_{<\alpha}$ that does not have a map to any object in $\mathscr{D}$. In other words, $\beta > \beta'$ for each object β' in $\mathscr{D}$. By uniqueness of maps in $\mathbf{Ord}_{<\alpha}$, there is a cocone $D \overset{\bullet}{\longrightarrow} \beta$. So β corresponds to an upper bound for the underlying partially ordered set of $\mathscr{D}$ in the underlying partially ordered set of $\mathbf{Ord}_{<\alpha}$. ☐

Let λ be a regular cardinal. A diagram $D : \mathscr{D} \to \mathscr{C}$ is λ-*sequential*, λ-*directed*, or λ-*filtered* if $\mathscr{D}$ is a small λ-sequential, λ-directed, or λ-filtered category, respectively. When $\lambda = \aleph_0$, we call λ-directed and λ-filtered diagrams *directed* and *finitely filtered* diagrams, respectively. A cocone $\alpha : D \overset{\bullet}{\longrightarrow} X$ is λ-*sequential*, λ-*directed*, or λ-*filtered* if D is a λ-sequential, λ-directed, or λ-filtered diagram, respectively.

Remark 2.1.10. Whenever λ and κ are regular cardinals with $\kappa > \lambda$, then every κ-filtered diagram is λ-filtered. So every functor that preserves λ-filtered colimits in particular preserves κ-filtered colimits.

The notation $\alpha : D \overset{\bullet}{\longrightarrow} X$ for a cocone can at times be cumbersome to work with. For this reason we will often instead use the notation $\{\alpha_d : Y_d \to X\}_d$ for a cocone $\alpha : D \overset{\bullet}{\longrightarrow} X$, where $Y_d = Dd$ and d ranges over the objects of $\mathscr{D}$. The trade-off with this notation is that the morphisms of the diagram D are implicit. Borrowing terminology from directed colimits, when $u : d_1 \to d_2$ is a morphism in $\mathscr{D}$, we will refer to $Du : Dd_1 \to Dd_2$ as *a connecting morphism* of the cocone $\{\alpha_d\}_d$. Unlike in the case of a directed cocone, a connecting morphism between objects d_1 and d_2 indexing a general cocone does not have to be unique.

Let $D : \mathscr{D} \to \mathscr{C}$ and let $E : \mathscr{E} \to \mathscr{C}$ be diagrams. A *map* $F : \alpha \to \beta$ *of cocones* $\alpha : D \overset{\bullet}{\longrightarrow} X$ and $\beta : E \overset{\bullet}{\longrightarrow} X$ is a functor $F : \mathscr{D} \to \mathscr{E}$ such that $EFd = Dd$ and $\beta_{Fd} = \alpha_d$ for each object d in $\mathscr{D}$. The cocone α is a *subcocone* of β if $F : \mathscr{D} \to \mathscr{E}$ is the inclusion of a subcategory. A map $F : \alpha \to \beta$ of cocones $\alpha : D \overset{\bullet}{\longrightarrow} X$ and $\beta : E \overset{\bullet}{\longrightarrow} X$ is *final* if $F : \mathscr{D} \to \mathscr{E}$ is a final functor.

Let X be an object in a category $\mathscr{C}$ and let $D : \mathscr{D} \to \mathscr{C}$ be a diagram. Recall, the comma category $D \downarrow X$ has an object for each pair (d, f) of an object d in $\mathscr{D}$ and a morphism $f : Dd \to X$ in $\mathscr{C}$. A morphism $u : (d_1, f) \to (d_2, g)$ in $D \downarrow X$ is a map $u : d_1 \to d_2$ in $\mathscr{D}$ such that $g \circ Du = f$. Often we can refer to just the map $f : Dd \to X$ as an object of $D \downarrow X$ without confusion. The comma category comes with a functor $\Phi : D \downarrow X \to \mathscr{C}$ that sends objects (d, f) to Dd and sends morphisms $u : (d_1, f) \to (d_2, g)$ to $Du : Dd_1 \to Dd_2$. We call Φ the *canonical diagram* of X relative to D. There is a cocone $\varphi : \Phi \overset{\bullet}{\longrightarrow} X$ defined by $\varphi_{(d,f)} = f : Dd \to X$. The cocone $\varphi : \Phi \overset{\bullet}{\longrightarrow} X$ is the *canonical cocone* of X relative to the diagram D. If $\mathscr{D}$ is a subcategory of $\mathscr{C}$ and $D : \mathscr{D} \hookrightarrow \mathscr{C}$ is the subcategory inclusion functor, then we will say φ is the canonical cocone of X relative to $\mathscr{D}$. If $\mathcal{D}$ is a collection of objects in $\mathscr{C}$ and $\mathscr{D} = \mathbf{Full}(\mathcal{D})$, then we will say φ is the canonical cocone of X relative to $\mathcal{D}$.

Let $\mathcal{X}$ be a collection of morphisms in a category $\mathscr{C}$ and let X be an object in $\mathscr{C}$. An $\mathcal{X}$-*map* is a map in $\mathcal{X}$.

Definition 2.1.11. We will say a cocone $\alpha : D \overset{\bullet}{\longrightarrow} X$ is an $\mathcal{X}$-*cocone* if $\alpha_d : Dd \to X$ is an $\mathcal{X}$-map for each object d in $\mathscr{D}$. When λ is a regular cardinal and α is both a λ-filtered cocone and an $\mathcal{X}$-cocone, we will say α is an $(\mathcal{X}, \lambda)$-*cocone.*

For a diagram $D : \mathscr{D} \to \mathscr{C}$, let $D \downarrow_{\mathcal{X}} X$ be the full subcategory of the comma category $D \downarrow X$ on the objects (d, f) such that $f : Dd \to X$ is an $\mathcal{X}$-map in $\mathscr{C}$.

Definition 2.1.12. Let $D : \mathscr{D} \to \mathscr{C}$ be a diagram. The *canonical* $\mathcal{X}$-*diagram of* X *with respect to* D is the restriction $\Phi|_{\mathcal{X}} : D \downarrow_{\mathcal{X}} X \to \mathscr{C}$ of the canonical diagram $\Phi : D \downarrow X \to \mathscr{C}$ to the subcategory $D \downarrow_{\mathcal{X}} X$. The *canonical $\mathcal{X}$-cocone of X with respect to D* is the restriction of the canonical cocone $\varphi : \Phi \overset{\centerdot}{\longrightarrow} X$ to the diagram $\Phi|_{\mathcal{X}}$.

By the universal property of colimits, a cocone $\{f_d : X_d \to X\}_d$ defines a map $g : \operatorname{colim}_d X_d \to X$. At times, we will write g as $\operatorname{colim}_d f_d$, since the above cocone defines a colimiting cocone $\{f_d \to g\}_d$ in the arrow category $\mathscr{C}^{\mathbf{2}}$.

Definition 2.1.13. Let $\mathcal{X}$ be a collection of maps in a category $\mathscr{C}$ and let $\mathscr{D}$ be a small category.

- A cocone $\{f_d : X_d \to X\}_d$ in $\mathscr{C}$ is *$\mathcal{X}$-tight* if the colimit $\operatorname{colim}_d X_d$ exists and the map $g : \operatorname{colim}_d X_d \to X$ defined by the cocone is an $\mathcal{X}$-map.
- An endofunctor $F : \mathscr{C} \to \mathscr{C}$ *preserves $\mathcal{X}$-tightness* of cocones of shape $\mathscr{D}$ if whenever $\alpha : D \overset{\centerdot}{\longrightarrow} C$ is an $\mathcal{X}$-tight cocone on a diagram $D : \mathscr{D} \to \mathscr{C}$, the cocone $F\alpha : FD \overset{\centerdot}{\longrightarrow} FC$ is $\mathcal{X}$-tight.

We will sometimes use an analogous condition for a functor $F : \mathscr{A} \to \mathscr{B}$ between categories. When $\mathcal{X}$ is a collection of maps in $\mathscr{A}$, $\mathcal{Y}$ is a collection of maps in $\mathscr{B}$, and $\mathscr{D}$ is a small category, we say that F *sends $\mathcal{X}$-tight cocones of shape $\mathscr{D}$ to $\mathcal{Y}$-tight cocones* if $F\alpha : FD \overset{\centerdot}{\longrightarrow} FC$ is a $\mathcal{Y}$-tight cocone whenever $\alpha : D \to C$ is an $\mathcal{X}$-tight cocone on a diagram $D : \mathscr{D} \to \mathscr{A}$. Colimiting cocones in a category $\mathscr{C}$ are exactly the $\mathcal{X}$-tight cocones for the collection $\mathcal{X}$ of isomorphisms in $\mathscr{C}$.

The following results show that λ-directed and λ-filtered diagrams are often interchangeable.

Proposition 2.1.14 ([AR94, 1.21])**.** *For every regular cardinal λ and every small λ-filtered category $\mathscr{C}$, there is a final functor $D : \mathscr{D} \to \mathscr{C}$ on a λ-directed category $\mathscr{D}$.*

Corollary 2.1.15. *Let λ be a regular cardinal. A category has λ-filtered colimits if and only if it has λ-directed colimits.*

Corollary 2.1.16. *Let λ be a regular cardinal. A functor $F : \mathscr{A} \to \mathscr{B}$ preserves λ-directed colimits if and only if F preserves λ-filtered colimits.*

Remark 2.1.17. Although it is true that a category has $\aleph_0$-directed colimits if and only if it has $\aleph_0$-sequential colimits and that a functor preserves $\aleph_0$-directed colimits if and only if it preserves $\aleph_0$-sequential colimits, this does not have to be true for regular cardinals $\lambda > \aleph_0$. [AR94, 1.7, 1.21].

2.1.4 Monads and Comonads

Definition 2.1.18.

- A *pointed endofunctor* on a category $\mathscr{C}$ is a pair (R, η) consisting of a functor $\mathrm{R} : \mathscr{C} \to \mathscr{C}$ and a natural transformation $\eta : Id \to \mathrm{R}$.

- A *copointed endofunctor* on a category $\mathscr{C}$ is a pair $(\mathrm{L}, \varepsilon)$ consisting of a functor $\mathrm{L} : \mathscr{C} \to \mathscr{C}$ and a natural transformation $\varepsilon : \mathrm{L} \to Id$.

We will call the natural transformations η and ε the unit and counit maps of R and L, respectively. We will refer to an endofunctor $\mathrm{F} : \mathscr{C} \to \mathscr{C}$ itself as a (co)pointed endofunctor when a (co)unit map for F exists.

A *map* $\theta : (\mathrm{R}, \eta) \to (\mathrm{R}', \eta')$ *of pointed endofunctors* is a natural transformation $\theta : \mathrm{R} \to \mathrm{R}'$ such that $\eta' \circ \theta = \eta$. A *map* $\theta : (\mathrm{L}, \varepsilon) \to (\mathrm{L}', \varepsilon')$ *of copointed endofunctors* is a natural transformation $\theta : \mathrm{L} \to \mathrm{L}'$ such that $\theta \circ \varepsilon = \varepsilon'$.

Definition 2.1.19.

- A *monad* on a category $\mathscr{C}$ is a triple (R, η, μ) consisting of a pointed endofunctor (R, η) and a map $\mu : \mathrm{RR} \to \mathrm{R}$ such that the following diagrams commute.

$$\begin{tikzcd}
\mathrm{R} \arrow[r, "\eta\mathrm{R}"] \arrow[dr, "id"'] & \mathrm{RR} \arrow[d, "\mu"] & \mathrm{R} \arrow[l, "\mathrm{R}\eta"'] \arrow[dl, "id"] \\
 & \mathrm{R} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{RRR} \arrow[r, "\mathrm{R}\mu"] \arrow[d, "\mu\mathrm{R}"'] & \mathrm{RR} \arrow[d, "\mu"] \\
\mathrm{RR} \arrow[r, "\mu"] & \mathrm{R}
\end{tikzcd}$$

- A *comonad* on a category $\mathscr{C}$ is a triple $(\mathrm{L}, \varepsilon, \delta)$ consisting of a copointed endofunctor $(\mathrm{L}, \varepsilon)$ and a map $\delta : \mathrm{L} \to \mathrm{LL}$ such that the following diagrams commute.

$$\begin{tikzcd}
 & \mathrm{L} \arrow[dl, "id"'] \arrow[d, "\delta"] \arrow[dr, "id"] & \\
\mathrm{L} & \mathrm{LL} \arrow[l, "\varepsilon\mathrm{L}"'] \arrow[r, "\mathrm{L}\varepsilon"] & \mathrm{L}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{L} \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathrm{LL} \arrow[d, "\delta\mathrm{L}"] \\
\mathrm{LL} \arrow[r, "\mathrm{L}\delta"] & \mathrm{LLL}
\end{tikzcd}$$

We will call the natural transformations μ and δ the multiplication and comultiplication maps of R and L, respectively. We will refer to an endofunctor $\mathrm{F} : \mathscr{C} \to \mathscr{C}$ itself as a (co)monad when maps exist that give F the structure of a (co)monad.

A *map* $\theta : (\mathrm{R}, \eta, \mu) \to (\mathrm{R}', \eta', \mu')$ *of monads* on $\mathscr{C}$ is a map of pointed endofunctors $\theta : (\mathrm{R}, \eta) \to (\mathrm{R}', \eta')$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{RR} \arrow[r, "\mathrm{R}\theta"] \arrow[d, "\mu"] & \mathrm{RR}' \arrow[r, "\theta\mathrm{R}'"] & \mathrm{R}'\mathrm{R}' \arrow[d, "\mu'"] \\
\mathrm{R} \arrow[rr, "\theta"] & & \mathrm{R}'
\end{tikzcd}$$

A *map* $\theta : (\mathrm{L}, \varepsilon, \delta) \to (\mathrm{L}', \varepsilon', \delta')$ *of comonads* on $\mathscr{C}$ is a map of copointed endofunctors $\theta : (\mathrm{L}, \varepsilon) \to (\mathrm{L}', \varepsilon')$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{L} \arrow[rr, "\theta"] \arrow[d, "\delta"] & & \mathrm{L}' \arrow[d, "\delta"] \\
\mathrm{LL} \arrow[r, "\mathrm{L}\theta"] & \mathrm{LL}' \arrow[r, "\theta\mathrm{L}'"] & \mathrm{L}'\mathrm{L}'
\end{tikzcd}$$

Note that $\theta\mathrm{R}' \circ \mathrm{R}\theta = \mathrm{R}'\theta \circ \theta\mathrm{R}$ and $\theta\mathrm{L}' \circ \mathrm{L}\theta = \mathrm{L}'\theta \circ \theta\mathrm{L}$, since functor composition in $\mathbf{End}(\mathscr{C})$ is monoidal.

Let $\mathbf{End}(\mathscr{C})$ be the functor category $\mathscr{C}^{\mathscr{C}}$ whose objects are endofunctors on $\mathscr{C}$ and whose morphisms are natural transformations. Let $\mathbf{pEnd}(\mathscr{C})$ be the category whose objects are pointed endofunctors on $\mathscr{C}$ and whose morphisms are maps of pointed endofunctors. The following observations will be useful.

Proposition 2.1.20. *The forgetful functor* $\mathbf{pEnd}(\mathscr{C}) \to \mathbf{End}(\mathscr{C})$ *creates large connected colimits.*

Proof. Given a connected, possibly large, diagram in $\mathbf{End}(\mathscr{C})$ that factors through $\mathbf{pEnd}(\mathscr{C})$, we can create a new diagram of endofunctors by adding the point Id and all of the unit maps to the original diagram. Then the original diagram is a final subdiagram of the new diagram. When the colimit of the new diagram in $\mathbf{End}(\mathscr{C})$ exists, the colimiting cocone includes a map from Id to the colimit. Taking this as the unit for the colimiting endofunctor, we get that the diagram is a colimiting diagram in $\mathbf{pEnd}(\mathscr{C})$.

If $D : \mathscr{D} \to \mathbf{End}(\mathscr{C})$ is a diagram that factors through $\mathbf{pEnd}(\mathscr{C})$ and if $\alpha : D \overset{\bullet}{\longrightarrow} T$ is a cocone in $\mathbf{End}(\mathscr{C})$, then T must be a pointed endofunctor and α must be a cocone in $\mathbf{pEnd}(\mathscr{C})$. Therefore, if the colimit of D exists in $\mathbf{pEnd}(\mathscr{C})$, then it is a colimit in $\mathbf{End}(\mathscr{C})$. □

Let $\mathbf{Cmd}(\mathscr{C})$ be the category whose objects are comonads on $\mathscr{C}$ and whose morphisms are maps of comonads.

Proposition 2.1.21. *The forgetful functor* $\mathbf{Cmd}(\mathscr{C}) \to \mathbf{End}(\mathscr{C})$ *creates colimits of the (possibly large) diagrams whose colimits exist in* $\mathbf{End}(\mathscr{C})$.

Proof. Let $D : \mathscr{D} \to \mathbf{End}(\mathscr{C})$ be a possibly large diagram that factors through $\mathbf{Cmd}(\mathscr{C})$ whose colimit in $\mathbf{End}(\mathscr{C})$ exists. Let $(Dd, \varepsilon^d, \delta^d)$ be the comonad structure on Dd for each object d in $\mathscr{D}$. Let $\mathrm{L} = \operatorname{colim} D$ with colimiting cocone $\zeta : D \overset{\bullet}{\longrightarrow} \mathrm{L}$. If we define

$\theta_d := \zeta_d\mathrm{L} \circ Dd\zeta_d \circ \delta^d : Dd \to \mathrm{LL}$ on each object d, then the fact that the connecting maps of the diagram D are maps of comonads implies that $\theta : D \overset{\cdot}{\longrightarrow} \mathrm{LL}$ is a cocone. This cocone defines a map $\delta : \mathrm{L} \to \mathrm{LL}$ out of the colimit satisfying the relations $\delta \circ \zeta_d = \zeta_d\mathrm{L} \circ Dd\zeta_d \circ \delta^d$. In the same way, we can define a map $\varepsilon : \mathrm{L} \to Id$ out of the colimit satisfying the relations $\varepsilon \circ \zeta_d = \varepsilon^d$. The fact that each Dd is a comonad and the universal property of the colimit imply the comonad axioms hold for $(\mathrm{L}, \varepsilon, \delta)$. So the colimit of D exists in $\mathbf{Cmd}(\mathscr{C})$. □

Corollary 2.1.22. *The category* $\mathbf{Cmd}(\mathscr{C})$ *is cocomplete when* $\mathscr{C}$ *is cocomplete.*

Remark 2.1.23. With the operation $\circ$ of endofunctor composition and the identity functor $Id : \mathscr{C} \to \mathscr{C}$, $(\mathbf{End}(\mathscr{C}), \circ, Id)$ is a strict monoidal category. A monad on $\mathscr{C}$ is a monoid in the strict monoidal category $(\mathbf{End}(\mathscr{C}), \circ, Id)$. Conversely, every monoid X in a strict monoidal category $(\mathscr{C}, \otimes, \mathtt{I})$ defines an endofunctor $X \otimes (-) : \mathscr{C} \to \mathscr{C}$ which is a monad on $\mathscr{C}$.

We will see in proposition 2.1.28 that monads and comonads always come from adjunctions. Conversely, every adjunction has an associated monad and comonad.

Proposition 2.1.24 ([Bor94a, 4.2.1])**.** *Suppose* $F : \mathscr{A} \to \mathscr{B}$ *is the left adjoint of a functor* $G : \mathscr{B} \to \mathscr{A}$.

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\top"] & \mathscr{B} \arrow[l, bend right, "G"'] \end{tikzcd}$$

Let $\xi : FG \to Id$ *and* $\nu : Id \to GF$ *be the counit and unit maps, respectively, for the adjunction.*

- *The endofunctor* $GF : \mathscr{A} \to \mathscr{A}$ *with the maps* $\nu : Id \to GF$ *and* $G\xi F : GFGF \to GF$ *is a monad* $(GF, \nu, G\xi F)$ *on* $\mathscr{A}$.

- *The endofunctor* $FG : \mathscr{B} \to \mathscr{B}$ *with the maps* $\xi : FG \to Id$ *and* $F\nu G : FG \to FGFG$ *is a comonad* $(FG, \xi, F\nu G)$ *on* $\mathscr{B}$.

2.1.5 Algebras and Coalgebras

A *coalgebra* for a copointed endofunctor $(\mathrm{L}, \varepsilon)$ on a category $\mathscr{C}$ is a pair $\langle X, k\rangle$ of an object X and a morphism $k : X \to \mathrm{L}X$ in $\mathscr{C}$ such that $\varepsilon_X \circ k = id_X$. An *algebra* for a pointed endofunctor (R, η) on $\mathscr{C}$ is a pair $\langle Y, s\rangle$ of an object Y and a morphism $s : \mathrm{R}Y \to Y$ such that $s \circ \eta_Y = id_Y$. We will call such pairs $\langle X, k\rangle$ and $\langle Y, s\rangle$ L-coalgebras and R-algebras, respectively. We will also say an object X in $\mathscr{C}$ is an L-coalgebra if a map $k : X \to \mathrm{L}X$ exists such that $\langle X, k\rangle$ is an L-coalgebra. Similarly, we will use the term "R-algebra" to refer to both the structure of an R-algebra and the property of having an R-algebra structure.

For a copointed endofunctor L on a category $\mathscr{C}$, we have a category of L-coalgebras, $\mathbf{Coalg}_{\mathrm{L}}$, whose objects are pairs $\langle X, k\rangle$ such that X is an object in $\mathscr{C}$ and $k : X \to \mathrm{L}X$ is a map exhibiting X as an L-coalgebra. A morphism $\langle X, k\rangle \to \langle Y, l\rangle$ in $\mathbf{Coalg}_{\mathrm{L}}$ is a map $f : X \to Y$ in $\mathscr{C}$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "k"] & Y \arrow[d, "l"] \\ \mathrm{L}X \arrow[r, "Lf"] & \mathrm{L}Y \end{tikzcd} \tag{2.1}$$

Composition in $\mathbf{Coalg}_{\mathrm{L}}$ is determined by composition in $\mathscr{C}$. The category $\mathbf{Coalg}_{\mathrm{L}}$ is usually not a subcategory of $\mathscr{C}$, since there may be multiple structure maps $X \to \mathrm{L}X$ that make X an L-coalgebra, but there is a forgetful functor $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathscr{C}$ which sends a pair $\langle X, k\rangle$ to X and a map $f : \langle X, k\rangle \to \langle Y, l\rangle$ to $f : X \to Y$. The category $\mathbf{Alg}_{\mathrm{R}}$ of R-algebras for a pointed endofunctor R is defined dually. It too comes with a forgetful functor $U_{\mathrm{R}} : \mathbf{Alg}_{\mathrm{R}} \to \mathscr{C}$.

Proposition 2.1.25.

- *The forgetful functor* $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathscr{C}$ *creates colimits.*
- *The forgetful functor* $U_{\mathrm{R}} : \mathbf{Alg}_{\mathrm{R}} \to \mathscr{C}$ *creates limits.*

Proof. The proof is analogous to 2.1.28. ☐

Corollary 2.1.26.

- *If $\mathscr{C}$ is cocomplete, then $\mathbf{Coalg}_{\mathrm{L}}$ is cocomplete.*
- *If $\mathscr{C}$ is complete, then $\mathbf{Alg}_{\mathrm{R}}$ is complete.*

In general, we will say that a category $\mathscr{A}$ is *over* a category $\mathscr{C}$ if there is a functor $A : \mathscr{A} \to \mathscr{C}$. We will say a functor $F : \mathscr{A} \to \mathscr{B}$ is *over* $\mathscr{C}$ if there is a functor $B : \mathscr{B} \to \mathscr{C}$ such that $BF = A$. We will use the notation $|A|$ to denote the collection of objects in the image of the functor $A : \mathscr{A} \to \mathscr{C}$. When the base category and the functor are clear from context, we will use $|\mathscr{A}|$ to denote the collection of objects in the image of A. With this notation, viewing $\mathbf{Coalg}_{\mathrm{L}}$ and $\mathbf{Alg}_{\mathrm{R}}$ as categories over $\mathscr{C}$ via the forgetful functors, $|\mathbf{Coalg}_{\mathrm{L}}|$ is the collection of L-coalgebras in $\mathscr{C}$ and $|\mathbf{Alg}_{\mathrm{R}}|$ is the collection of R-algebras in $\mathscr{C}$.

Remark 2.1.27. If $\mathscr{A}$ and $\mathscr{B}$ are categories over $\mathscr{C}$ and $F : \mathscr{A} \to \mathscr{B}$ is a functor over $\mathscr{C}$, then $|\mathscr{A}| \subseteq |\mathscr{B}|$. If in addition $G : \mathscr{B} \to \mathscr{A}$ is a functor over $\mathscr{C}$, then $|\mathscr{A}| = |\mathscr{B}|$.

A map of copointed endofunctors $\alpha : \mathrm{L} \to \mathrm{L}'$ on $\mathscr{C}$ induces a functor $\alpha_* : \mathbf{Coalg}_{\mathrm{L}} \to \mathbf{Coalg}_{\mathrm{L}'}$ over $\mathscr{C}$ which sends objects $\langle X, k\rangle$ to $\langle X, \alpha_X \circ k\rangle$ and sends morphisms $f : \langle X, k\rangle \to \langle Y, l\rangle$ to $f : \langle X, \alpha_X \circ k\rangle \to \langle Y, \alpha_Y \circ l\rangle$. A map of pointed endofunctors $\beta : \mathrm{R} \to \mathrm{R}'$ on $\mathscr{C}$ induces a functor $\beta^* : \mathbf{Alg}_{\mathrm{R}'} \to \mathbf{Alg}_{\mathrm{R}}$ over $\mathscr{C}$ which sends objects $\langle Y, s\rangle$ to $\langle Y, s \circ \beta_Y\rangle$ and morphisms $g : \langle Y, s\rangle \to \langle Z, t\rangle$ to $g : \langle Y, s \circ \beta_Y\rangle \to \langle Z, t \circ \beta_Z\rangle$.

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "f"] & \mathrm{L}X \arrow[r, "\alpha_X"] \arrow[d, "\mathrm{L}f"] & \mathrm{L}'X \arrow[d, "\mathrm{L}'f"] \\
Y \arrow[r, "l"] & \mathrm{L}Y \arrow[r, "\alpha_Y"] & \mathrm{L}'Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{R}Y \arrow[r, "\beta_Y"] \arrow[d, "\mathrm{R}g"] & \mathrm{R}'Y \arrow[r, "s"] \arrow[d, "\mathrm{R}'g"] & Y \arrow[d, "g"] \\
\mathrm{R}Z \arrow[r, "\beta_Z"] & \mathrm{R}'Z \arrow[r, "t"] & Z
\end{tikzcd}$$

By regarding a comonad $(\mathrm{L}, \varepsilon, \delta)$ on a category $\mathscr{C}$ as a copointed endofunctor, we of course still have the category $\mathbf{Coalg}_{\mathrm{L}}$ described in the previous section, but we also have the

Eilenberg-Moore category $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$. The objects in this category are again the pairs $\langle X, k\rangle$, where k is a map $X \to \mathrm{L}X$. But we now require the structure maps to satisfy two relations. Namely, we require that the following diagrams commute.

$$\begin{tikzcd}
 & X \arrow[dl, "id"'] \arrow[d, "k"] \\
X & \mathrm{L}X \arrow[l, "\varepsilon_X"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "k"] & \mathrm{L}X \arrow[d, "\delta_X"] \\
\mathrm{L}X \arrow[r, "\mathrm{L}k"] & \mathrm{LL}X
\end{tikzcd}$$

A morphism $\langle X, k\rangle \to \langle Y, l\rangle$ in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$ is a map $f : X \to Y$ in $\mathscr{C}$ making diagram (2.1) commute. Imposing the second condition on k amounts to the requirement that the coalgebra-structure map $k : X \to \mathrm{L}X$ of every object $\langle X, k\rangle$ in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$ is itself a map $k : \langle X, k\rangle \to \langle \mathrm{L}X, \delta_X\rangle$ in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$.

We now have that there is a forgetful functor $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}} \to \mathscr{C}$ and that $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$ is a full subcategory of $\mathbf{Coalg}_{\mathrm{L}}$ over $\mathscr{C}$. Since the map $\delta_X : \mathrm{L}X \to \mathrm{LL}X$ makes $\mathrm{L}X$ a coalgebra for the comonad L, every coalgebra X for the copointed endofunctor L is a retract of $\mathrm{L}X$, which is itself a coalgebra for the comonad L. Thus the retract closure of $|\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}|$ is equal to $|\mathbf{Coalg}_{\mathrm{L}}|$.

Dually, for any monad (R, η, μ), the Eilenberg-Moore category $\mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}}$ exists, there is a forgetful functor $\mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}} \to \mathscr{C}$, $\mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}}$ is a full subcategory of $\mathbf{Alg}_{\mathrm{R}}$ over $\mathscr{C}$, and the retract closure of $|\mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}}|$ is equal to $|\mathbf{Alg}_{\mathrm{R}}|$.

Even when L is a comonad, we will use the phrases "L-coalgebra" and "coalgebra of L" to refer to a coalgebra for the copointed endofunctor L. We will explicitly use the phrases "coalgebra for the comonad L", "comonad-coalgebra of L", or "L-comonad-coalgebra" whenever we want to specify the stronger notion of L-coalgebra. As for the term "L-coalgebra", "coalgebra for the comonad L" can refer to an object $\langle X, k\rangle$ in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$ or to an object X in $|\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}|$. We will use the same conventions for R-algebras.

Proposition 2.1.28.

- *The forgetful functor* $V_{\mathrm{L}} : \mathbf{Coalg}^{\mathrm{EM}}_{\mathrm{L}} \to \mathscr{C}$ *creates colimits and has a right adjoint* $\widetilde{\mathrm{L}} : \mathscr{C} \to \mathbf{Coalg}^{\mathrm{EM}}_{\mathrm{L}}$ *defined by sending objects* X *to* $\langle \mathrm{L}X, \delta_X \rangle$ *and sending morphisms* $f : X \to Y$ *to* $\mathrm{L}f : \langle \mathrm{L}X, \delta_X \rangle \to \langle \mathrm{L}Y, \delta_Y \rangle$*.*

- *The forgetful functor* $V_{\mathrm{R}} : \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}} \to \mathscr{C}$ *creates limits and has a left adjoint* $\widetilde{\mathrm{R}} : \mathscr{C} \to \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}}$ *defined by sending objects* X *to* $\langle \mathrm{R}X, \mu_X \rangle$ *and sending morphisms* $f : X \to Y$ *to* $\mathrm{R}f : \langle \mathrm{R}X, \mu_X \rangle \to \langle \mathrm{R}Y, \mu_Y \rangle$*.*

Proof. This is [Mac71, VI. §2, 1] along with an exercise in [Mac71, VI. §2]. □

Of course $\mathrm{L} = V_{\mathrm{L}}\widetilde{\mathrm{L}}$ and $\mathrm{R} = V_{\mathrm{R}}\widetilde{\mathrm{R}}$ in the above proposition.

Corollary 2.1.29.

- *If* $\mathscr{C}$ *is cocomplete, then* $\mathbf{Coalg}^{\mathrm{EM}}_{\mathrm{L}}$ *is cocomplete.*

- *If* $\mathscr{C}$ *is complete, then* $\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}}$ *is complete.*

A map of comonads $\alpha : \mathrm{L} \to \mathrm{L}'$ on $\mathscr{C}$ induces a functor $\alpha_* : \mathbf{Coalg}^{\mathrm{EM}}_{\mathrm{L}} \to \mathbf{Coalg}^{\mathrm{EM}}_{\mathrm{L}'}$ over $\mathscr{C}$ which sends objects $\langle X, k \rangle$ to $\langle X, \alpha_X \circ k \rangle$ and sends morphisms $f : \langle X, k \rangle \to \langle Y, l \rangle$ to $f : \langle X, \alpha_X \circ k \rangle \to \langle Y, \alpha_Y \circ l \rangle$. A map of monads $\beta : \mathrm{R} \to \mathrm{R}'$ on $\mathscr{C}$ induces a functor $\beta^* : \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}'} \to \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}}$ over $\mathscr{C}$ which sends objects $\langle Y, s \rangle$ to $\langle Y, s \circ \beta_Y \rangle$ and morphisms $g : \langle Y, s \rangle \to \langle Z, t \rangle$ to $g : \langle Y, s \circ \beta_Y \rangle \to \langle Z, t \circ \beta_Z \rangle$.

Proposition 2.1.30.

- *There is a bijective correspondence between maps of comonads* $\alpha : \mathrm{L} \to \mathrm{C}$ *on a category* $\mathscr{C}$ *and functors* $F : \mathbf{Coalg}^{\mathrm{EM}}_{\mathrm{L}} \to \mathbf{Coalg}^{\mathrm{EM}}_{\mathrm{C}}$ *over* $\mathscr{C}$*.*

- *There is a bijective correspondence between maps of monads* $\beta : \mathrm{R} \to \mathrm{F}$ *on* $\mathscr{C}$ *and functors* $G : \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{F}} \to \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}}$ *over* $\mathscr{C}$*.*

Proof. We will prove the statement about comonads. The proof for monads is dual. As we saw above, a map of comonads $\alpha : \mathrm{L} \to \mathrm{C}$ on $\mathscr{C}$ defines a functor $\alpha_* : \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}} \to \mathbf{Coalg}_{\mathrm{C}}^{\mathrm{EM}}$ over $\mathscr{C}$. Conversely, suppose $F : \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}} \to \mathbf{Coalg}_{\mathrm{C}}^{\mathrm{EM}}$ is a functor over $\mathscr{C}$. Then F sends an object $\langle X, k\rangle$ in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$ to an object $\langle X, \varphi^F_{\langle X,k\rangle}\rangle$ in $\mathbf{Coalg}_{\mathrm{C}}^{\mathrm{EM}}$. So $\varphi^F_{\langle X,k\rangle}$ is a map $X \to \mathrm{C}X$. The functor F sends a map $f : \langle X, k\rangle \to \langle Y, l\rangle$ to a map $\langle X, \varphi^F_{\langle X,k\rangle}\rangle \to \langle Y, \varphi^F_{\langle Y,l\rangle}\rangle$ in $\mathbf{Coalg}_{\mathrm{C}}^{\mathrm{EM}}$. Let $\beta^F : \mathrm{L} \to \mathrm{C}$ be the natural transformation given by $\beta^F_X = \mathrm{C}\varepsilon_X \circ \varphi^F_{\langle \mathrm{L}X, \delta_X\rangle}$.

We will show that the natural transformation β^F is a map of comonads. Indeed, the commutativity of the following diagram shows that β is a map of copointed endofunctors.

$$\begin{tikzcd}
\mathrm{L}X \arrow[r, "\varphi^F_{\langle \mathrm{L}X,\delta_X\rangle}"] \arrow[dr, "id"'] & \mathrm{CL}X \arrow[r, "\mathrm{C}\varepsilon_X"] \arrow[d, "\varepsilon_{\mathrm{L}X}"] & \mathrm{C}X \arrow[d, "\varepsilon_X"] \\
 & \mathrm{L}X \arrow[r, "\varepsilon_X"] & X
\end{tikzcd}$$

Since $\delta_X : \langle \mathrm{L}X, \delta_X\rangle \to \langle \mathrm{LL}X, \delta_{\mathrm{L}X}\rangle$ is a map in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$, the left diagram below commutes. Since $\langle \mathrm{L}X, \varphi^F_{\langle \mathrm{L}X,\delta_X\rangle}\rangle$ is an object in $\mathbf{Coalg}_{\mathrm{C}}^{\mathrm{EM}}$, the right diagram below commutes.

$$\begin{tikzcd}
\mathrm{L}X \arrow[r, "\delta_X"] \arrow[d, "\varphi^F_{\langle \mathrm{L}X,\delta_X\rangle}"'] & \mathrm{LL}X \arrow[d, "\varphi^F_{\langle \mathrm{LL}X,\delta_{\mathrm{L}X}\rangle}"] \\
\mathrm{CL}X \arrow[r, "\mathrm{C}\delta_X"] \arrow[dr, "id"'] & \mathrm{CLL}X \arrow[d, "\mathrm{C}\varepsilon_{\mathrm{L}X}"] \\
 & \mathrm{CL}X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{L}X \arrow[r, "\varphi^F_{\langle \mathrm{L}X,\delta_X\rangle}"] \arrow[d, "\varphi^F_{\langle \mathrm{L}X,\delta_X\rangle}"'] & \mathrm{CL}X \arrow[d, "\mathrm{C}\varphi^F_{\langle \mathrm{L}X,\delta_X\rangle}"] \\
\mathrm{CL}X \arrow[r, "\delta_{\mathrm{L}X}"] \arrow[d, "\mathrm{C}\varepsilon_X"'] & \mathrm{CCL}X \arrow[d, "\mathrm{CC}\varepsilon_X"] \\
\mathrm{C}X \arrow[r, "\delta_X"] & \mathrm{CC}X
\end{tikzcd}$$

It follows that β^F is a map of comonads.

When $\alpha : \mathrm{L} \to \mathrm{C}$ is a map of comonads and $F = \alpha_* : \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}} \to \mathbf{Coalg}_{\mathrm{C}}^{\mathrm{EM}}$, then $\varphi^F_{\langle X,k\rangle} = \alpha_X \circ k : X \to \mathrm{C}X$. So $\beta^{\alpha_*}_X = \mathrm{C}\varepsilon_X \circ \varphi^F_{\langle \mathrm{L}X,\delta_X\rangle} = \mathrm{C}\varepsilon_X \circ \alpha_{\mathrm{L}X} \circ \delta_X = \alpha_X \circ \mathrm{L}\varepsilon_X \circ \delta_X = \alpha_X$. Conversely, if $F : \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}} \to \mathbf{Coalg}_{\mathrm{C}}^{\mathrm{EM}}$ is a functor over $\mathscr{C}$, then $(\beta^F)_* : \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}} \to \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$ sends an object $\langle X, k\rangle$ in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$ to the object $\langle X, \beta^F_X \circ k\rangle$ in $\mathbf{Coalg}_{\mathrm{C}}^{\mathrm{EM}}$. The commutativity of the following diagram shows that $\beta^F_X \circ k = \varphi^F_{\langle X,k\rangle}$.

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "\varphi^F_{\langle X,k\rangle}"] & \mathrm{L}X \arrow[d, "\varphi^F_{\langle \mathrm{L}X,\delta_X\rangle}"] \arrow[rd, bend left, "\beta^F_X"] & \\
\mathrm{C}X \arrow[r, "\mathrm{C}k"] \arrow[rr, bend right, "id"'] & \mathrm{CL}X \arrow[r, "\mathrm{C}\varepsilon_X"] & \mathrm{C}X
\end{tikzcd}$$

So $\langle X, \beta^F_X \circ k\rangle = F\langle X, k\rangle$. So the correspondence is bijective. $\square$

2.1.6 Monadic Functors

We've shown that every adjunction

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend right, "F"']
\end{tikzcd} \tag{2.2}$$

defines a monad GF on $\mathscr{B}$ and that every monad R can be recovered from the adjunction $V_{\mathrm{R}} \vdash \widetilde{\mathrm{R}}$. The adjunctions $G \vdash F$ for which $\mathscr{A}$ already looks like a category of algebras for a monad are called *monadic* adjunctions.

Given the adjunction $F \dashv G$ of (2.2) with unit $\nu : I \to GF$ and counit $\xi : FG \to I$, there is a comparison functor $H_{GF} : \mathscr{A} \to \mathbf{Alg}^{\mathrm{EM}}_{GF}$ defined by sending an object A to the object $\langle GA, G\xi_A\rangle$ and sending a map $f : A_1 \to A_2$ to the map $Gf : \langle GA_1, G\xi_{A_1}\rangle \to \langle GA_2, G\xi_{A_2}\rangle$. The comparison functor is a functor over $\mathscr{A}$ because $V_{GF}H_{GF} = G$, where $V_{GF} : \mathbf{Alg}^{\mathrm{EM}}_{GF} \to \mathscr{B}$ is the forgetful functor.

Definition 2.1.31. The adjunction (2.2) is *monadic* if the comparison functor $H_{GF} : \mathscr{A} \to \mathbf{Alg}^{\mathrm{EM}}_{GF}$ is an equivalence of categories. A functor $G : \mathscr{A} \to \mathscr{B}$ is *monadic* if it is the right adjoint in a monadic adjunction.

There is a useful characterization of monadic functors. To state it, we need a definition.

Definition 2.1.32. A *split coequalizer* in a category $\mathscr{C}$ is a diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[l, bend right=60, "r"'] \arrow[r, "q"] & C \arrow[l, bend right=60, "s"']
\end{tikzcd}$$

such that $q \circ u = q \circ v$, $q \circ s = id$, $u \circ r = id$, and $v \circ r = s \circ q$.

We say that the diagram $A \overset{u}{\underset{v}{\rightrightarrows}} B$ has a split coequalizer in such a case. In the above situation, $q : B \to C$ is the coequalizer of u and v, and this colimit is absolute [Bor94b, 2.10.2].

Proposition 2.1.33 ([Bor94a, 4.4.4])**.** *A functor $G : \mathscr{A} \to \mathscr{B}$ is monadic if and only if the following conditions are satisfied.*

1. *G has a left adjoint $F : \mathscr{B} \to \mathscr{A}$.*

2. *G reflects isomorphisms.*

3. *If $u, v : A \to B$ are parallel maps in $\mathscr{A}$ such that the diagram*

$$\begin{tikzcd} GA \arrow[r, shift left, "Gu"] \arrow[r, shift right, "Gv"'] & GB \end{tikzcd}$$

   has a split coequalizer in $\mathscr{B}$, then the diagram $A \overset{u}{\underset{v}{\rightrightarrows}} B$ has a coequalizer in $\mathscr{A}$ that is preserved by G.

Using these conditions, it is easy to prove the following result.

Proposition 2.1.34. *Let T be a pointed endofunctor on a category $\mathscr{C}$. If the forgetful functor $U_{\mathrm{T}} : \mathbf{Alg}_{\mathrm{T}} \to \mathscr{C}$ has a left adjoint, then it is monadic.*

Proof. We just need to verify that U_{T} satisfies conditions (2) and (3) of 2.1.33. Suppose $f : \langle X, m \rangle \to \langle Y, n \rangle$ is a map in $\mathbf{Alg}_{\mathrm{T}}$ such that $f : X \to Y$ has an inverse $g : Y \to X$ in $\mathscr{C}$. Then $f \circ m \circ \mathrm{T}g = n \circ \mathrm{T}f \circ \mathrm{T}g = n = f \circ g \circ n$. So $m \circ \mathrm{T}g = g \circ n$, which means $g : \langle Y, n \rangle \to \langle X, m \rangle$ is a map in $\mathbf{Alg}_{\mathrm{T}}$.

Now, let $u, v : \langle X, m \rangle \to \langle Y, n \rangle$ be maps in $\mathbf{Alg}_{\mathrm{T}}$ and suppose

$$\begin{tikzcd} X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[l, bend right=60, "r"'] \arrow[r, "q"] & C \arrow[l, bend right=60, "s"'] \end{tikzcd}$$

is a split coequalizer in $\mathscr{C}$. Since split coequalizers are absolute colimits,

$$\begin{tikzcd} \mathrm{T}X \arrow[r, shift left, "\mathrm{T}u"] \arrow[r, shift right, "\mathrm{T}v"'] & \mathrm{T}Y \arrow[r, "q"] & \mathrm{T}C \end{tikzcd}$$

is a coequalizer diagram in $\mathscr{C}$. Since $q \circ n \circ \mathrm{T}u = q \circ u \circ m = q \circ v \circ m = q \circ n \circ \mathrm{T}v$, there is a unique map $l : \mathrm{T}C \to C$ such that $l \circ \mathrm{T}q = q \circ n$. The fact that $X \overset{u}{\underset{v}{\rightrightarrows}} Y \xrightarrow{q} C$ is a coequalizer diagram in $\mathscr{C}$ then implies that $l \circ \eta_C = id$, where $\eta : \mathrm{Id} \to \mathrm{T}$ is the unit map of T. Thus $q : \langle Y, n \rangle \to \langle C, l \rangle$ is a map in $\mathbf{Alg}_{\mathrm{T}}$. It is easy to check that $\langle X, m \rangle \overset{u}{\underset{v}{\rightrightarrows}} \langle Y, n \rangle \xrightarrow{q} \langle C, l \rangle$ is a coequalizer diagram in $\mathbf{Alg}_{\mathrm{T}}$. □

An important special case of a monadic functor is the inclusion of a reflective subcategory. We take our *reflective subcategories* to be full subcategories whose inclusion functors have a left adjoint. We call the left adjoint to the inclusion of a reflective subcategory $\mathscr{A} \hookrightarrow \mathscr{B}$ the *reflection* of $\mathscr{B}$ into $\mathscr{A}$. The monad defined by a reflective subcategory adjunction is an idempotent monad. Conversely, the Eilenberg-Moore category of an idempotent monad on a category $\mathscr{C}$ is always equivalent to a reflective subcategory of $\mathscr{C}$.

2.2 Free Objects

Let $\mathscr{A}$ and $\mathscr{B}$ be categories, let $U : \mathscr{A} \to \mathscr{B}$ be a functor, and let X be an object in $\mathscr{B}$. A *reflection* of X into $\mathscr{A}$ (or along U) is an object Y in $\mathscr{A}$ with a map $u : X \to UY$ such that for every object A in $\mathscr{A}$ and every map $v : X \to UA$, there is a unique map $f : Y \to A$ in $\mathscr{A}$ making the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "v"'] & UY \arrow[d, "Uf"] \\ & UA \end{tikzcd}$$

A reflection of an object X is unique up to unique isomorphism.

We will often call an object Y satisfying the above conditions the *free $\mathscr{A}$-object* on X. In many applications, U will be a functor that forgets structure, so we can think of Y as the object we get by freely adding the structure of an $\mathscr{A}$-object to the less-structured $\mathscr{B}$-object X.

Remark 2.2.1. When R is a pointed endofunctor on a category $\mathscr{C}$, we will specifically refer to a reflection of on object $X \in \mathrm{ob}(\mathscr{C})$ in $\mathbf{Alg}_{\mathrm{R}}$ as the *free* R-*algebra* on X. When R is a monad, the reflection of X into $\mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}}$ is equal to $\langle \mathrm{R}X, \mu_X \rangle$. So every object in the image of $\widetilde{\mathrm{R}} : \mathscr{C} \to \mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}}$ is a free R-monad-algebra. Continuing our convention of using the same terms for structures and properties, we will sometimes call an object $\mathrm{R}X$ in $\mathscr{C}$ a free R-monad-algebra.

By dualizing the above definitions, we get definitions for coreflections of objects, cofree objects, and cofree coalgebras for comonads.

Unsurprisingly, reflections in $\mathscr{A}$ have a close relationship to the existence of a left adjoint for U.

Proposition 2.2.2. *A functor $U : \mathscr{A} \to \mathscr{B}$ between categories has a left adjoint if and only if every object B in $\mathscr{B}$ has a reflection in $\mathscr{A}$.*

Proof. If a left adjoint $F : \mathscr{B} \to \mathscr{A}$ to U exists, then for each object X in $\mathscr{B}$, the unit map $\eta_X : X \to UFX$ of the adjunction at X satisfies the appropriate universal property. So FX is a reflection of X. Conversely, if every object in $\mathscr{B}$ has a reflection, we can define a function $F : ob(\mathscr{B}) \to ob(\mathscr{A})$ that sends each object to its reflection. For each object X in $\mathscr{B}$, let $\eta_X : X \to UFX$ be the universal map of the reflection. If $f : X \to Y$ is a map in $\mathscr{B}$, then the universal property of η_X guarantees the existence of a unique map $Ff : FX \to FY$ in $\mathscr{A}$

that makes the following diagram commute.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"] & UFX \arrow[d, "UFf"] \\
Y \arrow[r, "\eta_Y"] & UFY
\end{tikzcd}$$

So in this way, we define a functor $F : \mathscr{B} \to \mathscr{A}$. By construction, F satisfies the universal property of a left adjoint for U. $\square$

So in ideal conditions, we can construct a free $\mathscr{A}$-object on a $\mathscr{B}$-object by constructing a left adjoint for U. When $\mathscr{A}$ and $\mathscr{B}$ are locally small and $\mathscr{A}$ is complete, the adjoint functor theorems can be used for this purpose.

2.2.1 Free and Algebraically Free Monads

Many of the results in this section appear without detailed proof in [Kel80].

Let $\mathbf{Mnd}(\mathscr{C})$ be the category whose objects are monads on $\mathscr{C}$ and whose morphisms are maps of monads. The *free monad* on a pointed endofunctor (T, τ) is the reflection of (T, τ) along the functor $U : \mathbf{Mnd}(\mathscr{C}) \to \mathbf{pEnd}(\mathscr{C})$ that forgets the structure of the multiplication map on each monad. The remainder of this chapter is devoted characterizing endofunctors that have reflections in $\mathbf{Mnd}(\mathscr{C})$.

Since a left adjoint to $U : \mathbf{Mnd}(\mathscr{C}) \to \mathbf{pEnd}(\mathscr{C})$ rarely, if ever, exists, we cannot construct free monads on pointed endofunctors by constructing a left adjoint to U. Instead, in the main result of this section (2.2.8), we will show that to construct the free monad on a pointed endofunctor (T, τ), it suffices to construct a left adjoint to the forgetful functor $U_{\mathrm{T}} : \mathbf{Alg}_{\mathrm{T}} \to \mathscr{C}$. In section 2.4, we will discuss conditions on the endofunctor T that guarantee such a left adjoint exists. In section 2.4.4, we restrict to subcategories of $\mathbf{Mnd}(\mathscr{C})$ and $\mathbf{pEnd}(\mathscr{C})$ such that a left adjoint to the restriction of U does exist.

Remark 2.2.3. When $\mathscr{C}$ is a category with coproducts, the forgetful functor $V : \mathbf{pEnd}(\mathscr{C}) \to \mathbf{End}(\mathscr{C})$ from the category of pointed endofunctors to the category of endofunctors on $\mathscr{C}$ has a left adjoint $F : \mathbf{End}(\mathscr{C}) \to \mathbf{pEnd}(\mathscr{C})$. The free pointed endofunctor on an endofunctor $\mathrm{T} : \mathscr{C} \to \mathscr{C}$ is the objectwise coproduct $X \coprod \mathrm{Id}_{\mathscr{C}}$. Therefore, to construct free monads on endofunctors, it suffices to construct them on pointed endofunctors.

An important special case of a free monad on a pointed endofunctor is that of an algebraically-free monad. When $\theta : \mathrm{T} \to \mathrm{R}$ is a map from a pointed endofunctor on $\mathscr{C}$ to a monad on $\mathscr{C}$, there is a functor $\theta^\dagger : \mathbf{Alg}_\mathrm{R}^{\mathrm{EM}} \to \mathbf{Alg}_\mathrm{T}$ over $\mathscr{C}$ defined as the subcategory inclusion functor $\mathbf{Alg}_\mathrm{R}^{\mathrm{EM}} \hookrightarrow \mathbf{Alg}_\mathrm{R}$ followed by $\theta^* : \mathbf{Alg}_\mathrm{R} \to \mathbf{Alg}_\mathrm{T}$.

Definition 2.2.4. Let $\mathscr{C}$ be a category and let (T, τ) be a pointed endofunctor on $\mathscr{C}$. An *algebraically-free monad* on (T, τ) is a pair (R, θ) of a monad (R, η, μ) on $\mathscr{C}$ and a map $\theta : \mathrm{T} \to \mathrm{R}$ of pointed endofunctors such that the functor $\theta^\dagger : \mathbf{Alg}_\mathrm{R}^{\mathrm{EM}} \to \mathbf{Alg}_\mathrm{T}$ is an isomorphism of categories.

Surprisingly, to show (R, θ) is an algebraically-free monad on (T, τ), it actually suffices to show $\theta^\dagger : \mathbf{Alg}_\mathrm{R}^{\mathrm{EM}} \to \mathbf{Alg}_\mathrm{T}$ is an equivalence of categories.

Lemma 2.2.5. *If $\theta : \mathrm{T} \to \mathrm{R}$ is a map of pointed endofunctors on a category $\mathscr{C}$ to a monad R and if the functor $\theta^\dagger : \mathbf{Alg}_\mathrm{R}^{\mathrm{EM}} \to \mathbf{Alg}_\mathrm{T}$ is an equivalence of categories, then $\theta^\dagger$ is an isomorphism of categories.*

Proof. Since $\theta^\dagger$ is an equivalence, it must be full and faithful. So it suffices to show $\theta^\dagger$ is bijective on objects. Let $K : \mathbf{Alg}_\mathrm{T} \to \mathbf{Alg}_\mathrm{R}^{\mathrm{EM}}$ be an up-to-natural-isomorphism inverse for $\theta^\dagger$, and let $\alpha : \theta^\dagger K \to \mathrm{Id}$ and $\beta : \mathrm{Id} \to K\theta^\dagger$ be the natural isomorphisms. Then K is also full and faithful.

Let $\langle X, m\rangle$ be an object in $\mathbf{Alg}_{\mathrm{T}}$. Let $\langle Y, n\rangle = K\langle X, m\rangle$. Then $\theta^{\dagger}K\langle X, m\rangle = \langle Y, n\circ\theta_Y\rangle$ and $\alpha_{\langle X,m\rangle} : \langle Y, n\circ\theta_Y\rangle \to \langle X, m\rangle$ is an isomorphism in $\mathbf{Alg}_{\mathrm{T}}$. Let $p = \alpha_{\langle X,m\rangle}\circ n\circ \mathrm{R}\alpha^{-1}_{\langle X,m\rangle} : \mathrm{R}X \to X$. Then $p\circ\theta_X\circ \mathrm{T}\alpha_{\langle X,m\rangle} = \alpha_{\langle X,m\rangle}\circ n\circ\theta_Y = m\circ T\alpha_{\langle X,m\rangle}$. Thus $m = p\circ\theta_X$. So $\theta^{\dagger}\langle X, p\rangle = \langle X, m\rangle$. Thus $\theta^{\dagger}$ is surjective on objects.

$$\begin{tikzcd}
\mathrm{T}Y \arrow[r, "\theta_Y"] \arrow[d, "\mathrm{T}\alpha_{\langle X,m\rangle}"', "\cong"] & \mathrm{R}Y \arrow[r, "n"] \arrow[d, "\cong"', "\mathrm{R}\alpha_{\langle X,m\rangle}"] & Y \arrow[d, "\cong"', "\alpha_{\langle X,m\rangle}"] \\
\mathrm{T}X \arrow[r, "\theta_X"] \arrow[rr, bend right, "m"'] & \mathrm{R}X \arrow[r, dashed, "p"] & X
\end{tikzcd}$$

We note that if $u : \langle S, q\rangle \to \langle S, q\rangle$ is a map in $\mathbf{Alg}_{\mathrm{T}}$ such that Ku is the identity map on $K\langle S, q\rangle$, then $Ku = Kid$. So the faithfulness of K implies u is the identity map. This observation will help us prove that $\theta^{\dagger}$ is injective. Now, let $\langle X, m\rangle$ and $\langle Y, n\rangle$ be an objects in $\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}}$ such that $\theta^{\dagger}\langle X, m\rangle = \theta^{\dagger}\langle Y, n\rangle$. So $X = Y$ and $m\circ\theta_X = n\circ\theta_Y$ as objects and maps in $\mathscr{C}$. Let $\langle Z, p\rangle = K\langle X, m\circ\theta_X\rangle$. Then there is are isomorphisms $\beta_{\langle X,m\rangle} : \langle X, m\rangle \to \langle Z, p\rangle$ and $\beta_{\langle X,n\rangle} : \langle X, n\rangle \to \langle Z, p\rangle$. Let $u = \beta^{-1}_{\langle X,n\rangle}\circ\beta_{\langle X,m\rangle}$. So $u : \langle X, m\rangle \to \langle X, n\rangle$ is an isomorphism in $\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}}$. The naturality of β and the definition of u now imply that the following diagram commutes.

$$\begin{tikzcd}
\langle X, m\rangle \arrow[r, "u"] \arrow[d, "\beta_{\langle X,m\rangle}"'] & \langle X, n\rangle \arrow[dl, "\beta_{\langle X,n\rangle}"'] \arrow[d, "\beta_{\langle X,n\rangle}"] \\
K\theta^{\dagger}\langle X, m\rangle \arrow[r, "K\theta^{\dagger}u"] & K\theta^{\dagger}\langle X, n\rangle
\end{tikzcd}$$

It follows that $K\theta^{\dagger}u$ is the identity map on $\langle Z, p\rangle$. So, by our comments at the beginning of this paragraph, $\theta^{\dagger}u$ is the identity map on $\langle X, m\circ\theta_X\rangle$. But this means $u : X \to X$ is the identity map in $\mathscr{C}$. So $\langle X, m\rangle = \langle X, n\rangle$ in $\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}}$. ☐

Proposition 2.2.6. *An algebraically-free monad on a pointed endofunctor* (T, τ) *is a free monad on* (T, τ)*.*

Proof. If (R, θ) is an algebraically-free monad on T, then for each map of pointed endofunctors $\psi : \mathrm{T} \to \mathrm{R}'$ to a monad $(\mathrm{R}', \eta', \mu')$, there is a unique functor $K : \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}'} \to \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}}$ over $\mathscr{C}^{\mathbf{2}}$

making the following diagram commute.

$$\begin{tikzcd}
\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}'} \arrow[r, hook] \arrow[d, dashed, "\exists !"', "K"] & \mathbf{Alg}_{\mathrm{R}'} \arrow[dr, "\psi^*"] & \\
\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}} \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & \mathbf{Alg}_{\mathrm{R}} \arrow[r, "\theta^*"] & \mathbf{Alg}_{\mathrm{T}}
\end{tikzcd}$$

By 2.1.30, there is a unique map of monads $\rho : \mathrm{R} \to \mathrm{R}'$ such that $\rho^* = K$.

So we have two maps $\psi : \mathrm{T} \to \mathrm{R}'$ and $\rho \circ \theta : \mathrm{T} \to \mathrm{R}'$ of pointed endofunctors from T to R'. Since the functors $\psi^* : \mathbf{Alg}_{\mathrm{R}'} \to \mathbf{Alg}_{\mathrm{T}}$ and $(\rho \circ \theta)^* : \mathbf{Alg}_{\mathrm{R}'} \to \mathbf{Alg}_{\mathrm{T}}$ agree on $\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}'}$ and $\langle \mathrm{R}'X, \mu'_X \rangle$ is an object in $\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}'}$ for each object X, $\mu'_X \circ \psi_{\mathrm{R}'X} = \mu'_X \circ \rho_{\mathrm{R}'X} \circ \theta_{\mathrm{R}'X}$.

$$\begin{tikzcd}
\mathrm{T}X \arrow[r, "T\eta'_X"] \arrow[d, shift right, "\rho_X \circ \theta_X"'] \arrow[d, shift left, "\psi_X"] & \mathrm{T}\mathrm{R}'X \arrow[dr] \arrow[d, shift right, "\rho_{\mathrm{R}'X} \circ \theta_{\mathrm{R}'X}"'] \arrow[d, shift left, "\psi_{\mathrm{R}'X}"] & \\
\mathrm{R}'X \arrow[r, "\mathrm{R}'\eta'_X"] \arrow[rr, bend right, "id"'] & \mathrm{R}'\mathrm{R}'X \arrow[r, "\mu'_X"] & \mathrm{R}'X
\end{tikzcd}$$

So

$$\psi_X = \mu'_X \circ \mathrm{R}'\eta'_X \circ \psi_X = \mu'_X \circ \psi_{\mathrm{R}'X} \circ T\eta'_X = \mu'_X \circ \rho_{\mathrm{R}'X} \circ \theta_{\mathrm{R}'X} \circ T\eta'_X$$

$$= \mu'_X \circ \mathrm{R}'\eta'_X \circ \rho_X \circ \theta_X = \rho_X \circ \theta_X .$$

Thus $\psi = \rho \circ \theta$ and $\rho : \mathrm{R} \to \mathrm{R}'$ is the unique map of monads for which this equality holds. □

Frequently, the converse to 2.2.6 is also true.

Proposition 2.2.7 ([Kel80, 22.4])**.** *If $\mathscr{C}$ is a locally small, complete category, then a free monad* R *on a pointed endofunctor* T *is an algebraically-free monad.*

Proposition 2.2.8. *If (T, τ) is a pointed endofunctor on a category $\mathscr{C}$ and the forgetful functor $U_{\mathrm{T}} : \mathbf{Alg}_{\mathrm{T}} \to \mathscr{C}$ has a left adjoint $F : \mathscr{C} \to \mathbf{Alg}_{\mathrm{T}}$, then $U_{\mathrm{T}}F$ is an algebraically-free monad on (T, τ) whose unit and multiplication maps are determined by the adjunction, as in 2.1.24.*

Proof. Let $\mathrm{R} = U_{\mathrm{T}}F$ with unit map $\nu : \mathrm{Id} \to U_{\mathrm{T}}F$ and counit map $\xi : FU_{\mathrm{T}} \to Id$. There is a natural transformation $\psi : \mathrm{T}U_{\mathrm{T}} \to U_{\mathrm{T}}$ that is defined on each object $\langle X, m\rangle$ by $\psi_{\langle X,m\rangle} = m : \mathrm{T}X \to X$. Then the natural transformation

$$\mathrm{T} \xrightarrow{\mathrm{T}\nu} \mathrm{TR} = \mathrm{T}U_{\mathrm{T}}F \xrightarrow{\psi F} U_{\mathrm{T}}F = \mathrm{R}$$

is a map of pointed endofunctors. Let $\theta = \psi F \circ \mathrm{T}\nu$.

By 2.1.34, U_{T} is a monadic functor. So there is an equivalence of categories $H : \mathbf{Alg}_{\mathrm{T}} \to \mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}}$ that sends a T-algebra $\langle A, m\rangle$ to $\langle A, U_{\mathrm{T}}\xi_{\langle A,m\rangle}\rangle$. Since the following diagram commutes, $\theta^{\dagger}H = id$.

$$\begin{tikzcd}
\mathrm{T}U_{\mathrm{T}}\langle A,m\rangle \arrow[r, "\mathrm{T}\nu_{U_{\mathrm{T}}\langle A,m\rangle}"] \arrow[d, "m"] \arrow[dr, "id"] & \mathrm{T}U_{\mathrm{T}}FU_{\mathrm{T}}\langle A,m\rangle \arrow[r, "\psi_{FU_{\mathrm{T}}\langle A,m\rangle}"] \arrow[d, "\mathrm{T}U_{\mathrm{T}}\xi_{\langle A,m\rangle}"] & U_{\mathrm{T}}FU_{\mathrm{T}}\langle A,m\rangle \arrow[d, "U_{\mathrm{T}}\xi_{\langle A,m\rangle}"] \\
A \arrow[rr, bend right, "id"'] & \mathrm{T}U_{\mathrm{T}}\langle A,m\rangle \arrow[r, "\psi_{\langle A,m\rangle}"] & U_{\mathrm{T}}\langle A,m\rangle = A
\end{tikzcd}$$

Let $\widehat{H} : \mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}} \to \mathbf{Alg}_{\mathrm{T}}$ be an up-to-natural-isomorphism inverse for H. Then $H\theta^{\dagger} \cong H\theta^{\dagger}H\widehat{H} = H\widehat{H} \cong \mathrm{Id}$. So $\theta^{\dagger}$ is an equivalence of categories. By 2.2.5, $\theta^{\dagger}$ is an isomorphism and (R, θ) is the algebraically-free monad on (T, τ). □

We note that a consequence of the above proof is that the map $\psi F \circ \mathrm{T}\nu : \mathrm{T} \to \mathrm{R}$ is the universal map of the free monad.

By 2.2.2, to construct a left adjoint to the forgetful functor $U_{\mathrm{T}} : \mathbf{Alg}_{\mathrm{T}} \to \mathscr{C}$, it suffices to show that every object in $\mathscr{C}$ has a reflection along U_{T}. So the free monad on a pointed endofunctor T exists if and only if the free T-algebra exists on every object in $\mathscr{C}$.

2.3 Free Algebra and Monad Sequences

In the previous section, we saw that a free monad on a pointed endofunctor (T, τ) exists when the forgetful functor $U_{\mathrm{T}} : \mathbf{Alg}_{\mathrm{T}} \to \mathscr{C}$ has a left adjoint. In this section, we will define a *free monad sequence* for a pointed endofunctor (T, τ) on a cocomplete category $\mathscr{C}$. We will show that when the free monad sequence is *weakly convergent*, every object in $\mathscr{C}$ has a reflection in $\mathbf{Alg}_{\mathrm{T}}$ and thus U_{T} has a left adjoint. The weak convergence of the free monad sequence gives us more than just existence of the free monad on (T, τ) however. We actually get that the free monad sequence weakly converges to the free monad on (T, τ). So the weak convergence of the free monad sequence gives us the *constructive existence* of a free monad on (T, τ).

The evaluation of the free monad sequence for (T, τ) on an object A is the *free* T*-algebra sequence* for A. So the free monad sequence for (T, τ) converges objectwise if and only if the free T-algebra sequence on each object A in $\mathscr{C}$ converges. We will prove in 2.3.18 that the objectwise convergence of the free monad sequence for (T, τ) implies its weakly convergence. Proving this result takes some work. First we prove in 2.3.5 that for a well-pointed endofunctor (S, σ), an object A has a reflection in $\mathbf{Alg}_{\mathrm{S}}$ if the free S-algebra sequence for A converges. Then in section 2.3.2 we show that there is a well-pointed endofunctor (S, σ) on the category $\mathrm{T} \downarrow \mathscr{C}$ and that we can translate between free T-algebra sequences in $\mathscr{C}$ and free S-algebra sequences in $\mathrm{T} \downarrow \mathscr{C}$. This translation preserves the property of convergence. Furthermore, the existence of the reflection of an object in $\mathscr{C}$ into $\mathbf{Alg}_{\mathrm{T}}$ follows from the existence of the reflection of a related object in $\mathrm{T} \downarrow \mathscr{C}$ into $\mathbf{Alg}_{\mathrm{S}}$. Putting this all together, an object A in $\mathscr{C}$ has a reflection in $\mathbf{Alg}_{\mathrm{T}}$ if the free T-algebra sequence for A converges. This is exactly what we need to prove that the objectwise convergence of the free monad sequence for (T, τ) implies its weak convergence.

Sections 2.3.1 and 2.3.2 work up to our main results in theorems 2.3.18 and 2.3.22, where we prove that the objectwise convergence of the free monad sequence on (T, τ) implies the weak convergence of the free monad sequence and that the weak convergence of the free monad sequence implies the constructive existence of the free monad on (T,τ). In section 2.4, we will describe a smallness condition on the endofunctor T that ensures the free monad sequence for (T,τ) converges objectwise.

2.3.1 Free Algebra Sequences for Well-Pointed Endofunctors

We start this section by defining *well-pointed endofunctors*. We then define the *free* S-*algebra sequence* on an object in a category $\mathscr{C}$ for a well-pointed endofunctor (S,σ) on $\mathscr{C}$. We conclude by showing that an object in $\mathscr{C}$ has a reflection in $\mathbf{Alg}_{\mathrm{S}}$ when the free S-algebra sequence on the object converges.

Definition 2.3.1. A pointed endofunctor (S,σ) on a category $\mathscr{C}$ is *well-pointed* if the maps $\mathrm{S}\sigma : \mathrm{S} \to \mathrm{SS}$ and $\sigma\mathrm{S} : \mathrm{S} \to \mathrm{SS}$ are equal.

If a well-pointed endofunctor (S,σ) extends to a monad (S,σ,μ), then (S,σ,μ) is an idempotent monad and the maps $\sigma\mathrm{S} : \mathrm{S} \to \mathrm{SS}$ and $\mathrm{S}\sigma : \mathrm{S} \to \mathrm{SS}$ are natural isomorphisms [Bor94a, 4.2.3]. When (S,σ) does not have a monad structure, we don't know that $\sigma\mathrm{S}$ and $\mathrm{S}\sigma$ are natural isomorphisms. We can however still prove the following result.

Lemma 2.3.2 ([Kel80, 5.2])**.** *Let* (S,σ) *be a well-pointed endofunctor on a category* $\mathscr{C}$*. For every* S*-algebra* $\langle X,m\rangle$*, the maps* $m : \mathrm{S}X \to X$ *and* $\sigma_X : X \to \mathrm{S}X$ *are isomorphisms. Therefore* $m = \sigma_X^{-1}$ *is the unique* S*-algebra structure map for* X*.*

Proof. We know $m \circ \sigma_X = id_X$. Since S is well-pointed, $id_{\mathrm{S}X} = \mathrm{S}m \circ \mathrm{S}\sigma_X = \mathrm{S}m \circ \sigma_{\mathrm{S}X} = \sigma_X \circ m$. So σ_X is the inverse of m. □

Lemma 2.3.3. *If* (S,σ) *is a well-pointed endofunctor on a category* $\mathscr{C}$*, then the forgetful functor* $U_{\mathrm{S}} : \mathbf{Alg}_{\mathrm{S}} \to \mathscr{C}$ *is the inclusion of a full subcategory.*

Proof. By 2.3.2, if X is an S-algebra, then its S-algebra structure map $\mathrm{S}X \to X$ is unique. So U_{S} is the inclusion of a subcategory. Suppose $\langle X, m\rangle$ and $\langle Y, n\rangle$ are objects in $\mathbf{Alg}_{\mathrm{S}}$ and $f : X \to Y$ is a map in $\mathscr{C}$. Then, by the naturality of σ, $\mathrm{S}f \circ \sigma_X = \sigma_Y \circ f$. So $\sigma_Y^{-1} \circ \mathrm{S}f = f \circ \sigma_X^{-1}$. But, since m and n are isomorphisms, $m = \sigma_X^{-1}$ and $n = \sigma_Y^{-1}$. So f is a map $\langle X, m\rangle \to \langle Y, n\rangle$ in $\mathbf{Alg}_{\mathrm{S}}$. □

Let $\mathscr{C}$ be a cocomplete category and let (S, σ) be a well-pointed endofunctor on $\mathscr{C}$. We recall from 2.1.20 that $\mathbf{pEnd}(\mathscr{C})$ is closed under connected colimits. The *free monad sequence* for (S, σ) is the cocontinuous transfinite sequence $\mathrm{S}^\bullet : \mathbf{Ord} \to \mathbf{pEnd}(\mathscr{C})$ given by the following inductive definition. We will use the notation $(\mathrm{S}^\alpha, \sigma^\alpha) = \mathrm{S}^\bullet(\alpha)$ on objects of $\mathbf{Ord}$ and $\mathrm{S}_\alpha^\beta = \mathrm{S}^\bullet(\alpha \to \beta)$ on the maps of $\mathbf{Ord}$. We set S^0 equal to the identity functor $\mathrm{Id} : \mathscr{C} \to \mathscr{C}$, S^1 equal to S and S_0^1 equal to $\sigma : \mathrm{Id} \to \mathrm{S}$. If the pointed endofunctor $\mathrm{S}^\alpha : \mathscr{C} \to \mathscr{C}$ is defined on an ordinal α, then $\mathrm{S}^{\alpha+1} := \mathrm{S}\mathrm{S}^\alpha$ and we have a map $\mathrm{S}_\alpha^{\alpha+1} := \sigma\mathrm{S}^\alpha : \mathrm{S}^\alpha \to \mathrm{S}\mathrm{S}^\alpha$. If γ is a limit ordinal and maps $\mathrm{S}_\alpha^{\alpha+1} : \mathrm{S}^\alpha \to \mathrm{S}^{\alpha+1}$ are defined for all $\alpha < \gamma$, then $\mathrm{S}^\gamma := \mathrm{colim}_{\alpha<\gamma} \mathrm{S}^\alpha$ and we take S_α^γ to be the inclusion maps $\mathrm{S}^\alpha \to \mathrm{S}^\gamma$ of the colimiting cocone. For each ordinal α, the unit map $\sigma^\alpha : \mathrm{Id} \to \mathrm{S}^\alpha$ is the map $\mathrm{S}_0^\alpha : \mathrm{S}^0 \to \mathrm{S}^\alpha$.

$$\begin{tikzcd}
\mathrm{Id} \arrow[r, "\sigma"] \arrow[rrr, bend right, "\mathrm{S}_0^\alpha"'] & \mathrm{S} \arrow[r] & \cdots \arrow[r] & \mathrm{S}^\alpha \arrow[r, "\sigma\mathrm{S}^\alpha"] \arrow[rrr, bend right, "\mathrm{S}_\alpha^\gamma"'] & \mathrm{S}\mathrm{S}^\alpha \arrow[r] \arrow[rr, bend left, "\mathrm{S}_{\alpha+1}^\gamma"] & \cdots \arrow[r] & \mathrm{colim}_{\alpha<\gamma} \mathrm{S}^\alpha \arrow[r] & \cdots
\end{tikzcd}$$

Definition 2.3.4. Let α be an ordinal. A transfinite sequence $F : \mathbf{Ord} \to \mathscr{C}$ *converges at* α if for every ordinal $\beta > \alpha$, $F(\alpha \to \beta) : F(\alpha) \to F(\beta)$ is an isomorphism. We will just say F *converges* if it converges at α for some ordinal α.

The free monad sequence for a pointed endofunctor rarely converges, but, under mild assumptions, we will be able to show that it converges objectwise. We have a special name for the free monad sequence evaluated on an object. On an object X in $\mathscr{C}$, we will use the notation $\mathrm{S}^\bullet X$ for the transfinite sequence that sends ordinals α to $\mathrm{S}(\alpha)(X) = \mathrm{S}^\alpha X$ and sends maps $\alpha \to \beta$ to $\mathrm{S}(\alpha \to \beta)_X = \mathrm{S}^{\beta}_{\alpha X} : \mathrm{S}^\alpha X \to \mathrm{S}^\beta X$. The functor $\mathrm{S}^\bullet X : \mathbf{Ord} \to \mathscr{C}$ is the *free* S-*algebra sequence for* X.

We will show in section 2.4.2 that, under a smallness condition on S, the free S-algebra sequence $\mathrm{S}^\bullet X$ converges for each X in $\mathscr{C}$. The convergence of the free S-algebra sequence on an object X implies the existence of a reflection of X in $\mathbf{Alg}_{\mathrm{S}}$. In fact, the object the free S-algebra sequence converges to is the image of the reflection under U_{S}.

Proposition 2.3.5 ([Kel80, 5.3])**.** *If the free* S*-algebra sequence* $\mathrm{S}^\bullet X$ *converges at the ordinal* β*, then* $\left\langle \mathrm{S}^\beta X, \left(\mathrm{S}^{\beta+1}_{\beta\ \ X}\right)^{-1} \right\rangle$ *is the reflection of* X *in* $\mathbf{Alg}_{\mathrm{S}}$ *and* $\mathrm{S}^{\beta}_{0X} : X \to \mathrm{S}^\beta X$ *is the universal map of the reflection.*

Proof. Suppose Y is an S-algebra and $f : X \to Y$ is a map in $\mathscr{C}$. Then, by 2.3.2, $\sigma_Y : Y \to \mathrm{S}Y$ is an isomorphism. So $\sigma_Y^{-1} \circ \mathrm{S}f : \mathrm{S}X \to Y$ is a map such that $\sigma_Y^{-1} \circ \mathrm{S}f \circ \sigma_X = f$. If $g : \mathrm{S}X \to Y$ is another map such that $g \circ \sigma_X = f$, then the well-pointedness of S implies $\sigma_Y \circ g = \mathrm{S}g \circ \sigma_{\mathrm{S}X} = \mathrm{S}g \circ \mathrm{S}\sigma_X = \mathrm{S}f$. Since $\sigma_Y \circ \sigma_Y^{-1} \circ \mathrm{S}f = \mathrm{S}f = \sigma_Y \circ g$ and σ_Y is an isomorphism, $\sigma_Y^{-1} \circ \mathrm{S}f = g$. So $\sigma_Y^{-1} \circ \mathrm{S}f$ is the unique map $\mathrm{S}X \to Y$ such that $\sigma_Y^{-1} \circ \mathrm{S}f \circ \sigma_X = f$.

It follows by induction that for every α, the map $\mathrm{S}^{\alpha}_{0Y} : Y \to \mathrm{S}^\alpha Y$ is an isomorphism and $\mathrm{S}^{\alpha-1}_{0Y} \circ \mathrm{S}^\alpha f$ is the unique map $\mathrm{S}^\alpha X \to Y$ such that $\mathrm{S}^{\alpha-1}_{0Y} \circ \mathrm{S}^\alpha f \circ \mathrm{S}^{\alpha}_{0X} = f$. In particular, this holds for $\alpha = \beta$. Since $\mathrm{S}^\beta X$ and Y can both be uniquely identified with objects in the full subcategory $\mathbf{Alg}_{\mathrm{S}}$ of $\mathscr{C}$, $(\mathrm{S}^{\beta}_{0Y})^{-1} \circ \mathrm{S}^\beta f$ is the unique map of S-algebras such that $(\mathrm{S}^{\beta}_{0Y})^{-1} \circ \mathrm{S}^\beta f \circ \mathrm{S}^{\beta}_{0X} = f$. □

2.3.2 Construction of a Well-Pointed Endofunctor

Let (T, τ) be a pointed endofunctor on a cocomplete category $\mathscr{C}$. In section 2.3.3, we will define the *free* T*-algebra sequence* on an object in $\mathscr{C}$. Showing that an object A in $\mathscr{C}$

has a reflection in $\mathbf{Alg}_{\mathrm{T}}$ when the free T-algebra sequence for A converges is much more challenging than in the case of a well-pointed endofunctor. To handle this proof, we will define a well-pointed endofunctor (S, σ) on $\mathrm{T} \downarrow \mathscr{C}$ and reduce the problem of constructing a reflection for an object $A \in \mathrm{ob}(\mathscr{C})$ along $U_{\mathrm{T}} : \mathbf{Alg}_{\mathrm{T}} \to \mathscr{C}$ to the problem of constructing a reflection of an object in $\mathrm{T} \downarrow \mathscr{C}$ along $U_{\mathrm{S}} : \mathbf{Alg}_{\mathrm{S}} \to \mathrm{T} \downarrow \mathscr{C}$. Specifically, we will construct the left adjoints in the following diagram.

$$\begin{tikzcd}
\mathbf{Alg}_{\mathrm{T}} \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \mathbf{Alg}_{\mathrm{S}} \arrow[l, bend right] \arrow[r, hook, bend right, "U_{\mathrm{S}}"'] & \mathrm{T} \downarrow \mathscr{C} \arrow[r, bend right, "\tau^*"'] \arrow[r, phantom, "\perp"] & \mathscr{C}^{\mathbf{2}} \arrow[l, bend right, "\tau_!"'] \arrow[r, bend right, "\mathrm{dom}"'] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend right, "\Delta"']
\end{tikzcd} \tag{2.3}$$

We will see in 2.3.12 that the composite map $\mathbf{Alg}_{\mathrm{T}} \to \mathscr{C}$ is equal to U_{T}. So when $\tau_! \Delta A$ has a reflection in $\mathbf{Alg}_{\mathrm{S}}$, then A has a reflection in $\mathbf{Alg}_{\mathrm{T}}$.

As we saw in the previous section, the convergence of the free S-algebra sequence on the object $\tau_! \Delta A$ guarantees the existence of a reflection of $\tau_! \Delta A$ in $\mathrm{T} \downarrow \mathscr{C}$. To make this process more direct, in section 2.3.3, we will translate the free S-algebra sequence on $\tau_! \Delta A$ to the *free* T-*algebra sequence* on A. We will then show that the free T-algebra sequence on A converges if and only if the free S-algebra sequence on $\tau_! \Delta A$ converges.

We also note that if the free S-algebra sequence on every object in $\mathrm{T} \downarrow \mathscr{C}$ converges, then every object in $\mathrm{T} \downarrow \mathscr{C}$ has a reflection in $\mathbf{Alg}_{\mathrm{S}}$ and thus a left adjoint to $U_{\mathrm{S}} : \mathbf{Alg}_{\mathrm{S}} \to \mathscr{C}$ exists. We then have that the composite map $\mathscr{C} \to \mathbf{Alg}_{\mathrm{T}}$ is a left adjoint to U_{T} and so a free monad on (T, τ) exists. We will not emphasize this result, however, because it is stronger than necessary. The condition that the free S-algebra sequence on every object in $\mathrm{T} \downarrow \mathscr{C}$ converges is stronger than the condition that the free S-algebra sequence on every object in the image of $\tau_! \Delta$ converges. This latter convergence is sufficient to imply the existence of a free monad on (T, τ). We will however see in section 2.4 that when T satisfies a smallness condition, then we do get that the free monad sequence on (S, σ) converges objectwise.

Now, let $\mathscr{C}$ be a cocomplete category and let (T, τ) be a pointed endofunctor on $\mathscr{C}$. In the next few sections, it will be convenient to represent the objects of the comma category $\mathrm{T} \downarrow \mathscr{C}$ by triples (X, f, Y) of objects X and Y and a map $f : \mathrm{T}X \to Y$ in $\mathscr{C}$. The maps of $\mathrm{T} \downarrow \mathscr{C}$ are pairs $(x, y) : (X, f, Y) \to (A, g, B)$ of maps $x : X \to A$ and $y : Y \to B$ such that $y \circ f = g \circ \mathrm{T}x$.

Let $\Delta : \mathscr{C} \to \mathscr{C}$ be the functor that sends objects X to $id_X : X \to X$ and sends maps $f : X \to Y$ to $(f, f) : id_X \to id_Y$. Then $\mathrm{dom} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ is a right adjoint to Δ and $\mathrm{cod} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ is a left adjoint to Δ. We will name the unit map of the $\mathrm{cod} \dashv \Delta$ adjunction $\rho : \mathrm{Id} \to \Delta\,\mathrm{cod}$. On each object $f : X \to Y$, $\rho_f = (f, id_Y) : f \to id_Y$. Since $\mathrm{cod}\,\Delta = Id$, $(\Delta\,\mathrm{cod}, \rho)$ is a well-pointed endofunctor on $\mathscr{C}^{\mathbf{2}}$.

Lemma 2.3.6. *The category $\mathbf{Alg}_{\Delta\,\mathrm{cod}}$ is the full subcategory of $\mathscr{C}^{\mathbf{2}}$ on the objects $f : X \to Y$ that are isomorphisms in $\mathscr{C}$.*

Proof. A $\Delta\,\mathrm{cod}$-algebra is an object $f : X \to Y$ with a map $(u, v) : id_Y \to f$ such that $(u, v) \circ (f, id_Y) = (id_X, id_Y)$. But this means $v = id_Y$, $u \circ f = id_X$, and $f \circ u = v \circ id_Y$. So u is the inverse of f. Conversely, if f is an isomorphism, then $(f^{-1}, id_Y) : id_Y \to f$ is a $\Delta\,\mathrm{cod}$-algebra structure map for f. □

The natural transformation $\tau : \mathrm{Id} \to \mathrm{T}$ defines two functors $\tau^* : \mathrm{T} \downarrow \mathscr{C} \to \mathscr{C}^{\mathbf{2}}$ and $\tau_! : \mathscr{C}^{\mathbf{2}} \to \mathrm{T} \downarrow \mathscr{C}$. The functor $\tau^* : \mathrm{T} \downarrow \mathscr{C} \to \mathscr{C}^{\mathbf{2}}$ sends each object (X, f, Y) to $f \circ \tau_X : X \to Y$ and sends each morphism $(u, v) : (X, f, Y) \to (A, g, B)$ to $(u, v) : f \circ \tau_X \to g \circ \tau_A$. The functor $\tau_! : \mathscr{C}^{\mathbf{2}} \to \mathrm{T} \downarrow \mathscr{C}$ sends each object $f : X \to Y$ to the object (X, f', Y') defined by the following cocartesian diagram in $\mathscr{C}$.

$$\begin{tikzcd}
X \arrow[r, "\tau_X"] \arrow[d, "f"'] & \mathrm{T}X \arrow[d, "f'"] \\
Y \arrow[r, "\widehat{f}"] \arrow[ur, phantom, "\ulcorner", very near end] & Y'
\end{tikzcd}$$

Since pushouts are natural and τ is a natural transformation, this definition extends to morphisms. It is straightforward to check the following result.

Lemma 2.3.7. *The functor $\tau_!$ is the left adjoint of τ^*.*

The unit of the $\tau_! \dashv \tau^*$ adjunction is the map $\nu^\tau : \mathrm{Id} \to \tau^*\tau_!$ given on each object $f : X \to Y$ in $\mathscr{C}^{\mathbf{2}}$ by $\nu^\tau_f = (id_X, \widehat{f}) : f \to f' \circ \tau_X$. The counit of the adjunction is the map $\xi^\tau : \tau_!\tau^* \to \mathrm{Id}$ given on each object (X, f, Y) in $\mathrm{T} \downarrow \mathscr{C}$ by $\xi^\tau_{(X,f,Y)} = (id_X, g) : (X, (f \circ \tau_X)', Y') \to (X, f, Y)$, where g is the map out of Y' defined by the cocone $f : \mathrm{T}X \to Y$, $id : Y \to Y$.

$$\begin{tikzcd}
X \arrow[r, "\tau_X"] \arrow[d, "\tau_X"'] & \mathrm{T}X \arrow[dd, "(f\circ\tau_X)'"] \arrow[ddr, bend left, "f"] & \\
\mathrm{T}X \arrow[d, "f"'] & & \\
Y \arrow[r, "\widehat{f\circ\tau_X}"] \arrow[rr, bend right, "id"'] & Y' \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "g"] & Y
\end{tikzcd}$$

Lemma 2.3.8 ([Kel80, §14.1])**.** *The category $\mathrm{T} \downarrow \mathscr{C}$ is cocomplete.*

Proof. Let $\{(X_\alpha, f_\alpha, Y_\alpha)\}_\alpha$ be a diagram in $\mathrm{T} \downarrow \mathscr{C}$. Let $c : \operatorname{colim}_\alpha \mathrm{T}X_\alpha \to \mathrm{T}\operatorname{colim}_\alpha X_\alpha$ be the map defined by the cocone $\{\mathrm{T}X_\alpha \to \mathrm{T}(\operatorname{colim}_\alpha X_\alpha)\}_\alpha$ in $\mathscr{C}$. Let h and k be the colimiting cocone maps in the following cocartesian diagram in $\mathscr{C}$.

$$\begin{tikzcd}
\operatorname{colim}_\alpha \mathrm{T}X_\alpha \arrow[r, "c"] \arrow[d, "\operatorname{colim}_\alpha f_\alpha"'] & \mathrm{T}\operatorname{colim}_\alpha X_\alpha \arrow[d, "h"] \\
\operatorname{colim}_\alpha Y_\alpha \arrow[r, "k"] & Z \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

The triple $(\operatorname{colim}_\alpha X_\alpha, h, Z)$ is the colimit of the diagram $\{(X_\alpha, f_\alpha, Y_\alpha)\}_\alpha$ in $\mathrm{T} \downarrow \mathscr{C}$. ☐

Since $\mathrm{T} \downarrow \mathscr{C}$ is cocomplete, the category $\mathbf{End}(\mathrm{T} \downarrow \mathscr{C})$ is a cocomplete. Colimits are computed objectwise. We can therefore define an endofunctor S on $\mathrm{T} \downarrow \mathscr{C}$ as the pushout

$$\begin{tikzcd}
\tau_!\tau^* \arrow[r, "\tau_!\rho\tau_*"] \arrow[d, "\xi^\tau"] & \tau_!\Delta\operatorname{cod}\tau^* \arrow[d, "\zeta"] \\
\mathrm{Id} \arrow[r, "\sigma"] & \mathrm{S} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{2.4}$$

in $\mathbf{End}(\mathrm{T}\downarrow\mathscr{C})$.

A direct application of A.1.1 yields the following result.

Lemma 2.3.9 ([Kel80, 14.4])**.** *The pointed endofunctor* (S,σ) *on* $\mathrm{T}\downarrow\mathscr{C}$ *is well-pointed.*

Therefore the forgetful functor $U_{\mathrm{S}} : \mathbf{Alg}_{\mathrm{S}} \hookrightarrow \mathrm{T}\downarrow\mathscr{C}$ is the inclusion of a full subcategory. We note that $\mathbf{Alg}_{\mathrm{T}}$ is also a full subcategory of $\mathrm{T}\downarrow\mathscr{C}$, even though T is not a well-pointed endofunctor.

Lemma 2.3.10 ([Kel80, 14.4])**.** *The category* $\mathbf{Alg}_{\mathrm{S}}$ *is the repletion of* $\mathbf{Alg}_{\mathrm{T}}$ *in* $\mathrm{T}\downarrow\mathscr{C}$*.*

Proof. By A.1.2, an object (X,f,Y) in $\mathrm{T}\downarrow\mathscr{C}$ is an S-algebra if and only if the object $f\circ\tau_X : X\to Y$ in $\mathscr{C}^{\mathbf{2}}$ is a $\Delta\,$cod-algebra. In other words, by 2.3.6, (X,f,Y) is an S-algebra if and only if $f\circ\tau_X : X\to Y$ is an isomorphism in $\mathscr{C}$. But, if $f\circ\tau_X$ is an isomorphism, then X is an T-coalgebra and $Y\cong X$. So there is an isomorphism in $\mathrm{T}\downarrow\mathscr{C}$ between (X,f,Y) and an object $(X,g,X)=\langle X,g\rangle$ in $\mathbf{Alg}_{\mathrm{T}}$. Conversely, if there is an isomorphism $(u,v):(X,f,Y)\to(Z,g,Z)$ in $\mathrm{T}\downarrow\mathscr{C}$ to an object $\langle Z,g\rangle$ in $\mathbf{Alg}_{\mathrm{T}}$, then $f\circ\tau_X\circ u = f\circ Tu\circ\tau_Z = v\circ g\circ\tau_Z = v$. Since u and v are isomorphisms, $f\circ\tau_X$ is an isomorphism. So (X,f,Y) is an S-algebra. ☐

Lemma 2.3.11. *The subcategory inclusion functor* $\mathbf{Alg}_{\mathrm{T}}\hookrightarrow\mathbf{Alg}_{\mathrm{S}}$ *is an equivalence of categories. In particular, it has a left adjoint.*

Proof. Since $\mathbf{Alg}_{\mathrm{T}}$ and $\mathbf{Alg}_{\mathrm{S}}$ are both full subcategories of $\mathrm{T}\downarrow\mathscr{C}$, $\mathbf{Alg}_{\mathrm{T}}$ is a full subcategory of $\mathbf{Alg}_{\mathrm{S}}$. So the inclusion $\mathscr{A}\hookrightarrow\mathscr{B}$ is full and faithful. What it means for $\mathbf{Alg}_{\mathrm{S}}$ to be the repletion of $\mathbf{Alg}_{\mathrm{T}}$ is that the inclusion $\mathbf{Alg}_{\mathrm{T}}\hookrightarrow\mathbf{Alg}_{\mathrm{S}}$ is essentially surjective. So the inclusion is an equivalence of categories. ☐

Lemma 2.3.12. *The forgetful functor* $U_{\mathrm{T}} : \mathbf{Alg}_{\mathrm{T}}\to\mathscr{C}$ *is equal to the composite map* $\mathbf{Alg}_{\mathrm{T}}\to\mathscr{C}$ *in diagram (2.3).*

Proof. We know the subcategory inclusion functor $\mathbf{Alg}_{\mathrm{T}} \hookrightarrow \mathrm{T} \downarrow \mathscr{C}$ factors through $U_{\mathrm{S}} : \mathbf{Alg}_{\mathrm{S}} \hookrightarrow \mathrm{T} \downarrow \mathscr{C}$. The subcategory inclusion functor $\mathbf{Alg}_{\mathrm{T}} \hookrightarrow \mathrm{T} \downarrow \mathscr{C}$ sends an object $\langle X, m \rangle$ to the object (X, m, X) and a map $f : \langle X, m \rangle \to \langle Y, n \rangle$ to the map $(f, f) : (X, m, X) \to (Y, n, Y)$. Then $\tau^*(X, m, X) = m \circ \tau_X = id_X$ and $\tau^*(f, f) = (f, f) : id_X \to id_Y$. So $\operatorname{cod} \tau^*(X, m, X) = X$ and $\operatorname{cod} \tau^*(f, f) = f : X \to Y$. This is exactly what U_{T} does to objects and morphisms. □

We are now able to show that the reflection of an object $A \in \mathrm{ob}(\mathscr{C})$ along $U_{\mathrm{T}} : \mathbf{Alg}_{\mathrm{T}} \to \mathscr{C}$ exists when the free S-algebra sequence for $(A, id, \mathrm{T}A)$ converges.

Proposition 2.3.13. *Suppose the free* S*-algebra sequence for* $(A, id, \mathrm{T}A)$ *converges at* α*. Let* $(B, f, C) = \mathrm{S}^{\alpha}(A, id, \mathrm{T}A)$ *and let* $(u, v) = \sigma^{\alpha}_{(A, id, \mathrm{T}A)} : (A, id, \mathrm{T}A) \to (B, f, C)$*. Then* $f \circ \tau_B : B \to C$ *is an isomorphism,* $\langle B, (f \circ \tau_B)^{-1} \circ f \rangle$ *is the reflection of* A *in* $\mathbf{Alg}_{\mathrm{T}}$*, and* $u : A \to B$ *is the universal map of the reflection.*

Proof. By 2.3.5 and 2.3.3, (B, f, C) can be identified with the reflection of $(A, id, \mathrm{T}A)$ in $\mathbf{Alg}_{\mathrm{S}}$ and $(u, v) : (A, id, \mathrm{T}A) \to (B, f, C)$ is the universal map of this reflection. In 2.3.10, we saw that the composition $f \circ \tau_B : B \to C$ is an isomorphism. So $\langle B, g \circ f \rangle$ is an object in $\mathbf{Alg}_{\mathrm{T}}$ and $(id, g) : (B, f, C) \to (B, g \circ f, B)$ is an isomorphism in $\mathrm{T} \downarrow \mathscr{C}$, where $g = (f \circ \tau_B)^{-1}$. Thus $\langle B, g \circ f \rangle$ is the reflection of (A, id, A) in $\mathbf{Alg}_{\mathrm{T}}$ and the universal map is $(u, g \circ v) : (A, id, A) \to (B, g \circ f, B)$. By 2.3.12 and the observation that $\tau_! \Delta A = (A, id, \mathrm{T}A)$, the object $\langle B, g \circ f \rangle$ is the reflection of A in $\mathbf{Alg}_{\mathrm{T}}$. Since $\operatorname{dom} \tau^* \tau_! \Delta$ is the identity functor on $\mathscr{C}$ and the unit of the adjunction $\operatorname{dom} \tau^* \vdash \tau_! \Delta$ is the identity map, $u : A \to B$ is the universal map of the reflection. □

In particular, if the free S-algebra sequence on $(A, id, \mathrm{T}A)$ converges for each object A in $\mathscr{A}$, then the left adjoint to $U_{\mathrm{T}} : \mathbf{Alg}_{\mathrm{T}} \to \mathscr{C}$ exists.

2.3.3 Free Monad Sequences for Pointed Endofunctors

Let (T, τ) be a pointed endofunctor on a cocomplete category $\mathscr{C}$ and let (S, σ) be the well-pointed endofunctor on $\mathrm{T} \downarrow \mathscr{C}$ we defined in section 2.3.2. We were able to show in 2.3.13 that A has a reflection in $\mathbf{Alg}_{\mathrm{T}}$ when the free S-algebra sequence for $(A, id, \mathrm{T}A)$ converges.

In this section, we will translate the free S-algebra sequence for an object (A, f, B) to a transfinite sequence in $\mathscr{C}$. We define the *free* T*-algebra sequence for* A to be the sequence in $\mathscr{C}$ given by free S-algebra sequence for $(A, id, \mathrm{T}A)$ under this translation. The free T-algebra sequence for A converges exactly when the free S-algebra sequence for $(A, id, \mathrm{T}A)$ converges. So the convergence of the free T-algebra sequence for A guarantees that A has a reflection in $\mathbf{Alg}_{\mathrm{T}}$. Furthermore, as for the free S-algebra sequence, the object the free T-algebra sequence for A converges to is the image of the reflection of A under U_{T}.

Since the free T-algebra sequence is functorial, we will be able to use it to define a *free monad sequence on* (T, τ). We will show in 2.3.22 that when the free monad sequence for (T, τ) is *weakly convergent*, then the free monad on (T, τ) exists and it is given by the endofunctor the free monad sequence weakly converges to. We will then show in 2.3.18 that the convergence of the free T-algebra sequence on A for each A in $\mathscr{C}$ is enough to imply that the free monad sequence for (T, τ) weakly converges. In section 2.4, we will give a condition on the endofunctor $\mathrm{T} : \mathscr{C} \to \mathscr{C}$ that guarantees the convergence of the free T-algebra sequence on each object A in $\mathscr{C}$.

Lemma 2.3.14 ([Kel80, 17.1]). *For each object* (A, f, B) *in* $\mathrm{T} \downarrow \mathscr{C}$*,* $\mathrm{S}(A, f, B) = (B, g, C)$*, where* $g : \mathrm{T}B \to C$ *is the coequalizer in* $\mathscr{C}$ *of the maps* $\mathrm{T}f \circ \mathrm{T}\tau_A$ *and* $\mathrm{T}f \circ \tau_{\mathrm{T}A}$*.*

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, shift left, "\mathrm{T}\tau_A"] \arrow[r, shift right, "\tau_{\mathrm{T}A}"'] & \mathrm{TT}A \arrow[r, "\mathrm{T}f"] & \mathrm{T}B \arrow[r, dashed, "g"] & C
\end{tikzcd}$$

The unit $\sigma_{(A,f,B)} : (A, f, B) \to \mathrm{S}(A, f, B)$ *is equal to* $(f, g) \circ (\tau_A, \tau_B)$*.*

Proof. Let (A, f, B) be an object in $\mathrm{T} \downarrow \mathscr{C}$. The composite functor $\mathrm{dom}\,\tau^* : \mathrm{T} \downarrow \mathscr{C} \to \mathscr{C}$ sends objects (X, h, Y) to X and morphisms $(p, q) : (X, h, Y) \to (Z, k, W)$ to $p : X \to Z$. One can compute directly that $\mathrm{dom}\,\tau^*\mathrm{S}(A, f, B) = B$ and that evaluating diagram (2.4) on (A, f, B) and then applying $\mathrm{dom}\,\tau^*$ yields the cocartesian square

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}(f\circ\tau_A)"] \arrow[d, "id"'] & \mathrm{T}B \arrow[d, "id"] \\
\mathrm{T}A \arrow[r, "\mathrm{T}(f\circ\tau_A)"'] & \mathrm{T}B. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

So $\mathrm{S}(A, f, B) = (B, g, C)$ for some map $g : \mathrm{T}B \to C$ in $\mathscr{C}$ and $\sigma_{(A,f,B)} = (f \circ \tau_A, z) : (A, f, B) \to (B, g, C)$ for some map $z : B \to C$.

Let X be an object in $\mathscr{C}$. There are bijective correspondences between the following classes.

- Maps $C \to X$ in $\mathscr{C}$.
- Pairs of objects (B, h, X) and maps $(id, t) : (B, g, C) \to (B, h, X)$ in $\mathrm{T} \downarrow \mathscr{C}$.
- Triples of objects (B, h, X) in $\mathrm{T} \downarrow \mathscr{C}$, maps $s : B \to X$ in $\mathscr{C}$, and maps $u : \mathrm{T}B \to X$ in $\mathscr{C}$ such that the following diagram in $\mathrm{T} \downarrow \mathscr{C}$ commutes.

$$\begin{tikzcd}
\tau_!\tau^*(A, f, B) \arrow[r, "\tau_!\rho_{\tau_*(A,f,B)}"] \arrow[d, "\xi^\tau_{(A,f,B)}"'] & \tau_!\Delta\,\mathrm{cod}\,\tau^*(A, f, B) \arrow[d, "(id,u)"] \\
(A, f, B) \arrow[r, "(f\circ\tau_A,s)"] & (B, h, X)
\end{tikzcd}$$

- Triples of objects (B, h, X) in $\mathrm{T} \downarrow \mathscr{C}$, maps $s : B \to X$ in $\mathscr{C}$, and maps $v : B \to X$ in $\mathscr{C}$ such that the following diagram in $\mathscr{C}^{\mathbf{2}}$ commutes.

$$\begin{tikzcd}
\tau^*(A, f, B) \arrow[r, "\rho_{\tau_*(A,f,B)}"] \arrow[rr, bend right, "\tau^*(f\circ\tau_A,s)"'] & \Delta\,\mathrm{cod}\,\tau^*(A, f, B) \arrow[r, "(id,v)"] & \tau^*(B, h, X)
\end{tikzcd}$$

The map $\rho_{\tau^*(A,f,B)}$ is equal to $(f\circ\tau_A, id) : f\circ\tau_A \to id_B$. So there is a bijective correspondence between the class in the last bullet point and triples of objects (B,h,X) in $\mathrm{T}\downarrow\mathscr{C}$, maps $s : B\to X$ in $\mathscr{C}$, and maps $v : B \to X$ in $\mathscr{C}$ such that $h\circ\mathrm{T}(f\circ\tau_A) = v\circ f$ and the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f\circ\tau_A"] \arrow[d, "f\circ\tau_A"'] & B \arrow[r, "id"] \arrow[d, "id"] & B \arrow[d, "h\circ\tau_B"] \\
B \arrow[r, "id"] \arrow[rr, bend right, "s"'] & B \arrow[r, "v"] & X
\end{tikzcd} \tag{2.5}$$

But then $s = v = h\circ\tau_B$. So maps $C\to X$ in $\mathscr{C}$ are in bijective correspondence with maps $h : \mathrm{T}B \to X$ such that $h\circ\mathrm{T}(f\circ\tau_A) = h\circ\tau_B\circ f = h\circ\mathrm{T}f\circ\tau_{\mathrm{T}A}$. Thus $g : \mathrm{T}B\to C$ is the coequalizer of the maps $\mathrm{T}f\circ\mathrm{T}\tau_A$ and $\mathrm{T}f\circ\tau_{\mathrm{T}A}$. By the relation in diagram (2.5) above, the unit map $(f\circ\tau_A, z) : (A,f,B)\to(B,g,C)$ must satisfy the equation $z = g\circ\tau_B$. ☐

Lemma 2.3.15. *If* $\big((A_\alpha, f_\alpha, B_\alpha)\big)_\alpha$ *is a sequence in* $\mathrm{T}\downarrow\mathscr{C}$*, then* $\operatorname{dom}\tau^*\big(\operatorname{colim}_\alpha(A_\alpha, f_\alpha, B_\alpha)\big) = \operatorname{colim}_\alpha A_\alpha$.

Proof. We saw in 2.3.8 that the colimit of the $(A_\alpha, f_\alpha, B_\alpha)$'s in $\mathrm{T}\downarrow\mathscr{C}$ is equal to $(\operatorname{colim} A_\alpha, h, Z)$, where h and Z are determined by a pushout square. ☐

Now, using 2.3.14, and 2.3.15, we can describe $\mathrm{S}^\beta(A,f,B)$ for any ordinal β as follows. This construction takes place in the category $\mathscr{C}$. Let $\mathrm{X}_0 = A$ and $\mathrm{X}_1 = B$. Let $\pi_0 = f : \mathrm{TX}_0 \to \mathrm{X}_1$.

Suppose objects X_α and $\mathrm{X}_{\alpha+1}$ and a map $\pi_\alpha : \mathrm{TX}_\alpha \to \mathrm{X}_{\alpha+1}$ are defined for an ordinal α. Then the object $\mathrm{X}_{\alpha+2}$ with the map $\pi_{\alpha+1} : \mathrm{TX}_{\alpha+1}\to\mathrm{X}_{\alpha+2}$ is the coequalizer of the maps $\mathrm{T}\pi_\alpha\circ\tau\mathrm{TX}_\alpha$ and $\mathrm{T}\pi_\alpha\circ\mathrm{T}\tau\mathrm{X}_\alpha$.

$$\begin{tikzcd}
\mathrm{TX}_\alpha \arrow[r, shift left, "\tau\mathrm{TX}_\alpha"] \arrow[r, shift right, "\mathrm{T}\tau\mathrm{X}_\alpha"'] & \mathrm{TTX}_\alpha \arrow[r, "\mathrm{T}\pi_\alpha"] & \mathrm{TX}_{\alpha+1} \arrow[r, dashed, "\pi_{\alpha+1}"] & \mathrm{X}_{\alpha+2}
\end{tikzcd} \tag{2.6}$$

Let γ be a limit ordinal and suppose X_α and π_α are defined for each $\alpha < \gamma$. We have maps

$$x_\alpha^{\alpha+1} := \pi_\alpha \circ \tau\mathrm{X}_\alpha : \mathrm{X}_\alpha \to \mathrm{X}_{\alpha+1}$$

for each ordinal $\alpha < \gamma$. These maps make $(\mathrm{X}_\alpha)_{\alpha<\gamma}$ a γ-sequence. Let X_γ be the colimit of this γ-sequence and let $\{x_\alpha^\gamma : \mathrm{X}_\alpha \to \mathrm{X}_\gamma\}_{\alpha<\gamma}$ be the colimiting cocone. Similarly, the maps

$$\mathrm{T}x_\alpha^{\alpha+1} : \mathrm{TX}_\alpha \to \mathrm{TX}_{\alpha+1}$$

make $(\mathrm{TX}_\alpha)_{\alpha<\gamma}$ a γ-sequence with colimiting cocone $\{g_\alpha : \mathrm{TX}_\alpha \to colim_{\mathrm{T}x_\alpha^{\alpha+1}}\mathrm{TX}_\alpha\}_\alpha$. But we also have maps

$$y_\alpha := T\pi_\alpha \circ \tau\mathrm{TX}_\alpha : \mathrm{TX}_\alpha \to \mathrm{TX}_{\alpha+1}.$$

So we get two induced maps from $colim_{\mathrm{T}x_\alpha^{\alpha+1}}\mathrm{TX}_\alpha$ to itself. Let y be the map induced by $g_{\alpha+1} \circ y_\alpha$. The map induced by $g_{\alpha+1} \circ \mathrm{T}x_\alpha^{\alpha+1} = g_\alpha$ is the identity map. We also have a map $c : colim_{\mathrm{T}x_\alpha^{\alpha+1}}\mathrm{TX}_\alpha \to \mathrm{TX}_\gamma$ defined by the cocone $\{\mathrm{T}x_\alpha^\gamma : \mathrm{TX}_\alpha \to \mathrm{TX}_\gamma\}_{\alpha<\gamma}$. This is shown in the below diagram.

$$\begin{tikzcd}
\mathrm{TX}_\alpha \arrow[r, shift left, "\tau\mathrm{TX}_\alpha"] \arrow[r, shift right, "\mathrm{T}\tau\mathrm{X}_\alpha"'] \arrow[d, "g_\alpha"] & \mathrm{TTX}_\alpha \arrow[r, "\mathrm{T}\pi_\alpha"] & \mathrm{TX}_{\alpha+1} \arrow[d, "g_{\alpha+1}"] \arrow[dr, bend left, "\mathrm{T}x_{\alpha+1}^\gamma"] & \\
\underset{\mathrm{T}x_\alpha^{\alpha+1}}{colim}\mathrm{TX}_\alpha \arrow[rr, shift left, "y"] \arrow[rr, shift right, "id"'] & & \underset{\mathrm{T}x_\alpha^{\alpha+1}}{colim}\mathrm{TX}_\alpha \arrow[r, "c"] & \mathrm{TX}_\gamma
\end{tikzcd} \tag{2.7}$$

We define $\pi_\gamma : \mathrm{TX}_\gamma \to \mathrm{X}_{\gamma+1}$ to be the coequalizer of the maps $c \circ y$ and c in the bottom row. The map π_γ defines a map $x_\gamma^{\gamma+1} := \pi_\gamma \circ (\tau\mathrm{X}_\gamma) : \mathrm{X}_\gamma \to \mathrm{X}_{\gamma+1}$.

By composing the successor maps $x_\alpha^{\alpha+1} : X_\alpha \to X_{\alpha+1}$ and colimiting cocone maps $x_\alpha^\gamma : X_\alpha \to X_\gamma$, we get a map $x_\alpha^{\alpha'} : X_\alpha \to X_{\alpha'}$ for each pair of ordinals $\alpha \leq \alpha'$. The maps $x_\alpha^{\alpha'} : \mathrm{X}_\alpha \to \mathrm{X}_{\alpha'}$ define a transfinite sequence $(\mathrm{X}_\alpha)_\alpha$. We have that $\mathrm{S}^\beta(A, f, B) = (X_\beta, \pi_\beta, X_{\beta+1})$ for each ordinal β.

Now, if we instead set $\mathrm{X}_0 = A$, $\mathrm{X}_1 = \mathrm{T}A$, and $\pi_0 = id : \mathrm{T}X_0 \to \mathrm{X}_1$ and then rerun the above construction in the category $\mathscr{C}$, the transfinite sequence $(\mathrm{X}_\alpha)_\alpha$ with connecting maps $x_\alpha^\beta : \mathrm{X}_\alpha \to \mathrm{X}_\beta$ is the *free* T-*algebra sequence for* A.

Remark 2.3.16. If (T, τ) is a well-pointed endofunctor, then on any object A, the free T-algebra sequence on A defined above agrees with the one defined in section 2.3.1.

Proposition 2.3.17 ([Kel80, 17.3]). *The free* T-*algebra sequence for* A *converges if and only if the free* S-*algebra sequence for* $(A, id, \mathrm{T}A)$ *converges.*

Proof. We know $\mathrm{S}^\alpha(A, id, \mathrm{T}A) = (X_\alpha, \pi_\alpha, X_{\alpha+1})$ and $\mathrm{S}_\alpha^\beta(A, id, \mathrm{T}A) = (x_\alpha^\beta, x_{\alpha+1}^{\beta+1}) : (X_\alpha, \pi_\alpha, X_{\alpha+1}) \to (X_\beta, \pi_\beta, X_{\beta+1})$ for each α and $\beta > \alpha$. □

The free T-algebra sequence on an object of $\mathscr{C}$ is functorial. In fact, we can run the above construction in the category of endofunctors on $\mathscr{C}$. We set $\mathrm{X}_0 = Id : \mathscr{C} \to \mathscr{C}$, $\mathrm{X}_1 = \mathrm{T} : \mathscr{C} \to \mathscr{C}$, and we let $\pi_0 : \mathrm{T}\mathrm{X}_0 \to \mathrm{X}_1$ be the identity natural transformation. We then rerun the above construction in the category $\mathbf{End}(\mathscr{C})$. We get a transfinite sequence $(\mathrm{X}_\alpha)_\alpha$ of endofunctors with connecting natural transformations $x_\alpha^\beta : \mathrm{X}_\alpha \to \mathrm{X}_\beta$. We will write this sequence as a functor $\mathrm{X}_\bullet^\mathrm{T} : \mathbf{Ord} \to \mathbf{pEnd}(\mathscr{C})$, where $\mathrm{X}_\bullet^\mathrm{T}(\alpha) = (X_\alpha, x_0^\alpha)$ on objects α and $\mathrm{X}_\bullet^\mathrm{T}(\alpha \to \beta) = x_\alpha^\beta$ on morphisms $\alpha \to \beta$. The transfinite sequence $\mathrm{X}_\bullet^\mathrm{T}$ is the *free monad sequence for* (T, τ).

If we evaluate each endofunctor in the free monad sequence for (T, τ) on an object A, then we get the free T-algebra sequence for A. When the free T-algebra sequence for A converges for each A, we will say that the free monad sequence for (T, τ) *converges objectwise*.

The objectwise convergence of the free monad sequence for (T, τ) implies that the free monad on (T, τ) exists. We will show in section 2.4.3 that, under a smallness condition on T, the free monad sequence $\mathrm{X}_\bullet^\mathrm{T}$ converges objectwise.

Theorem 2.3.18. *If the free monad sequence on* (T, τ) *converges objectwise, then the free monad on* (T, τ) *exists.*

Proof. Let A be an object in $\mathscr{C}$. Then the free T-algebra sequence on A converges. By 2.3.17, the free S-algebra sequence on the object $(A, id, \mathrm{T}A)$ in $\mathrm{T} \downarrow \mathscr{C}$ converges. So, by 2.3.13, A has a reflection in $\mathbf{Alg}_{\mathrm{T}}$. Since this holds for each object in $\mathscr{C}$, by 2.2.2, the forgetful functor $U_{\mathrm{T}} : \mathbf{Alg}_{\mathrm{T}} \to \mathscr{C}$ has a left adjoint. $\square$

2.3.4 Weakly Convergent Free Monad Sequences

Let $\mathscr{C}$ be a cocomplete category, let (T, τ) be a pointed endofunctor on $\mathscr{C}$. Let $(\mathrm{X}_\alpha)_\alpha$ be the free monad sequence for (T, τ) with connecting maps $x_\alpha^\beta : \mathrm{X}_\alpha \to \mathrm{X}_\beta$.

Even though $\mathbf{End}(\mathscr{C})$ is a cocomplete category, since $(\mathrm{X}_\alpha)_\alpha$ is a large diagram in $\mathbf{End}(\mathscr{C})$, its colimit does not have to exist. When the colimit R of $(\mathrm{X}_\alpha)_\alpha$ does exist, we will use the notation $x_\alpha^\infty : X_\alpha \to \mathrm{R}$ for the maps in the colimiting cocone. We will use the notation R' for the colimit of the transfinite sequence $(\mathrm{TX}_\alpha)_\alpha$ with connecting maps $\mathrm{T}x_\alpha^\beta : \mathrm{TX}_\alpha \to \mathrm{TX}_\beta$ when this colimit exists. The cocone $\{\mathrm{T}x_\alpha^\infty : \mathrm{TX}_\alpha \to \mathrm{TR}\}_\alpha$ defines a map $c' : \mathrm{R}' \to \mathrm{TR}$. The cocone $\{x_{\alpha+1}^\infty \circ \pi_\alpha : \mathrm{TX}_\alpha \to \mathrm{R}\}_\alpha$ defines a map $\pi' : \mathrm{R}' \to \mathrm{R}$.

Definition 2.3.19. The free monad sequence $(\mathrm{X}_\alpha)_\alpha$ is *weakly convergent* if the colimits $\mathrm{R} = \mathrm{colim}_\alpha \mathrm{X}_\alpha$ and $\mathrm{R}' = \mathrm{colim}_\alpha \mathrm{TX}_\alpha$ exist and are objectwise colimits in $\mathscr{C}$ and if there is a map $m : \mathrm{TR} \to \mathrm{R}$ satisfying the following conditions.

1. The map $m : \mathrm{TR} \to \mathrm{R}$ is the coequalizer in $\mathbf{End}(\mathscr{C})$ of the maps $\mathrm{T}m \circ \tau\mathrm{TR}$ and $\mathrm{T}m \circ \mathrm{T}\tau\mathrm{R}$ in the following diagram.

$$\begin{tikzcd}
\mathrm{TR} \arrow[r, shift left, "\tau\mathrm{TR}"] \arrow[r, shift right, "\mathrm{T}\tau\mathrm{R}"'] & \mathrm{TTR} \arrow[r, "\mathrm{T}m"] & \mathrm{TR}
\end{tikzcd}$$

2. The following diagram in $\mathbf{End}(\mathscr{C})$ is cocartesian.

$$\begin{tikzcd}
\mathrm{R}' \arrow[r, "c'"] \arrow[d, "\pi'"'] & \mathrm{TR} \arrow[d, "m"] \\
\mathrm{R} \arrow[r, "id"] & \mathrm{R} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Proposition 2.3.20. *If the free monad sequence for* (T, τ) *weakly converges, then the equations* $m \circ \tau\mathrm{R} = id$ *and* $m \circ \mathrm{T}x_0^\infty = x_1^\infty$ *hold.*

Proof. An inductive argument proves that the following diagram commutes for each ordinal α.

$$\begin{tikzcd}
\mathrm{T} \arrow[r, "\mathrm{T}x_0^\alpha"] \arrow[dr, "x_1^{\alpha+1}"'] & \mathrm{TX}_\alpha \arrow[d, "\pi_\alpha"] & \mathrm{X}_\alpha \arrow[l, "\tau\mathrm{X}_\alpha"'] \arrow[dl, "x_\alpha^{\alpha+1}"] \\
& \mathrm{X}_{\alpha+1} &
\end{tikzcd}$$

Then the following diagram commutes, where $i : \mathrm{T} \to \mathrm{R}'$ is the inclusion map of the colimiting cocone for $\mathrm{R}' = \mathrm{colim}_\alpha\, \mathrm{T}X_\alpha$.

$$\begin{tikzcd}
\mathrm{T} \arrow[r, "i"] \arrow[dr, "x_1^\infty"'] & \mathrm{R}' \arrow[d, "\pi'"] & \mathrm{R} \arrow[l, "\mathrm{colim}_\alpha\, \tau\mathrm{X}_\alpha"'] \arrow[dl, "id"] \\
& \mathrm{R} &
\end{tikzcd}$$

It is easy to check that $c' \circ i = \mathrm{T}x_0^\infty : \mathrm{T} \to \mathrm{TR}$ and that $c' \circ \mathrm{colim}_\alpha\, \tau\mathrm{X}_\alpha$ is the map defined by the cocone $\{\mathrm{T}x_\alpha^\infty \circ \tau\mathrm{X}_\alpha = \tau\mathrm{R} \circ x_\alpha^\infty : \mathrm{X}_\alpha \to \mathrm{TR}\}_\alpha$. So $c' \circ \mathrm{colim}_\alpha\, \tau\mathrm{X}_\alpha = \tau\mathrm{R}$. By 2.3.19 (2), $m \circ \mathrm{T}x_0^\infty = m \circ c' \circ i = \pi' \circ i = x_1^\infty$ and $m \circ \tau\mathrm{R} = m \circ c' \circ \mathrm{colim}_\alpha\, \tau\mathrm{X}_\alpha = \pi' \circ \mathrm{colim}_\alpha\, \tau\mathrm{X}_\alpha = id$. □

We have the following adaptation of 2.3.5 to weak convergence.

Proposition 2.3.21. *If* (T, τ) *is a well-pointed endofunctor on* $\mathscr{C}$ *and if the free monad sequence for* (T, τ) *converges weakly to* R*, then for each object* A *in* $\mathscr{C}$*,* $\tau_{\mathrm{R}A}$ *is an isomorphism,* $\langle \mathrm{R}A, \tau_{\mathrm{R}A}^{-1} \rangle$ *is the reflection of* A *in* $\mathbf{Alg}_\mathrm{T}$ *and* ${x_0^\infty}_A : A \to \mathrm{R}A$ *is the universal map of the reflection.*

Proof. Let A be an object in $\mathscr{C}$. As we saw in the proof of 2.3.5, for each map $f : A \to B$ in $\mathscr{C}$ to an T-algebra B and each ordinal α, there is a unique map $g_\alpha := (x_0^\alpha)_B^{-1} \circ \mathrm{X}_\alpha f : \mathrm{X}_\alpha A \to B$ such that $g_\alpha \circ {x_0^\alpha}_A = f$. It follows that the colimit $g = \operatorname{colim}_\alpha g_\alpha$ of the sequence $(g_\alpha)_\alpha$ is the unique map $\mathrm{R}A \to B$ such that $g \circ {x_0^\infty}_A = f$.

By 2.3.2 and 2.3.20, $\tau_{\mathrm{R}A}$ is an isomorphism. Also by 2.3.2, τ_B is an isomorphism. It follows that g is a map of T-algebras. So $g : \mathrm{R}A \to B$ must be the unique map of T-algebras such that $g \circ {x_0^\infty}_A = f$. □

Theorem 2.3.22. *If the free monad sequence for* (T, τ) *converges weakly, then there is a map* $\mu : \mathrm{RR} \to \mathrm{R}$ *such that* $(\mathrm{R}, x_0^\infty, \mu)$ *is the free monad on* (T, τ) *with universal map* $x_1^\infty : \mathrm{T} \to \mathrm{R}$.

Proof. First, we note that if $\{(X_\alpha, f_\alpha, Y_\alpha)\}_\alpha$ is a large diagram in $\mathrm{T} \downarrow \mathscr{C}$ and the colimit of $(f_\alpha)_\alpha$ in $\mathscr{C}^{\mathbf{2}}$ exists, then the colimit of $\{(X_\alpha, f_\alpha, Y_\alpha)\}_\alpha$ exists and is given by $(\operatorname{colim}_\alpha X_\alpha, h, Z)$ in the following cocartesian square, where $c : \operatorname{colim}_\alpha \mathrm{T}X_\alpha \to \mathrm{T} \operatorname{colim}_\alpha X_\alpha$ is the map defined by the cocone $\{\mathrm{T}X_\alpha \to \mathrm{T}(\operatorname{colim}_\alpha X_\alpha)\}_\alpha$ in $\mathscr{C}$.

$$\begin{tikzcd}
\operatorname{colim}_\alpha \mathrm{T}X_\alpha \arrow[r, "c"] \arrow[d, "\operatorname{colim}_\alpha f_\alpha"'] & \mathrm{T}\operatorname{colim}_\alpha X_\alpha \arrow[d, "h"] \\
\operatorname{colim}_\alpha Y_\alpha \arrow[r, "k"] & Z \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Let (S, σ) be the well-pointed endofunctor on the comma category $\mathrm{T} \downarrow \mathscr{C}$ constructed in section 2.3.2. Let A be an object in $\mathscr{C}$. The sequence $\big((\mathrm{X}_\alpha A, \pi_{\alpha A}, \mathrm{X}_{\alpha+1} A)\big)_\alpha$ is the free S-algebra sequence on $(A, id, \mathrm{T}A)$. The colimit of the transfinite sequence $(\pi_{\alpha A})_\alpha$ in $\mathscr{C}^{\mathbf{2}}$ with connecting maps $\big(\mathrm{T}x^\beta_{\alpha A}, x^{\beta+1}_{\alpha+1 A}\big) : \pi_{\alpha A} \to \pi_{\beta A}$ exists and is equal to $\pi'_A : \mathrm{R}'A \to \mathrm{R}A$. By our comments above and by (2) of definition 2.3.19, $(\mathrm{R}A, m_A, \mathrm{R}A)$ is the colimit of $\big((\mathrm{X}_\alpha A, \pi_{\alpha A}, \mathrm{X}_{\alpha+1} A)\big)_\alpha$ in $\mathrm{T} \downarrow \mathscr{C}$.

By 2.3.14 and 2.3.19 (1), $\mathrm{S}(\mathrm{R}A, m_A, \mathrm{R}A) = (\mathrm{R}A, m_A, \mathrm{R}A)$ and the unit map $\sigma_{(\mathrm{R}A, m_A, \mathrm{R}A)} : (\mathrm{R}A, m_A, \mathrm{R}A) \to \mathrm{S}(\mathrm{R}A, m_A, \mathrm{R}A)$ is the composition

$$(\mathrm{R}A, m_A, \mathrm{R}A) \xrightarrow{(\tau_{\mathrm{R}A}, \tau_{\mathrm{R}A})} (\mathrm{TR}A, \mathrm{T}m_A, \mathrm{TR}A) \xrightarrow{(m_A, m_A)} (\mathrm{R}A, m_A, \mathrm{R}A)$$

So, by 2.3.20, $\sigma_{(\mathrm{R}A, m_A, \mathrm{R}A)} = id : (\mathrm{R}A, m_A, \mathrm{R}A) \to (\mathrm{R}A, m_A, \mathrm{R}A)$. Thus $(\mathrm{R}A, m_A, \mathrm{R}A)$ is an S-algebra.

By 2.3.21, $({x_0^\infty}_A, {x_1^\infty}_A) : (A, id, \mathrm{T}A) \to (\mathrm{R}A, m_A, \mathrm{R}A)$ is the universal map of the reflection of $(A, id, \mathrm{T}A)$ into $\mathbf{Alg}_\mathrm{S}$. Let $\langle X, n \rangle$ be an object in $\mathbf{Alg}_\mathrm{T}$ and let $f : A \to X$ be a map in $\mathscr{C}$. We will show that ${x_0^\infty}_A : A \to \mathrm{R}A$ is the universal map of the reflection of A into $\mathbf{Alg}_\mathrm{T}$. By 2.3.10, (X, n, X) is an S-algebra. Since $(f, n \circ \mathrm{T}f) : (A, id, \mathrm{T}A) \to (X, n, X)$ is a map in $\mathrm{T} \downarrow \mathscr{C}$, there is a unique map $(u, v) : (\mathrm{R}A, m_A, \mathrm{R}A) \to (X, n, X)$ which is a map of S-algebras such that $(u, v) \circ ({x_0^\infty}_A, {x_1^\infty}_A) = (f, n \circ \mathrm{T}f)$. Because (u, v) is a map in $\mathbf{Alg}_\mathrm{S}$ between objects in $\mathbf{Alg}_\mathrm{T}$ and $\mathbf{Alg}_\mathrm{T}$ is a full subcategory of $\mathbf{Alg}_\mathrm{S}$, $u = v$. Because $\mathbf{Alg}_\mathrm{S}$ is a full subcategory of $\mathrm{T} \downarrow \mathscr{C}$, (u, u) is the unique map in $\mathrm{T} \downarrow \mathscr{C}$ such that $(u, u) \circ ({x_0^\infty}_A, {x_1^\infty}_A) = (f, n \circ \mathrm{T}f)$. But this is exactly the condition that $u : \langle \mathrm{R}A, m_A \rangle \to \langle X, n \rangle$ is a unique map in $\mathbf{Alg}_\mathrm{T}$ such that $u \circ {x_1^\infty}_A = n \circ \mathrm{T}f$. If $s : \langle \mathrm{R}A, m_A \rangle \to \langle X, n \rangle$ is a map in $\mathbf{Alg}_\mathrm{T}$ such that $s \circ {x_0^\infty}_A = f$, then the commutativity of the following diagram and the uniqueness of u shows that $s = u$.

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}{x_0^\infty}_A"] \arrow[rr, bend left, "{x_1^\infty}_A"] \arrow[dr, "\mathrm{T}f"'] & \mathrm{TR}A \arrow[r, "m_A"] \arrow[d, "\mathrm{T}s"] & \mathrm{R}A \arrow[d, "s"] \\
& \mathrm{T}X \arrow[r, "n"] & X
\end{tikzcd}$$

So $\langle \mathrm{R}A, m_A \rangle$ is the reflection of A in $\mathbf{Alg}_\mathrm{T}$ and ${x_0^\infty}_A : A \to \mathrm{R}A$ is the universal map of the reflection.

Since this holds for every object A, by 2.2.2, the functor $A \mapsto \langle \mathrm{R}A, m_A \rangle$ is the left adjoint to $U_\mathrm{T} : \mathbf{Alg}_\mathrm{T} \to \mathscr{C}$ with unit $x_0^\infty : Id \to \mathrm{R}$. So there is a map $\mu : \mathrm{RR} \to \mathrm{R}$ such that $(\mathrm{R}, x_0^\infty, \mu)$

is the free monad on (T, τ). By the proof of 2.2.8, the universal map of the free monad is $m \circ \mathrm{T}x_0^\infty = x_1^\infty$. □

When the free monad sequence for (T, τ) is weakly convergent, we will say the free monad on (T, τ) *exists constructively.*

Theorem 2.3.23. *If the free monad sequence for* (T, τ) *converges objectwise, then it converges weakly.*

Proof. For each A, let $\alpha(A)$ be the first ordinal at which the free T-algebra sequence for A converges. We define a pointed endofunctor (R, η) on $\mathscr{C}$ as follows. On each object A, let $\mathrm{R}A = \mathrm{X}_{\alpha(A)}A$ and let $\eta_A = {x_0^{\alpha(A)}}_A : A \to \mathrm{R}A$. Let $f : A \to B$ be a map in $\mathscr{C}$. Let $\beta = \max\{\alpha(A), \alpha(B)\}$. We define $\mathrm{R}f : \mathrm{R}A \to \mathrm{R}B$ as the map $\mathrm{X}_\beta f : \mathrm{X}_\beta A \to \mathrm{X}_\beta B$ composed with the isomorphisms $\mathrm{X}_{\alpha(A)}A \cong \mathrm{X}_\beta A$ and $\mathrm{X}_\beta B \cong \mathrm{X}_{\alpha(B)}B$. Then $\mathrm{R} : \mathscr{C} \to \mathscr{C}$ is an endofunctor and $\eta : Id \to \mathrm{R}$ is a natural transformation. If we define $m_A = \left(x_{\alpha(A)}^{\alpha(A)+1}\right)_A^{-1} \circ \pi_{\alpha(A)_A} : \mathrm{T}\mathrm{X}_{\alpha(A)}A \to \mathrm{X}_{\alpha(A)}A$ on each object A, then m is a natural transformation $\mathrm{TR} \to \mathrm{R}$.

Just as for (small) diagrams, an endofunctor is the colimit of a large connected diagram in $\mathbf{pEnd}(\mathscr{C})$ if it is the objectwise colimit of the large diagram. So $(\mathrm{R}, \eta) = \operatorname{colim} \mathrm{X}_\bullet^\mathrm{T}$. Then $\eta = x_0^\infty$. Since the large diagram $(\mathrm{T}\mathrm{X}_\alpha)_\alpha$ also converges objectwise, its colimit exists.

It is easy to check that conditions (1) and (2) of definition 2.3.19 hold on each object A. Therefore they hold in $\mathbf{End}(\mathscr{C})$. □

2.4 Objectwise Convergence of Free Monad Sequences

In this section, we will describe a smallness condition that we can place on a pointed endofunctor to guarantee that its free monad sequence converges objectwise. This main result is theorem 2.4.23. The variations of this smallness condition play a central role in nearly all of the results in this thesis.

Because *left proper orthogonal factorization systems* are key to defining the smallness condition, we spend section 2.4.1 defining them and reviewing key facts about them. We then proceed in a similar manner to section 2.3. It is much easier to show that the smallness condition on a pointed endofunctor (S,σ) implies the free monad sequence for (S,σ) converges objectwise when we know that (S,σ) is well-pointed. So we handle this case first in section 2.4.2. Section 2.4.3 then uses the same reduction we used in sections 2.3.2 and 2.3.3. We show that when (T,τ) is a pointed endofunctor on a category $\mathscr{C}$ that satisfies the smallness condition, then the well-pointed endofunctor (S,σ) on $\mathrm{T}\downarrow\mathscr{C}$ that we constructed in section 2.3.2 satisfies a related smallness condition. So the free monad sequence for (S,σ) converges objectwise and thus the free monad sequence for (T,τ) converges objectwise.

Finally, in section 2.4.4, we describe a particular subcategory of small endofunctors in $\mathbf{pEnd}(\mathscr{C})$. Not only do all of the objects in this subcategory have reflections in $\mathbf{Mnd}(\mathscr{C})$, but their reflections are themselves endofunctors that satisfy the same smallness condition. So the forgetful functor $U : \mathbf{Mnd}(\mathscr{C}) \to \mathbf{pEnd}(\mathscr{C})$ restricts to a functor that has a left adjoint. This fact will play an important role in some results in later chapters.

2.4.1 Orthogonal Factorization Systems

Let $\mathcal{X}$ be a collection of maps in a category $\mathscr{C}$. A map f in $\mathscr{C}$ has the *left lifting property* with respect to $\mathcal{X}$ if for each $g \in \mathcal{X}$, there is a map $l : Y \to A$ that makes the following diagram commute. A map g in $\mathscr{C}$ has the *right lifting property* with respect to $\mathcal{X}$ if for each $f \in \mathcal{X}$, there is a map $l : Y \to A$ that makes the following diagram commute.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & A \arrow[d, "g"] \\
Y \arrow[r, "v"'] \arrow[ur, dashed, "l"] & B
\end{tikzcd} \tag{2.8}$$

When a lift l in diagram (2.8) exists, we will say l is a *solution to the lifting problem* $(u,v) : f \to g$. We will use the notation ${}^{\square}\mathcal{X}$ for the collection of maps in $\mathscr{C}$ with the left lifting property with respect to $\mathcal{X}$ and $\mathcal{X}^{\square}$ for the collection of maps with the right lifting property with respect to $\mathcal{X}$.

For any collection of maps $\mathcal{X}$, $\mathcal{X}^{\square} = \big({}^{\square}(\mathcal{X}^{\square})\big)^{\square}$ and ${}^{\square}\mathcal{X} = {}^{\square}\big(({}^{\square}\mathcal{X})^{\square}\big)$. We also have the following properties.

- Every isomorphism in $\mathscr{C}$ is contained in both $\mathcal{X}^{\square}$ and ${}^{\square}\mathcal{X}$.
- The collections $\mathcal{X}^{\square}$ and ${}^{\square}\mathcal{X}$ are stable under retracts in the arrow category $\mathscr{C}^{\mathbf{2}}$.
- The collection $\mathcal{X}^{\square}$ is stable under composition and the collection ${}^{\square}\mathcal{X}$ is stable under transfinite composition in $\mathscr{C}$.
- The collection $\mathcal{X}^{\square}$ is stable under pullbacks and the collection ${}^{\square}\mathcal{X}$ is stable under pushouts in $\mathscr{C}$.
- The collection $\mathcal{X}^{\square}$ is stable under products and the collection ${}^{\square}\mathcal{X}$ is stable under coproducts in the arrow category $\mathscr{C}^{\mathbf{2}}$.

Definition 2.4.1. An *orthogonal factorization system* on a category $\mathscr{C}$ is a pair $(\mathcal{L},\mathcal{R})$ of collections of maps in $\mathscr{C}$ such that

- $\mathcal{L}^{\square} = \mathcal{R}$ and $\mathcal{L} = {}^{\square}\mathcal{R}$,
- if $f \in \mathcal{L}$ and $g \in \mathcal{R}$, then the lift l in diagram (2.8) is unique, and
- every map f in $\mathscr{C}$ factors as $f = i \circ p$, where $i \in \mathcal{R}$ and $p \in \mathcal{L}$.

The factorization $f = i \circ p$ in an orthogonal factorization system is always functorial. Indeed, in the commutative square on the left, if f and g factor as $f = i_f \circ p_f$ and $g = i_g \circ p_g$

for $i_f, i_g \in \mathcal{R}$ and $p_f, p_g \in \mathcal{L}$, then there is a unique lift in the right rectangle.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & A \arrow[d, "g"] \\
Y \arrow[r, "v"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "i_f"] & A \arrow[r, "i_g"] & Eg \arrow[d, "p_g"] \\
Ef \arrow[r, "p_f"'] \arrow[urr, dashed, "\exists!"] & Y \arrow[r, "v"'] & B
\end{tikzcd}$$

When $\mathscr{D}$ is a small category and $\mathcal{X}$ is a collection of maps in $\mathscr{C}$, we will use the notation $\mathcal{X}^{\mathscr{D}}$ for the collection of maps in the functor category $\mathscr{C}^{\mathscr{D}}$ that are objectwise $\mathcal{X}$-maps. Using the functoriality of orthogonal factorization systems and the uniqueness of lifts in orthogonal factorization systems, it is easy to prove the following.

Proposition 2.4.2. *If* $(\mathcal{E}, \mathcal{M})$ *is an orthogonal factorization system on* $\mathscr{C}$ *and* $\mathscr{D}$ *is a small category, then* $(\mathcal{E}^{\mathscr{D}}, \mathcal{M}^{\mathscr{D}})$ *is an orthogonal factorization system on* $\mathscr{C}^{\mathscr{D}}$.

Proposition 2.4.3. *Let* $(\mathcal{L}, \mathcal{R})$ *be an orthogonal factorization system on a category* $\mathscr{C}$.

- *If* $D : \mathscr{D} \to \mathscr{C}^{\mathbf{2}}$ *is a diagram whose objects are in* $\mathcal{L}$*, then* $\operatorname{colim} D \in \mathcal{L}$ *when it exists.*
- *If* $D : \mathscr{D} \to \mathscr{C}^{\mathbf{2}}$ *is a diagram whose objects are in* $\mathcal{R}$*, then* $\lim D \in \mathcal{R}$ *when it exists.*

Proof. The proof is somewhat similar to the proof of 2.1.25. □

Definition 2.4.4. An orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathscr{C}$ is *left proper* if every map in $\mathcal{E}$ is an epimorphism, *right proper* if every map in $\mathcal{M}$ is a monomorphism, and *proper* if it is both left proper and right proper.

Proposition 2.4.5. *Let* $(\mathcal{E}, \mathcal{M})$ *be an orthogonal factorization system on* $\mathscr{C}$ *and let* $h = g \circ f$.

- *If* $(\mathcal{E}, \mathcal{M})$ *is right proper and* $h \in \mathcal{E}$*, then* $g \in \mathcal{E}$.
- *If* $(\mathcal{E}, \mathcal{M})$ *is left proper,* $h \in \mathcal{E}$*, and* $f \in \mathcal{E}$*, then* $g \in \mathcal{E}$.
- *If* $(\mathcal{E}, \mathcal{M})$ *is left proper and* $h \in \mathcal{M}$*, then* $f \in \mathcal{M}$.

- *If $(\mathcal{E}, \mathcal{M})$ is right proper, $h \in \mathcal{M}$, and $g \in \mathcal{M}$, then $f \in \mathcal{M}$.*

An easy example of a left proper orthogonal factorization system that exists on any category $\mathscr{C}$ is when $\mathcal{E}$ is the collection of all isomorphisms in $\mathscr{C}$ and $\mathcal{M}$ is the collection of all maps in $\mathscr{C}$. Dually, when $\mathcal{M}$ is the collection of all isomorphisms in $\mathscr{C}$ and $\mathcal{E}$ is the collection of all maps in $\mathscr{C}$, then $(\mathcal{E}, \mathcal{M})$ is a right proper orthogonal factorization system. Under fairly weak conditions on $\mathscr{C}$, we can get less trivial proper orthogonal factorization systems on $\mathscr{C}$. To state this result, we will need some definitions.

Definition 2.4.6. A map f in a category $\mathscr{C}$ is a *strong epimorphism* if it is an epimorphism and a lift l exists in diagram (2.8) whenever g is a monomorphism. A map g in $\mathscr{C}$ is a *strong monomorphism* if it is a monomorphism and a lift l exists in diagram (2.8) whenever f is an epimorphism.

In a given category $\mathscr{C}$, we will use the notation $\mathcal{M}^{\downarrowtail}$, $\mathcal{M}^{s\downarrowtail}$, $\mathcal{E}^{\twoheaddownarrow}$, and $\mathcal{E}^{s\twoheaddownarrow}$ for the classes of monomorphisms, strong monomorphisms, epimorphisms, and strong epimorphisms, respectively.

Definition 2.4.7. Let $\mathcal{M}$ be a subcollection of the monomorphisms in a category $\mathscr{C}$. An $\mathcal{M}$-*subobject* of an object X is an $\mathcal{M}$-map $Y \to X$. Two $\mathcal{M}$-subobjects $f : Y \to X$ and $g : Z \to X$ of X are in the same isomorphism class if there is an isomorphism $h : Y \to Z$ such that $f = g \circ h$. The category $\mathscr{C}$ is $\mathcal{M}$-*well-powered* if for each object X in $\mathscr{C}$, the collection of $\mathcal{M}$-subobjects of X has only a set of isomorphism classes.

Definition 2.4.8. Let $\mathcal{E}$ be a subcollection of the epimorphisms in a category $\mathscr{C}$. An $\mathcal{E}$-*quotient* of an object X is an $\mathcal{E}$-map $X \to Y$. Two $\mathcal{E}$-quotients $f : X \to Y$ and $g : X \to Z$ of X are in the same isomorphism class if there is an isomorphism $h : Y \to Z$ such that $g = h \circ f$.

The category $\mathscr{C}$ is *$\mathcal{E}$-well-copowered* if for each object X in $\mathscr{C}$, the collection of $\mathcal{E}$-quotients of X has only a set of isomorphism classes.

We will say a left proper orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathscr{C}$ is *well-copowered* if $\mathscr{C}$ is $\mathcal{E}$-well-copowered.

As we already saw, we will say a category is well-powered when it is $\mathcal{M}^{\downarrowtail}$-well-powered. Similarly, we will say a category is well-copowered when it is $\mathcal{E}^{\twoheaddownarrow}$-well-copowered.

Proposition 2.4.9 ([Bor94b, 4.4.3])**.**

- *If $\mathscr{C}$ is a complete well-powered category, then $(\mathcal{E}^{s\twoheaddownarrow}, \mathcal{M}^{\downarrowtail})$ is a proper orthogonal factorization system on $\mathscr{C}$.*
- *If $\mathscr{C}$ is a cocomplete well-copowered category, then $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail})$ is a proper orthogonal factorization system on $\mathscr{C}$.*

Dualizing the notation in section 2.1.3, for any object X in $\mathscr{C}$ and any collection $\mathcal{X}$ of maps in $\mathscr{C}$, $X \downarrow_{\mathcal{X}} \mathscr{C}$ will denote the full subcategory of the comma category $X \downarrow \mathscr{C}$ on the objects (f, Y) such that $f : X \to Y$ is an $\mathcal{X}$-map in $\mathscr{C}$.

Lemma 2.4.10. *Let $(\mathcal{E}, \mathcal{M})$ be a left proper strong factorization system on a well-copowered category $\mathscr{C}$. For each object X in $\mathscr{C}$, the category $X \downarrow_{\mathcal{E}} \mathscr{C}$ is equivalent to a partially ordered set.*

Proof. Since X has only a set's worth of $\mathcal{E}$-quotients, $X \downarrow_{\mathcal{E}} \mathscr{C}$ only has a set of isomorphism classes of objects. Let $u, v : (f, C_1) \to (g, C_2)$ be parallel maps in $X \downarrow_{\mathcal{E}} \mathscr{C}$. Since $u \circ f = g = v \circ f$ and f is an epimorphism, $u = v$. So whenever there is a map $(C_1, f) \to (C_2, g)$ between two objects, this map is unique.

We can now identify the existence of a map $(f,C_1)\to(g,C_2)$ with the relation $(f,C_1)\le(g,C_2)$. The fact that $X\downarrow_{\mathcal{E}}\mathscr{C}$ is a category means that $\le$ is reflexive and transitive. It remains to show that $\le$ is antisymmetric. If there is a map $u:(f,C_1)\to(g,C_2)$ and a map $v:(g,C_2)\to(f,C_1)$, then $v\circ u\circ f=f$ and $u\circ v\circ g=g$. So $v\circ u=id$ and $u\circ v=id$. □

Remark 2.4.11. If $\mathscr{C}$ is $\mathcal{E}$-well-copowered, then every transfinite sequence $X:\mathbf{Ord}\to\mathscr{C}$ whose maps $X(0\to\alpha):X(0)\to X(\alpha)$ are $\mathcal{E}$-maps must converge. Indeed, by 2.4.10, X can be identified with a large sequence in a partially ordered set. Since this sequence will exhaust all possible values in the partially ordered set otherwise, it must converge.

2.4.2 Convergence Results for Well-Pointed Endofunctors

Our goal in this section is to describe conditions under which the free monad sequence on a well-pointed endofunctor (S,σ) converges objectwise. As we saw in section 2.3.3, the objectwise convergence of this sequence implies the free monad on (S,σ) exists.

Let (S,σ) be a well-pointed endofunctor on a cocomplete category $\mathscr{C}$. Suppose $(\mathcal{E},\mathcal{M})$ and $(\mathcal{E}',\mathcal{M}')$ are left proper orthogonal factorization systems on $\mathscr{C}$ and that $\mathscr{C}$ is both $\mathcal{E}$-well-copowered and $\mathcal{E}'$-well-copowered.

To construct the free monad on (S,σ), we will need a sort of smallness condition on the functor $\mathrm{S}:\mathscr{C}\to\mathscr{C}$. Let λ be a regular cardinal. For the remainder of this section, we require that

$$\mathrm{S}:\mathscr{C}\to\mathscr{C}\text{ preserves }\mathcal{E}\text{-tightness of }(\mathcal{M}',\lambda)\text{-cocones.}$$

Refer to section 2.1.3 for the definitions of these terms. This is the most general smallness condition we work with.

Remark 2.4.12. We could actually be a little bit more general by only requiring that S preserves $\mathcal{E}$-tightness of λ-sequential $\mathcal{M}'$-cocones. This condition will be sufficient to prove objectwise convergence of the free monad sequence and by 2.1.17, this condition is more general than the one above. Since we wish to keep notation consistent throughout this thesis, and since filtered cocones are essential to the content of chapters 6 and 7, we choose to use the stronger condition.

We will use various strengthenings of the smallness condition throughout this thesis. One strengthening is the condition that $\mathrm{S} : \mathscr{C} \to \mathscr{C}$ preserves $\mathcal{E}$-tightness of all λ-filtered cocones. A special case is when S preserves colimits of λ-filtered cocones. Another is that $\mathrm{S} : \mathscr{C} \to \mathscr{C}$ sends $\mathcal{E}$-tight $(\mathcal{M}, \lambda)$-cocones to $\mathcal{E}$-tight $\mathcal{M}$-cocones.

The following proposition gives a condition for the convergence of the free S-algebra sequence on an object X. As we saw in 2.3.5, when this sequence converges, it converges to a free S-algebra on X.

Proposition 2.4.13 ([Kel80, 6.1])**.** *Let X be an object in $\mathscr{C}$. Suppose there is a limit ordinal γ such that the following condition is satisfied: For every ordinal $\beta \geq \gamma$, if the cocone $\left\{\mathrm{S}^{\beta}_{\alpha X} : \mathrm{S}^{\alpha}X \to \mathrm{S}^{\beta}X\right\}_{\alpha<\gamma}$ is $\mathcal{E}$-tight, then the cocone $\left\{\mathrm{S}\mathrm{S}^{\beta}_{\alpha X} : \mathrm{S}\mathrm{S}^{\alpha}X \to \mathrm{S}\mathrm{S}^{\beta}X\right\}_{\alpha<\gamma}$ is $\mathcal{E}$-tight. Then the free* S*-algebra sequence for X converges.*

Proof. Since $\mathrm{S}^{\bullet}X : \mathbf{Ord} \to \mathscr{C}$ is cocontinuous, $\left\{\mathrm{S}^{\gamma}_{\alpha X} : \mathrm{S}^{\alpha}X \to \mathrm{S}^{\gamma}X\right\}_{\alpha<\gamma}$ is a colimiting cocone. So it is an $\mathcal{E}$-tight cocone in particular. By our hypothesis, the cocone $\left\{\mathrm{S}^{\gamma+1}_{\alpha+1X} : \mathrm{S}^{\alpha+1}X \to \mathrm{S}\mathrm{S}^{\gamma+1}X\right\}_{\alpha<\gamma}$ is $\mathcal{E}$-tight. But this cocone is a final subcocone of $\left\{\mathrm{S}^{\gamma+1}_{\alpha}{}_{X} : \mathrm{S}^{\alpha}X \to \mathrm{S}\mathrm{S}^{\gamma+1}X\right\}_{\alpha<\gamma}$. So $\{\mathrm{S}^{\gamma+1}_{\alpha}{}_{X}\}_{\alpha<\gamma}$ is an $\mathcal{E}$-tight cocone. Continuing in this way, a transfinite inductive argument shows that $\left\{\mathrm{S}^{\beta}_{\alpha X} : \mathrm{S}^{\alpha}X \to \mathrm{S}^{\beta}X\right\}_{\alpha<\gamma}$ is an $\mathcal{E}$-tight cocone for each $\beta \geq \gamma$. In other words, $\mathrm{S}^{\beta}_{\gamma X} : \mathrm{S}^{\gamma}X \to \mathrm{S}^{\beta}X$ is an $\mathcal{E}$-map for each $\beta \geq \gamma$.

We've shown that for each $\beta \geq \gamma$, the transfinite sequence $\mathrm{S}^{\bullet}X : \mathbf{Ord} \to \mathscr{C}$, sends the map $\gamma \to \beta$ to an $\mathcal{E}$-map. So after reindexing, our remarks in 2.4.11 apply to $\mathrm{S}^{\bullet}X$. So the free S-algebra sequence for X converges. □

If $\mathrm{S} : \mathscr{C} \to \mathscr{C}$ preserves $\mathcal{E}$-tightness of all λ-filtered cocones, then it is easy to see that the hypothesis of the above proposition holds. Indeed, we can take γ to be a regular ordinal with $|\gamma| > \lambda$ to ensure that any cocone indexed by $\mathbf{Ord}_{<\gamma}$ is λ-filtered. If we only know that S preserves $\mathcal{E}$-tightness of $(\mathcal{M}', \lambda)$-cocones, then we will need to do some more work to show the hypothesis of proposition 2.4.13 holds. The following two technical lemmas make use of our smallness condition on S to show exactly that.

Remark 2.4.14. If $\Lambda : \mathbf{Ord} \to \mathbf{Ord}$ is a strictly increasing cocontinuous functor, then the restriction $\Lambda : \mathbf{Ord}_{<\gamma} \to \mathbf{Ord}_{<\Lambda(\gamma)}$ is a final functor for every limit ordinal γ. Since Λ is strictly increasing, $\Lambda(\alpha) < \Lambda(\gamma)$ for every ordinal $\alpha < \gamma$. Since Λ is also cocontinuous, $\alpha \leq \Lambda(\alpha)$ for every ordinal $\alpha < \Lambda(\gamma)$. The result is then an immediate consequence of A.2.3.

We will use the successor functor $^{+} : \mathbf{Ord} \to \mathbf{Ord}$ that sends each ordinal α to $\alpha + 1$ and sends each map $\alpha \to \beta$ to the map $\alpha + 1 \to \beta + 1$. There is a natural transformation $\varsigma : \mathrm{Id} \to {}^{+}$ defined on each ordinal α as the map $\alpha \to \alpha + 1$. We will abbreviate $\alpha + 1$ as α^{+} so that ${}^{+}(\alpha) = \alpha^{+}$.

Lemma 2.4.15 ([Kel80, 4.1])**.** *For every functor $F : \mathbf{Ord} \to \mathscr{C}$, there is a strictly increasing cocontinuous functor $\Lambda : \mathbf{Ord} \to \mathbf{Ord}$, a functor $G : \mathbf{Ord} \to \mathscr{C}$, and a natural transformation $\theta : G \to F\Lambda^{+}$ such that the following conditions hold.*

1. *For every ordinal $\beta > \Lambda(\alpha^{+})$, the map*

$$F(\Lambda(\alpha^{+}) \to \beta) \circ \theta_{\alpha} : G(\alpha) \to F(\beta)$$

   is an $\mathcal{M}'$-map.

2. *For any endofunctor $\mathrm{T} : \mathscr{C} \to \mathscr{C}$ and any limit ordinal γ, the map*

$$\operatorname*{colim}_{\alpha<\gamma} \mathrm{T}\theta_{\alpha} : \operatorname*{colim}_{\alpha<\gamma} \mathrm{T}G(\alpha) \to \operatorname*{colim}_{\alpha<\gamma} \mathrm{T}F\Lambda(\alpha^{+})$$

is an isomorphism.

Proof. We will define $\Lambda : \mathbf{Ord} \to \mathbf{Ord}$ inductively. Let $\Lambda(0) = 0$. Since we want Λ to be cocontinuous, it is enough to define it on successor ordinals. Let α be an ordinal and suppose $\Lambda(\alpha)$ is defined. For each $\beta > \Lambda(\alpha)$, we apply the $(\mathcal{E}', \mathcal{M}')$ factorization to the map $F(\Lambda(\alpha) \to \beta) : F\Lambda(\alpha) \to F(\beta)$. We end up with a large sequence of $\mathcal{E}'$-quotients of $F\Lambda(\alpha)$ indexed by $\beta > \Lambda(\alpha)$. By our observation in 2.4.11, this sequence converges. Let $\Lambda(\alpha + 1)$ be the first β at which this happens. So there is a factorization

$$\begin{tikzcd}
F\Lambda(\alpha) \arrow[r, "\psi_\alpha"] \arrow[rr, bend left, "F\Lambda(\alpha\to\alpha+1)"] & G(\alpha) \arrow[r, "\theta_\alpha"] & F\Lambda(\alpha+1)
\end{tikzcd}$$

with $\psi_\alpha \in \mathcal{E}'$ and $\theta_\alpha \in \mathcal{M}'$.

The convergence of the sequence of $\mathcal{E}'$-quotients of $F\Lambda(\alpha)$ implies that $F(\Lambda(\alpha+1) \to \beta) \circ \theta_\alpha : G(\alpha) \to F(\beta)$ is an $\mathcal{M}'$-map for each $\beta > \Lambda(\alpha+1)$. The maps $\psi_{\alpha+1} \circ \theta_\alpha : G(\alpha) \to G(\alpha+1)$ and $\psi_\gamma \circ F(\Lambda(\alpha) \to \Lambda(\gamma)) \circ \theta_\alpha : G(\alpha) \to G(\gamma)$ for a limit ordinal γ make G a functor $G : \mathbf{Ord} \to \mathscr{C}$. Then $\psi : F\Lambda \to G$ and $\theta : G \to F\Lambda^+$ are natural transformations.

The following diagram of natural transformations commutes.

$$\begin{tikzcd}
G \arrow[r, "G\varsigma"] \arrow[d, "\theta"] & G^+ \arrow[d, "\theta^+"] \\
F\Lambda^+ \arrow[r, "F\Lambda^+\varsigma"] \arrow[ur, "\psi^+"] & F\Lambda^{++}
\end{tikzcd}$$

We apply T to the above diagram, restrict to $\mathbf{Ord}_{<\gamma}$ for a limit ordinal γ, and then take the colimit of the entire diagram over $\mathbf{Ord}_{<\gamma}$. Since

$$\operatorname*{colim}_{\alpha<\gamma} \mathrm{T}G\varsigma : \operatorname*{colim}_{\alpha<\gamma} \mathrm{T}G(\alpha) \to \operatorname*{colim}_{\alpha<\gamma} \mathrm{T}G(\alpha)$$

and

$$\operatorname*{colim}_{\alpha<\gamma} \mathrm{T}F\Lambda^+\varsigma : \operatorname*{colim}_{\alpha<\gamma} \mathrm{T}F\Lambda(\alpha) \to \operatorname*{colim}_{\alpha<\gamma} \mathrm{T}F\Lambda(\alpha)$$

are identity maps, we know that $\operatorname{colim}_{\alpha<\gamma} \mathrm{T}\psi$ is an inverse for $\operatorname{colim}_{\alpha<\gamma} \mathrm{T}\theta$. □

Lemma 2.4.16 ([Kel80, 4.2])**.** *Let $F : \mathbf{Ord} \to \mathscr{C}$ be a functor. There is a limit ordinal γ such that for each $\beta \geq \gamma$, if the cocone*

$$\Big\{F(\alpha \to \beta) : F(\alpha) \to F(\beta)\Big\}_{\alpha<\gamma}$$

is $\mathcal{E}$-tight, then the cocone

$$\Big\{SF(\alpha \to \beta) : SF(\alpha) \to SF(\beta)\Big\}_{\alpha<\gamma}$$

is $\mathcal{E}$-tight.

Proof. Let Λ, G, and $\theta : G \to F\Lambda^+$ be the objects defined in 2.4.15. Let γ_0 be a regular ordinal with $|\gamma_0| > \lambda$. Then γ_0 and $\Lambda(\gamma_0)$ are limit ordinals.

Suppose $\big\{F(\alpha \to \beta) : F(\alpha) \to F(\beta)\big\}_{\alpha<\Lambda(\gamma_0)}$ is $\mathcal{E}$-tight. Then by our observation in 2.4.14, $\big\{F(\Lambda(\alpha) \to \beta) : F\Lambda(\alpha) \to F(\beta)\big\}_{\alpha<\gamma_0}$ is $\mathcal{E}$-tight. So the cocone $\big\{F(\Lambda(\alpha^+) \to \beta) : F\Lambda(\alpha^+) \to F(\beta)\big\}_{\alpha<\gamma_0}$ is also $\mathcal{E}$-tight. An application of 2.4.15 (2) with T = Id to this cocone now tells us that the cocone

$$\Big\{F(\Lambda(\alpha^+) \to \beta) \circ \theta_\alpha : G(\alpha) \to F(\beta)\Big\}_{\alpha<\gamma_0}$$

is $\mathcal{E}$-tight. We know this last cocone is an $\mathcal{M}'$-cocone. Furthermore, since γ_0 is a regular ordinal with $|\gamma_0| > \lambda$, no λ-small diagram in $\mathbf{Ord}_{<\gamma_0}$ is final. So the last cocone is a λ-filtered cocone.

By our assumption on S, the cocone

$$\Big\{SF(\Lambda(\alpha^+) \to \beta) \circ S\theta_\alpha : SG(\alpha) \to SF(\beta)\Big\}_{\alpha<\gamma_0}$$

is $\mathcal{E}$-tight. Now 2.4.15 (2) with T = S tells us that $\big\{SF(\Lambda(\alpha^+) \to \beta) : SF\Lambda(\alpha^+) \to SF(\beta)\big\}_{\alpha<\gamma_0}$ and therefore $\big\{SF(\Lambda(\alpha) \to \beta) : SF\Lambda(\alpha) \to SF(\beta)\big\}_{\alpha<\gamma_0}$ is $\mathcal{E}$-tight. So by 2.4.14, $\big\{SF(\alpha \to \beta) : SF(\alpha) \to SF(\beta)\big\}_{\alpha<\Lambda(\gamma_0)}$ is an $\mathcal{E}$-tight cocone. □

We now have all of the prerequisites for the proof of our main result of this section.

Proposition 2.4.17. *On each object X in $\mathscr{C}$, the free* S*-algebra sequence for X converges.*

Proof. By lemma 2.4.16, S satisfies the hypothesis of proposition 2.4.13 for every object X in $\mathscr{C}$. □

Remark 2.4.18. It is easy to check that all of the above propositions go through under the weaker assumption that S preserves $\mathcal{E}$-tightness of λ-sequential $\mathcal{M}'$-cocones.

2.4.3 Convergence Results for Pointed Endofunctors

Let (T, τ) be a pointed endofunctor on a cocomplete category $\mathscr{C}$. We will show in 2.4.23 that the smallness condition of the previous section on the endofunctor T guarantees the objectwise converges of the free monad sequence on (T, τ). Our method for showing this result is to reduce it to the case of a well-pointed endofunctor.

Let (S, σ) be the well-pointed endofunctor on $\mathrm{T} \downarrow \mathscr{C}$ constructed in the section 2.3.2. We need to show that the free monad sequence on (S, σ) converges objectwise. Then by 2.3.17, the free monad sequence on (T, τ) converges objectwise. We already saw that $\mathrm{T} \downarrow \mathscr{C}$ is a cocomplete category. So, by 2.4.17, to construct a free monad on (S, σ), we only need that $\mathrm{S} : \mathrm{T} \downarrow \mathscr{C} \to \mathrm{T} \downarrow \mathscr{C}$ satisfies the smallness condition with respect to two orthogonal factorization systems on $\mathrm{T} \downarrow \mathscr{C}$.

In 2.4.21 we will see how an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathscr{C}$ defines an orthogonal factorization system $(\mathcal{E}_\mathrm{T}, \mathcal{M}_\mathrm{T})$ on $\mathrm{T} \downarrow \mathscr{C}$. We will then show in 2.4.22 that when $\mathrm{T} : \mathscr{C} \to \mathscr{C}$ preserves $\mathcal{E}$-tightness of $(\mathcal{M}', \lambda)$-cocones for a regular cardinal λ and two orthogonal factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ on $\mathscr{C}$, then $\mathrm{S} : \mathrm{T} \downarrow \mathscr{C} \to \mathrm{T} \downarrow \mathscr{C}$ preserves $\mathcal{E}_T$-tightness of $(\mathcal{M}'_\mathrm{T}, \lambda)$-cocones.

Let $(\mathcal{E}, \mathcal{M})$ be an orthogonal factorization system on a category $\mathscr{C}$ and let (T, τ) be a pointed endofunctor on $\mathscr{C}$. We will use the notation $\mathcal{M}_{\mathrm{T}}$ for the class of maps (u, v) in $\mathrm{T} \downarrow \mathscr{C}$ such that both u and v are in $\mathcal{M}$. We will use the notation $\mathcal{E}_{\mathrm{T}}$ for the class of maps $(u, v) : (X, f, Y) \to (A, g, B)$ in $\mathrm{T} \downarrow \mathscr{C}$ such that $u \in \mathcal{E}$ and v factors as the map p defined by the pushout in $\mathscr{C}$ of $\mathrm{T}u$ along f followed by a map $r \in \mathcal{E}$, as shown in the following diagram.

$$\begin{tikzcd}
\mathrm{T}X \arrow[r, "\mathrm{T}u"] \arrow[d, "f"] & \mathrm{T}A \arrow[d, "h"] \arrow[dr, bend left, "g"] & \\
Y \arrow[r, "p"] \arrow[rr, bend right, "v"'] & Z \arrow[r, dashed, "r"] \arrow[ul, phantom, "\ulcorner", very near start] & B
\end{tikzcd} \tag{2.9}$$

Lemma 2.4.19 ([Kel80, 15.3])**.** *If the orthogonal factorization system* $(\mathcal{E}, \mathcal{M})$ *is a left proper, then every map in* $\mathcal{E}_{\mathrm{T}}$ *is an epimorphism in* $\mathrm{T} \downarrow \mathscr{C}$*.*

Proof. Let $(u, v) : (X, f, Y) \to (A, g, B)$ be a map in $\mathcal{E}_{\mathrm{T}}$. We will factor (u, v) as in diagram (2.9). Suppose (a, b) and (c, d) are parallel maps $(A, g, B) \to (C, k, D)$ in $\mathrm{T} \downarrow \mathscr{C}$ such that $(a, b) \circ (u, v) = (c, d) \circ (u, v)$. Then $a \circ u = c \circ u : X \to C$ and $b \circ v = d \circ v : Y \to D$ as maps in $\mathscr{C}$. Since $u \in \mathcal{E}$, it is an epimorphism and $a = c$. So it remains to show that $b = d$. Since r is an epimorphism, it suffices to show $b \circ r = d \circ r$. Since Z is a colimit, two maps $b \circ r$ and $d \circ r$ out of Z agree if and only if $b \circ r \circ p = d \circ r \circ p$ and $b \circ r \circ h = d \circ r \circ h$. We've already determined that the first equality holds. Since $a = c$, we know $b \circ r \circ h = b \circ g = k \circ \mathrm{T}a = k \circ \mathrm{T}c = d \circ g = d \circ r \circ h$. ☐

Lemma 2.4.20 ([Kel80, 15.2])**.** *Every map in* $\mathrm{T} \downarrow \mathscr{C}$ *factors as a map* $(u, v) : (X, f, Y) \to (A, g, B)$ *in* $\mathcal{E}_{\mathrm{T}}$ *followed by a map* $(s, t) : (A, g, B) \to (C, k, D)$ *in* $\mathcal{M}_{\mathrm{T}}$*.*

Proof. Let $(x, y) : (X, f, Y) \to (C, k, D)$ be a map in $\mathrm{T} \downarrow \mathscr{C}$ First, we factor $x : X \to C$ in $\mathscr{C}$ as a map $u : X \to A$ in $\mathcal{E}$ followed by a map $s : A \to X$ in $\mathcal{M}$. Next, we factor v as in the following diagram.

$$\begin{tikzcd}
\mathrm{T}X \arrow[r, "\mathrm{T}u"] \arrow[d, "f"] & \mathrm{T}A \arrow[d, "h"] \arrow[dr, bend left, "k\circ\mathrm{T}s"] & \\
Y \arrow[r, "p"] \arrow[rr, bend right, "v"'] & Z \arrow[r, dashed, "q"] \arrow[ul, phantom, "\ulcorner", very near start] & D
\end{tikzcd}$$

Then we factor q as a map $r : Z \to B$ in $\mathcal{E}$ followed by a map $t : B \to D$ in $\mathcal{M}$. So $(u, r \circ p) : (X, f, Y) \to (A, r \circ h, B)$ is a map in $\mathcal{E}_{\mathrm{T}}$ and $(s, t) : (A, r \circ h, B) \to (C, k, D)$ is a map in $\mathcal{M}_{\mathrm{T}}$. ☐

Proposition 2.4.21 ([Kel80, 15.1, 15.2])**.** *If the orthogonal factorization system* $(\mathcal{E}, \mathcal{M})$ *is left proper, then the pair* $(\mathcal{E}_{\mathrm{T}}, \mathcal{M}_{\mathrm{T}})$ *is a left proper orthogonal factorization system on* $\mathrm{T}\downarrow\mathscr{C}$*.*

Proof. By lemmas 2.4.19 and 2.4.20, we only need to check $\mathcal{E}_{\mathrm{T}} = {}^{\square}\mathcal{M}_{\mathrm{T}}$ and $\mathcal{E}_{\mathrm{T}}^{\square} = \mathcal{M}_{\mathrm{T}}$. Actually, since 2.4.20 tells us a factorization for $(\mathcal{E}_{\mathrm{T}}, \mathcal{M}_{\mathrm{T}})$ exists, it suffices to show $\mathcal{E}_{\mathrm{T}} \subseteq {}^{\square}\mathcal{M}_{\mathrm{T}}$. Consider the following commutative diagram in $\mathrm{T}\downarrow\mathscr{C}$.

$$\begin{tikzcd}
(X, f, Y) \arrow[r, "{(a,b)}"] \arrow[d, "{(u,v)}"'] & (U, j, V) \arrow[d, "{(s,t)}"] \\
(A, g, B) \arrow[r, "{(c,d)}"] & (C, k, D)
\end{tikzcd} \tag{2.10}$$

Suppose $(u, v) \in \mathcal{E}_{\mathrm{T}}$ and $(s, t) \in \mathcal{M}_{\mathrm{T}}$. Then a solution $l : A \to U$ to the lifting problem $(a, c) : u \to s$ exists. Using the notation of diagram (2.9), the maps $j \circ \mathrm{T}l : \mathrm{T}A \to V$ and $b : Y \to V$ define a map $m : Z \to V$ such that $m \circ h = j \circ \mathrm{T}l$ and the following diagram commutes.

$$\begin{tikzcd}
Y \arrow[rr, "b"] \arrow[d, "p"] & & V \arrow[d, "t"] \\
Z \arrow[r, "r"'] \arrow[urr, dashed, "m"] & B \arrow[r, "d"'] & D
\end{tikzcd}$$

Since $r \in \mathcal{E}$ and $t \in \mathcal{M}$, a solution $n : B \to V$ to the lifting problem $(m, d) : r \to t$ exists. So $(l, n) : (A, g, B) \to (U, j, V)$ is a lift in diagram (2.10). ☐

Proposition 2.4.22 ([Kel80, 15.5])**.** *Let* (T, τ) *be a pointed endofunctor on a category* $\mathscr{C}$*. Let* $(\mathcal{E}, \mathcal{M})$ *and* $(\mathcal{E}', \mathcal{M}')$ *be left proper orthogonal factorization systems on* $\mathscr{C}$*. If there is a regular cardinal* λ *such that* $\mathrm{T} : \mathscr{C} \to \mathscr{C}$ *preserves* $\mathcal{E}$*-tightness of* $(\mathcal{M}', \lambda)$*-cocones, then the well-pointed endofunctor* (S, σ) *on* $\mathrm{T} \downarrow \mathscr{C}$ *constructed in section 2.3.2 preserves* $\mathcal{E}_{\mathrm{T}}$*-tightness of* $(\mathcal{M}'_{\mathrm{T}}, \lambda)$*-cocones.*

Proof. By 2.4.3 and the fact that colimits commute with each other, to show that $\mathrm{S} : \mathrm{T} \downarrow \mathscr{C} \to \mathrm{T} \downarrow \mathscr{C}$ preserves $\mathcal{E}_{\mathrm{T}}$-tightness of $(\mathcal{M}'_{\mathrm{T}}, \lambda)$-cocones, it suffices to show that each of the endofunctors Id, $\tau_! \tau^*$, and $\tau_! \Delta \operatorname{cod} \tau^*$ on $\mathrm{T} \downarrow \mathscr{C}$ have this property. This is trivial for Id.

Let $\mathcal{E}^{\mathbf{2}}$ be the collection of maps (u, v) in $\mathscr{C}^{\mathbf{2}}$ such that $u \in \mathcal{E}$ and $v \in \mathcal{E}$. Similarly, let $\mathcal{M}^{\mathbf{2}}$ be the collection of maps (u, v) in $\mathscr{C}^{\mathbf{2}}$ such that both u and v are in $\mathcal{M}$.

First, we note that $\tau_! : \mathscr{C}^{\mathbf{2}} \to \mathrm{T} \downarrow \mathscr{C}$ sends $\mathcal{E}^{\mathbf{2}}$-tight $(\mathcal{M}'^{\mathbf{2}}, \lambda)$-cocones in $\mathscr{C}^{\mathbf{2}}$ to $\mathcal{E}_{\mathrm{T}}$-tight cocones in $\mathrm{T} \downarrow \mathscr{C}$. Indeed, let $\{(u_\alpha, v_\alpha) : f_\alpha \to f\}_\alpha$ be an $\mathcal{E}^{\mathbf{2}}$-tight $(\mathcal{M}'^{\mathbf{2}}, \lambda)$-cocone in $\mathscr{C}^{\mathbf{2}}$. Let $X_\alpha = \operatorname{dom} f_\alpha$, $Y_\alpha = \operatorname{cod} f_\alpha$, $X = \operatorname{dom} f$ and $Y = \operatorname{cod} f$. Let $(u, v) : \operatorname{colim}_\alpha f_\alpha \to f$ be the map defined by the cocone $\{(u_\alpha, v_\alpha)\}_\alpha$. Then the smallness condition on T means that the map $\operatorname{colim}_\alpha \mathrm{T} X_\alpha \to \mathrm{T} X$ defined by the cocone $\{\mathrm{T} u_\alpha : \mathrm{T} X_\alpha \to \mathrm{T} X\}_\alpha$ is in $\mathcal{E}$. Also 2.4.5 implies that the colimiting cocone $\{X_\alpha \to \operatorname{colim}_\alpha X_\alpha\}_\alpha$ is an $\mathcal{E}$-tight $(\mathcal{M}', \lambda)$-cocone. So $\operatorname{colim}_\alpha \mathrm{T} X_\alpha \to \mathrm{T}(\operatorname{colim}_\alpha \mathrm{T} X_\alpha)$ is an $\mathcal{E}$-map and therefore an epimorphism. By 2.4.5, the map $\mathrm{T} u : \mathrm{T} \operatorname{colim}_\alpha X_\alpha \to \mathrm{T} X$ is in $\mathcal{E}$. It now follows from 2.4.3 that, as a map in $\mathscr{C}^{\mathbf{2}}$, $\tau_!(u, v) : \tau_! \operatorname{colim} f_\alpha \to \tau_! f$ is an $\mathcal{E}^{\mathbf{2}}$-map. It follows from 2.4.3, 2.4.5, and the fact that colimits commute that $\{\tau_!(u_\alpha, v_\alpha)\}_\alpha$ is an $\mathcal{E}_{\mathrm{T}}$-tight cocone.

Now we know it is sufficient to show that the functors $\tau^* : \mathrm{T} \downarrow \mathscr{C} \to \mathscr{C}^{\mathbf{2}}$ and $\Delta \operatorname{cod} \tau^* : \mathrm{T} \downarrow \mathscr{C} \to \mathscr{C}^{\mathbf{2}}$ send $\mathcal{E}_{\mathrm{T}}$-tight $(\mathcal{M}'_{\mathrm{T}}, \lambda)$-cocones to $\mathcal{E}^{\mathbf{2}}$-tight $(\mathcal{M}'^{\mathbf{2}}, \lambda)$-tight cocones. Clearly, establishing this fact for the former functor is enough to show it holds for the latter. Let $\{(u_\alpha, v_\alpha) : (X_\alpha, f_\alpha, Y_\alpha) \to (X, f, Y)\}_\alpha$ be an $\mathcal{E}_{\mathrm{T}}$-tight $(\mathcal{M}'_{\mathrm{T}}, \lambda)$-cocone in $\mathrm{T} \downarrow \mathscr{C}$. Then, in particular, the maps $u_\alpha : X_\alpha \to X$ are in $\mathcal{M}'$. Let $u : \operatorname{colim}_\alpha X_\alpha \to X$ and $v : \operatorname{colim}_\alpha Y_\alpha \to Y$ be the maps in $\mathscr{C}$ defined by the cocones $\{u_\alpha\}_\alpha$ and $\{v_\alpha\}_\alpha$, respectively. The map $\operatorname{colim}_\alpha(X_\alpha, f_\alpha, Y_\alpha) \to (X, f, Y)$ in $\mathrm{T} \downarrow \mathscr{C}$ defined by the cocone $\{(u_\alpha, v_\alpha) : (X_\alpha, f_\alpha, Y_\alpha) \to (X, f, Y)\}_\alpha$ is the pair (u, k) shown in the following diagram, where $g = \operatorname{colim}_\alpha(X_\alpha, f_\alpha, Y_\alpha)$.

$$\begin{tikzcd}
\operatorname{colim}_\alpha \mathrm{T}X_\alpha \arrow[r, "c"] \arrow[d, "\operatorname{colim}_\alpha f_\alpha"'] & \mathrm{T}(\operatorname{colim}_\alpha X_\alpha) \arrow[r, "\mathrm{T}u"] \arrow[d, "g"] & \mathrm{T}X \arrow[d, "f"] \\
\operatorname{colim}_\alpha Y_\alpha \arrow[r, "h"] \arrow[rr, bend right, "v"'] & Z \arrow[r, "k"] \arrow[ul, phantom, "\ulcorner", very near start] & Y
\end{tikzcd}$$

So the fact that this cocone is $\mathcal{E}_{\mathrm{T}}$-tight means that $u \in \mathcal{E}$ and k factors as $k = r \circ p$, where $r \in \mathcal{E}$ and p is the pushout of $\mathrm{T}u$ along g. So we now know that $\{u_\alpha : X_\alpha \to X\}_\alpha$ is an $\mathcal{E}$-tight $(\mathcal{M}', \lambda)$-cocone in $\mathscr{C}$. Thus $\mathrm{T}u \circ c : \operatorname{colim} \mathrm{T}X_\alpha \to \mathrm{T}X$ is in $\mathcal{E}$. Since $\mathcal{E}$ is stable under pushouts, $p \circ h$ must be an $\mathcal{E}$-map. Thus $v = r \circ p \circ h \in \mathcal{E}$. So $(u, v) \in \mathcal{E}^{\mathbf{2}}$. □

We now have all of the components for the proof of our main theorem.

Theorem 2.4.23 ([Kel80, 15.6])**.** *Let $\mathscr{C}$ be a cocomplete category equipped with well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$. If (T, τ) is a pointed endofunctor on $\mathscr{C}$ and $\mathrm{T} : \mathscr{C} \to \mathscr{C}$ preserves $\mathcal{E}$-tightness of $(\mathcal{M}', \lambda)$-cocones for a regular cardinal λ, then the free monad sequence on (T, τ) converges objectwise and thus the free monad on (T, τ) exists.*

Proof. By 2.3.8 and 2.4.21, $\mathrm{T} \downarrow \mathscr{C}$ is a cocomplete category with well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}_{\mathrm{T}}, \mathcal{M}_{\mathrm{T}})$ and $(\mathcal{E}'_{\mathrm{T}}, \mathcal{M}'_{\mathrm{T}})$ By 2.4.22, (S, σ) preserves $\mathcal{E}_{\mathrm{T}}$-tightness of $(\mathcal{M}'_{\mathrm{T}}, \lambda)$-cocones. So, by 2.4.17, the free monad sequence for (S, σ) converges objectwise. Thus, by 2.3.17, the free monad sequence for (T, τ) converges objectwise, and so the free monad on (T, τ) exists by 2.3.18. □

Remark 2.4.24. If T preserves $\mathcal{E}$-tightness of λ-sequential $\mathcal{M}'$-cocones, then the proof of 2.4.22 shows that S preserves $\mathcal{E}_{\mathrm{T}}$-tightness of λ-sequential $\mathcal{M}'_{\mathrm{T}}$-cocones. Then theorem 2.4.23 can be adapted to show that the free monad sequence on (T, τ) converges objectwise.

2.4.4 A Special Case of the Smallness Condition on Endofunctors

Let $(\mathcal{E}, \mathcal{M})$ be a left proper orthogonal factorization system on $\mathscr{C}$. Let $(\mathcal{E}, \lambda)$-$\mathbf{pEnd}(\mathscr{C})$ and $(\mathcal{E}, \lambda)$-$\mathbf{Mnd}(\mathscr{C})$ be the full subcategories of $\mathbf{pEnd}(\mathscr{C})$ and $\mathbf{Mnd}(\mathscr{C})$, respectively, on the endofunctors that preserve $\mathcal{E}$-tightness of λ-filtered cocones.

Proposition 2.4.25. *If there is a regular cardinal λ such that the functor $\mathrm{T} : \mathscr{C} \to \mathscr{C}$ preserves $\mathcal{E}$-tightness of λ-filtered cocones, then the free monad on T preserves $\mathcal{E}$-tightness of λ-filtered cocones.*

Proof. First, we will show that the category $(\mathcal{E}, \lambda)$-$\mathbf{pEnd}(\mathscr{C})$ is closed under connected colimits. Since $\mathscr{C}$ is cocomplete, $\mathbf{pEnd}(\mathscr{C})$ is closed under connected colimits and the colimits are computed objectwise. Let $D : \mathscr{D} \to \mathbf{pEnd}(\mathscr{C})$ be a connected diagram such that for each object d in $\mathscr{D}$, $Dd : \mathscr{C} \to \mathscr{C}$ preserves $\mathcal{E}$-tightness of λ-filtered cocones. Let $\{f_\alpha : Y_\alpha \to Z\}_\alpha$ be an $\mathcal{E}$-tight λ-filtered cocone in $\mathscr{C}$. For each object d in $\mathscr{D}$, $\{Dd(f_\alpha) : Dd(Y_\alpha) \to Dd(Z)\}_\alpha$ is an $\mathcal{E}$-tight cocone in $\mathscr{C}$. Since $\mathcal{E}$ is the left class of an orthogonal factorization system, it is closed under colimits. Therefore $\{\mathrm{colim}_d\, Dd(f_\alpha) : \mathrm{colim}_d\, Dd(Y_\alpha) \to \mathrm{colim}_d\, Dd(Z)\}_\alpha$ is an $\mathcal{E}$-tight cocone in $\mathscr{C}$. So the endofunctor colim D in $\mathbf{pEnd}(\mathscr{C})$ preserves $\mathcal{E}$-tightness of λ-filtered cocones. Since $(\mathcal{E}, \lambda)$-$\mathbf{pEnd}(\mathscr{C})$ is a full subcategory of $\mathbf{pEnd}(\mathscr{C})$, colim D is the colimit of D as a diagram in $(\mathcal{E}, \lambda)$-$\mathbf{pEnd}(\mathscr{C})$.

Next, we note that the composition of two endofunctors in $(\mathcal{E}, \lambda)$-$\mathbf{pEnd}(\mathscr{C})$ is an endofunctor in $(\mathcal{E}, \lambda)$-$\mathbf{pEnd}(\mathscr{C})$. Indeed, the composition AB of two pointed endofunctors (A, α) and (B, β) is pointed by $A\beta \circ \alpha B : \mathrm{Id} \to AB$. Suppose A and B are endofunctors on $\mathscr{C}$ that preserve $\mathcal{E}$-tightness of λ-filtered cocones. Let $\{f_\alpha : Y_\alpha \to Z\}_\alpha$ be an $\mathcal{E}$-tight λ-filtered cocone in $\mathscr{C}$. Then $\{B(f_\alpha) : \mathrm{S}Y_\alpha \to \mathrm{S}Z\}_\alpha$ is an $\mathcal{E}$-tight λ-filtered cocone in $\mathscr{C}$. So $\{AB(f_\alpha) : \mathrm{S}Y_\alpha \to \mathrm{S}Z\}_\alpha$ is an $\mathcal{E}$-tight λ-filtered cocone in $\mathscr{C}$.

The free monad sequence of T starts with an endofunctor T in $(\mathcal{E},\lambda)$-**pEnd**$(\mathscr{C})$. If for an ordinal α the endofunctors X_α and $\mathrm{X}_{\alpha+1}$ are in $(\mathcal{E},\lambda)$-**pEnd**$(\mathscr{C})$, then diagram (2.6) is in $(\mathcal{E},\lambda)$-**pEnd**$(\mathscr{C})$. If γ is a limit ordinal and X_α is in $(\mathcal{E},\lambda)$-**pEnd**$(\mathscr{C})$ for each $\alpha < \gamma$, then X_γ must be in $(\mathcal{E},\lambda)$-**pEnd**$(\mathscr{C})$, since the diagram of X_α's is connected.

Now, let $\mathrm{R} : \mathscr{C} \to \mathscr{C}$ be the free monad on T, which exists by 2.4.23. As we saw in the proof of 2.3.18, on each object Y, there is an ordinal β_Y such that the transfinite sequence $(\mathrm{X}_\alpha Y)_\alpha$ converges at β_Y and $\mathrm{R}Y = \mathrm{X}_{\beta_Y} Y$. Let $\{f_\alpha : Y_\alpha \to Z\}_\alpha$ be an $\mathcal{E}$-tight λ-filtered cocone in $\mathscr{C}$ and let $\beta' = \sup\{\beta_Z, \beta_{Y_\alpha} \,|\, \alpha\}$. Then the cocone $\{\mathrm{R}f_\alpha : \mathrm{R}Y_\alpha \to \mathrm{R}Z\}_\alpha$ is equal to the cocone $\{\mathrm{X}_{\beta'} f_\alpha : \mathrm{X}_{\beta'} Y_\alpha \to \mathrm{X}_{\beta'} Z\}_\alpha$, which must be $\mathcal{E}$-tight, since $\mathrm{X}_{\beta'}$ is an endofunctor in $(\mathcal{E},\lambda)$-**pEnd**$(\mathscr{C})$. □

Let $G : (\mathcal{E},\lambda)$-**Mnd**$(\mathscr{C}) \to (\mathcal{E},\lambda)$-**pEnd**$(\mathscr{C})$ be the functor that forgets the multiplication map. By the above proposition and theorem 2.4.23, every endofunctor in $(\mathcal{E},\lambda)$-**pEnd**$(\mathscr{C})$ has a reflection in $(\mathcal{E},\lambda)$-**Mnd**$(\mathscr{C})$. So, by 2.2.2, there is an adjunction

$$\begin{tikzcd}(\mathcal{E},\lambda)\text{-}\mathbf{Mnd}(\mathscr{C}) \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\perp"] && (\mathcal{E},\lambda)\text{-}\mathbf{pEnd}(\mathscr{C}). \arrow[ll, bend right, "F"']\end{tikzcd} \tag{2.11}$$

Note that because $(\mathcal{E},\lambda)$-**Mnd**$(\mathscr{C})$ is not a coreflective subcategory of **Mnd**$(\mathscr{C})$, the above adjunction does not on its own imply that free monads exist on the objects of $(\mathcal{E},\lambda)$-**pEnd**$(\mathscr{C})$. We still need the fact that the endofunctors in **pEnd**$(\mathscr{C})$ that preserve $\mathcal{E}$-tightness of λ-filtered cocones have reflections in **Mnd**$(\mathscr{C})$.

2.5 Free Monoids

Typically, the existence of free monoids on objects in a strict monoidal category $(\mathscr{C}, \otimes, I)$ requires some strong assumptions on how the functors $(-)\otimes X : \mathscr{C} \to \mathscr{C}$ and $X \otimes (-) : \mathscr{C} \to \mathscr{C}$ behave with respect to certain colimits. The classic result that the free monoid on an object exists when $(-) \otimes X : \mathscr{C} \to \mathscr{C}$ and $X \otimes (-) : \mathscr{C} \to \mathscr{C}$ preserve coproducts for each object X in $\mathscr{C}$ is an example. For the application we are interested in, the functors $(-) \otimes X : \mathscr{C} \to \mathscr{C}$ preserve connected colimits, but not coproducts. This is the case in the strict monoidal category of endofunctors under composition. We will prove that a free monoid on a pointed object (T, τ) in a strict monoidal category $(\mathscr{C}, \otimes, I)$ exists when the functor $(-)\otimes X : \mathscr{C} \to \mathscr{C}$ preserves connected colimits for each object X and the *free monoid sequence* for (T, τ) *weakly converges.*

Let $(\mathscr{C}, \otimes, I)$ be a monoidal category. For us, a *pointed object* in $(\mathscr{C}, \otimes, I)$ is a pair (T, τ) of an object T and a map $\tau : I \to T$ in $\mathscr{C}$. We note that I is usually not the terminal object in $\mathscr{C}$. A *map* $f : (X, \eta^X) \to (Y, \eta^Y)$ *of pointed objects* is a map $f : X \to Y$ in $\mathscr{C}$ such that $f \circ \eta^X = \eta^Y$. A *map* $f : (X, \eta^X, \mu^X) \to (Y, \eta^Y, \mu^Y)$ *of monoids* in $\mathscr{C}$ is defined in the same way as a map of monads. So f is a map that satisfies the equations $\mu^Y \circ (Y \otimes f) \circ (f \otimes X) = f \circ \mu_X$ and $f \circ \eta^X = \eta^Y$. Let $\mathbf{pObj}(\mathscr{C})$ be the category whose objects are pointed objects in $(\mathscr{C}, \otimes, \mathtt{I})$ and whose morphisms are maps of pointed objects. Let $\mathbf{Mon}(\mathscr{C})$ be the category whose objects are monoids and whose morphisms are maps of monoids. The *free monoid* on a pointed object (T, τ) is the reflection of (T, τ) along the forgetful functor $U : \mathbf{Mon}(\mathscr{C}) \to \mathbf{pObj}(\mathscr{C})$.

2.5.1 Weakly Convergent Free Monoid Sequences

Let $(\mathscr{C}, \otimes, I)$ be a cocomplete strict monoidal category and let (T, τ) be a pointed object in $(\mathscr{C}, \otimes, I)$. We will define a *free monoid sequence* on (T, τ) in a similar way to the free monad sequence in section 2.3.3. We will also define *weak convergence* of free monoid sequences in an analogous way to the weak convergence of free monad sequences. We will then show that, when the free monoid sequence on (T,τ) weakly converges and $(-)\otimes X : \mathscr{C} \to \mathscr{C}$ preserves certain large connected colimits for each object X in $\mathscr{C}$, then the free monoid on (T,τ) exists.

The construction of the *free monoid sequence* takes place in $(\mathscr{C},\otimes,I)$. Let $X_0 = I$, let $X_1 = T$, and let $\pi_0 = \tau : I \to T$. Suppose objects X_α and $X_{\alpha+1}$ and a map $\pi_\alpha : T\otimes X_\alpha \to X_{\alpha+1}$ are defined for an ordinal α. We define $\pi_{\alpha+1} : T\otimes X_{\alpha+1} \to X_{\alpha+2}$ to be the coequalizer of the maps $(T\otimes\pi_\alpha)\circ(\tau\otimes T\otimes X_\alpha)$ and $(T\otimes\pi_\alpha)\circ(T\otimes\tau\otimes X_\alpha)$.

$$\begin{tikzcd}
T\otimes X_\alpha \arrow[r, shift left, "\tau\otimes T\otimes X_\alpha"] \arrow[r, shift right, "T\otimes\tau\otimes X_\alpha"'] & T\otimes T\otimes X_\alpha \arrow[r, "T\otimes\pi_\alpha"] & T\otimes X_{\alpha+1} \arrow[r, dashed, "\pi_{\alpha+1}"] & X_{\alpha+2}
\end{tikzcd}$$

If γ is a limit ordinal and X_α and π_α are defined for each $\alpha < \gamma$, then we define $\{x_\alpha^\gamma : X_\alpha \to X_\gamma\}_{\alpha<\gamma}$ to be the colimiting cocone of the γ-sequence $(X_\alpha)_{\alpha<\gamma}$ with connecting maps

$$x_\alpha^{\alpha+1} := \pi_\alpha \circ (\tau\otimes X_\alpha) : X_\alpha \to X_{\alpha+1}.$$

The two maps

$$\begin{tikzcd}
T\otimes X_\alpha \arrow[r, shift left, "\tau\otimes T\otimes X_\alpha"] \arrow[r, shift right, "T\otimes\tau\otimes X_\alpha"'] & T\otimes T\otimes X_\alpha \arrow[r, "T\otimes\pi_\alpha"] & T\otimes X_{\alpha+1} \arrow[r, "x_{\alpha+1}^\gamma"] & T\otimes X_\gamma
\end{tikzcd}$$

from $T\otimes X_\alpha$ to $T\otimes X_\gamma$ define two maps

$$\begin{tikzcd}
\underset{T\otimes x_\alpha^{\alpha+1}}{colim}(T\otimes X_\alpha) \arrow[r, shift left, "y"] \arrow[r, shift right, "id"'] & \underset{T\otimes x_\alpha^{\alpha+1}}{colim}(T\otimes X_\alpha) \arrow[r, "c"] & T\otimes X_\gamma,
\end{tikzcd}$$

where c is the map defined by the cocone $\{T\otimes x_\alpha^\gamma : T\otimes X_\alpha \to T\otimes X_\gamma\}_{\alpha<\gamma}$. And we define $\pi_\gamma : T\otimes X_\gamma \to X_{\gamma+1}$ to be the colimit of this last diagram.

By composing the maps $x_\alpha^{\alpha+1}$ and x_α^γ, we get a unique map $x_\alpha^\beta : X_\alpha \to X_\beta$ for each pair of ordinals with $\alpha \leq \beta$. The transfinite sequence $(X_\alpha)_\alpha$ with connecting maps x_α^β is the *free monoid sequence* or *free $\otimes$-monoid sequence* on (T,τ).

Lemma 2.5.1. *Suppose for each object A in $\mathscr{C}$, the endofunctor $(-) \otimes A : \mathscr{C} \to \mathscr{C}$ preserves connected colimits. If $(X_\alpha)_\alpha$ is the free monoid sequence on (T, τ), then $(X_\alpha \otimes (-))_\alpha$ is the free monad sequence on the pointed endofunctor $(T \otimes (-), \tau \otimes (-))$.*

Proof. Let $\mathbf{End}_\otimes(\mathscr{C})$ be the subcategory of $\mathbf{End}(\mathscr{C})$ whose objects and morphisms are in the image of the functor $\mathscr{C} \to \mathbf{End}_\otimes(\mathscr{C})$ defined on objects by $X \mapsto X \otimes (-)$ and on morphisms by $f \mapsto f \otimes (-)$. The composition $A \otimes (-) \circ B \otimes (-)$ of two endofunctors $A \otimes (-) : \mathscr{C} \to \mathscr{C}$ and $B \otimes (-) : \mathscr{C} \to \mathscr{C}$ in $\mathbf{End}_\otimes(\mathscr{C})$ is the endofunctor $(A \otimes B) \otimes (-) : \mathscr{C} \to \mathscr{C}$ in $\mathbf{End}_\otimes(\mathscr{C})$. Since $(-) \otimes X : \mathscr{C} \to \mathscr{C}$ preserves connected colimits for each object X in $\mathscr{C}$ and colimits of endofunctors are computed objectwise, the colimit in $\mathbf{End}(\mathscr{C})$ of a connected diagram of functors in $\mathbf{End}_\otimes(\mathscr{C})$ is in $\mathbf{End}_\otimes(\mathscr{C})$. So $\mathbf{End}_\otimes(\mathscr{C})$ is closed under connected colimits that exist in $\mathbf{End}(\mathscr{C})$.

In a similar manner to the proof of 2.4.25, an inductive argument now shows that the free monad sequence on the pointed endofunctor $T \otimes (-)$ is a transfinite sequence in $\mathbf{End}_\otimes(\mathscr{C})$. Let $(X_\alpha \otimes (-))_\alpha$ be this transfinite sequence. This is the image of the free monoid sequence $(X_\alpha)_\alpha$ on (T, τ) under the functor $X \mapsto X \otimes (-)$. □

When the colimit Y of the free monoid sequence $(X_\alpha)_\alpha$ does exist, we will use the notation $x_\alpha^\infty : X_\alpha \to Y$ for the maps in the colimiting cocone. We will use the notation Y' for the colimit of the transfinite sequence $(T \otimes X_\alpha)_\alpha$ with connecting maps $T \otimes x_\alpha^\beta : T \otimes X_\alpha \to T \otimes X_\beta$ when this colimit exists. The cocone $\{T \otimes x_\alpha^\infty : T \otimes X_\alpha \to T \otimes Y\}_\alpha$ defines a map $c' : Y' \to TY$. The cocone $\{x_{\alpha+1}^\infty \circ \pi_\alpha : T \otimes X_\alpha \to Y\}_\alpha$ defines a map $\pi' : Y' \to Y$.

Definition 2.5.2. The free monoid sequence $(\mathrm{X}_\alpha)_\alpha$ is *weakly convergent* if the colimit $Y = \operatorname{colim}_\alpha X_\alpha$ exists, the colimit $Y' = \operatorname{colim}_\alpha T \otimes X_\alpha$ exists, and there is a map $m : T \otimes Y \to Y$ satisfying the following conditions.

1. The map $m : T \otimes Y \to Y$ is the coequalizer of the maps $(T \otimes m) \circ (\tau \otimes T \otimes Y)$ and $(T \otimes m) \circ (T \otimes \tau \otimes Y)$ in the following diagram.

$$\begin{tikzcd}
T \otimes Y \arrow[r, shift left, "\tau\otimes T\otimes Y"] \arrow[r, shift right, "T\otimes\tau\otimes Y"'] & T \otimes T \otimes Y \arrow[r, "T\otimes m"] & T \otimes Y
\end{tikzcd}$$

2. The following diagram is cocartesian.

$$\begin{tikzcd}
Y' \arrow[r, "c'"] \arrow[d, "\pi'"'] & T \otimes Y \arrow[d, "m"] \\
Y \arrow[r, "id"] \arrow[ur, phantom, very near end, "\ulcorner"] & Y
\end{tikzcd}$$

Lemma 2.5.3. *Let $(X_\alpha)_\alpha$ be the free monoid sequence on (T, τ). Suppose for each object A in $\mathscr{C}$, the endofunctor $(-) \otimes A : \mathscr{C} \to \mathscr{C}$ preserves connected colimits and preserves the colimits of the large diagrams $(X_\alpha)_\alpha$ and $(T \otimes X_\alpha)_\alpha$. Then the free monoid sequence on (T, τ) is weakly convergent if and only if the free monad sequence for the pointed endofunctor $T \otimes (-) : \mathscr{C} \to \mathscr{C}$ is weakly convergent.*

Proof. By 2.5.1, $(X_\alpha \otimes (-))_\alpha$ is the free monad sequence on the pointed endofunctor $(T \otimes (-), \tau \otimes (-))$.

Suppose the free monad sequence $(X_\alpha \otimes (-))_\alpha$ converges weakly. By evaluating the endofunctors on $\mathtt{I}$, we get that $\operatorname{colim}_\alpha X_\alpha$ exists, $\operatorname{colim}_\alpha (T \otimes X_\alpha)$ exists, and, since connected colimits of pointed endofunctors are computed objectwise, conditions (1) and (2) of definition 2.5.2 hold.

Conversely, suppose the free monoid sequence $(X_\alpha)_\alpha$ is weakly convergent. Then the endofunctor $\mathrm{R} := \operatorname{colim}_\alpha (X_\alpha \otimes (-))$ exists and equals $(\operatorname{colim}_\alpha X_\alpha) \otimes (-)$ and the endofunctor $\mathrm{R}' := \operatorname{colim}_\alpha \big((T \otimes X_\alpha) \otimes (-)\big)$ exists and equals $\big(\operatorname{colim}_\alpha (T \otimes X_\alpha)\big) \otimes (-)$. Both of the endofunctor colimits are computed objectwise. Since $(-) \otimes A$ preserves colimits for each object A, conditions (1) and (2) of definition 2.3.19 hold. □

Theorem 2.5.4. *Let* $(X_\alpha)_\alpha$ *be the free monoid sequence on* (T, τ)*. Suppose for each object* A *in* $\mathscr{C}$*, the endofunctor* $(-) \otimes A : \mathscr{C} \to \mathscr{C}$ *preserves connected colimits and preserves the colimits of the large diagrams* $(X_\alpha)_\alpha$ *and* $(T \otimes X_\alpha)_\alpha$*. If the free monoid sequence on* (T, τ) *is weakly convergent, then the free monoid on* (T, τ) *exists and is equal to the object the free monoid sequence weakly converges to.*

Proof. By 2.5.1, $(X_\alpha \otimes (-))_\alpha$ is the free monad sequence on the pointed endofunctor $(T \otimes (-), \tau \otimes (-))$. By 2.5.3, the free monad sequence $(X_\alpha \otimes (-))_\alpha$ weakly converges. Furthermore, as we saw in the proof of 2.5.3, $\operatorname{colim}_\alpha(X_\alpha \otimes (-)) = (\operatorname{colim}_\alpha X_\alpha) \otimes (-)$. So the free monad (R, η, μ) on the pointed endofunctor $(T \otimes (-), \tau \otimes (-))$ exists and $\mathrm{R} = \operatorname{colim}_\alpha(X_\alpha \otimes (-)) = (\operatorname{colim}_\alpha X_\alpha) \otimes (-)$. So $(\mathrm{R}\mathtt{I}, \eta_{\mathtt{I}}, \mu_{\mathtt{I}})$ is the free monoid on (T, τ). $\square$

Chapter 3: Algebraic Weak Factorization Systems and Algebraic Model Categories

Algebraic weak factorization systems (AWFSs) were introduced in [GT06] under the name natural weak factorization systems as a generalization to WFSs of structure present in orthogonal factorization systems. AWFSs replace a WFSs $(\mathcal{L}, \mathcal{R})$ with a choice of functorial factorization (L, R) and place the structure of a comonad on L and the structure of a monad on R. In [Gar08] and [Gar07], Garner applied the results of [Kel80] to the construction of free AWFSs on left algebraic weak factorization systems. Some other sources that discuss algebraic weak factorization systems and related topics are [Rie11], [BG16a], and [BG16b].

Garner's method for constructing AWFSs can be thought of as an adaptation of the small object argument for sets to a version of the small object argument for diagrams. Previous work in the thesis [RB99] showed how to place the structure of a comonad on the cofibrant replacement functor and the structure of a monad on the fibrant replacement functor in a model category. Interestingly, both sources get monad structures by eliminating redundancy in the small object argument. But, while [RB99] eliminates redundancy by manually omitting redundant cells, [Gar07] eliminates redundancy by taking coequalizers and forcing redundant cells to be equal.

Some useful properties of algebraic weak factorization systems come from the fact that they encode the structure of lifts in a weak factorization system, rather than just property of having a lift. We then have categories of algebras and coalgebras for the right and left endofunctors in an AWFS that lift the right and left classes of the underlying model category. As we saw in 2.1.25, these categories have forgetful functors that create limits and colimits, respectively. So in this sense, if $(\mathcal{L}, \mathcal{R})$ is the underlying weak factorization system (WFS), then we have a cocomplete category of objects in $\mathcal{L}$ and a complete category of objects in $\mathcal{R}$. Furthermore, by keeping track of the structure of lifts, we can show that diagrams in the category of coalgebras have natural lifts with respect to diagrams in the category of algebras for an AWFS. We give a more rigorous introduction to the basic properties of AWFSs in 3.1.

In the rest of this chapter, we give an exposition of Garner's construction of free AWFSs, along with some original results. We prove a more general version of Garner's result and also fix an issue with his argument. Garner's approach is to put a monoidal structure on the category of left algebraic weak factorization systems (LAWFSs) such that the AWFSs are exactly the monoids in the category of LAWFSs. Garner then claims that that when $(\mathrm{L}_1, \mathrm{R}_1)$ satisfies a smallness condition, then by [Kel80], the free monad sequence for R_1 converges. If we give ourselves this result, then the free monoid sequence for $(\mathrm{L}_1, \mathrm{R}_1)$ converges. So another result of [Kel80] shows that the free monoid (aka AWFS) on the LAWFS $(\mathrm{L}_1, \mathrm{R}_1)$ exists. The problem with this argument is that the smallness condition on $(\mathrm{L}_1, \mathrm{R}_1)$ implies objectwise convergence, not convergence. Since there isn't really an analog of objectwise convergence for a free monoid sequence, the argument does not go through. To fix this, we use our definitions of weakly convergent free monad sequences and free monoid sequences in chapter 2. The objectwise convergence of the free monad sequence on R_1 implies the weak convergence of this sequence. We show in 3.2.15 that the weak convergence of the free monad sequence for R_1 implies the weak convergence of the free monoid sequence for $(\mathrm{L}_1, \mathrm{R}_1)$. We then can apply 2.5.4 to show that the free monoid on $(\mathrm{L}_1, \mathrm{R}_1)$ exists.

We call the smallness condition we place on our LAWFSs *compactness*. The surprising fact that we only need to look at $\mathcal{E}_{/\cong}$-tight $\mathcal{M}'_{/\cong}$-cocones, rather than $\mathcal{E}^{\mathbf{2}}$-tight $\mathcal{M}'^{\mathbf{2}}$-cocones to get the existence of free AWFSs is mentioned in [Gar07, p 31] and is similar to the monomorphism hypothesis of [BR13]. Our definition of compactness generalizes the smallness conditions considered by Garner, since compactness only requires that the right functor R of a LAWFS (L, R) preserves $\mathcal{E}_{/\cong}$-tightness of $(\mathcal{M}'_{/\cong}, \lambda)$-cocones, rather than preserving colimits of $(\mathcal{M}'_{/\cong}, \lambda)$-cocones. This extra leeway in what counts as a compact functor ends up being very useful for later results.

Our result for the existence of free AWFSs on compact comonads is in some sense a sort of generalization of the functorial version of Chorny's generalized small object argument [Cho06, 1.1]. We fall short of a true generalization, since we work with a comonad instead of a pointed endofunctor and since our compactness condition involves the entire category $\mathscr{C}$, rather than just domains of the maps in the colimiting class. However, we are substantially more general in that Chorny's condition corresponds to $(\mathcal{E}, \mathcal{M}')$-compactness, when $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ are both the (isomorphism, any map) orthogonal factorization systems. With some effort, one could make our compactness conditions more closely resemble a true generalization of Chorny's condition.

After reviewing Garner's algebraic small object argument, we prove some results in section 3.3 specific to $\mathcal{E}$-compact LAWFSs. We are able to show that the free AWFS on an $\mathcal{E}$-compact LAWFS is $\mathcal{E}$-compact. We then show that the reflections of Garner's algebraic small object argument can be replaced with fully-defined left adjoints in this context. We show in 3.3.4 that a model category with $\mathcal{E}$-compact LAWFSs is an $\mathcal{E}$-compact algebraic model category, getting the map of AWFSs required in the definition of an algebraic model category for free.

The most important results in this section are 3.2.16, 3.2.18, 3.2.19, 3.3.3, and 3.3.4, but we will also frequently use results from sections 3.1.3 and 3.2.5 in later chapters.

3.1 Properties of Algebraic Weak Factorization Systems

3.1.1 Functorial Factorizations

Recall, in the notation of section 2.1.1, $\mathscr{C}^{\mathbf{2}}$ is the arrow category of $\mathscr{C}$ and $\mathscr{C}^{\mathbf{3}}$ is the category of composable arrows in $\mathscr{C}$. As we noted in section 2.1.1, maps $f \to g$ in $\mathscr{C}^{\mathbf{2}}$ are pairs (u, v) of maps $u : \operatorname{dom} f \to \operatorname{dom} g$ and $v : \operatorname{cod} f \to \operatorname{cod} g$ in $\mathscr{C}$ such that $v \circ f = g \circ u$. At times, it will be convenient to use the notation $\vec{r} : f \to g$ for a map from f to g in $\mathscr{C}^{\mathbf{2}}$, rather than representing the map by a pair.

A *functorial factorization* on $\mathscr{C}$ is a section for the composition functor $\mathrm{comp} : \mathscr{C}^{\mathbf{3}} \to \mathscr{C}^{\mathbf{2}}$. More explicitly, a functorial factorization on $\mathscr{C}$ consists of three functors $\mathrm{L} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$, $\mathrm{R} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$, and $E : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ such that $\operatorname{dom} \mathrm{L} = \operatorname{dom}$, $\operatorname{cod} \mathrm{L} = E = \operatorname{dom} \mathrm{R}$, and $\operatorname{cod} \mathrm{R} = \operatorname{cod}$. So, to a map $(u, v) : f \to g$, this functorial factorization assigns a map

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\mathrm{L}f"'] & C \arrow[d, "\mathrm{L}g"] \\
Ef \arrow[r, "E(u,v)"] \arrow[d, "\mathrm{R}f"'] & Eg \arrow[d, "\mathrm{R}g"] \\
B \arrow[r, "v"] & D
\end{tikzcd}$$

in $\mathscr{C}^{\mathbf{3}}$. We will write a functorial factorization as a pair (L, R), where E is understood. We will refer to the functor E as the *middomain functor* associated to (L, R).

A *map of functorial factorizations* $\zeta : (\mathrm{L}, \mathrm{R}) \to (\mathrm{L}', \mathrm{R}')$ with middomain functors E and E', respectively, is a natural transformation $\zeta : E \to E'$ such that the following diagram commutes for each object $f : X \to Y$ in $\mathscr{C}^{\mathbf{2}}$.

$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{L}f"'] \arrow[dr, "\mathrm{L}'f"] & \\
Ef \arrow[rr, "\zeta_f"] \arrow[dr, "\mathrm{R}f"'] & & E'f \arrow[dl, "\mathrm{R}'f"] \\
 & Y &
\end{tikzcd}$$

Every functorial factorization (L, R) comes equipped with an counit map $\vec{\varepsilon}$ for L and a unit map $\vec{\eta}$ for R. The natural transformations $\vec{\varepsilon} : \mathrm{L} \to I$ and $\vec{\eta} : I \to \mathrm{R}$ are defined on each f by the following diagrams.

$$\begin{tikzcd}
{} \arrow[r, "id"] \arrow[d, "\mathrm{L}f"'] \arrow[dr, phantom, "\vec{\varepsilon}_f"] & {} \arrow[d, "f"] & {} \arrow[r, "\mathrm{L}f"] \arrow[d, "f"'] \arrow[dr, phantom, "\vec{\eta}_f"] & {} \arrow[d, "\mathrm{R}f"] \\
{} \arrow[r, "\mathrm{R}f"'] & {} & {} \arrow[r, "id"'] & {}
\end{tikzcd} \tag{3.1}$$

So L is a copointed endofunctor and R is a pointed endofunctor on $\mathscr{C}^{\mathbf{2}}$.

A map f is an L-coalgebra with structure map $\vec{k} : f \to \mathrm{L}f$ if and only if a lift exists in the right square of diagram (3.1). A map f is an R-algebra with structure map $\vec{s} : \mathrm{R}f \to f$ if and only if a lift exists in the left square of diagram (3.1). Indeed, since L is a domain-preserving functor and R is a codomain-preserving functor, the maps $\vec{k}$ and $\vec{s}$ each consist of only one nonidentity map in $\mathscr{C}$. We will refer to these maps as k and s, so that $\vec{k} = (id, k)$ and $\vec{s} = (s, id)$ are described on each f by the following commutative diagrams.

$$\begin{tikzcd}
{} \arrow[r, "id"] \arrow[d, "f"'] \arrow[dr, phantom, "\vec{k}"] & {} \arrow[d, "\mathrm{L}f"] & {} \arrow[r, "s"] \arrow[d, "\mathrm{R}f"'] \arrow[dr, phantom, "\vec{s}"] & {} \arrow[d, "f"] \\
{} \arrow[r, "k"'] & {} & {} \arrow[r, "id"'] & {}
\end{tikzcd}$$

We assume all weak factorization systems are functorial. Specifically, a *weak factorization system* $(\mathcal{L}, \mathcal{R})$ is a pair of collections $\mathcal{L}$ and $\mathcal{R}$ of objects in $\mathscr{C}^{\mathbf{2}}$ such that the following conditions hold.

1. $\mathcal{L}^{\square} = \mathcal{R}$ and $\mathcal{L} = {}^{\square}\mathcal{R}$.

2. There is a functorial factorization system (L,R) on $\mathscr{C}$ such that $\mathrm{L}f \in \mathcal{L}$ and $\mathrm{R}f \in \mathcal{R}$ for each object f in $\mathscr{C}^{\mathbf{2}}$.

If $(\mathcal{L},\mathcal{R})$ is a weak factorization system and (L,R) is a functorial factorization satisfying condition (2), then we will say (L,R) is an *associated functorial factorization* of $(\mathcal{L},\mathcal{R})$. In fact, we show in 3.1.2 that condition (1) of the definition is redundant.

We recall the notation $|F|$ or $|\mathscr{A}|$ of section 2.1.5 for the collection of objects in the image of a functor $F : \mathscr{A} \to \mathscr{B}$.

Proposition 3.1.1. *If* (L,R) *is a functorial factorization, then* $|\mathbf{Alg}_{\mathrm{R}}| \subseteq |\mathbf{Coalg}_{\mathrm{L}}|^{\square}$ *and* $|\mathbf{Coalg}_{\mathrm{L}}| \subseteq {}^{\square}|\mathbf{Alg}_{\mathrm{R}}|$.

Proof. Let $\langle f, \vec{k} \rangle$ be an L-coalgebra and let $\langle g, \vec{s} \rangle$ be an R-algebra. If $(u,v) : f \to g$ is a map in $\mathscr{C}^{\mathbf{2}}$, then the map $s \circ E(u,v) \circ k$ shown below is a solution to the lifting problem $(u,v) : f \to g$.

$$\begin{tikzcd}
 & {} \arrow[r, "u"] \arrow[dl, "f"'] \arrow[d, "\mathrm{L}f"] & {} \arrow[d, "Lg"] \arrow[dr, "id"] & \\
{} \arrow[r, "k"] \arrow[dr, "id"'] & {} \arrow[r, "E(u{,}v)"] \arrow[d, "\mathrm{R}f"] & {} \arrow[r, "s"] \arrow[d, "Rg"] & {} \arrow[dl, "g"] \\
 & {} \arrow[r, "v"] & {} &
\end{tikzcd} \tag{3.2}$$

□

Proposition 3.1.2. *If* (L,R) *is a functorial factorization on* $\mathscr{C}$ *such that* $\mathrm{L}f \in |\mathbf{Coalg}_{\mathrm{L}}|$ *and* $\mathrm{R}f \in |\mathbf{Alg}_{\mathrm{R}}|$ *for each object* f *in* $\mathscr{C}^{\mathbf{2}}$*, then* $|\mathbf{Alg}_{\mathrm{R}}| = |\mathbf{Coalg}_{\mathrm{L}}|^{\square}$ *and* $|\mathbf{Coalg}_{\mathrm{L}}| = {}^{\square}|\mathbf{Alg}_{\mathrm{R}}|$.

Proof. By 3.1.1, $|\mathbf{Alg}_{\mathrm{R}}| \subseteq |\mathbf{Coalg}_{\mathrm{L}}|^{\square}$ and $|\mathbf{Coalg}_{\mathrm{L}}| \subseteq {}^{\square}|\mathbf{Alg}_{\mathrm{R}}|$. If $f \in {}^{\square}|\mathbf{Alg}_{\mathrm{R}}|$ and $\mathrm{R}f \in |\mathbf{Alg}_{\mathrm{R}}|$, then a lift exists in the right square of diagram (3.1). Similarly, if $f \in |\mathbf{Coalg}_{\mathrm{L}}|^{\square}$ and $\mathrm{L}f \in |\mathbf{Coalg}_{\mathrm{L}}|$, then a lift exists in the left square of diagram (3.1). □

Therefore a functorial factorization system (L,R) satisfying the hypothesis of 3.1.2 defines a weak factorization system $(\mathcal{L},\mathcal{R})$ with $\mathcal{L} = |\mathbf{Coalg}_{\mathrm{L}}|$ and $\mathcal{R} = |\mathbf{Alg}_{\mathrm{R}}|$. Conversely, if $(\mathcal{L},\mathcal{R})$ is a weak factorization system and (L,R) is a choice of functorial factorization for $(\mathcal{L},\mathcal{R})$, then $|\mathbf{Coalg}_{\mathrm{L}}| = \mathcal{L}$ and $|\mathbf{Alg}_{\mathrm{R}}| = \mathcal{R}$. The inclusion $|\mathbf{Coalg}_{\mathrm{L}}| \subseteq \mathcal{L}$ holds because $\mathrm{L}f \in \mathcal{L}$ for each f and $\mathcal{L}$ is retract-closed. For each $f \in \mathcal{L}$, since $\mathrm{R}f \in \mathcal{R}$, a lift exists in the right square of diagram (3.1) and thus $f \in |\mathbf{Coalg}_{\mathrm{L}}|$. A similar argument shows $|\mathbf{Alg}_{\mathrm{R}}| = \mathcal{R}$.

When a functorial factorization system (L,R) satisfies the hypothesis of 3.1.2, we will call $(\mathcal{L},\mathcal{R})$, with $\mathcal{L} = |\mathbf{Coalg}_{\mathrm{L}}|$ and $\mathcal{R} = |\mathbf{Alg}_{\mathrm{R}}|$, the *associated weak factorization system* of (L,R). In definition 3.1.5, we will put additional algebraic structure on a functorial factorization that in particular guarantees it satisfies the hypothesis of 3.1.2.

Remark 3.1.3. The lift constructed in proposition 3.1.1 is natural with respect to maps of L-coalgebras and maps of R-algebras. Let $(a,b) : \langle f,\vec{k}\rangle \to \langle g,\vec{l}\rangle$ be a map of L-coalgebras, let $(c,d) : \langle p,\vec{s}\rangle \to \langle q,\vec{t}\rangle$ be a map of R-algebras. Suppose $(u,v) : f \to p$ and $(x,y) : g \to q$ are maps in $\mathscr{C}^{\mathbf{2}}$ such that $(x,y) \circ (a,b) = (c,d) \circ (u,v)$. Of course $E(x,y) \circ E(a,b) = E(c,d) \circ E(u,v)$. Furthermore, what it means for (a,b) to be a map of L-coalgebras is that $E(a,b) \circ k = l \circ b$. Similarly, $t \circ E(c,d) = c \circ s$. Therefore $c \circ s \circ E(u,v) \circ k = t \circ E(x,y) \circ l \circ b$.

We will make frequent use of the following proposition.

Proposition 3.1.4. *Let $\mathcal{L}$ and $\mathcal{R}$ be classes of maps in $\mathscr{C}$ such that $\mathcal{L} = {}^{\square}\mathcal{R}$. If (L,R) is a functorial factorization on $\mathscr{C}$ such that $\mathrm{L}f \in \mathcal{L}$ for each f and $|\mathbf{Alg}_{\mathrm{R}}| \subseteq \mathcal{R}$, then $|\mathbf{Alg}_{\mathrm{R}}| = \mathcal{R}$.*

Proof. If $f \in \mathcal{R}$, then a solution to the lifting problem $(id, \mathrm{R}f) : \mathrm{L}f \to f$ exists. So $f \in |\mathbf{Alg}_{\mathrm{R}}|$. Thus $\mathcal{R} \subseteq |\mathbf{Alg}_{\mathrm{R}}|$. □

3.1.2 Algebraic Weak Factorization Systems

When L is a domain-preserving comonad on $\mathscr{C}^{\mathbf{2}}$, then the comultiplication map $\vec{\delta} : \mathrm{L} \to \mathrm{LL}$ can be expressed as $\vec{\delta} = (id, \delta)$, where δ is a natural transformation $obj(\mathscr{C}^{\mathbf{2}}) \to morph(\mathscr{C})$. Similarly, when R is a codomain-preserving monad on $\mathscr{C}^{\mathbf{2}}$, we can express the multiplication map $\vec{\mu} : \mathrm{RR} \to \mathrm{R}$ as $\vec{\mu} = (\mu, id)$.

Definition 3.1.5. An *algebraic weak factorization system* (AWFS) on $\mathscr{C}$ is a functorial factorization (L, R) on $\mathscr{C}$ such that $(\mathrm{L}, \vec{\varepsilon})$ is equipped with the structure of a comonad $(\mathrm{L}, \vec{\varepsilon}, \vec{\delta})$, $(\mathrm{R}, \vec{\eta})$ is equipped with the structure of a monad $(\mathrm{R}, \vec{\eta}, \vec{\mu})$, and the following distributivity condition is satisfied.

Let $\Xi : \mathrm{LR} \to \mathrm{RL}$ be the natural transformation that assigns the map $(\delta_f, \mu_f) : \mathrm{LR}f \to \mathrm{RL}f$ to each f. We require that the following diagrams commute.

$$\begin{tikzcd}
\mathrm{LR} \arrow[rr, "\Xi"] \arrow[d, "\vec{\delta}\mathrm{R}"] & & \mathrm{RL} \arrow[d, "\mathrm{R}\vec{\delta}"] \\
\mathrm{LLR} \arrow[r, "\mathrm{L}\Xi"] & \mathrm{LRL} \arrow[r, "\Xi\mathrm{L}"] & \mathrm{RLL}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{LRR} \arrow[r, "\Xi\mathrm{R}"] \arrow[d, "\mathrm{L}\vec{\mu}"] & \mathrm{RLR} \arrow[r, "\mathrm{R}\Xi"] & \mathrm{RRL} \arrow[d, "\vec{\mu}\mathrm{L}"] \\
\mathrm{LR} \arrow[rr, "\Xi"] & & \mathrm{RL}
\end{tikzcd}$$

Remark 3.1.6. The two distributivity conditions in the definition of an algebraic weak factorization system are equivalent. Indeed, the equation $\Xi\mathrm{L} \circ \mathrm{L}\Xi \circ \vec{\delta}\mathrm{R} = \mathrm{R}\vec{\delta} \circ \Xi$ encodes the two equations $\delta_{\mathrm{L}f} \circ \delta_f = E(id, \delta_f) \circ \delta_f$ and $\delta_f \circ \mu_f = \mu_{\mathrm{L}f} \circ E(\delta_f, \mu_f) \circ \delta_{\mathrm{R}f}$ for each $f \in \mathrm{ob}(\mathscr{C}^{\mathbf{2}})$, where E be the middomain functor of the AWFS (L, R). The first of the two equations is equivalent to $\vec{\delta}\mathrm{L} \circ \vec{\delta} = \mathrm{L}\vec{\delta} \circ \vec{\delta}$, which is already the associativity condition on the comultiplication of the comonad $(\mathrm{L}, \vec{\varepsilon}, \vec{\delta})$. Similarly, the equation $\Xi \circ \mathrm{L}\vec{\mu} = \vec{\mu}\mathrm{L} \circ \mathrm{R}\Xi \circ \Xi\mathrm{R}$ encodes the two equations $\delta_f \circ \mu_f = \mu_{\mathrm{L}f} \circ E(\delta_f, \mu_f) \circ \delta_{\mathrm{R}f}$ and $\mu_f \circ \mu_{\mathrm{R}f} = \mu_f \circ E(\mu_f, id)$ for each $f \in \mathrm{ob}(\mathscr{C}^{\mathbf{2}})$, the second of which is equivalent to the associativity of multiplication for $(\mathrm{R}, \vec{\eta}, \vec{\mu})$.

Remark 3.1.7. We will see in proposition 3.2.3 that the distributivity condition naturally arises from considering the structure of a RAWFS (L,R) on a LAWFS such that (L,R) is still a LAWFS.

We will also need to work with the following weakenings of an AWFS.

1. A functorial factorization (L,R) is a *left algebraic weak factorization system* (LAWFS) on $\mathscr{C}$ if $(\mathrm{L},\vec{\varepsilon})$ is equipped with the structure of a comonad $(\mathrm{L},\vec{\varepsilon},\vec{\delta})$.
2. A functorial factorization (L,R) is a *right algebraic weak factorization system* (RAWFS) on $\mathscr{C}$ if $(\mathrm{R},\vec{\eta})$ is equipped with the structure of a monad $(\mathrm{R},\vec{\eta},\vec{\mu})$.

In section 3.1.3 we work out various relations between L-coalgebras and R-algebras when we only have the structure of an LAWFS or RAWFS, rather than a full AWFS.

In the case of a full AWFS (L,R), we note that because $\vec{\delta}_f : \mathrm{L}f \to \mathrm{LL}f$ makes $\mathrm{L}f$ an L-coalgebra and $\vec{\mu}_f : \mathrm{RR}f \to \mathrm{R}f$ makes $\mathrm{R}f$ and R-algebra, (L,R) satisfies the hypothesis of 3.1.2. This proves the following.

Proposition 3.1.8. *If* (L,R) *is an AWFS on* $\mathscr{C}$*, then* $(|\mathbf{Coalg}_{\mathrm{L}}|,|\mathbf{Alg}_{\mathrm{R}}|)$ *is a weak factorization system on* $\mathscr{C}$*.*

An AWFS has more structure than just that of a weak factorization system, however. As remark 3.1.3 makes clear, for any diagrams $D_{\mathrm{L}} : \mathscr{D} \to \mathscr{C}^{\mathbf{2}}$ and $D_{\mathrm{R}} : \mathscr{D} \to \mathscr{C}^{\mathbf{2}}$, which factor through $\mathbf{Coalg}_{\mathrm{L}}$ and $\mathbf{Alg}_{\mathrm{R}}$, respectively, and any natural transformation $\vec{\theta} : D_{\mathrm{L}} \to D_{\mathrm{R}}$, there is natural transformation $\lambda : \mathrm{cod}\ D_{\mathrm{L}} \to \mathrm{dom}\ D_{\mathrm{R}}$ which encodes a lift in each diagram of the following form, where $\vec{\theta}_d = (\theta_d^0, \theta_d^1)$.

$$\begin{tikzcd}
\cdot \arrow[r, "\theta_d^0"] \arrow[d, "D_{\mathrm{L}}d"'] & \cdot \arrow[d, "D_{\mathrm{R}}d"] \\
\cdot \arrow[r, "\theta_d^1"'] \arrow[ur, dashed, "\lambda_d"] & \cdot
\end{tikzcd}$$

So we have natural lifts rather than only knowing that a solution to each lifting problem $(u, v) : f \to g$ exists independently when f is an L-coalgebra and g is an R-algebra.

Definition 3.1.9. Let (L, R) and $(\mathrm{L}', \mathrm{R}')$ be AWFSs with middomain functors E and E', respectively. A *map of algebraic weak factorization systems* $\zeta : (\mathrm{L}, \mathrm{R}) \to (\mathrm{L}', \mathrm{R}')$ is a map of functorial factorizations such that $(id, \zeta) : \mathrm{L} \to \mathrm{L}'$ is a map of comonads and $(\zeta, id) : \mathrm{R} \to \mathrm{R}'$ is a map of monads.

A *map of LAWFSs* $\zeta : (\mathrm{L}, \mathrm{R}) \to (\mathrm{L}', \mathrm{R}')$ is a map of functorial factorizations such that $(id, \zeta) : \mathrm{L} \to \mathrm{L}'$ is a map of comonads. A *map of RAWFSs* $\zeta : (\mathrm{L}, \mathrm{R}) \to (\mathrm{L}', \mathrm{R}')$ is a map of functorial factorizations such that $(\zeta, id) : \mathrm{R} \to \mathrm{R}'$ is a map of monads.

Definition 3.1.10. Let $\mathscr{C}$ be a bicomplete category and let $\mathcal{W}$ be a collection of maps in $\mathscr{C}$ that satisfies the 2 out of 3 property. An *algebraic model category* on $\mathscr{C}$ with weak equivalences $\mathcal{W}$ is a map of AWFSs $(\mathrm{C}_t, \mathrm{F}) \to (\mathrm{C}, \mathrm{F}_t)$ on $\mathscr{C}$ such that $|\mathbf{Coalg}_{\mathrm{C}_t}| = |\mathbf{Coalg}_{\mathrm{C}}| \cap \mathcal{W}$ and $|\mathbf{Alg}_{\mathrm{F}_t}| = |\mathbf{Alg}_{\mathrm{F}}| \cap \mathcal{W}$.

From the definition it is clear that the underlying weak factorization systems $(|\mathbf{Coalg}_{\mathrm{C}_t}|, |\mathbf{Alg}_{\mathrm{F}}|)$ and $(|\mathbf{Coalg}_{\mathrm{C}}|, |\mathbf{Alg}_{\mathrm{F}_t}|)$ of an algebraic model category on $\mathscr{C}$ are the weak factorization systems of a model category on $\mathscr{C}$.

To better describe the structure of algebraic weak factorization systems and algebraic model categories, we will need to define categorical lifts. Similar to how the operations $(-)^{\square}$ and ${}^{\square}(-)$ on collections describe the relations between the left and right classes in a weak factorization system, we will define operations $(-)^{\boxslash}$ and ${}^{\boxslash}(-)$ on categories that describe the relations between categories of algebras and coalgebras for an AWFS.

3.1.3 Categorical Lifts

Let $\mathscr{A}$ be a category over $\mathscr{C}^{\mathbf{2}}$. So there is a functor $A : \mathscr{A} \to \mathscr{C}^{\mathbf{2}}$. We will define a new category $A^{\boxslash}$ over $\mathcal{C}^{\mathbf{2}}$. When the context is clear, we will also use the notation $\mathscr{A}^{\boxslash}$.

An object in $\mathscr{A}^{\boxslash}$ is a pair $\langle p, \varphi \rangle$, where p is an object in $\mathscr{C}^{\mathbf{2}}$ and φ is a coherent choice of lift in the following square for every object f in $\mathscr{A}$ and every map $(u, v) : A(f) \to p$ in $\mathscr{C}^{\mathbf{2}}$.

$$\begin{tikzcd}
{} \arrow[r, "u"] \arrow[d, "A(f)"'] & {} \arrow[d, "p"] \\
{} \arrow[r, "v"'] \arrow[ur, dashed, "\varphi(f,u,v)" description] & {}
\end{tikzcd}$$

The coherence condition on φ specifies that $\varphi(g, s, t) \circ b = \varphi(f, u, v)$ for every map $(a', b') : f \to g$ in $\mathscr{A}$ and every commutative diagram of the following form, where $(a, b) = A(a', b') : A(f) \to A(g)$.

$$\begin{tikzcd}
{} & {} \arrow[r, "u"] \arrow[dl, "a"'] \arrow[dd, "A(f)"] & {} \arrow[dd, "p"] \\
{} \arrow[urr, "s"] \arrow[dd, "A(g)"'] & & \\
& {} \arrow[r, "v"] \arrow[dl, "b"'] & {} \\
{} \arrow[urr, "t"'] & &
\end{tikzcd}$$

A morphism $\langle p, \varphi \rangle \to \langle q, \psi \rangle$ in $\mathscr{A}^{\boxslash}$ is a map $(x, y) : p \to q$ in $\mathscr{C}^{\mathbf{2}}$ such that for every commutative diagram of the following form, $x \circ \varphi(f, u, v) = \psi(f, s, t)$.

$$\begin{tikzcd}
{} \arrow[r, "u"] \arrow[drr, "s"] \arrow[dd, "A(f)"'] & {} \arrow[dr, "x"] \arrow[dd, "p"] & \\
& & {} \arrow[dd, "q"] \\
{} \arrow[r, "v"] \arrow[drr, "t"'] & {} \arrow[dr, "y"] & \\
& & {}
\end{tikzcd}$$

By forgetting the choice of natural lift, we get a forgetful functor $\mathscr{A}^{\boxslash} \to \mathscr{C}^{\mathbf{2}}$.

We define the category ${}^{\boxslash}\mathscr{A}$ over $\mathscr{C}^{\mathbf{2}}$ dually. Namely, an object in ${}^{\boxslash}\mathscr{A}$ is a pair $\langle f, \varphi \rangle$, where f is an object in $\mathscr{C}^{\mathbf{2}}$ and φ is coherent choice of lift in each diagram of the following form.

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "A(p)"] \\ \cdot \arrow[r, "v"'] \arrow[ur, dashed, "\varphi(u,v,p)" description] & \cdot \end{tikzcd}$$

A morphism $\langle f, \varphi \rangle \to \langle g, \psi \rangle$ in ${}^{\boxslash}\mathscr{A}$ is a map $(a, b) : f \to g$ in $\mathscr{C}^{\mathbf{2}}$ compatible with the lifts φ and ψ.

Proposition 3.1.11. *Let $\mathscr{A}$ be a category over $\mathscr{C}^{\mathbf{2}}$.*

1. *The forgetful functor ${}^{\boxslash}\mathscr{A} \to \mathscr{C}^{\mathbf{2}}$ creates colimits.*

2. *The forgetful functor $\mathscr{A}^{\boxslash} \to \mathscr{C}^{\mathbf{2}}$ creates limits.*

Proof. Let A be the functor $\mathscr{A} \to \mathscr{C}^{\mathbf{2}}$ and let $U : {}^{\boxslash}\mathscr{A} \to \mathscr{C}^{\mathbf{2}}$ be the forgetful functor. Let $D : \mathscr{D} \to {}^{\boxslash}\mathscr{A}$ be a diagram such that the colimit of UD exists in $\mathscr{C}^{\mathbf{2}}$. Let $(\alpha, \beta) : UD \overset{\cdot}{\longrightarrow} \operatorname{colim} UD$ be the colimiting cocone in $\mathscr{C}^{\mathbf{2}}$. By the definition of ${}^{\boxslash}\mathscr{A}$, there is a coherent lift φ_d such that $Dd = \langle UDd, \varphi_d \rangle$ for each object d in $\mathscr{D}$.

Suppose p is an object in $\mathscr{A}$ and $(u, v) : \operatorname{colim} UD \to Ap$ is a map in $\mathscr{C}^{\mathbf{2}}$. For each d, the lifting problem $(u \circ \alpha_d, v \circ \beta_d) : UDd \to Ap$ has the solution $\varphi_d(u \circ \alpha_d, v \circ \beta_d, p)$. These lifts are natural with respect to morphisms in the image of D by the definition of morphisms in ${}^{\boxslash}\mathscr{A}$. So $d \mapsto \varphi_d(u \circ \alpha_d, v \circ \beta_d, p)$ defines a cocone $\operatorname{cod} UD \overset{\cdot}{\longrightarrow} \operatorname{dom} Ap$. Therefore there is an induced map

$$\psi(u, v, p) : \operatorname{cod}(\operatorname{colim} UD) \cong \operatorname{colim}(\operatorname{cod} UD) \to \operatorname{dom} Ap$$

out of the colimit, which is a solution to the lifting problem $(u, v) : \operatorname{colim} UD \to Ap$. If $(x, y) = A(x', y') : Ap \to Aq$ for a map $(x', y') : p \to q$ in $\mathscr{A}$, then for each d, $x \circ \varphi_d(u, v, p) = \varphi_d(xu, yv, q)$. Thus $x \circ \psi(u, v, p) = \psi(xu, yv, q)$. So $\langle \operatorname{colim} UD, \psi \rangle$ is an object in ${}^{\boxslash}\mathscr{A}$ and $(\alpha, \beta) : D \overset{\cdot}{\longrightarrow} \langle \operatorname{colim} UD, \psi \rangle$ is a cocone in ${}^{\boxslash}\mathscr{A}$. The uniqueness of maps out of the colimiting cocone $\operatorname{colim} UD$ in $\mathscr{C}^{\mathbf{2}}$ shows that the cocone (α, β) in ${}^{\boxslash}\mathscr{A}$ is a colimiting cocone.

The proof of (2) is dual. □

Corollary 3.1.12. *Let $\mathscr{A}$ be a category over $\mathscr{C}^{\mathbf{2}}$.*

1. *If $\mathscr{C}^{\mathbf{2}}$ is cocomplete, then ${}^{\boxslash}\mathscr{A}$ is cocomplete.*

2. *If $\mathscr{C}^{\mathbf{2}}$ is complete, then $\mathscr{A}^{\boxslash}$ is complete.*

Proposition 3.1.13.

1. *If (L, R) is an LAWFS on $\mathscr{C}$, then $\mathbf{Alg}_{\mathrm{R}}$ is a retract of $\mathbf{Coalg}_{\mathrm{L}}^{\boxslash}$ over $\mathscr{C}^{\mathbf{2}}$.*

2. *If (L, R) is an RAWFS on $\mathscr{C}$, then $\mathbf{Coalg}_{\mathrm{L}}$ is a retract of ${}^{\boxslash}\mathbf{Alg}_{\mathrm{R}}$ over $\mathscr{C}^{\mathbf{2}}$.*

Proof. We will construct the first retraction. The second one is dual. Let $E : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ be the middomain functor of (L, R). Suppose $\langle f, \vec{k} \rangle$ is an R-algebra. We define a natural lift $\phi_{\langle f, \vec{k} \rangle}$ for $\langle f, \vec{k} \rangle$ as follows. If $\langle h, \vec{m} \rangle$ is an L-coalgebra and $(a, b) : h \to f$ is a morphism of $\mathscr{C}^{\mathbf{2}}$, then $\phi_{\langle f, \vec{k} \rangle}\big(\langle h, \vec{m} \rangle, a, b\big) \coloneqq k \circ E(a, b) \circ m$, where $\vec{k} = (k, id)$ and $\vec{m} = (id, m)$. We now define the functor F as follows.

$$\begin{tikzcd}
F: & \mathbf{Alg}_{\mathrm{R}} \arrow[r] & \mathbf{Coalg}_{\mathrm{L}}^{\boxslash} \\
& \langle f, \vec{k} \rangle \arrow[r, mapsto] \arrow[d, "{(u,v)}"] & \big\langle f, \phi_{\langle f, \vec{k} \rangle} \big\rangle \arrow[d, "{(u,v)}"] \\
& \langle g, \vec{l} \rangle \arrow[r, mapsto] & \big\langle g, \phi_{\langle g, \vec{l} \rangle} \big\rangle
\end{tikzcd}$$

Let G be the functor defined as below

$$\begin{tikzcd}
G: & \mathbf{Coalg}_{\mathrm{L}}^{\boxslash} \arrow[r] & \mathbf{Alg}_{\mathrm{R}} \\
& \langle f, \varphi \rangle \arrow[r, mapsto] \arrow[d, "{(u,v)}"] & \Big\langle f, \vec{\varphi}\big(\langle \mathrm{L}f, \vec{\delta}_f \rangle, id, \mathrm{R}f\big) \Big\rangle \arrow[d, "{(u,v)}"] \\
& \langle g, \psi \rangle \arrow[r, mapsto] & \Big\langle g, \vec{\psi}\big(\langle \mathrm{L}g, \vec{\delta}_g \rangle, id, \mathrm{R}g\big) \Big\rangle,
\end{tikzcd}$$

where $\vec{\varphi}\big(\langle \mathrm{L}f, \vec{\delta}_f\rangle, id, \mathrm{R}f\big) = \big(\varphi\big(\langle \mathrm{L}f, \vec{\delta}_f\rangle, id, \mathrm{R}f\big), id\big) : \mathrm{R}f \to f$ and we use the same convention for $\vec{\psi}$.

Clearly, $G \circ F$ sends morphisms to themselves. Let $\langle f, \vec{k}\rangle$ be an R-algebra. Then $G \circ F\big(\langle f, \vec{k}\rangle\big) = \Big\langle f, \vec{\phi}_{\langle f, \vec{k}\rangle}\big(\langle \mathrm{L}f, \vec{\delta}_f\rangle, id, \mathrm{R}f\big)\Big\rangle = \big\langle f, \big(k \circ E(id, \mathrm{R}f) \circ \delta_f, id\big)\big\rangle$. But $E(id, \mathrm{R}f) \circ \delta_f = id_{Ef}$. So $G \circ F\big(\langle f, \vec{k}\rangle\big) = \langle f, \vec{k}\rangle$.

From our definitions of F and G, it is clear that they are functors over $\mathscr{C}^{\mathbf{2}}$. ☐

Corollary 3.1.14.

1. *If* (L, R) *is an LAWFS on* $\mathscr{C}$*, then* $|\mathbf{Coalg}_{\mathrm{L}}^{\boxslash}| = |\mathbf{Alg}_{\mathrm{R}}|$.

2. *If* (L, R) *is an RAWFS on* $\mathscr{C}$*, then* $|{}^{\boxslash}\mathbf{Alg}_{\mathrm{R}}| = |\mathbf{Coalg}_{\mathrm{L}}|$.

Proof. This is just an application of 2.1.27 to the result in 3.1.13. ☐

Proposition 3.1.15.

1. *If* (L, R) *is an LAWFS on* $\mathscr{C}$*, then there is an isomorphism of categories* $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}\boxslash} \cong \mathbf{Alg}_{\mathrm{R}}$ *over* $\mathscr{C}^{\mathbf{2}}$.

2. *If* (L, R) *is an RAWFS on* $\mathscr{C}$*, then there is an isomorphism of categories* ${}^{\boxslash}\mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}} \cong \mathbf{Coalg}_{\mathrm{L}}$ *over* $\mathscr{C}^{\mathbf{2}}$.

Proof of (1). Consider the functors F and G from the proof of proposition 3.1.13. We have a restriction functor $\mathbf{Coalg}_{\mathrm{L}}^{\boxslash} \to \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}\boxslash}$. Composing with F gives a functor $F' : \mathbf{Alg}_{\mathrm{R}} \to \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}\boxslash}$. Since $\langle \mathrm{L}f, \vec{\delta}_f\rangle$ is a coalgebra for the comonad L, G extends to a functor $G' : \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}\boxslash} \to \mathbf{Alg}_{\mathrm{R}}$.

Just as in the previous proof, F' and G' are functors over $\mathscr{C}^{\mathbf{2}}$ and $G' \circ F' = Id_{\mathbf{Alg}_{\mathrm{R}}}$.

Note that on morphisms, $F' \circ G'$ agrees with the identity functor. Suppose $\langle f, \psi \rangle$ is an object in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}\boxslash}$. Let

$$\chi = \phi_{\big\langle f, \vec{\psi}\big(\langle \mathrm{L}f, \vec{\delta}_f \rangle, id, \mathrm{R}f\big)\big\rangle}.$$

So $F' \circ G'\big(\langle f, \psi \rangle\big) = \langle f, \chi \rangle$. Let $V_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}} \to \mathscr{C}^{\mathbf{2}}$ be the forgetful functor. On a square $(a, b) : V_{\mathrm{L}}\big(\langle g, \vec{l} \rangle\big) \to f$,

$$\chi\big(\langle g, \vec{l} \rangle, a, b\big) = \psi\big(\langle \mathrm{L}f, \vec{\delta}_f \rangle, id, \mathrm{R}f\big) \circ E(a, b) \circ l$$

Because $\langle g, \vec{l} \rangle$ is an object in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$, $\vec{l} = (id, l) : \langle g, \vec{l} \rangle \to \langle \mathrm{L}g, \vec{\delta}_g \rangle$ is a map in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$. Therefore $\big(a, E(a, b) \circ l\big) : g \to \mathrm{L}f$ is a map in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$. Since ψ is a natural lift with respect to maps in $\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}}$,

$$\psi\big(\langle \mathrm{L}f, \vec{\delta}_f \rangle, id, \mathrm{R}f\big) \circ E(a, b) \circ l = \psi\big(\langle g, \vec{l} \rangle, a, b\big).$$

So $\chi = \psi$ and $F' \circ G' = Id_{\mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}\boxslash}}$.

The proof of (2) is dual. $\square$

A much weaker version of the above proposition is given in [BG16a, §2.7].

Proposition 3.1.16.

1. *If* (L, R) *is a LAWFS on* $\mathscr{C}$*, then* $|\mathbf{Coalg}_{\mathrm{L}}^{\boxslash}| = |\mathbf{Coalg}_{\mathrm{L}}|^{\square}$.
2. *If* (L, R) *is a RAWFS on* $\mathscr{C}$*, then* $|{}^{\boxslash}\mathbf{Alg}_{\mathrm{R}}| = {}^{\square}|\mathbf{Alg}_{\mathrm{R}}|$.

Proof. (1) Suppose $f \in |\mathbf{Coalg}_{\mathrm{L}}|^{\square}$. Since L is a comonad, $\mathrm{L}f \in |\mathbf{Coalg}_{\mathrm{L}}|$ and a solution exists to the lifting problem $(id, \mathrm{R}f) : \mathrm{L}f \to f$. Thus $f \in |\mathbf{Alg}_{\mathrm{R}}|$. The reverse inclusion is immediate. The second case is dual. $\square$

3.2 Free AWFSs

Let $\mathbf{AWFS}(\mathscr{C})$ be the category whose objects are AWFSs and whose maps are maps of AWFSs. Let $\mathbf{LAWFS}(\mathscr{C})$ be the category whose objects are LAWFSs and whose maps are maps of LAWFSs. There is a functor

$$\mathbf{AWFS}(\mathscr{C}) \xrightarrow{\mathbb{G}_1} \mathbf{LAWFS}(\mathscr{C}) \tag{3.3}$$

that forgets the multiplicative structure and distributivity rules of AWFSs and forgets that a map $\zeta : (\mathrm{L}, \mathrm{R}) \to (\mathrm{C}, \mathrm{F})$ of AWFSs is a map $(\zeta, id) : \mathrm{R} \to \mathrm{F}$ of monads. A *free AWFS* on a LAWFS $(\mathrm{L}_1, \mathrm{R}_1)$ is a reflection of $(\mathrm{L}_1, \mathrm{R}_1)$ along $\mathbb{G}_1$. In section 3.2.3, we will see how to apply the results of section 2.5 to the construction of free AWFSs on LAWFSs. We will see that when (L, R) is the free AWFS on the LAWFS $(\mathrm{L}_1, \mathrm{R}_1)$, then $\mathbf{Alg}_{\mathrm{R}_1} \cong \mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}}$ over $\mathscr{C}^{\mathbf{2}}$ and $|\mathbf{Alg}_{\mathrm{R}_1}| = |\mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}}| = |\mathbf{Alg}_{\mathrm{R}}|$.

In section 3.2.4, we will see how to construct free LAWFSs on comonads. Let $\mathbf{Cmd}(\mathscr{C}^{\mathbf{2}})$ be the category whose objects are comonads on $\mathscr{C}^{\mathbf{2}}$ and whose morphisms the maps of comonads. Let $\mathbb{G}_2 : \mathbf{LAWFS}(\mathscr{C}) \to \mathbf{Cmd}(\mathscr{C}^{\mathbf{2}})$ be the forgetful functor that sends each LAWFS (L, R) to the comonad L and sends each map $\zeta : (\mathrm{L}, \mathrm{R}) \to (\mathrm{C}, \mathrm{F})$ of LAWFSs to the map $(id, \zeta) : \mathrm{L} \to \mathrm{C}$ of comonads. The free LAWFS $(\mathrm{L}_1, \mathrm{R}_1)$ on a comonad L_0 is the reflection of L_0 along the functor $\mathbb{G}_2$. When $(\mathrm{L}_1, \mathrm{R}_1)$ is the free LAWFS on a comonad L_0, we have that $\mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}\boxslash} \cong \mathbf{Alg}_{\mathrm{R}_1}$ and $|\mathbf{Coalg}_{\mathrm{L}_0}^{\boxslash}| = |\mathbf{Alg}_{\mathrm{R}_1}|$.

Finally, in section 3.2.5, we show how to construct free comonads on diagrams in $\mathscr{C}^{\mathbf{2}}$. Let $\mathbf{CAT}$ be the metacategory whose objects are the categories in our Grothendieck universe and whose morphisms are functors. Then $\mathbf{CAT}/\mathcal{C}^{\mathbf{2}}$ is the metacategory whose objects are functors $F : \mathscr{A} \to \mathscr{C}^{\mathbf{2}}$ and whose morphisms K from $F : \mathscr{A} \to \mathscr{C}^{\mathbf{2}}$ to $G : \mathscr{B} \to \mathscr{C}^{\mathbf{2}}$ are functors $K : \mathscr{A} \to \mathscr{B}$ such that $GK = F$. There is a functor $\mathbb{G}_3 : \mathbf{Cmd}(\mathscr{C}^{\mathbf{2}}) \to \mathbf{CAT}/\mathscr{C}^{\mathbf{2}}$ that sends each comonad L on $\mathscr{C}^{\mathbf{2}}$ to the forgetful functor $V_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}} \to \mathscr{C}^{\mathbf{2}}$ and sends each map of comonads $\vec{\zeta} : \mathrm{L} \to \mathrm{C}$ to the functor $\vec{\zeta}_* : \mathbf{Coalg}_{\mathrm{L}}^{\mathrm{EM}} \to \mathbf{Coalg}_{\mathrm{C}}^{\mathrm{EM}}$ over $\mathscr{C}^{\mathbf{2}}$. A free comonad L_0^I on a diagram $I : \mathscr{I} \to \mathscr{C}^{\mathbf{2}}$ is the reflection of I along $\mathbb{G}_3$. When L_0^I is the free comonad on I, $I^{\boxslash} \cong \mathbf{Coalg}_{\mathrm{L}_0^I}^{\mathrm{EM}\boxslash}$ over $\mathscr{C}^{\mathbf{2}}$.

Putting this all together, we will have a method for constructing free AWFSs on diagrams $I : \mathscr{I} \to \mathscr{C}^{\mathbf{2}}$. This is the reflection of I along the composite functor $\mathbb{G}_3\mathbb{G}_2\mathbb{G}_1$.

$$\mathbf{AWFS}(\mathscr{C}) \xrightarrow{\mathbb{G}_1} \mathbf{LAWFS}(\mathscr{C}) \xrightarrow{\mathbb{G}_2} \mathbf{Cmd}(\mathscr{C}^{\mathbf{2}}) \xrightarrow{\mathbb{G}_3} \mathbf{CAT}/\mathscr{C}^{\mathbf{2}}$$

The free AWFS $(\mathrm{L}^I, \mathrm{R}^I)$ on $I : \mathscr{I} \to \mathscr{C}^{\mathbf{2}}$ can be viewed as a categorical analog to the small object argument on a set of maps in $\mathscr{C}$. We have in this case that $I^{\boxslash} \cong \mathbf{Alg}_{\mathrm{R}^I}^{\mathrm{EM}}$ over $\mathscr{C}^{\mathbf{2}}$ and that $|I^{\boxslash}| = |\mathbf{Alg}_{\mathrm{R}^I}|$.

3.2.1 Monoidal Structure on the Category of LAWFSs

In order to apply our results for the existence of free monads on pointed endofunctors to the existence of free AWFSs on LAWFSs, we need to define a strict monoidal structure on the category left algebraic weak factorization systems. We will describe the monoidal structure in this section.

Let $\mathscr{C}$ be a category. It will often be convenient to represent an object of $\mathbf{LAWFS}(\mathscr{C})$ by a letter X rather than a pair (L, R). When we want to specify the left and right factors of a LAWFS X, we will often use the notation $X = (\mathrm{L}^X, \mathrm{R}^X)$.

Given two LAWFSs (L, R) and (C, F) on $\mathscr{C}$ with middomain functors E and E', respectively, let $(\mathrm{L}, \mathrm{R}) \circledast (\mathrm{C}, \mathrm{F})$ be the functorial factorization on $\mathscr{C}$ defined on each object $f : X \to Y$ as the object

$$X \xrightarrow{\mathrm{LF}f \circ \mathrm{C}f} E\mathrm{F}f \xrightarrow{\mathrm{RF}f} Y$$

in $\mathscr{C}^{\mathbf{3}}$ and on each morphism $(u,v) : f \to g$ from $f : X \to Y$ to $g : Z \to W$ as the map

$$\begin{tikzcd}
X \arrow[r, "\mathrm{LF}f \circ \mathrm{C}f"] \arrow[d, "u"] & E\mathrm{F}f \arrow[r, "\mathrm{RF}f"] \arrow[d, "E(E'(u,v),v)"] & Y \arrow[d, "v"] \\
Z \arrow[r, "\mathrm{LF}g \circ \mathrm{C}g"'] & E\mathrm{F}g \arrow[r, "\mathrm{RF}g"'] & W
\end{tikzcd}$$

in $\mathscr{C}^{\mathbf{3}}$.

Let $\mathtt{I} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{3}}$ be the functorial factorization defined on objects $f : X \to Y$ in $\mathscr{C}^{\mathbf{2}}$ by

$$X \xrightarrow{id} X \xrightarrow{f} Y$$

Then $(\mathrm{L},\mathrm{R}) \circledast \mathtt{I} = (\mathrm{L},\mathrm{R})$ and $\mathtt{I} \circledast (\mathrm{L},\mathrm{R}) = (\mathrm{L},\mathrm{R})$. The equality $\mathtt{I} \circledast \mathtt{I} = \mathtt{I}$ makes $\mathtt{I}$ a LAWFS. There is a unit map $\eta : \mathtt{I} \to X$ defined on each f by $\eta_f = \mathrm{L}f : \operatorname{dom} f \to Ef$ and it is easy to check that this is a map of LAWFSs.

Proposition 3.2.1 ([Gar07, §3.2]). *The operation $\circledast$ defines a bifunctor*

$$\circledast : \mathbf{LAWFS}(\mathscr{C}) \times \mathbf{LAWFS}(\mathscr{C}) \longrightarrow \mathbf{LAWFS}(\mathscr{C}).$$

Proof. Let (L,R) and (C,F) be LAWFSs on $\mathscr{C}$ with middomain functors E and E', respectively. We will show that the functorial factorization $(\mathrm{L},\mathrm{R}) \circledast (\mathrm{C},\mathrm{F})$ is a LAWFS. Let $\mathrm{S}f = \mathrm{LF}f \circ \mathrm{C}f$. Then S is a domain-preserving endofunctor on $\mathscr{C}^{\mathbf{2}}$ with counit map $\vec{\varepsilon}^{\mathrm{S}} = (id, \mathrm{RF}f) : \mathrm{S}f \to f$. It suffices to show that $(\mathrm{S}, \vec{\varepsilon}^{\mathrm{S}})$ extends to a comonad. Let $\vec{\delta}^{\mathrm{L}} = (id, \delta^{\mathrm{L}})$ and $\vec{\delta}^{\mathrm{C}} = (id, \delta^{\mathrm{C}})$ be the comultiplication maps of L and C, respectively. The comultiplication map for S is the map $\vec{\delta}^{\mathrm{S}}_f = (id, \delta^{\mathrm{S}}_f) : \mathrm{LF}f \circ \mathrm{C}f \to \mathrm{LF}(\mathrm{LF}f \circ \mathrm{C}f) \circ \mathrm{C}(\mathrm{LF}f \circ \mathrm{C}f)$, where

$$\delta^{\mathrm{S}}_f = E\big(E'(id, \mathrm{LF}f) \circ \delta^{\mathrm{C}}_f, id\big) \circ \delta^{\mathrm{L}}_{\mathrm{F}f} \tag{3.4}$$

is the map shown in the following diagram.

$$\begin{tikzcd}
{} \arrow[r, "id"] \arrow[d, "\mathrm{C}f"'] & {} \arrow[r, "id"] \arrow[d, "\mathrm{C}f"'] & {} \arrow[r, "id"] \arrow[d, "\mathrm{CC}f"'] & {} \arrow[d, "\mathrm{C}(\mathrm{LF}f \circ \mathrm{C}f)"] \\
{} \arrow[r, "id"] \arrow[d, "\mathrm{LF}f"'] & {} \arrow[r, "\delta^{\mathrm{C}}_f"] \arrow[d, "\mathrm{LLF}f"'] & {} \arrow[r, "E'(id{,}\mathrm{LF}f)"] & {} \arrow[d, "\mathrm{LF}(\mathrm{LF}f \circ \mathrm{C}f)"] \\
{} \arrow[r, "\delta^{\mathrm{L}}_{Ff}"] \arrow[dr, bend right, "id"'] & {} \arrow[rr, "E\left(E'(id{,}\mathrm{LF}f) \circ \delta^{\mathrm{C}}_f{,}id\right)"] \arrow[d, "\mathrm{RLF}f"'] & & {} \arrow[d, "\mathrm{RF}(\mathrm{LF}f \circ \mathrm{C}f)"] \\
& {} \arrow[rr, "id"] & & {}
\end{tikzcd}$$

From the diagram, $\mathrm{RF}(\mathrm{LF}f \circ \mathrm{C}f) \circ \delta^{\mathrm{S}}_f = id$. So $\vec{\varepsilon}^{\mathrm{S}}_{\mathrm{S}f} \circ \vec{\delta}^{\mathrm{S}}_f = id_{\mathrm{S}f}$. Applying E' to the map $\vec{\varepsilon}^{\mathrm{S}}_f = (id, \mathrm{RF}f) : \mathrm{S}f \to f$ in $\mathscr{C}^{\mathbf{2}}$ gives a map $E'(id, \mathrm{RF}f) : E'\mathrm{S}f \to E'f$ in $\mathscr{C}$. Then

$$E'(id, \mathrm{RF}f) \circ E'(id, \mathrm{LF}f) \circ \delta^{\mathrm{C}}_f = E'(id, \mathrm{F}f) \circ \delta^{\mathrm{C}}_f = id.$$

Now, applying E to the map $\big(E'(id, \mathrm{RF}f), \mathrm{RF}f\big) : \mathrm{FS}f \to \mathrm{F}f$ gives us

$$E\big(E'(id, \mathrm{RF}f), \mathrm{RF}f\big) \circ E\big(E'(id, \mathrm{LF}f) \circ \delta^{\mathrm{C}}_f, id\big) \circ \delta^{\mathrm{L}}_{\mathrm{F}f} = E(id, \mathrm{RF}f) \circ \delta^{\mathrm{L}}_{\mathrm{F}f} = id.$$

So $\mathrm{S}\vec{\varepsilon}^{\mathrm{S}}_f \circ \vec{\delta}^{\mathrm{S}}_f = id_{\mathrm{S}f}$.

One can check that a map of LAWFSs $\zeta : (\mathrm{C}, \mathrm{F}) \to (\mathrm{C}', \mathrm{F}')$ defines a map of LAWFSs $(\mathrm{L}, \mathrm{R}) \circledast \zeta : (\mathrm{L}, \mathrm{R}) \circledast (\mathrm{C}, \mathrm{F}) \to (\mathrm{L}, \mathrm{R}) \circledast (\mathrm{C}', \mathrm{F}')$ and that a map of LAWFSs $\theta : (\mathrm{L}, \mathrm{R}) \to (\mathrm{L}', \mathrm{R}')$ defines a map of LAWFSs $\theta \circledast (\mathrm{C}, \mathrm{F}) : (\mathrm{L}, \mathrm{R}) \circledast (\mathrm{C}, \mathrm{F}) \to (\mathrm{L}', \mathrm{R}') \circledast (\mathrm{C}, \mathrm{F})$. ☐

Corollary 3.2.2. *The category* $(\mathbf{LAWFS}(\mathscr{C}), \circledast, \mathtt{I})$ *is strict monoidal.*

Proof. Since functor composition is a strict monoidal product, $(\mathrm{R}^{\mathrm{X}}\mathrm{R}^{\mathrm{Y}})\mathrm{R}^{\mathrm{Z}}f = \mathrm{R}^{\mathrm{X}}(\mathrm{R}^{\mathrm{Y}}\mathrm{R}^{\mathrm{Z}}f)$ on each object f. So $(\mathrm{X} \circledast \mathrm{Y}) \circledast \mathrm{Z} = \mathrm{X} \circledast (\mathrm{Y} \circledast \mathrm{Z})$. ☐

Proposition 3.2.3 ([Gar07, §3.2])**.** *There is an isomorphism of categories*

$$\mathbf{Mon}_{\circledast}(\mathbf{LAWFS}(\mathscr{C})) \cong \mathbf{AWFS}(\mathscr{C}).$$

Proof. A monoid in $(\mathbf{LAWFS}(\mathscr{C}), \circledast, \mathtt{I})$ is an object $\mathrm{X} = (\mathrm{L}, \mathrm{R})$ with maps $\eta : \mathtt{I} \to \mathrm{X}$ and $\mu : \mathrm{X} \circledast \mathrm{X} \to \mathrm{X}$ in $\mathbf{LAWFS}(\mathscr{C})$ satisfying the monoid equations. The monoid equations translate exactly to the requirement that $(\mathrm{R}, \vec{\eta}, \vec{\mu})$ is a monad. As we already noted, the unit map $\mathtt{I} \to \mathrm{X}$ is a map in $\mathbf{LAWFS}(\mathscr{C})$. Requiring that the multiplication map $\mu : \mathrm{X} \circledast \mathrm{X} \to \mathrm{X}$ is a map in $\mathbf{LAWFS}(\mathscr{C})$ is equivalent to the requirement that X satisfy the distributivity conditions of an AWFS. To see this, we will need some notation. Let $\mathrm{S}f = \mathrm{LR}f \circ \mathrm{L}f$ and let $\mathrm{T}f = \mathrm{RR}f$ on each object f in $\mathscr{C}^{\mathbf{2}}$. Then $(\mathrm{S}, \mathrm{T}) = \mathrm{X} \circledast \mathrm{X}$ is a LAWFS. The comultiplication map $\delta^{\mathrm{S}}_f : \mathrm{S}f \to \mathrm{SS}f$ of (S, T) is defined by equation (3.4) of proposition 3.2.1, with $(\mathrm{C}, \mathrm{F}) = (\mathrm{L}, \mathrm{R})$. The middomain functor of (S, T) sends maps $(u, v) : f \to g$ in $\mathscr{C}^{\mathbf{2}}$ to maps $E\big(E(u, v), id\big) : E\mathrm{R}f \to E\mathrm{R}g$ in $\mathscr{C}$. Then the requirement that $\mu : \mathrm{X} \circledast \mathrm{X} \to \mathrm{X}$ is a map in $\mathbf{LAWFS}(\mathscr{C})$ is the requirement that the following diagram commutes for each f.

$$\begin{tikzcd}
(\mathrm{S}f, \mathrm{T}f) \arrow[rr, "\mu_f"] \arrow[d, "\delta^{\mathrm{S}}_f"] & & (\mathrm{L}f, \mathrm{R}f) \arrow[d, "\delta_f"] \\
(\mathrm{SS}f, \mathrm{T}f \circ \mathrm{TS}f) \arrow[r, "E\big(E(id,\mu_f),\mu_f\big)"] & (\mathrm{SL}f, \mathrm{R}f \circ \mathrm{TL}f) \arrow[r, "\mu_{\mathrm{L}f}"] & (\mathrm{LL}f, \mathrm{R}f \circ \mathrm{RL}f)
\end{tikzcd}$$

We note that

$$E\big(E(id, \mu_f), \mu_f\big) \circ \delta^{\mathrm{S}}_f = E\big(E(id, \mu_f), \mu_f\big) \circ E\big(E(id, \mathrm{LR}f) \circ \delta_f, id\big) \circ \delta_{\mathrm{R}f}$$

$$= E(\delta_f, \mu_f) \circ \delta_{\mathrm{R}f}$$

So the above requirement is equivalent to the condition $\delta_f \circ \mu_f = \mu_{\mathrm{L}f} \circ E(\delta_f, \mu_f) \circ \delta_{\mathrm{R}f}$, which, by 3.1.6, is equivalent to the distributivity conditions.

A map of monoids in $(\mathbf{LAWFS}(\mathscr{C}), \circledast, \mathtt{I})$ is a map in $\mathbf{LAWFS}(\mathscr{C})$ that respects the multiplication maps of the monoids. In other words, it is a map of LAWFSs and a map of RAWFSs. So it must be a map of AWFSs and vice versa. □

Remark 3.2.4. Dually, we can define an operation $\odot$ that sends functorial factorizations (L,R) and (C,F), with middomain functors E and E' respectively, to the functorial factorization $(\mathrm{L},\mathrm{R})\odot(\mathrm{C},\mathrm{F})$ defined on each object $f : X \to Y$ in $\mathscr{C}^{\mathbf{2}}$ as the object

$$\begin{tikzcd} X \arrow[r, "\mathrm{LC}f"] & E\mathrm{C}f \arrow[r, "\mathrm{F}f\circ\mathrm{RC}f"] & Y \end{tikzcd}$$

in $\mathscr{C}^{\mathbf{3}}$ and defined on each morphism $(u,v) : f \to g$ in $\mathscr{C}^{\mathbf{2}}$ from $f : X \to Y$ to $g : Z \to W$ as the morphism

$$\begin{tikzcd} X \arrow[r, "\mathrm{LC}f"] \arrow[d, "u"] & E\mathrm{C}f \arrow[r, "\mathrm{F}f\circ\mathrm{RC}f"] \arrow[d, "{E(u,E'(u,v))}" description] & Y \arrow[d, "v"] \\ Z \arrow[r, "\mathrm{LC}g"'] & E\mathrm{C}g \arrow[r, "\mathrm{F}g\circ\mathrm{RC}g"'] & W \end{tikzcd}$$

in $\mathscr{C}^{\mathbf{3}}$.

The dual of 3.2.1 shows that $(\mathrm{L},\mathrm{R})\odot(\mathrm{C},\mathrm{F})$ of two RAWFSs (L,R) and (C,F) is a RAWFS. So the category $\mathbf{RAWFS}(\mathscr{C})$ of RAWFSs and maps of RAWFSs has a strict monoidal structure $(\mathbf{RAWFS}(\mathscr{C}),\odot,\perp)$, where $\perp$ is the functorial factorization that sends an object $f : X \to Y$ in $\mathscr{C}^{\mathbf{2}}$ to the object

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "id"] & Y \end{tikzcd}$$

in $\mathscr{C}^{\mathbf{3}}$.

We do not have that the $\otimes$-product of two RAWFSs is a RAWFS or that the $\odot$-product of two LAWFSs is a LAWFS.

Remark 3.2.5. In [Gar07, §3.2], Garner shows the operations $\otimes$ and $\odot$ above define a *two-fold monoidal structure* in the sense of [BFSV03, 1.3] on the category $\mathbf{FF}(\mathscr{C})$ whose objects are functorial factorizations on $\mathscr{C}$ and whose morphisms are maps of functorial factorizations. This means in particular that $(\mathbf{FF}(\mathscr{C}),\otimes,\mathtt{I})$ is a strict monoidal category and that there is a natural transformation

$$z_{A,B,C,D} : (A \odot B) \circledast (C \odot D) \to (A \circledast C) \odot (B \circledast D)$$

satisfying some coherence conditions.

Then the LAWFSs in $\mathbf{FF}(\mathscr{C})$ are the comonoids with respect to $(\odot, \bot)$ and the RAWFSs are the monoids with respect to $(\circledast, \mathtt{I})$. Although showing the axioms of a two-fold monoidal category are satisfied takes more work, the proof of 3.2.1 becomes easier in this context. It is just a matter of proving that, for comonoids $(X, \varepsilon^X, \delta^X)$ and $(Y, \varepsilon^Y, \delta^Y)$, the map

$$\varepsilon^X \circledast \varepsilon^Y : X \circledast Y \to \bot \circledast \bot = \bot$$

and the composition

$$X \circledast Y \xrightarrow{\delta^X \circledast \delta^Y} (X \odot X) \circledast (Y \odot Y) \xrightarrow{z_{X,X,Y,Y}} (X \circledast Y) \odot (X \circledast Y)$$

make $X \circledast Y$ a comonoid with respect to $(\circledast, \mathtt{I})$.

The main part of the proof of 3.2.3 was unpacking the requirement that $\mu : X \circledast X \to X$ is a map in $\mathbf{LAWFS}(\mathscr{C})$. This is easily stated in this context as the requirement that the following diagram commutes.

$$\begin{tikzcd}
X \circledast X \arrow[r, "\delta \odot \delta"] \arrow[d, "\mu"] & (X \odot X) \circledast (X \odot X) \arrow[r, "z_{X,X,X,X}"] & (X \circledast X) \odot (X \circledast X) \arrow[d, "\mu \odot \mu"] \\
X \arrow[rr, "\delta"] & & X \odot X
\end{tikzcd}$$

This doesn't give us a shortcut for proving the equivalence though.

Let $\mathbf{CDP}(\mathscr{C}^\mathbf{2})$ be the full subcategory of $\mathbf{End}(\mathscr{C}^\mathbf{2})$ on the codomain-preserving endofunctors.

Proposition 3.2.6.

1. *The subcategory inclusion functor* $\mathbb{K}_1 : \mathbf{CDP}(\mathscr{C}^\mathbf{2}) \hookrightarrow \mathbf{End}(\mathscr{C}^\mathbf{2})$ *creates colimits of the possibly large connected diagrams in* $\mathbf{CDP}(\mathscr{C}^\mathbf{2})$ *whose colimits exist in* $\mathbf{End}(\mathscr{C}^\mathbf{2})$*.*

2. *The functor* $\mathbb{K}_2 : \mathbf{FF}(\mathscr{C}) \to \mathbf{CDP}(\mathscr{C}^\mathbf{2})$ *that sends each functorial factorization* (L, R) *to* R *and sends each map of functorial factorizations* $\zeta : (\mathrm{L}, \mathrm{R}) \to (\mathrm{C}, \mathrm{F})$ *to* $(\zeta, id) : \mathrm{R} \to \mathrm{F}$ *creates colimits of the possibly large connected diagrams in* $\mathbf{FF}(\mathscr{C})$ *whose colimits exist in* $\mathbf{CDP}(\mathscr{C}^\mathbf{2})$*.*

3. *The forgetful functor* $\mathbb{K}_3 : \mathbf{LAWFS}(\mathscr{C}) \to \mathbf{FF}(\mathscr{C})$ *creates colimits of the possibly large connected diagrams in* $\mathbf{LAWFS}(\mathscr{C})$ *whose colimits exist in* $\mathbf{FF}(\mathscr{C})$*.*

Proof. (1) If $D : \mathscr{D} \to \mathbf{End}(\mathscr{C}^\mathbf{2})$ is a possibly large, connected diagram of codomain-preserving endofunctors whose colimit exists, then the colimit must be a codomain-preserving endofunctor. The colimit is a colimit in $\mathbf{CDP}(\mathscr{C}^\mathbf{2})$ since $\mathbf{CDP}(\mathscr{C}^\mathbf{2})$ is a full subcategory of $\mathbf{End}(\mathscr{C}^\mathbf{2})$.

(2) If $((\mathrm{L}_\alpha, \mathrm{R}_\alpha))_\alpha$ is a possibly large connected diagram in $\mathbf{FF}(\mathscr{C})$ such that the colimit R of $(\mathrm{R}_\alpha)_\alpha$ exists in $\mathbf{CDP}(\mathscr{C}^\mathbf{2})$, then the inclusion map $\mathrm{R}_\alpha \to \mathrm{R}$ of the colimiting cocone along with L_α for any α defines a left factor L for R so that (L, R) is a functorial factorization. Different choices of α will yield the same L. It easily follows that (L, R) is the colimit of $((\mathrm{L}_\alpha, \mathrm{R}_\alpha))_\alpha$ in $\mathbf{FF}(\mathscr{C})$.

(3) This is an immediate consequence of 2.1.21. □

So the functor $\mathbb{K} = \mathbb{K}_1\mathbb{K}_2\mathbb{K}_3 : \mathbf{LAWFS}(\mathscr{C}) \to \mathbf{End}(\mathscr{C}^\mathbf{2})$ creates colimits of the possibly large connected diagrams in $\mathbf{LAWFS}(\mathscr{C})$ whose colimits exist in $\mathbf{End}(\mathscr{C}^\mathbf{2})$.

Proposition 3.2.7 ([Gar08, 4.18])**.** *When* $\mathscr{C}$ *is a cocomplete category, the categories* $\mathbf{FF}(\mathscr{C})$ *and* $\mathbf{LAWFS}(\mathscr{C})$ *are cocomplete.*

Proof. As we saw in 3.2.6 (2), the colimit of a connected diagram in $\mathbf{FF}(\mathscr{C})$ is the objectwise colimit, which must exist, since $\mathscr{C}$ is cocomplete. We can get the colimit of any diagram $D : \mathscr{D} \to \mathbf{FF}(\mathscr{C})$ of functorial factorizations as follows. Create a new diagram $D' : \mathscr{D}' \to \mathbf{FF}(\mathscr{C})$ by adding the object $\mathtt{I}$ to $\mathscr{D}$ and adding the unit map $\eta : \mathtt{I} \to Dd$ for each functorial factorization Dd in the diagram. The diagram D' is connected, so the objectwise colimit is a functorial factorization. It is easy to check that this object is the colimit of the original diagram in $\mathbf{FF}(\mathscr{C})$.

The above paragraph along with 3.2.6 (3) implies that $\mathbf{LAWFS}(\mathscr{C})$ is cocomplete. □

Proposition 3.2.8 ([Gar08, 4.18])**.** *For each LAWFS Y, the endofunctor $(-)\circledast Y : \mathbf{LAWFS}(\mathscr{C}) \to \mathbf{LAWFS}(\mathscr{C})$ preserves colimits of the possibly large connected diagrams $D : \mathscr{D} \to \mathbf{LAWFS}(\mathscr{C})$ such that the colimit of the diagram $\mathbb{K}D$ in $\mathbf{End}(\mathscr{C}^{\mathbf{2}})$ exists objectwise.*

Proof. Let $D : \mathscr{D} \to \mathbf{LAWFS}(\mathscr{C})$ be a possibly large diagram such that the colimit of $\mathbb{K}D$ in $\mathbf{End}(\mathscr{C}^{\mathbf{2}})$ exists. By 3.2.6, the colimit of D in $\mathbf{LAWFS}(\mathscr{C})$ exists. Let $\theta : D \overset{\bullet}{\longrightarrow} \operatorname{colim} D$ be the colimiting cocone. We will use the notation $Dd = (\mathrm{L}^{Dd}, \mathrm{R}^{Dd})$ and $\operatorname{colim} D = (\mathrm{L}, \mathrm{R})$. Let $Y = (\mathrm{L}^{Y}, \mathrm{R}^{Y})$ be a LAWFS. The cocone $\{\theta_d \circledast Y : Dd \circledast Y \to (\operatorname{colim} D) \circledast Y\}_d$ is sent by $\mathbb{K}$ to the cocone $\{(\theta_d \mathrm{R}^{Y}, id) : \mathrm{R}^{Dd}\mathrm{R}^{Y} \to (\operatorname{colim}_d \mathrm{R}^{Dd})\mathrm{R}^{Y}\}_d$ in $\mathbf{End}(\mathscr{C})$. Since the colimit of $\mathbb{K}D$ is computed objectwise, $(\operatorname{colim}_d \mathrm{R}^{Dd})\mathrm{R}^{Y} = \operatorname{colim}_d(\mathrm{R}^{Dd}\mathrm{R}^{Y})$. Therefore the cocone $\{(\theta_d \mathrm{R}^{Y}, id) : \mathrm{R}^{Dd}\mathrm{R}^{Y} \to (\operatorname{colim}_d \mathrm{R}^{Dd})\mathrm{R}^{Y}\}_d$ is colimiting. Because by 3.2.6, the functor $\mathbb{K} : \mathbf{LAWFS}(\mathscr{C}) \to \mathbf{End}(\mathscr{C})$ reflects the colimit of D, the cocone $\{\theta_d \circledast Y : Dd \circledast Y \to (\operatorname{colim} D) \circledast Y\}_d$ must be a colimiting cocone in $\mathbf{LAWFS}(\mathscr{C})$. □

3.2.2 Compact Objects

This section will make frequent use of the definitions and notation in 2.1.3 and 2.4.1 so the reader may wish to review these sections.

To show that an object (L, R) in $\mathbf{LAWFS}(\mathscr{C})$ has a reflection in $\mathbf{AWFS}(\mathscr{C})$, we need to show that (L, R) satisfies a smallness condition. The smallness condition on (L, R) is just a smallness condition on R. The smallness condition on R, which we call *strong compactness*, is more general than the smallness condition introduced in 2.4.2. To describe this condition, it will be useful to have some notation.

Definition 3.2.9. Let $\mathcal{X}$ be a collection of maps in a category $\mathscr{C}$.

- Let $\mathcal{X}^{\mathbf{2}}$ be the collection of maps (u, v) in $\mathscr{C}^{\mathbf{2}}$ such that $u \in \mathcal{X}$ and $v \in \mathcal{X}$.
- Let $\mathcal{X}_{/\cong}$ be the collection of maps (u, v) in $\mathscr{C}^{\mathbf{2}}$ such that $u \in \mathcal{X}$ and v is an isomorphism.
- Let $\mathcal{V}_{/\cong}$ be the collection of maps (u, v) in $\mathscr{C}^{\mathbf{2}}$ such that v is an isomorphism

Definition 3.2.10 (Compact Functors)**.** Let λ be a regular cardinal and let $\mathscr{C}$ be a category equipped with left proper, orthogonal factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$.

- An endofunctor $F : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is $(\mathcal{E}, \mathcal{M}', \lambda)$*-compact* if F sends $\mathcal{E}_{/\cong}$-tight $(\mathcal{M}'_{/\cong}, \lambda)$-filtered cocones to $\mathcal{E}^{\mathbf{2}}$-tight cocones.
- An endofunctor $F : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is *strongly* $(\mathcal{E}, \mathcal{M}', \lambda)$*-compact* if F sends $\mathcal{E}_{/\cong}$-tight $(\mathcal{M}'_{/\cong}, \lambda)$-filtered cocones to $\mathcal{E}_{/\cong}$-tight cocones.

An endofunctor $F : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is *(strongly)* $(\mathcal{E}, \mathcal{M}')$*-compact* if it is (strongly) $(\mathcal{E}, \mathcal{M}', \lambda)$-compact for some regular cardinal λ.

When $(\mathcal{E}', \mathcal{M}')$ is the (isomorphism, any map) left proper, orthogonal factorization system on $\mathscr{C}$, then we will say an endofunctor F on $\mathscr{C}^{\mathbf{2}}$ is *(strongly)* $(\mathcal{E}, \lambda)$*-compact* if it is (strongly) $(\mathcal{E}, \mathcal{M}', \lambda)$-compact. It is $\mathcal{E}$*-compact* if it is $(\mathcal{E}, \lambda)$-compact for some regular cardinal λ.

Remark 3.2.11*.* If a functor $F : \mathscr{C} \to \mathscr{C}$ preserves $\mathscr{E}$-tightness of λ-filtered cocones for a regular cardinal λ, then the functor $F^{\mathbf{2}} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is strongly $(\mathcal{E}, \lambda)$-compact.

Definition 3.2.12. Let $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ be left proper, orthogonal factorization systems on a category $\mathscr{C}$.

- A functorial factorization (L, R) on $\mathscr{C}$ is $(\mathcal{E}, \mathcal{M}')$-*compact* if $\mathrm{L} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is an $(\mathcal{E}, \mathcal{M}')$-compact functor.
- A map $q : (\mathrm{L}, \mathrm{R}) \to (\mathrm{L}', \mathrm{R}')$ of functorial factorizations is $(\mathcal{E}, \mathcal{M}')$-*compact* if both (L, R) and $(\mathrm{L}', \mathrm{R}')$ are $(\mathcal{E}, \mathcal{M}')$-compact.

As for endofunctors, we say a functorial factorization or map of functorial factorizations is $\mathcal{E}$-*compact* when $(\mathcal{E}', \mathcal{M}')$ is the (isomorphism, any map) orthogonal factorization system.

The above definitions apply to AWFSs, LAWFSs, and algebraic model categories, so it makes sense to talk about these things being $(\mathcal{E}, \mathcal{M}')$-compact.

Remark 3.2.13. The functor R in a functorial factorization (L, R) is strongly $(\mathcal{E}, \mathcal{M}')$-compact if and only if L is $(\mathcal{E}, \mathcal{M}')$-compact. So we could just as well define an $(\mathcal{E}, \mathcal{M}')$-compact functorial factorization (L, R) to be one for which R is strongly $(\mathcal{E}, \mathcal{M}')$-compact.

3.2.3 Free AWFSs on LAWFSs

We are now ready to prove that free AWFSs exist on certain compact LAWFSs. We will also show that free AWFSs on compact LAWFSs are *algebraically free*.

We will now start adding the assumption that $\mathscr{C}$ is a locally small category. Let $\mathscr{C}$ be a cocomplete, locally small category equipped with well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$.

Proposition 3.2.14. *If* $(\mathrm{T}, \vec{\tau})$ *is a strongly* $(\mathcal{E}, \mathcal{M}')$*-compact, codomain-preserving endofunctor on* $\mathscr{C}^{\mathbf{2}}$*, then the free monad sequence on* $(\mathrm{T}, \vec{\eta})$ *converges objectwise.*

Proof. We will show that the free T-algebra sequence on each object f in $\mathscr{C}^{\mathbf{2}}$ converges. Let $f : A \to B$ be an object in $\mathscr{C}^{\mathbf{2}}$. Let $(\mathrm{X}_\alpha)_\alpha$ be the free monad sequence on $(\mathrm{T}, \vec{\tau})$. So $(\mathrm{X}_\alpha f)_\alpha$ is the free T-algebra sequence on f.

By 3.2.6 (1), the colimit in $\mathbf{End}(\mathscr{C}^{\mathbf{2}})$ of a connected diagram of codomain-preserving endofunctors is a codomain-preserving endofunctor. Since compositions of codomain-preserving endofunctors are also codomain-preserving endofunctors, the free monad sequence on $(T, \vec{\tau})$ is a sequence of codomain-preserving endofunctors. Every codomain-preserving endofunctor F on $\mathscr{C}^{\mathbf{2}}$ restricts to an endofunctor $\widehat{\mathrm{F}}$ on the comma category $\mathscr{C} \downarrow B$. So $(\widehat{\mathrm{X}}_\alpha)_\alpha$ is a sequence of endofunctors on $\mathscr{C} \downarrow B$. When $(\mathrm{F}, \vec{\eta})$ is a pointed endofunctor with $\vec{\eta} = (\eta, id)$, $(\widehat{\mathrm{F}}, \eta)$ is a pointed endofunctor on $\mathscr{C} \downarrow B$. Then $(\widehat{\mathrm{X}}_\alpha)_\alpha$ is the free monad sequence for $(\widehat{\mathrm{T}}, \tau)$ on $\mathscr{C} \downarrow B$ and $(\widehat{\mathrm{X}}_\alpha f)_\alpha$ is the free T-algebra sequence on the object f in $\mathscr{C} \downarrow B$.

We note that the category $\mathscr{C} \downarrow B$ is cocomplete and that the well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ lift to well-copowered, left proper, orthogonal factorization systems $(\widehat{\mathcal{E}}, \widehat{\mathcal{M}})$ and $(\widehat{\mathcal{E}'}, \widehat{\mathcal{M}'})$, respectively, on $\mathscr{C} \downarrow B$. Then our assumption that T is strongly $(\mathcal{E}, \mathcal{M}')$-compact is exactly the requirement that $\widehat{\mathrm{T}}$ preserves $\widehat{\mathcal{E}}$-tightness of $(\widehat{\mathcal{M}'}, \lambda)$-cocones for some regular cardinal λ. Therefore, by 2.4.23, the free $\widehat{\mathrm{T}}$-algebra sequence $(\widehat{\mathrm{X}}_\alpha f)_\alpha$ on f converges. But this means the free T-algebra sequence $(\mathrm{X}_\alpha f)_\alpha$ on f converges. □

The basic argument for the above proof is outlined in [Gar07, p31].

Proposition 3.2.15. *If* (L, R) *is an* $(\mathcal{E}, \mathcal{M}')$*-compact LAWFS, then the free* $\circledast$*-monoid sequence on* (L, R) *is weakly convergent.*

Proof. Let $(\mathrm{X}_\alpha)_\alpha$ be the free monoid sequence on (T, η), where $\mathrm{T} = (\mathrm{L}, \mathrm{R})$ and $\mathrm{X}_\alpha = (\mathrm{L}^{\mathrm{X}_\alpha}, \mathrm{R}^{\mathrm{X}_\alpha})$ for each α. From the definition of the monoid product $\circledast$, it is clear that $(\mathrm{R}^{\mathrm{X}_\alpha})_\alpha$

is the free monad sequence on $(\mathrm{R}, \vec{\eta})$, where $\vec{\eta} = (\eta, id)$. Since R is strongly $(\mathcal{E}, \mathcal{M}')$-compact, by 3.2.14, the free monad sequence $(\mathrm{R}^{\mathrm{X}_\alpha})_\alpha$ converges objectwise and by 2.3.23, $(\mathrm{R}^{\mathrm{X}_\alpha})_\alpha$ is weakly convergent. So the objectwise colimits $\mathrm{F} = \mathrm{colim}_\alpha \mathrm{R}^{\mathrm{X}_\alpha}$ and $\mathrm{F}' = \mathrm{colim}_\alpha \mathrm{R}^{\mathrm{X}_\alpha}$ exist and there is a map $(m, id) : \mathrm{RF} \to \mathrm{F}$ such that conditions (1) and (2) of 2.3.19 hold. By 3.2.6, the colimits $(\mathrm{C}, \mathrm{F}) = \mathrm{colim}_\alpha X_\alpha$ and $(\mathrm{C}', \mathrm{F}') = \mathrm{colim}_\alpha \mathrm{T} \circledast \mathrm{X}_\alpha$ exist in $\mathbf{LAWFS}(\mathscr{C})$. If we can show that $m : \mathrm{T} \circledast (\mathrm{C}, \mathrm{F}) \to (\mathrm{C}, \mathrm{F})$ is a map in $\mathbf{LAWFS}(\mathscr{C})$, then it will follow from 3.2.6 that $(X_\alpha)_\alpha$ is a weakly convergent free monoid sequence in $(\mathbf{LAWFS}, \circledast, \mathtt{I})$.

To show that $m : \mathrm{T} \circledast (\mathrm{C}, \mathrm{F}) \to (\mathrm{C}, \mathrm{F})$ is a map in $\mathbf{LAWFS}(\mathscr{C})$, it suffices to show on each object f in $\mathscr{C}^{\mathbf{2}}$ that the following diagrams commute

$$\begin{tikzcd} & \mathtt{I} \arrow[dl, "(id{,}\mathrm{S}f)"'] \arrow[dr, "(id{,}\mathrm{C}f)"] & \\ \mathrm{S}f \arrow[rr, "m_f"] & & \mathrm{C}f \end{tikzcd} \qquad \begin{tikzcd} \mathrm{S}f \arrow[rr, "m_f"] \arrow[d, "\vec{\delta}^{\mathrm{S}}_f"'] & & \mathrm{C}f \arrow[d, "\vec{\delta}^{\mathrm{C}}_f"] \\ \mathrm{SS}f \arrow[r, "\mathrm{S}m_f"] & \mathrm{SC}f \arrow[r, "m_{\mathrm{C}f}"] & \mathrm{CC}f, \end{tikzcd}$$

where $\mathrm{S}f = \mathrm{LF}f \circ \mathrm{C}f$ and $\vec{\delta}^{\mathrm{S}}$ is the comultiplication map of S defined in 3.2.1. Because $(\mathrm{R}^{\mathrm{X}_\alpha})_\alpha$ converges objectwise, there is an ordinal β such that $\mathrm{X}_\beta f = (\mathrm{L}^{\mathrm{X}_\beta} f, \mathrm{R}^{\mathrm{X}_\beta} f) = (\mathrm{C}f, \mathrm{F}f)$ and, as we saw in 2.3.23, $(m_f, id) = (\pi_{\beta_f}, id) : \mathrm{RL}^{\mathrm{X}_\beta} f \to \mathrm{L}^{\mathrm{X}_\beta} f$. But $\pi_\beta : \mathrm{T} \circledast \mathrm{X}_\beta \to \mathrm{X}_\beta$ is a map in $\mathbf{LAWFS}(\mathscr{C})$, since it is a map in the free monoid sequence for (T, τ). Therefore $\pi_{\beta_f} = m_f$ satisfies the relations in the above diagrams. □

Theorem 3.2.16. *If* (L, R) *is an* $(\mathcal{E}, \mathcal{M}')$*-compact LAWFS, then the free AWFS on* (L, R) *exists and is equal to the colimit of the free monoid sequence for* (L, R).

Proof. We know from 3.2.15 that the free monoid sequence $(\mathrm{X}_\alpha)_\alpha$ for (L, R) is weakly convergent. We know from 3.2.8 and the fact that the free monad sequence on R converges objectwise that for every LAWFS Y, the endofunctor $(-) \circledast \mathrm{Y} : \mathbf{LAWFS}(\mathscr{C}) \to \mathbf{LAWFS}(\mathscr{C})$ preserves connected colimits and preserves the colimits of the large diagrams $(\mathrm{X}_\alpha)_\alpha$ and $(\mathrm{T} \circledast \mathrm{X}_\alpha)_\alpha$. Thus, by 2.5.4, (L, R) has a reflection along the forgetful functor

$\mathbf{Mon}_{\circledast}(\mathbf{LAWFS}(\mathscr{C})) \to \mathbf{LAWFS}(\mathscr{C})$. This reflection is the colimit of the free monoid sequence for (L, R). By 3.2.3, this is equivalent to (L, R) having a reflection along the forgetful functor $\mathbb{G}_1 : \mathbf{AWFS}(\mathscr{C}) \to \mathbf{LAWFS}(\mathscr{C})$. □

Remark 3.2.17. By remark 2.4.24, we could have gotten the same result in theorem 3.2.16 if we replaced $(\mathcal{E}, \mathcal{M}')$-compactness of (L, R) with the condition that $\mathrm{R} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ preserves $\mathcal{E}_{/\cong}$-tightness of λ-sequential $\mathcal{M}'_{/\cong}$-cocones for some regular cardinal λ.

We will say an AWFS (L, R) is an *algebraically free* AWFS on an LAWFS $(\mathrm{L}_1, \mathrm{R}_1)$ if there is a map $\zeta : (\mathrm{L}_1, \mathrm{R}_1) \to (\mathrm{L}, \mathrm{R})$ of LAWFSs such that $(\zeta, id)^{\dagger} : \mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}} \to \mathbf{Alg}_{\mathrm{R}_1}$ is an isomorphism of categories over $\mathscr{C}^{\mathbf{2}}$. Since the functor $(\zeta, id)^{\dagger} : \mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}} \to \mathbf{Alg}_{\mathrm{R}_1}$ factors through $\mathbf{Alg}_{\mathrm{R}}$ over $\mathscr{C}^{\mathbf{2}}$, a consequence of algebraic freeness is that $|\mathbf{Alg}_{\mathrm{R}}| = |\mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}}|$.

Unlike in section 2.2.1, we will not be able to show that every free AWFS on a locally small cocomplete category is algebraically free. We will, however, be able to show that whenever $(\mathrm{L}_1, \mathrm{R}_1)$ is an $(\mathcal{E}, \mathcal{M}')$-compact LAWFS on a locally small cocomplete category, the free AWFS on $(\mathrm{L}_1, \mathrm{R}_1)$ is algebraically free.

Proposition 3.2.18. *The free AWFS on an $(\mathcal{E}, \mathcal{M}')$-compact LAWFS on $\mathscr{C}$ is algebraically free.*

Proof. Let $(\mathrm{L}_1, \mathrm{R}_1)$ be an $(\mathcal{E}, \mathcal{M}')$-compact LAWFS on $\mathscr{C}$. Let (L, R) be the free AWFS on $(\mathrm{L}_1, \mathrm{R}_1)$ with universal map $\zeta : (\mathrm{L}_1, \mathrm{R}_1) \to (\mathrm{L}, \mathrm{R})$. As we saw in the proof of 3.2.15, applying the functor $\mathbb{K} : \mathbf{LAWFS}(\mathscr{C}) \to \mathbf{End}(\mathscr{C}^{\mathbf{2}})$ to the free monoid sequence on $(\mathrm{L}_1, \mathrm{R}_1)$ yields the free monad sequence on R_1. Since R_1 is strongly $(\mathcal{E}, \mathcal{M}')$-compact, by 3.2.14, the free monad sequence converges objectwise. So R is the free monad on R_1 with universal map $(\zeta, id) : \mathrm{R}_1 \to \mathrm{R}$. The result follows from 2.2.7. □

We have the following important consequence of the existence of free AWFSs and their algebraic-freeness. We will often use this in combination with 3.1.4.

Proposition 3.2.19. *Let* $\mathcal{L}$ *and* $\mathcal{R}$ *be classes of maps in* $\mathscr{C}$ *such that* $\mathcal{L} = {}^{\square}\mathcal{R}$*. If* $(\mathrm{L}_1, \mathrm{R}_1)$ *is an* $(\mathcal{E}, \mathcal{M}')$*-compact LAWFS on* $\mathscr{C}$ *such that* $|\mathbf{Alg}_{\mathrm{R}_1}| = \mathcal{R}$*, then* $|\mathbf{Coalg}_{\mathrm{L}}| = \mathcal{L}$ *and* $|\mathbf{Alg}_{\mathrm{R}}| = \mathcal{R}$ *for the free AWFS* (L, R) *on* $(\mathrm{L}_1, \mathrm{R}_1)$*.*

Proof. Existence of the free AWFS (L, R) on $(\mathrm{L}_1, \mathrm{R}_1)$ is 3.2.16. The equality $|\mathbf{Alg}_{\mathrm{R}}| = |\mathbf{Alg}_{\mathrm{R}_1}| = \mathcal{R}$ comes from 3.2.18. Then by 3.1.13 and 3.1.16, $|\mathbf{Coalg}_{\mathrm{L}}| = |{}^{\boxslash}\mathbf{Alg}_{\mathrm{R}}| = {}^{\square}|\mathbf{Alg}_{\mathrm{R}}| = {}^{\square}\mathcal{R} = \mathcal{L}$. ☐

3.2.4 Free LAWFSs on Comonads

Let $\mathscr{C}$ be a cocomplete category. We will show that every comonad on $\mathscr{C}^{\mathbf{2}}$ has a reflection along the forgetful functor $\mathbb{G}_2 : \mathbf{LAWFS}(\mathscr{C}) \to \mathbf{Cmd}(\mathscr{C}^{\mathbf{2}})$.

Lemma 3.2.20. *Let* $(\mathrm{L}_0, \vec{\varepsilon}^{\mathrm{L}_0}, \vec{\delta}^{\mathrm{L}_0})$ *be a comonad on* $\mathscr{C}^{\mathbf{2}}$ *and let* $(\varepsilon^t, \varepsilon^b) = \vec{\varepsilon}^{\mathrm{L}_0}$*. The endofunctor* $\mathrm{L}_1 : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ *defined on each object* f *as the pushout of* $\mathrm{L}_0 f$ *along* ε^t_f *has the structure of a comonad.*

Proof. Let $(\delta^t, \delta^b) = \vec{\delta}^{\mathrm{L}_0}$. On each object f in $\mathscr{C}^{\mathbf{2}}$, let $\mathrm{R}_1 f$ be the map, shown in the below left diagram, defined by the cocone consisting of the maps ε^b_f and f. Since pushouts are functorial, $(\mathrm{L}_1, \mathrm{R}_1)$ is a functorial factorization. Let $E_1 : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ be the middomain functor for $(\mathrm{L}_1, \mathrm{R}_1)$. Let δ_f be the map, shown in the below right diagram, defined by the cocone of maps $E_1(\varepsilon^t_f, \alpha_f) \circ \alpha_{\mathrm{L}_0 f} \circ \delta^b_f$ and $\mathrm{L}_1\mathrm{L}_1 f$.

$$\begin{tikzcd}
{} \arrow[r, "\varepsilon^t_f"] \arrow[d, "\mathrm{L}_0 f"] \arrow[dr, phantom, "\ulcorner", very near end] & {} \arrow[r, "id"] \arrow[d, "\mathrm{L}_1 f"] & {} \arrow[d, "f"] \\
{} \arrow[r, "\alpha_f"] \arrow[rr, bend right, "\varepsilon^b_f"'] & {} \arrow[r, dashed, "\mathrm{R}_1 f"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[r, "\varepsilon^t_f"] \arrow[d, "\mathrm{L}_0 f"] \arrow[dr, phantom, "\ulcorner", very near end] & {} \arrow[r, "id"] \arrow[d, "\mathrm{L}_1 f"] & {} \arrow[d, "\mathrm{L}_1\mathrm{L}_1 f"] \\
{} \arrow[r, "\alpha_f"] \arrow[rr, bend right, "E_1(\varepsilon^t_f, \alpha_f) \circ \alpha_{\mathrm{L}_0 f} \circ \delta^b_f"'] & {} \arrow[r, dashed, "\delta_f"] & {}
\end{tikzcd}$$

Using the universal property of pushouts and the comonad structure on $(\mathrm{L}_0, \vec{\varepsilon}^{\mathrm{L}_0}, \vec{\delta}^{\mathrm{L}_0})$, the reader can check that the endofunctor $\mathrm{L}_1 : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is a comonad with counit map $\vec{\varepsilon}_f = (id, \mathrm{R}_1 f)$ and comultiplication map $\vec{\delta}_f = (id, \delta_f)$ on each object f in $\mathscr{C}^{\mathbf{2}}$. □

An immediate consequence of the above proof is that the maps $(\varepsilon^t_f, \alpha_f) : \mathrm{L}_0 f \to \mathrm{L}_1 f$ define a map of comonads $\mathrm{L}_0 \to \mathrm{L}_1$.

We define $\mathbb{F}_2(\mathrm{L}_0)$ to be the LAWFS $(\mathrm{L}_1, \mathrm{R}_1)$ with comonad structure $(\mathrm{L}_1, \vec{\varepsilon}, \vec{\delta})$. It is easy to check the following result.

Proposition 3.2.21 ([Gar08, 4.7])**.** *The LAWFS* $\mathbb{F}_2(\mathrm{L}_0)$ *is the reflection of* L_0 *along* $\mathbb{G}_2$. *The universal map of this reflection is given by* $(\varepsilon^t_f, \alpha_f) : \mathrm{L}_0 f \to \mathrm{L}_1 f$ *on each object* f *in* $\mathscr{C}^{\mathbf{2}}$.

So every comonad on $\mathscr{C}^{\mathbf{2}}$ has a reflection in $\mathbf{LAWFS}(\mathscr{C})$ and thus $\mathbb{F}_2 : \mathbf{Cmd}(\mathscr{C}^{\mathbf{2}}) \to \mathbf{LAWFS}(\mathscr{C})$ is the left adjoint to $\mathbb{G}_2$.

When we want to combine 3.2.16 and 3.2.21 to construct a free AWFS on a comonad, we need to know that the free LAWFS of 3.2.21 is a compact LAWFS. Suppose $\mathscr{C}$ is equipped with well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$.

Proposition 3.2.22. *If* L_0 *is an* $(\mathcal{E}, \mathcal{M}')$*-compact comonad on* $\mathscr{C}^{\mathbf{2}}$*, then* $\mathbb{F}_2(\mathrm{L}_0)$ *is an* $(\mathcal{E}, \mathcal{M}')$*-compact LAWFS on* $\mathscr{C}$*.*

Proof. Let $P : \mathscr{C}^{\mathbf{out}} \to \mathscr{C}$ be the functor that sends a span in $\mathscr{C}$ to its colimit. Because colimits commute, P preserves colimits. By 2.4.3, using the notation of 2.4.2, P sends $\mathcal{E}^{\mathbf{out}}$-maps to $\mathcal{E}$-maps. Thus P sends $\mathcal{E}^{\mathbf{out}}$-tight cocones to $\mathcal{E}$-tight cocones. If $\{(u_\alpha, v_\alpha) : f_\alpha \to f\}_\alpha$ is an $\mathcal{E}_{/\cong}$-tight $(\mathcal{M}'_{/\cong}, \lambda)$-cocone for some regular cardinal λ, then $\{\mathrm{L}_0(u_\alpha, v_\alpha) : \mathrm{L}_0 f_\alpha \to \mathrm{L}_0 f\}_\alpha$ is an $\mathcal{E}^{\mathbf{2}}$-tight cocone. It follows that $\{\mathrm{L}_1(u_\alpha, v_\alpha) : \mathrm{L}_1 f_\alpha \to \mathrm{L}_1 f\}_\alpha$ is an $\mathcal{E}^{\mathbf{2}}$-tight cocone. So L_1 is $(\mathcal{E}, \mathcal{M}')$-compact and thus $(\mathrm{L}_1, \mathrm{R}_1)$ is $(\mathcal{E}, \mathcal{M}')$-compact. □

Proposition 3.2.23. *If* $(\mathrm{L}_1, \mathrm{R}_1)$ *is the free LAWFS on the comonad* L_0*, then there is an isomorphism of categories* $\mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}\boxslash} \cong \mathbf{Coalg}_{\mathrm{L}_1}^{\mathrm{EM}\boxslash}$.

Proof. Let $V_{\mathrm{L}_0} : \mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}} \to \mathscr{C}^{\mathbf{2}}$ be the forgetful functor and let $\psi : V_{\mathrm{L}_0} \to \mathrm{L}_0 V_{\mathrm{L}_0}$ be the natural transformation defined by $\psi_{\langle f, \vec{k} \rangle} = \vec{k}$ on each object $\langle f, \vec{k} \rangle$ in $\mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}}$. Let p be an object in $|\mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}\boxslash}|$ and let $\theta : V_{\mathrm{L}_0} \overset{\centerdot}{\longrightarrow} p$ be the canonical cocone of p relative to V_{L_0}.

The fact that $\langle \mathrm{L}_0 p, \vec{\delta}_p \rangle$ is an object in $\mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}}$ and the commutativity of the following diagram shows that there is a bijective correspondence between solutions to the lifting problem $\vec{\varepsilon}_p : \mathrm{L}_0 p \to p$ and natural solutions to the lifting problem $\theta : V_{\mathrm{L}_0} \overset{\centerdot}{\longrightarrow} p$. Furthermore, this correspondence respects the naturality of maps in $\mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}\boxslash}$.

$$\begin{tikzcd}
V_{\mathrm{L}_0} \arrow[r, "\psi"] & \mathrm{L}_0 V_{\mathrm{L}_0} \arrow[r, "\mathrm{L}_0\theta"] \arrow[d, "\vec{\varepsilon} V_{\mathrm{L}_0}"] & \mathrm{L}_0 p \arrow[d, "\vec{\varepsilon}_p"] \\
 & V_{\mathrm{L}_0} \arrow[r, "\theta"] & p
\end{tikzcd}$$

The same is true of $\mathbf{Coalg}_{\mathrm{L}_1}^{\mathrm{EM}}$. Since there is a bijective correspondence between the lifts in the following diagrams, this means there is a bijective correspondence between the objects of the categories $\mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}\boxslash}$ and $\mathbf{Coalg}_{\mathrm{L}_1}^{\mathrm{EM}\boxslash}$.

$$\begin{tikzcd}
{} \arrow[r, "\varepsilon_p^t"] \arrow[d, "\mathrm{L}_0 p"'] & {} \arrow[d, "p"] & & {} \arrow[r, "id"] \arrow[d, "\mathrm{L}_1 p"'] & {} \arrow[d, "p"] \\
{} \arrow[r, "\varepsilon_p^b"] \arrow[ur, dashed, "s_p"] & {} & & {} \arrow[r, "\mathrm{R}_1 p"] \arrow[ur, dashed, "t_p"] & {}
\end{tikzcd}$$

This bijection extends to morphisms as well and defines inverse functors $\mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}\boxslash} \to \mathbf{Coalg}_{\mathrm{L}_1}^{\mathrm{EM}\boxslash}$ and $\mathbf{Coalg}_{\mathrm{L}_1}^{\mathrm{EM}\boxslash} \to \mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}\boxslash}$ over $\mathscr{C}^{\mathbf{2}}$. In fact, the latter functor is defined by the map $\mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}} \to \mathbf{Coalg}_{\mathrm{L}_1}^{\mathrm{EM}}$ of categories over $\mathscr{C}^{\mathbf{2}}$ that is defined by the map of comonads $\mathrm{L}_0 \to \mathrm{L}_1$. ☐

3.2.5 The Algebraic Small Object Argument

The small object argument begins with a set $\mathcal{I}$ of morphisms in a category $\mathscr{C}$ and constructs a weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathscr{C}$ such that $\mathcal{I}^{\square} = \mathcal{R}$. We will use theorem 3.2.16 and proposition 3.2.21 to define a similar construction on diagrams in the arrow category $\mathscr{C}^{\mathbf{2}}$. When $I : \mathscr{I} \to \mathscr{C}^{\mathbf{2}}$ is a functor on a small category and $\mathscr{C}$ satisfies a certain smallness property, we will show how to construct a free AWFS $(\mathrm{L}^I, \mathrm{R}^I)$ on I such that $I^{\boxslash} \cong \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}^I}$ and $|I^{\boxslash}| = |\mathbf{Alg}_{\mathrm{R}^I}|$.

Let $\mathscr{C}$ be a cocomplete, locally small category equipped with well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$. The reflection $(\mathrm{L}^I, \mathrm{R}^I)$ of a functor $I : \mathscr{I} \to \mathscr{C}^{\mathbf{2}}$ along $\mathbb{G}_3\mathbb{G}_2\mathbb{G}_1$ is the free AWFS on I when it exists.

$$\mathbf{AWFS}(\mathscr{C}) \xrightarrow{\mathbb{G}_1} \mathbf{LAWFS}(\mathscr{C}) \xrightarrow{\mathbb{G}_2} \mathbf{Cmd}(\mathscr{C}^{\mathbf{2}}) \xrightarrow{\mathbb{G}_3} \mathbf{CAT}/\mathscr{C}^{\mathbf{2}}$$

We handled reflections along $\mathbb{G}_1$ in section 3.2.3. The reflection of a LAWFS X along $\mathbb{G}_1$ is given by the colimit of the free monoid sequence for X when the free monoid sequence is weakly convergent. We will use the notation $\mathbb{F}_1(\mathrm{X})$ for this reflection when it exists. By theorem 3.2.16, the reflection $\mathbb{F}_1(\mathrm{X})$ exists when X is an $(\mathcal{E}, \mathcal{M}')$-compact LAWFS.

We defined reflections along $\mathbb{G}_2$ in section 3.2.4. We saw in 3.2.20 and 3.2.21 that the reflection of a comonad L_0 along $\mathbb{G}_2$ is determined objectwise by a cocartesian square. We use the notation $(\mathrm{L}_1, \mathrm{R}_1) = \mathbb{F}_2(\mathrm{L}_0)$ for the reflection of L_0 and since $\mathscr{C}$ is cocomplete, this reflection always exists.

The reflection of $I : \mathscr{I} \to \mathscr{C}^{\mathbf{2}}$ along $\mathbb{G}_3$ is given by the *density comonad*. The *density comonad* on I is the left Kan extension of I along itself. We will use the notation $\mathbb{F}_3(I) = \mathrm{L}_0^I$ for the codensity comonad on I when it exists. Explicitly, on each object f in $\mathscr{C}^{\mathbf{2}}$,

$$\mathrm{L}_0^I f = \operatorname*{colim}_{I(i) \to f} I(i),$$

where the colimit is indexed by the comma category $I \downarrow f$. We present the details of this construction and prove that it is actually a comonad in A.4. Because $\mathscr{C}^{\mathbf{2}}$ is a locally small, cocomplete category, we have the following existence result.

Proposition 3.2.24. *When $\mathscr{I}$ is a small category, the density comonad on I exists.*

Proposition 3.2.25 ([Gar08, 4.6])**.** *The density comonad L_0^I is the reflection of I along $\mathbb{G}_3$.*

Proof. The universal natural transformation of the left Kan extension $\alpha : I \to \mathrm{L}_0^I I$ defines a functor $\widetilde{I} : \mathscr{I} \to \mathbf{Coalg}_{\mathrm{L}_0^I}^{\mathrm{EM}}$ such that $V_{\mathrm{L}_0^I}\widetilde{I} = I$. We will show that $\widetilde{I}$ is the universal map of the reflection. Let $(\mathrm{C}, \varepsilon, \delta)$ be a comonad on $\mathscr{C}^{\mathbf{2}}$ and let $V_{\mathrm{C}} : \mathbf{Coalg}_{\mathrm{C}}^{\mathrm{EM}} \to \mathscr{C}^{\mathbf{2}}$ be the forgetful functor. Suppose $K : \mathscr{I} \to \mathbf{Coalg}_{\mathrm{C}}^{\mathrm{EM}}$ is a functor such that $V_{\mathrm{C}}K = I$. The unit map ν of the $\widetilde{\mathrm{C}} \dashv V_{\mathrm{C}}$ adjunction gives a natural transformation $V_{\mathrm{C}}\nu : V_{\mathrm{C}} \to V_{\mathrm{C}} \circ \widetilde{C}V_{\mathrm{C}}$. So $V_{\mathrm{C}}\nu K$ is a natural transformation $I \to \mathrm{C}V_{\mathrm{C}}K = \mathrm{C}I$ and one can check that $\varepsilon I \circ V_{\mathrm{C}}\nu = id$ and $\delta I \circ V_{\mathrm{C}}\nu = \mathrm{C}V_{\mathrm{C}}\nu \circ V_{\mathrm{C}}\nu$. By A.4.4, there is a unique map $\gamma : \mathrm{L}_0^I \to \mathrm{C}$ of comonads such that $\gamma I \circ \alpha = V_{\mathrm{C}}\nu K$. The reader can check that this means $\gamma_* \widetilde{I} = K$. □

Proposition 3.2.26. *Let $I : \mathscr{I} \to \mathscr{C}^{\mathbf{2}}$ be a small category over $\mathscr{C}^{\mathbf{2}}$ with a reflection $\mathrm{L}_0^I = \mathbb{F}_3(I)$ in $\mathbf{Cmd}(\mathscr{C}^{\mathbf{2}})$. There is an isomorphism $I^{\boxslash} \cong \mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}\boxslash}$ of categories over $\mathscr{C}^{\mathbf{2}}$.*

Proof. Let $\theta : I \overset{\bullet}{\longrightarrow} f$ be the canonical cocone of f relative to I. Since $\mathrm{L}_0^I f$ is by definition the colimit of the canonical diagram $I \downarrow f \to \mathscr{C}^{\mathbf{2}}$, a solution to the natural lifting problem θ is exactly the data of a solution to the lifting problem $\vec{\varepsilon}_f : \mathrm{L}_0^I f \to f$. It is easy to check that this correspondence extends to a bijection of morphisms and is functorial. In other words, it defines inverse functors $I^{\boxslash} \to \mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}\boxslash}$ and $\mathbf{Coalg}_{\mathrm{L}_0}^{\mathrm{EM}\boxslash} \to I^{\boxslash}$. □

To construct a free AWFS on a functor $I : \mathscr{I} \to \mathscr{C}^{\mathbf{2}}$, we need all of the reflections to exist in succession. So we need $\mathscr{I}$ to be a small category and we need an additional condition that will guarantee that the LAWFS $\mathbb{F}_2\mathbb{F}_3(I)$ is $(\mathcal{E},\mathcal{M}')$-compact. A sufficient condition is that the category $\mathscr{C}$ *permits the algebraic small object argument.* We say this is the case when for each object X in $\mathscr{C}$, there is a regular cardinal λ such that the functor

$$\mathscr{C}(X,-):\mathscr{C}\to\mathbf{Set}$$

sends $\mathcal{E}$-tight $(\mathcal{M}',\lambda)$-cocones to $\mathcal{E}^{\downarrow}$-tight cocones, where $\mathcal{E}^{\downarrow}$ is the collection of epimorphisms in the category $\mathbf{Set}$.

Proposition 3.2.27. *If $(\mathcal{E},\mathcal{M})$ is either proper or the (isomorphism, any map) orthogonal factorization system and if $\mathscr{C}$ permits the algebraic small object argument, then the reflection $\mathbb{F}_3(I)$ of a diagram $I:\mathscr{I}\to\mathscr{C}^{\mathbf{2}}$ is an $(\mathcal{E},\mathcal{M}')$-compact comonad.*

Proof. Before proving the result, we make the following observation. The natural isomorphism $\mathscr{C}(\mathcal{A}\cdot C_1,C_2)\cong\mathbf{Set}(\mathcal{A},\mathscr{C}(C_1,C_2))$ implies that for each object C in $\mathscr{C}$, the copower functor $(-)\cdot C:\mathbf{Set}\to\mathscr{C}$ preserves colimits. So if $\{x_\alpha:\mathcal{A}_\alpha\to\mathcal{A}\}_\alpha$ is a colimiting cocone of sets, then $\{x_\alpha\cdot C:\mathcal{A}_\alpha\cdot C\to\mathcal{A}\cdot C\}_\alpha$ is a colimiting cocone in $\mathscr{C}$. Now, suppose $(\mathcal{E},\mathcal{M})$ is a proper orthogonal factorization system on $\mathscr{C}$. If C is an object in $\mathscr{C}$ and $p:\mathcal{A}\to\mathcal{B}$ is a surjection of sets, then the map $p\cdot C:\mathcal{A}\cdot C\to\mathcal{B}\cdot C$ is a split epimorphism. Indeed, there is a map $s:\mathcal{B}\to\mathcal{A}$ such that $p\circ s=id$. Thus $(p\cdot C)\circ(s\cdot C)=id$. Since every left class $\mathcal{E}$ of a proper orthogonal factorization system $(\mathcal{E},\mathcal{M})$ contains all split epimorphisms, $p\cdot C\in\mathcal{E}$. It follows that if $\{x_\alpha:\mathcal{A}_\alpha\to\mathcal{A}\}_\alpha$ is an $\mathcal{E}^{\downarrow}$-tight cocone of sets, then $\{x_\alpha\cdot C:\mathcal{A}_\alpha\cdot C\to\mathcal{A}\cdot C\}_\alpha$ is an $\mathcal{E}$-tight cocone in $\mathscr{C}$.

The comonad $\mathrm{L}_0^I=\mathbb{F}_3(I)$ is described on each object f by the coend formula $\mathrm{L}_0^I f=\int^{i\in\mathscr{I}}\mathscr{C}(I(i),f)\cdot I(i)$. Because $\mathscr{I}$ is a small category, there is a regular cardinal λ such that for every object X in the image of $\operatorname{dom} I:\mathscr{I}\to\mathscr{C}$, the functor $\mathscr{C}(X,-)$ sends $\mathcal{E}$-tight $(\mathcal{M}',\lambda)$-cocones to $\mathcal{E}^{\downarrow}$-tight cocones. Let $\{\vec{u}_\alpha:f_\alpha\to f\}_\alpha$ be an $\mathcal{E}_{/\cong}$-tight $(\mathcal{M}'_{/\cong},\lambda)$-cocone in

$\mathscr{C}^{\mathbf{2}}$. Then the cocone $\{(\vec{u}_\alpha)_* : \mathscr{C}^{\mathbf{2}}(I(i), f_\alpha) \to \mathscr{C}^{\mathbf{2}}(I(i), f)\}_\alpha$ is $\mathcal{E}^{\downarrow}$-tight for each object i in $\mathscr{I}$. By our remarks in the previous paragraph, the cocone

$$\{(\vec{u}_\alpha)_* \cdot I(i) : \mathscr{C}^{\mathbf{2}}(I(i), f_\alpha) \cdot I(i) \to \mathscr{C}^{\mathbf{2}}(I(i), f) \cdot I(i)\}_\alpha$$

is $\mathcal{E}^{\mathbf{2}}$-tight for each object i in $\mathscr{I}$. By 2.4.3, the cocone $\{\mathrm{L}_0^I \vec{u}_\alpha : \mathrm{L}_0^I f_\alpha \to \mathrm{L}_0^I f\}_\alpha$ is $\mathcal{E}^{\mathbf{2}}$-tight. $\square$

Now, putting everything together, we can identify a large class of functors $I : \mathscr{I} \to \mathscr{C}^{\mathbf{2}}$ that have reflections in $\mathbf{AWFS}(\mathscr{C})$.

Theorem 3.2.28. *If $\mathscr{C}$ permits the algebraic small object argument and the orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ is proper or equal to the (isomorphism, any map) orthogonal factorization system, then every functor $I : \mathscr{I} \to \mathscr{C}^{\mathbf{2}}$ on a small category $\mathscr{I}$ has a reflection $(\mathrm{L}^I, \mathrm{R}^I) = \mathbb{F}_1\mathbb{F}_2\mathbb{F}_3(I)$ in $\mathbf{AWFS}(\mathscr{C})$ such that $I^{\boxslash} \cong \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}^I}$.*

Proof. By 3.2.25 and 3.2.27, the reflection $\mathrm{L}_0^I = \mathbb{F}_3(I)$ of I exists and is $(\mathcal{E}, \mathcal{M}')$-compact. By 3.2.21 and 3.2.22, the reflection $(\mathrm{L}_1^I, \mathrm{R}_1^I) = \mathbb{F}_2(\mathrm{L}_0^I)$ exists and is $(\mathcal{E}, \mathcal{M}')$-compact. By 3.2.16, the reflection $(\mathrm{L}^I, \mathrm{R}^I) = \mathbb{F}_1((\mathrm{L}_1^I, \mathrm{R}_1^I))$ exists.

By 3.2.26, 3.2.23, 3.1.15, and 3.2.18,

$$I^{\boxslash} \cong \mathbf{Coalg}^{\mathrm{EM}\boxslash}_{\mathrm{L}_0^I} \cong \mathbf{Coalg}^{\mathrm{EM}\boxslash}_{\mathrm{L}_1^I} \cong \mathbf{Alg}_{\mathrm{R}_1^I} \cong \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}^I}$$

$\square$

In particular, in the above theorem, when $\mathcal{R}$ is a class of maps in $\mathscr{C}$ such that $|I^{\boxslash}| = \mathcal{R}$, then $({}^{\square}\mathcal{R}, \mathcal{R})$ is a weak factorization system and $(\mathrm{L}^I, \mathrm{R}^I)$ is an associated functorial factorization.

3.3 $\mathcal{E}$-Compactness

When our LAWFSs are $\mathcal{E}$-compact, rather than the full generality of being $(\mathcal{E}, \mathcal{M}')$-compact, there are additional results we can prove. The main advantage of this case is that the composition of strongly $\mathcal{E}$-compact endofunctors is a strongly $\mathcal{E}$-compact endofunctor. This means that the $\circledast$-product of two $\mathcal{E}$-compact LAWFSs is an $\mathcal{E}$-compact LAWFSs. Using 2.4.25, this allows us to prove that the free monad on an $\mathcal{E}$-compact LAWFS is $\mathcal{E}$-compact. We will then show how to replace the forgetful functors $\mathbb{G}_1$, $\mathbb{G}_2$, and $\mathbb{G}_3$ with forgetful functors that have left adjoints. We will also be able to show that when the weak factorization systems of a model category have associated $\mathcal{E}$-compact AWFSs, then the model category is an algebraic model category.

3.3.1 Free AWFSs on $\mathcal{E}$-Compact LAWFSs

Let $\mathscr{C}$ be a cocomplete, locally small category equipped with well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$.

Proposition 3.3.1. *If* X *is an* $\mathcal{E}$*-compact LAWFS and* Y *is an* $(\mathcal{E}, \mathcal{M}')$*-compact LAWFS, then* $\mathrm{X} \circledast \mathrm{Y}$ *is an* $(\mathcal{E}, \mathcal{M}')$*-compact LAWFS.*

Proof. Let $(\mathrm{L}^{\mathrm{X}}, \mathrm{R}^{\mathrm{X}}) = \mathrm{X}$ and $(\mathrm{L}^{\mathrm{Y}}, \mathrm{R}^{\mathrm{Y}}) = \mathrm{Y}$. There are regular cardinals λ and κ such that R^{X} preserves $\mathcal{E}_{/\cong}$-tightness of $(\mathcal{V}_{/\cong}, \lambda)$-cocones and R^{Y} preserves $\mathcal{E}_{/\cong}$-tightness of $(\mathcal{M}'_{/\cong}, \kappa)$-cocones. Let ι be regular cardinal larger that λ and κ. Since every ι-filtered diagram is λ-filtered, R^{X} preserves $\mathcal{E}_{/\cong}$-tightness of $(\mathcal{V}_{/\cong}, \iota)$-cocones. Similarly, R^{Y} preserves $\mathcal{E}_{/\cong}$-tightness of $(\mathcal{M}'_{/\cong}, \iota)$-cocones. Thus $\mathrm{R}^{\mathrm{X}}\mathrm{R}^{\mathrm{Y}}$ preserves $\mathcal{E}_{/\cong}$-tightness of $(\mathcal{M}'_{/\cong}, \iota)$-cocones. ☐

Corollary 3.3.2. *If* X *and* Y *are* $\mathcal{E}$*-compact LAWFSs, then* $\mathrm{X} \circledast \mathrm{Y}$ *is an* $\mathcal{E}$*-compact LAWFS.*

Proposition 3.3.3. *Let λ be a regular cardinal. The free AWFS on an $(\mathcal{E}, \lambda)$-compact LAWFS is $(\mathcal{E}, \lambda)$-compact.*

Proof. Let $(\mathrm{L}_1, \mathrm{R}_1)$ be an $(\mathcal{E}, \lambda)$-compact LAWFS. By 3.2.16, the free AWFS (L, R) on $(\mathrm{L}_1, \mathrm{R}_1)$ exists and is equal to the colimit of the free monoid sequence on $(\mathrm{L}_1, \mathrm{R}_1)$. We've seen that applying the large-connected-colimit-preserving functor $\mathbb{K}$ to the free monoid sequence on $(\mathrm{L}_1, \mathrm{R}_1)$ gives us the free monad sequence on R_1. Let $(\mathrm{X}_\alpha)_\alpha$ be the free monad sequence on $(\mathrm{R}_1, \vec{\eta}^{\mathrm{R}_1})$. Then the colimit of this sequence exists and is equal to R.

We will mirror the proof of 3.2.14. Let B be an object in $\mathscr{C}$. The free monad sequence on $(\mathrm{R}_1, \vec{\eta}^{\mathrm{R}_1})$ is a sequence of codomain-preserving endofunctors and R is a codomain-preserving endofunctor. Every codomain-preserving endofunctor F on $\mathscr{C}^{\mathbf{2}}$ restricts to an endofunctor $\widehat{\mathrm{F}}$ on the comma category $\mathscr{C} \downarrow B$. So $(\widehat{\mathrm{X}}_\alpha)_\alpha$ is a sequence of endofunctors on $\mathscr{C} \downarrow B$. It follows that $(\widehat{\mathrm{X}}_\alpha)_\alpha$ is the free monad sequence for $(\widehat{\mathrm{R}}_1, \eta^{\mathrm{R}_1})$ on $\mathscr{C} \downarrow B$ and that the colimit of this sequence is $\widehat{\mathrm{R}}$.

As we mentioned in 3.2.14, the category $\mathscr{C} \downarrow B$ is cocomplete and the well-copowered, left proper, orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ lifts to a well-copowered, left proper, orthogonal factorization system $(\widehat{\mathcal{E}}, \widehat{\mathcal{M}})$ on $\mathscr{C} \downarrow B$. Then the fact that R_1 is strongly $(\mathcal{E}, \lambda)$-compact implies that $\widehat{\mathrm{R}}_1$ preserves $\widehat{\mathcal{E}}$-tightness of λ-filtered cocones. So, by 2.4.23, $\widehat{\mathrm{R}}$ is the free monad on $\widehat{\mathrm{R}}_1$. By 2.4.25, $\widehat{\mathrm{R}}$ preserves $\widehat{\mathcal{E}}$-tightness of λ-filtered cocones. Since this holds for every object B in $\mathscr{C}$, R must preserve $\mathcal{E}_{/\cong}$-tightness of $(\mathcal{V}_{/\cong}, \lambda)$-cocones. So R is strongly $(\mathcal{E}, \lambda)$-compact and thus (L, R) is $(\mathcal{E}, \lambda)$-compact. □

Let $\mathbf{AWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C})$ be the full subcategory of $\mathbf{AWFS}(\mathscr{C})$ on the AWFSs that are $(\mathcal{E}, \lambda)$-compact. Similarly, let $\mathbf{LAWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C})$ be the full subcategory of $\mathbf{LAWFS}(\mathscr{C})$ on the LAWFSs that are $(\mathcal{E}, \lambda)$-compact. Then the forgetful functor $\mathbb{G}_1 : \mathbf{AWFS}(\mathscr{C}) \to \mathbf{LAWFS}(\mathscr{C})$ restricts to a forgetful functor $\mathbf{AWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C}) \to \mathbf{LAWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C})$, which, by abuse of notation, we will still call $\mathbb{G}_1$. Because $\mathbf{AWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C})$ is a full subcategory of $\mathbf{AWFS}(\mathscr{C})$, 3.3.3 implies that every object in $\mathbf{LAWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C})$ has a reflection in $\mathbf{AWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C})$ given by $\mathbb{F}_1$. So $\mathbb{F}_1$ is the left adjoint to $\mathbb{G}_1$.

$$\begin{tikzcd}
\mathbf{AWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C}) \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathbb{G}_1"'] & \mathbf{LAWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C}) \arrow[l, bend right, "\mathbb{F}_1"']
\end{tikzcd}$$

We know from 3.3.3 that every object in the image of $\mathbb{F}_1$ is a free AWFS, but as we noted at the end of section 2.4.4, the existence of the above adjunction on its own is not enough to imply that fact.

Let $\mathbf{Cmd}^{\mathcal{E}}_{\lambda}(\mathscr{C}^{\mathbf{2}})$ be the full subcategory of $\mathbf{Cmd}(\mathscr{C}^{\mathbf{2}})$ on $(\mathcal{E},\lambda)$-compact comonads. It follows from 3.2.22 that the functors $\mathbb{G}_2$ and $\mathbb{F}_2$ restrict to functors $\mathbb{G}_2 : \mathbf{LAWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C}) \to \mathbf{Cmd}^{\mathcal{E}}_{\lambda}(\mathscr{C}^{\mathbf{2}})$ and $\mathbb{F}_2 : \mathbf{Cmd}^{\mathcal{E}}_{\lambda}(\mathscr{C}) \to \mathbf{LAWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C})$ of the same name. It is easy to check that the new $\mathbb{F}_2$ is a left adjoint to the new $\mathbb{G}_2$.

Let $\mathbf{CAT}/_{LK^{\mathcal{E}}_{\lambda}}\mathscr{C}^{\mathbf{2}}$ be the full sub-metacategory of $\mathbf{CAT}/\mathscr{C}^{\mathbf{2}}$ on the functors $A : \mathscr{A} \to \mathscr{C}^{\mathbf{2}}$ such that the left Kan extension $\mathrm{Lan}_A(A) : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ of A along itself exists and is an $(\mathcal{E},\lambda)$-compact comonad. Not only is A an object in $\mathbf{CAT}/_{LK^{\mathcal{E}}_{\lambda}}\mathscr{C}^{\mathbf{2}}$ when $\mathscr{A}$ is small and $\mathscr{C}$ permits the algebraic small object argument, but also the forgetful functor $V_{\mathrm{L}} : \mathbf{Coalg}^{\mathrm{EM}}_{\mathrm{L}} \to \mathscr{C}^{\mathbf{2}}$ of an $(\mathcal{E},\lambda)$-compact comonad $\mathrm{L} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is an object in the category. This follows from A.4.4, using a similar approach to the proof of 3.2.25. So $\mathbb{G}_3$ and $\mathbb{F}_3$ restrict to adjunctions on $\mathbf{Cmd}^{\mathcal{E}}_{\lambda}(\mathscr{C})$ and $\mathbf{CAT}/_{LK^{\mathcal{E}}_{\lambda}}\mathscr{C}^{\mathbf{2}}$, which, by abuse of notation, we call by the same names. So we get the following adjunctions.

$$\begin{tikzcd}
\mathbf{AWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C}) \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathbb{G}_1"'] & \mathbf{LAWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C}) \arrow[l, bend right, "\mathbb{F}_1"'] \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathbb{G}_2"'] & \mathbf{Cmd}^{\mathcal{E}}_{\lambda}(\mathscr{C}^{\mathbf{2}}) \arrow[l, bend right, "\mathbb{F}_2"'] \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathbb{G}_3"'] & \mathbf{CAT}/_{LK^{\mathcal{E}}_{\lambda}}\mathscr{C}^{\mathbf{2}} \arrow[l, bend right, "\mathbb{F}_3"']
\end{tikzcd}$$

3.3.2 Maps of $\mathcal{E}$-Compact AWFSs

Let $\mathscr{C}$ be a bicomplete, locally small category equipped with a well-copowered, left proper, orthogonal factorization system $(\mathcal{E}, \mathcal{M})$. We will also assume $\mathcal{C}$ is a model category with weak equivalences $\mathcal{W}$, cofibrations $\mathcal{C}$, and fibrations $\mathcal{F}$.

Proposition 3.3.4. *If the weak factorization systems* $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ *and* $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ *each have an associated functorial factorization that is an* $\mathcal{E}$*-compact LAWFS, then there is an* $\mathcal{E}$*-compact algebraic model category on* $\mathscr{C}$ *whose underlying model category is the one with weak equivalences* $\mathcal{W}$*, cofibrations* $\mathcal{C}$*, and fibrations* $\mathcal{F}$*.*

Proof. Let $\mathrm{X} = (\mathrm{L}_t, \mathrm{R})$ and $\mathrm{Y} = (\mathrm{L}, \mathrm{R}_t)$ be the $\mathcal{E}$-compact LAWFSs associated to the weak factorization systems $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$, respectively. As we mentioned in section 3.2.1, there is a unit map $\eta : \mathtt{I} \to \mathrm{Y}$ in the category $\mathbf{LAWFS}(\mathscr{C})$. By 3.2.1, the map $\mathrm{X} \circledast \eta : \mathrm{X} = \mathrm{X} \circledast \mathtt{I} \to \mathrm{X} \circledast \mathrm{Y}$ is a map in $\mathbf{LAWFS}(\mathscr{C})$. Since the $\circledast$-product of two $\mathcal{E}$-compact LAWFSs is $\mathcal{E}$-compact, $\mathrm{X} \circledast \eta : \mathrm{X} \to \mathrm{X} \circledast \mathrm{Y}$ is a map in $\mathbf{LAWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C})$. Thus $\mathbb{F}_1(\mathrm{X} \circledast \eta) : \mathbb{F}_1(\mathrm{X}) \to \mathbb{F}_1(\mathrm{X} \circledast \mathrm{Y})$ is a map in $\mathbf{AWFS}^{\mathcal{E}}_{\lambda}(\mathscr{C})$.

We know that for each object f in $\mathscr{C}^{\mathbf{2}}$, $\mathrm{L}_t f \in \mathcal{C} \cap \mathcal{W}$, $\mathrm{R} f \in \mathcal{F}$, $\mathrm{L} f \in \mathcal{C}$, and $\mathrm{R}_t f \in \mathcal{F} \cap \mathcal{W}$. Since $\mathrm{X} \circledast \mathrm{Y} f = (\mathrm{L}_t \mathrm{R}_t f \circ \mathrm{L} f, \mathrm{R}\mathrm{R}_t f)$, $\mathrm{X} \circledast \mathrm{Y}$ is an associated functorial factorization for the weak factorization system $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$. Now, by 3.1.2, $|\mathbf{Coalg}_{\mathrm{S}}| = \mathcal{C}$ and $|\mathbf{Alg}_{\mathrm{T}}| = \mathcal{F} \cap \mathcal{W}$, where $(\mathrm{S}, \mathrm{T}) = \mathrm{X} \circledast \mathrm{Y}$. Also $|\mathbf{Coalg}_{\mathrm{L}_t}| = \mathcal{C} \cap \mathcal{W}$ and $|\mathbf{Alg}_{\mathrm{R}}| = \mathcal{F}$. □

Chapter 4: Transferring Algebraic Model Structures

We prove two major results in this chapter, theorem 4.1.8 and theorem 4.1.12. Theorem 4.1.8 states that we can lift a compact algebraic model structure along a right adjoint when the adjunction is compact and when an acyclicity condition is satisfied. We use these conditions to prove in theorem 4.1.12 that the projective model structure on $\mathscr{C}^{\mathscr{D}}$ exists and is algebraic when $\mathscr{D}$ is small and the model category on $\mathscr{C}$ is a compact algebraic model category. When these theorems are applied to $\mathcal{E}$-compact algebraic model categories, the algebraic model categories they produce are $\mathcal{E}$-compact.

We prove a few results about transferring model structures in the enriched context in section 4.2. We were not able to prove that the model structure constructed in 4.2.3 is an enriched model structure.

4.1 Transferring Algebraic Model Structures

4.1.1 Compact Adjunctions

We will be able to lift an AWFS along a right adjoint when both the original the adjunction and the original AWFS satisfy compactness conditions. We will use this result to prove that compact algebraic model categories lift along compact right adjoints when an *acyclicity condition* is satisfied.

Definition 4.1.1 (Compact Adjunctions)**.** Let $\mathscr{A}$ be a cocomplete category equipped with well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_1', \mathcal{M}_1')$. Let $\mathscr{B}$ be a cocomplete category equipped with well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}_2, \mathcal{M}_2)$ and $(\mathcal{E}_2', \mathcal{M}_2')$. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor with a right adjoint $G : \mathscr{B} \to \mathscr{A}$. The adjunction $F \dashv G$ is $(\mathcal{E}_1, \mathcal{M}_1'; \mathcal{E}_2, \mathcal{M}_2')$-*compact* if the following conditions are satisfied.

1. There is a regular cardinal λ such that G sends $\mathcal{E}_2$-tight $(\mathcal{M}_2', \lambda)$-cocones to $\mathcal{E}_1$-tight cocones.
2. $G(\mathcal{M}_2') \subseteq \mathcal{M}_1'$ and $F(\mathcal{E}_1) \subseteq \mathcal{E}_2$.

The following result shows that when $(\mathcal{E}_1, \mathcal{M}_1) = (\mathcal{E}_1', \mathcal{M}_1')$ and $(\mathcal{E}_2, \mathcal{M}_2) = (\mathcal{E}_2', \mathcal{M}_2')$, condition (2) can be reduced to only checking one of the subset inclusions.

Proposition 4.1.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be categories equipped with orthogonal factorization systems $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \mathcal{M}_2)$, respectively. Let $G : \mathscr{A} \to \mathscr{B}$ be a functor with a left adjoint $F : \mathscr{B} \to \mathscr{A}$.*

$$F(\mathcal{E}_2) \subseteq \mathcal{E}_1 \textit{ if and only if } G(\mathcal{M}_1) \subseteq \mathcal{M}_2$$

While this result is well-known for adjunctions between the categories $\mathscr{A}$ and $\mathscr{B}$, a similar proof shows that it holds for adjunctions between the arrow categories.

Proposition 4.1.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be categories equipped with orthogonal factorization systems $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \mathcal{M}_2)$, respectively. Let $G : \mathscr{A}^{\mathbf{2}} \to \mathscr{B}^{\mathbf{2}}$ be a functor with a left adjoint $F : \mathscr{B}^{\mathbf{2}} \to \mathscr{A}^{\mathbf{2}}$. Then*

$$F(\mathcal{E}_2) \subseteq \mathcal{E}_1 \textit{ if and only if } G(\mathcal{M}_1) \subseteq \mathcal{M}_2,$$

where we now view $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{M}_1$, and $\mathcal{M}_2$ as collections of objects.

Proof. Let $\vec{\nu} : I \to GF$ and $\vec{\xi} : FG \to I$ be the unit and counit maps, respectively, for the adjunction. Suppose $F(\mathcal{E}_2) \subseteq \mathcal{E}_1$. Let $g \in \mathcal{M}_1$. Suppose $f \in \mathcal{E}_2$ and $(u,v) : f \to Gg$ is a map in $\mathscr{B}^{\mathbf{2}}$. Then a solution to the lifting problem $\vec{\xi}_g \circ F(u,v) : Ff \to g$ exists. Therefore, a solution to the lifting problem $G\vec{\xi}_g \circ GF(u,v) \circ \vec{\nu}_f : f \to Gg$ exists. But the commutativity of the following diagram shows that the lifting problem $(u,v) : f \to Gg$ has a solution.

$$\begin{tikzcd}
f \arrow[r, "{(u,v)}"] \arrow[d, "\vec{\nu}_f"'] & Gg \arrow[d, "\vec{\nu}_{Gg}"] \arrow[dr, bend left, "id"] & \\
GFf \arrow[r, "{GF(u,v)}"] & GFGg \arrow[r, "G\vec{\xi}_g"] & Gg
\end{tikzcd}$$

The proof of the converse is similar. □

The following proposition can be used to prove condition (2) of the definition holds for the most common proper orthogonal factorization systems.

Proposition 4.1.4 ([Bor94b, 4.3.9])**.** *A left adjoint preserves epimorphisms and strong epimorphisms. A right adjoint preserves monomorphisms and strong monomorphisms.*

4.1.2 Lifting an Algebraic Model Structure Along a Right Adjoint

A cofibrantly generated model category can be lifted along a right adjoint when the right adjoint satisfies a colimit-preservation condition and an acyclicity condition is satisfied [GS07, 3.6]. We extend this result to compact algebraic model categories in this section. Our work also generalizes a result of [GKR20].

Let $\mathscr{A}$ be a cocomplete category equipped with well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_1', \mathcal{M}_1')$. Let $\mathscr{B}$ be a cocomplete category equipped with well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}_2, \mathcal{M}_2)$ and $(\mathcal{E}_2', \mathcal{M}_2')$. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor with a left adjoint $G : \mathscr{B} \to \mathscr{A}$ and let $\nu : I \to GF$ and $\xi : FG \to I$ be the unit and counit maps of this adjunction. Suppose that $F \dashv G$, is an $(\mathcal{E}_1, \mathcal{M}_1'; \mathcal{E}_2, \mathcal{M}_2')$-compact adjunction.

We will abuse notation and also refer to the functor $F^{\mathbf{2}} : \mathscr{A}^{\mathbf{2}} \to \mathscr{B}^{\mathbf{2}}$ that sends objects f to Ff and maps $(u,v) : f \to g$ to $(Fu, Fv) : Ff \to Fg$ as F. We will similarly use G for both the functor $\mathscr{B} \to \mathscr{A}$ and the functor $\mathscr{B}^{\mathbf{2}} \to \mathscr{A}^{\mathbf{2}}$ between arrow categories. As functors between the arrow categories, F is still left adjoint to G.

$$\begin{tikzcd}
\mathscr{A}^{\mathbf{2}} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathscr{B}^{\mathbf{2}} \arrow[ll, bend left, "G"]
\end{tikzcd}$$

Let $\vec{\nu} = (\nu\,\mathrm{dom}, \nu\,\mathrm{cod}) : I \to GF$ and $\vec{\xi} = (\xi\,\mathrm{dom}, \xi\,\mathrm{cod}) : FG \to I$ be the unit and counit maps of the adjunction between arrow categories.

Let (L, R) be an $(\mathcal{E}_1, \mathcal{M}'_1)$-compact AWFS on $\mathscr{A}$. We will use the factorization (L, R) to construct a new AWFS (X, Y) on $\mathscr{B}$. We start with the endofunctor $F\mathrm{L}G : \mathscr{B}^{\mathbf{2}} \to \mathscr{B}^{\mathbf{2}}$. This is a comonad with comultiplication and counit maps given by

$$F\mathrm{L}G \xrightarrow{F\vec{\delta}G} F\mathrm{LL}G \xrightarrow{F\mathrm{L}\vec{\nu}\mathrm{L}G} F\mathrm{L}GF\mathrm{L}G$$

and

$$F\mathrm{L}G \xrightarrow{F\vec{\varepsilon}G} FG \xrightarrow{\vec{\xi}} I.$$

Lemma 4.1.5. *The comonad $F\mathrm{L}G : \mathscr{B}^{\mathbf{2}} \to \mathscr{B}^{\mathbf{2}}$ is $(\mathcal{E}_2, \mathcal{M}'_2)$-compact.*

Proof. There is a regular cardinal λ such that the adjunction $F \dashv G : \mathscr{A} \to \mathscr{B}$ is $(\mathcal{E}_1, \mathcal{M}'_1; \mathcal{E}_2, \mathcal{M}'_2, \lambda)$-compact and the comonad $\mathrm{L} : \mathscr{A}^{\mathbf{2}} \to \mathscr{A}^{\mathbf{2}}$ is $(\mathcal{E}_1, \mathcal{M}'_1, \lambda)$-compact. Let $\{(p_\alpha, q_\alpha) : f_\alpha \to g\}$ be an $\mathcal{E}_{2/\cong}$-tight $(\mathcal{M}'_{2/\cong}, \lambda)$-cocone in $\mathscr{B}^{\mathbf{2}}$. Since $G : \mathscr{B} \to \mathscr{A}$ sends $\mathcal{E}_2$-tight $(\mathcal{M}'_2, \lambda)$-filtered cocones to $\mathcal{E}_1$-tight λ-filtered cocones and since $G(\mathcal{M}'_2) \subseteq \mathcal{M}'_1$, $\{G(p_\alpha, q_\alpha) : Gf_\alpha \to Gg\}$ is an $\mathcal{E}_{1/\cong}$-tight $(\mathcal{M}'_{1/\cong}, \lambda)$-cocone. Therefore $\{\mathrm{L}G(p_\alpha, q_\alpha) : \mathrm{L}Gf_\alpha \to \mathrm{L}Gg\}$ is an $\mathcal{E}_1^{\mathbf{2}}$-tight cocone. The functor F preserves colimits since it is a left adjoint. Because $F(\mathcal{E}_1) \subseteq \mathcal{E}_2$, $\{F\mathrm{L}G(p_\alpha, q_\alpha) : F\mathrm{L}Gf_\alpha \to F\mathrm{L}Gg\}$ is an $\mathcal{E}_2^{\mathbf{2}}$-tight cocone. □

By 3.2.21, there is a reflection $(\mathrm{X}_1, \mathrm{Y}_1) \coloneqq \mathbb{F}_2(F\mathrm{L}G)$ of $F\mathrm{L}G$ in $\mathbf{LAWFS}(\mathscr{B})$. Since $F\mathrm{L}G$ is an $(\mathcal{E}_2, \mathcal{M}_2')$-compact functor, by 3.2.22, (X_1, Y_1) is an $(\mathcal{E}_2, \mathcal{M}_2')$-compact LAWFS. Thus by 3.2.16, the reflection $(X, Y) \coloneqq \mathbb{F}_1(X_1, Y_1)$ of (X_1, Y_1) in $\mathbf{AWFS}(\mathscr{B})$ exists.

Remark 4.1.6. When $(\mathcal{E}_1', \mathcal{M}_1')$ and $(\mathcal{E}_2', \mathcal{M}_2')$ are the (isomorphism, any map) orthogonal factorization systems, the comonad $F\mathrm{L}G$ and the LAWFS $(\mathrm{X}_1, \mathrm{Y}_1)$ are $\mathcal{E}_2$-compact. Then, by 3.3.3, the AWFS (X, Y) is $\mathcal{E}_2$-compact.

Proposition 4.1.7. $G^{-1}|\mathbf{Alg}_{\mathrm{R}}| = |\mathbf{Alg}_{\mathrm{Y}}|$.

Proof. We know from 3.2.18 that $|\mathbf{Alg}_{\mathrm{Y}}| = |\mathbf{Alg}_{\mathrm{Y}}^{\mathrm{EM}}| = |\mathbf{Alg}_{\mathrm{Y}_1}|$. Therefore it suffices to show that $G^{-1}|\mathbf{Alg}_{\mathrm{R}}| = |\mathbf{Alg}_{\mathrm{Y}_1}|$.

Let $E : \mathscr{A}^{\mathbf{2}} \to \mathscr{A}$ and $E_1 : \mathscr{B}^{\mathbf{2}} \to \mathscr{B}$ be the middomain functors for (L, R) and (X_1, Y_1), respectively. Note that on each object $f : A \to B$ in $\mathscr{C}^{\mathbf{2}}$, the counit map $\vec{\xi}_f = (\xi_A, \xi_B) : FGf \to f$ has the "vertical factorization" show below, where $(\xi_A, a) : F\mathrm{L}Gf \to X_1 f$ is the universal map of the $\mathbb{F}_2$-reflection.

$$\begin{tikzcd}
FGA \arrow[r, "\xi_A"] \arrow[d, "F\mathrm{L}Gf"] & A \arrow[d, "\mathrm{X}_1 f"] \\
FE(Gf) \arrow[r, "a"] \arrow[d, "F\mathrm{R}Gf"] & E_1 f \arrow[d, "\mathrm{Y}_1 f"] \arrow[ul, phantom, "\ulcorner", very near start] \\
FGB \arrow[r, "\xi_B"] & B
\end{tikzcd}$$

Suppose $f : A \to B$ is in $|\mathbf{Alg}_{\mathrm{Y}_1}|$. Then there is a solution l to the lifting problem $(id, Y_1 f) : X_1 f \to f$. After applying G, we get that the following diagram is commutative.

$$\begin{tikzcd}
GA \arrow[r, "\nu_{GA}"] \arrow[d, "\mathrm{L}Gf"] & GFGA \arrow[r, "G\xi_A"] \arrow[d, "GF\mathrm{L}Gf"] & GA \arrow[r, "id"] \arrow[d, "GX_1 f"'] & GA \arrow[d, "Gf"] \\
E(Gf) \arrow[r, "\nu_{EGf}"] \arrow[d, "\mathrm{R}Gf"] & GFE(Gf) \arrow[r, "Ga"] \arrow[d, "GF\mathrm{R}Gf"] & GE_1 f \arrow[r, "GY_1 f"] \arrow[ur, "Gl"] & GB \\
GB \arrow[r, "\nu_{GB}"] & GFGB \arrow[urr, bend right, "G\xi_B"] & &
\end{tikzcd}$$

Since $G\xi \circ \nu G = id_G$, this means there is a solution to the lifting problem $(id_{GA}, \mathrm{R}Gf) : \mathrm{L}Gf \to Gf$. So Gf is an object in $|\mathbf{Alg}_{\mathrm{R}}|$.

Suppose conversely, that Gf is an object in $|\mathbf{Alg}_{\mathrm{R}}|$. Then there is a solution k to the lifting problem $(id_{GA}, \mathrm{R}Gf) : \mathrm{L}Gf \to Gf$. From the below diagram we see that $\xi_A \circ Fk$ is a solution to the lifting problem $(\xi_A, \xi_B \circ F\mathrm{R}Gf) : F\mathrm{L}Gf \to f$.

$$\begin{tikzcd}
FGA \arrow[r, "id"] \arrow[d, "F\mathrm{L}Gf"'] & FGA \arrow[r, "\xi_A"] \arrow[d, "FGf"] & A \arrow[d, "f"] \\
FE(Gf) \arrow[r, "F\mathrm{R}Gf"] \arrow[ur, "Fk"] & FGB \arrow[r, "\xi_B"] & B
\end{tikzcd}$$

But the universal property of cocartesian squares then implies there is a solution to the lifting problem $(id, Y_1 f) : X_1 f \to f$. □

Theorem 4.1.8. *Suppose the categories $\mathscr{A}$ and $\mathscr{B}$ are complete in addition to being cocomplete. If $\zeta : (\mathrm{L}_t, \mathrm{R}) \to (\mathrm{L}, \mathrm{R}_t)$ is an $(\mathcal{E}_1, \mathcal{M}'_1)$-compact algebraic model category on $\mathscr{A}$ with weak equivalences $\mathcal{W}$, then there is a map $\theta : (X_t, Y) \to (X, Y_t)$ of AWFSs on $\mathscr{B}$ such that $|\mathbf{Alg}_{\mathrm{Y}}| = G^{-1}|\mathbf{Alg}_{\mathrm{R}}|$ and $|\mathbf{Alg}_{\mathrm{Y}_t}| = G^{-1}|\mathbf{Alg}_{\mathrm{R}_t}|$. If the acyclicity condition, $|\mathbf{Coalg}_{\mathrm{X}_t}| \subseteq G^{-1}\mathcal{W}$, is satisfied, then θ is an algebraic model category with weak equivalences $G^{-1}\mathcal{W}$.*

Proof. Let (X_t, Y) and (X, Y_t) be the LAWFSs $\mathbb{F}_1\mathbb{F}_2(F\mathrm{L}_tG)$ and $\mathbb{F}_1\mathbb{F}_2(F\mathrm{L}G)$, respectively. To get a map $\theta : (X_t, Y) \to (X, Y_t)$ of AWFSs, it suffices to produce a map of comonads $F\mathrm{L}_tG \to F\mathrm{L}G$, which we can then apply the reflection $\mathbb{F}_1\mathbb{F}_2$ to. The map $F\vec{\zeta}G : F\mathrm{L}_tG \to F\mathrm{L}G$, with $\vec{\zeta} := (id, \zeta)$, will do, since the following diagrams commute.

$$\begin{tikzcd}
F\mathrm{L}_tG \arrow[dd, "F\vec{\zeta}G"'] \arrow[dr, "F\vec{\varepsilon}G"] & & \\
 & FG \arrow[r, "\xi"] & I \\
F\mathrm{L}G \arrow[ur, "F\vec{\varepsilon}G"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
F\mathrm{L}_tG \arrow[r, "F\vec{\delta}G"] \arrow[d, "F\vec{\zeta}G"] & F\mathrm{L}_t\mathrm{L}_tG \arrow[r, "F\mathrm{L}_t\vec{\nu}\mathrm{L}_tG"] \arrow[d, "F\vec{\zeta}\vec{\zeta}G"] & F\mathrm{L}_tGF\mathrm{L}_tG \arrow[d, "F\vec{\zeta}GF\vec{\zeta}G"] \\
F\mathrm{L}G \arrow[r, "F\vec{\delta}G"] & F\mathrm{L}\mathrm{L}G \arrow[r, "F\mathrm{L}\vec{\nu}\mathrm{L}G"] & F\mathrm{L}GF\mathrm{L}G
\end{tikzcd}$$

It therefore remains to show that, with weak equivalences $G^{-1}\mathcal{W}$, θ is an algebraic model structure. The 2-out-of-3 property on $\mathcal{W}$ implies that $G^{-1}\mathcal{W}$ has the 2-out-of-3 property. We know

$$|\mathbf{Alg}_{\mathrm{Y}_t}| = G^{-1}|\mathbf{Alg}_{\mathrm{R}_t}| = G^{-1}\big(|\mathbf{Alg}_{\mathrm{R}}| \cap \mathcal{W}\big) = G^{-1}|\mathbf{Alg}_{\mathrm{R}}| \cap G^{-1}\mathcal{W}.$$

So $|\mathbf{Alg}_{\mathrm{Y}_t}| = |\mathbf{Alg}_{\mathrm{Y}}| \cap G^{-1}\mathcal{W}$. By the acyclicity condition and the existence of θ, $|\mathbf{Coalg}_{\mathrm{X}_t}| \subseteq |\mathbf{Coalg}_{\mathrm{X}}| \cap G^{-1}\mathcal{W}$. Now, suppose $g \in |\mathbf{Coalg}_{\mathrm{X}}| \cap G^{-1}\mathcal{W}$. Factoring g as $X_t g$ followed by Yg, the 2-out-of-3 property of $G^{-1}\mathcal{W}$ implies that Yg is in $G^{-1}\mathcal{W}$. But this means Yg is in $G^{-1}|\mathbf{Alg}_{\mathrm{R}}| \cap G^{-1}\mathcal{W} = |\mathbf{Alg}_{\mathrm{Y}_t}|$. So the lifting problem $(X_t g, id) : g \to Yg$ has a solution. Thus $|\mathbf{Coalg}_{\mathrm{X}_t}| = |\mathbf{Coalg}_{\mathrm{X}}| \cap G^{-1}\mathcal{W}$. □

Proposition 4.1.9 (Acyclicity Condition)**.** *Assuming the hypotheses of theorem 4.1.8 and defining* $(X_t, Y) = \mathbb{F}_1\mathbb{F}_2(F\mathrm{L}_t G)$*, the following conditions are equivalent.*

1. $|\mathbf{Coalg}_{\mathrm{X}_t}| \subseteq G^{-1}\mathcal{W}$.

2. *There is a model category on* $\mathscr{B}$ *with weak equivalences* $G^{-1}\mathcal{W}$ *and fibrations* $G^{-1}|\mathbf{Alg}_{\mathrm{R}}|$.

3. ${}^{\square}(G^{-1}|\mathbf{Alg}_{\mathrm{R}}|) \subseteq G^{-1}\mathcal{W}$.

Proof. (1) ⇒ (2) is theorem 4.1.8. (2) ⇒ (3). If $\mathcal{C}_t$ is the class of maps in a model category with the left lifting property with respect to the fibrations, then $\mathcal{C}_t \subseteq \mathcal{W}$. So ${}^{\square}(G^{-1}|\mathbf{Alg}_{\mathrm{R}}|) \subseteq G^{-1}\mathcal{W}$. (3) ⇒ (1). Suppose ${}^{\square}G^{-1}|\mathbf{Alg}_{\mathrm{R}}| \subseteq G^{-1}\mathcal{W}$. Since $G^{-1}|\mathbf{Alg}_{\mathrm{R}}| = |\mathbf{Alg}_{\mathrm{Y}}|$, we know $|\mathbf{Coalg}_{\mathrm{X}_t}| = {}^{\square}|\mathbf{Alg}_{\mathrm{Y}}| \subseteq G^{-1}\mathcal{W}$. □

Remark 4.1.10. With the model structure constructed on $\mathscr{B}$ in theorem 4.1.8, $F \dashv G$ is a Quillen adjunction.

Remark 4.1.11. When $(\mathcal{E}_1', \mathcal{M}_1')$ and $(\mathcal{E}_2', \mathcal{M}_2')$ are the (isomorphism, any map) orthogonal factorization systems, by 4.1.6, the algebraic model category θ of theorem 4.1.8 is $\mathcal{E}_2$-compact.

4.1.3 Projective Algebraic Model Structures

Riehl proved in [Rie11, 4.5] that when $\mathscr{D}$ is small and $\mathscr{C}$ has a cofibrantly generated algebraic model structure, then the projective model structure on $\mathscr{C}^{\mathscr{D}}$ exists and is a cofibrantly generated algebraic model category. We generalize this result to compact algebraic model categories in this section.

Let $\mathscr{C}$ be a bicomplete category with well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$. Let $\zeta : (\mathrm{L}_t, \mathrm{R}) \to (\mathrm{L}, \mathrm{R}_t)$ be an $(\mathcal{E}, \mathcal{M}')$-compact algebraic model category with weak equivalences $\mathcal{W}$. Let $\mathscr{D}$ be a small category.

Theorem 4.1.12 (Projective Algebraic Model Structure)**.** *There is an induced algebraic model category on the functor category $\mathscr{C}^{\mathscr{D}}$ whose underlying model structure is the projective model structure.*

Proof. We follow the proof of [Rie11, 4.5] closely until the last few paragraphs. Because AWFSs are functorial factorizations, the AWFS $(\mathrm{L}_t, \mathrm{R})$ defines a functorial factorization $(\mathrm{L}_t^{\mathscr{D}}, \mathrm{R}^{\mathscr{D}})$ on the functor category $\mathscr{C}^{\mathscr{D}}$. The functorial factorization $(\mathrm{L}_t^{\mathscr{D}}, \mathrm{R}^{\mathscr{D}})$ is the objectwise $(\mathrm{L}_t, \mathrm{R})$-factorization. In fact, we can describe $\mathrm{L}_t^{\mathscr{D}} : (\mathscr{C}^{\mathscr{D}})^{\mathbf{2}} \to (\mathscr{C}^{\mathscr{D}})^{\mathbf{2}}$ by the composition

$$(\mathscr{C}^{\mathscr{D}})^{\mathbf{2}} \xrightarrow{\cong} (\mathscr{C}^{\mathbf{2}})^{\mathscr{D}} \xrightarrow{\mathrm{L}_{t*}} (\mathscr{C}^{\mathbf{2}})^{\mathscr{D}} \xrightarrow{\cong} (\mathscr{C}^{\mathscr{D}})^{\mathbf{2}},$$

where ${\mathrm{L}_t}^*(\alpha) = \mathrm{L}_t\alpha$ for each functor $\alpha : \mathscr{D} \to \mathscr{C}^{\mathbf{2}}$. If $\vec{\varepsilon} : \mathrm{L}_t \to \mathrm{Id}$ is the counit map, then $\vec{\varepsilon}\alpha : \mathrm{L}_t\alpha \to \alpha$ defines a counit map for $\mathrm{L}_t^{\mathscr{D}}$. In a similar way, the comultiplication map of L_t defines a comultiplication map for $\mathrm{L}_t^{\mathscr{D}}$ which makes it a comonad. The endofunctor $\mathrm{R}^{\mathscr{D}}$ has a similar description and has the structure of a monad [Rie11, §4.2]. Although these factorizations are objectwise, the $\mathrm{L}_t^{\mathscr{D}}$-coalgebras do not need to be objectwise L_t-coalgebras.

Let $\mathscr{D}_0$ be the discrete category on the objects of $\mathscr{D}$. As we saw above, the AWFSs $(\mathrm{L}_t, \mathrm{R})$ and $(\mathrm{L}, \mathrm{R}_t)$ define AWFSs $(\mathrm{L}_t^{\mathscr{D}_0}, \mathrm{R}^{\mathscr{D}_0})$ and $(\mathrm{L}^{\mathscr{D}_0}, \mathrm{R}_t^{\mathscr{D}_0})$ on $\mathscr{C}^{\mathscr{D}_0}$. It is straightforward to check that the objectwise map $\zeta^{\mathscr{D}_0} : (\mathrm{L}_t^{\mathscr{D}_0}, \mathrm{R}^{\mathscr{D}_0}) \to (\mathrm{L}^{\mathscr{D}_0}, \mathrm{R}_t^{\mathscr{D}_0})$ is a map of AWFSs. Unlike for the category $\mathscr{D}$, since $\mathscr{D}_0$ is discrete, $\mathrm{L}_t^{\mathscr{D}_0}$-coalgebras are exactly the objectwise L_t-coalgebras and likewise for $\mathrm{R}^{\mathscr{D}_0}$, $\mathrm{L}^{\mathscr{D}_0}$, and $\mathrm{R}_t^{\mathscr{D}_0}$. Thus $\zeta^{\mathscr{D}_0}$ is an algebraic model category whose weak equivalences, fibrations, and cofibrations are the objectwise weak equivalences, fibrations, and cofibrations.

As discussed in 2.4.2, we get objectwise well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}^{\mathscr{D}_0}, \mathcal{M}^{\mathscr{D}_0})$ and $(\mathcal{E}'^{\mathscr{D}_0}, \mathcal{M}'^{\mathscr{D}_0})$ on $\mathscr{C}^{\mathscr{D}_0}$ and objectwise well-copowered, left proper, orthogonal factorization systems $(\mathcal{E}^{\mathscr{D}}, \mathcal{M}^{\mathscr{D}})$ and $(\mathcal{E}'^{\mathscr{D}}, \mathcal{M}'^{\mathscr{D}})$ on $\mathscr{C}^{\mathscr{D}}$. There is a regular cardinal λ such that L_t and L are $(\mathcal{E}, \mathcal{M}', \lambda)$-compact. Clearly, $\mathrm{L}_t^{\mathscr{D}_0}$ and $\mathrm{L}^{\mathscr{D}_0}$ send $(\mathcal{E}_{/\cong})^{\mathscr{D}_0}$-tight $\big((\mathcal{M}'_{/\cong})^{\mathscr{D}_0}, \lambda\big)$-cocones to $(\mathcal{E}_{/\cong})^{\mathscr{D}_0}$-tight cocones, since they both do so on each object d in $\mathscr{D}_0$. So $\zeta^{\mathscr{D}_0}$ is an $(\mathcal{E}^{\mathscr{D}_0}, \mathcal{M}'^{\mathscr{D}_0})$-compact algebraic model category.

Let $N : \mathscr{D}_0 \hookrightarrow \mathscr{D}$ be the subcategory inclusion functor. Let $\mathrm{Lan}_N(-) : \mathscr{C}^{\mathscr{D}_0} \to \mathscr{C}^{\mathscr{D}}$ be the functor defined by taking left Kan extensions along N. As described in [Rie11, 4.5], the functor $\mathrm{Lan}_N(-)$ has a right adjoint which is the restriction functor $N^* : \mathscr{C}^{\mathscr{D}} \to \mathscr{C}^{\mathscr{D}_0}$. We will use theorem 4.1.8 to transfer the objectwise model structure on $\mathscr{C}^{\mathscr{D}_0}$ to a model structure on $\mathscr{C}^{\mathscr{D}}$. Because a map $\alpha : F \to G$ in $\mathscr{C}^{\mathscr{D}}$ is an objectwise fibration if and only if $N^*(\alpha) = \alpha N$ is an objectwise fibration in $\mathscr{C}^{\mathscr{D}_0}$ and because the same is true for weak equivalences, this new model structure on $\mathscr{C}^{\mathscr{D}}$ will be the desired projective model structure.

First we will show that the $\mathrm{Lan}_N(-) \dashv N^*$ adjunction is $(\mathcal{E}^{\mathscr{D}_0}, \mathcal{M}'^{\mathscr{D}_0}; \mathcal{E}^{\mathscr{D}}, \mathcal{M}'^{\mathscr{D}})$-compact. Since the aforementioned orthogonal factorization systems are just objectwise collections $N^*(\mathcal{M}^{\mathscr{D}}) \subseteq \mathcal{M}^{\mathscr{D}_0}$ and $N^*(\mathcal{M}'^{\mathscr{D}}) \subseteq \mathcal{M}'^{\mathscr{D}_0}$. So, by 4.1.2, condition (2) of definition 4.1.1 holds. We also have that $N^*(\mathcal{E}^{\mathscr{D}}) \subseteq \mathcal{E}^{\mathscr{D}_0}$. Since colimits are computed objectwise in both functor categories, N^* preserves colimits. Thus N^* sends $\mathcal{E}^{\mathscr{D}}$-tight $(\mathcal{M}'^{\mathscr{D}}, \kappa)$-cocones to $\mathcal{E}^{\mathscr{D}_0}$-tight cocones for some regular cardinal κ.

It remains to show that the $\mathrm{Lan}_N(-) \dashv N^*$ adjunction satisfies the acyclicity condition. We will prove condition 4.1.9 (3) holds. Let $\mathcal{R} = |\mathbf{Alg}_{\mathrm{R}^{\mathscr{D}_0}}|$. As we've already noted, $\mathcal{R}$ is the collection of objectwise R-algebras in $\mathscr{C}^{\mathscr{D}_0}$. Let $\rho : F \to G$ be a map in $\mathscr{C}^{\mathscr{D}}$ that is in $^{\boxslash}(N^{*-1}\mathcal{R})$. Since $\mathrm{R}^{\mathscr{D}}\rho$ is an objectwise R-algebra, the lifting problem $(\mathrm{L}_t^{\mathscr{D}}\rho, id) : \rho \to \mathrm{R}^{\mathscr{D}}\rho$ has a solution. So ρ is a retract of the objectwise L_t-coalgebra $\mathrm{L}_t^{\mathscr{D}}\rho$. But an objectwise L_t-coalgebra is in particular an objectwise weak equivalence. Since $N^{*-1}\mathcal{W}$ is closed under retracts, ρ must be in $N^{*-1}\mathcal{W}$. □

Remark 4.1.13. When $(\mathcal{E}', \mathcal{M}')$ is the (isomorphism, any map) orthogonal factorization system, by 4.1.11, the projective algebraic model category on $\mathscr{C}^{\mathscr{D}}$ is $\mathcal{E}^{\mathscr{D}}$-compact.

4.2 Enriched Algebraic Model Structures

4.2.1 Monoidal Projective Algebraic Model Structures

Let $(\mathscr{V}, \otimes, I)$ be a bicomplete closed symmetric monoidal category. We will use the notation $\underline{\mathscr{V}}(a, b)$ for the internal hom objects of $\mathscr{V}$.

Suppose $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ are well-copowered, left proper, orthogonal factorization systems on $\mathscr{V}$. Let $\zeta : (\mathrm{L}_t, \mathrm{R}) \to (\mathrm{L}, \mathrm{R}_t)$ be an $(\mathcal{E}, \mathcal{M}')$-compact algebraic model category with weak equivalences $\mathcal{W}$ whose underlying model category is monoidal with respect to $\otimes$. Let $\mathscr{D}$ be a small $\mathscr{V}$-enriched category.

Proposition 4.2.1. *There is an induced algebraic model category on the functor category $\mathscr{V}^{\mathscr{D}}$ which is a $\mathcal{V}$-model category and whose underlying model structure is the projective model structure.*

Proof. By 4.1.12, there is an algebraic model category on $\mathscr{V}^{\mathscr{D}}$ whose underlying model structure is the projective model structure. The category $\mathscr{V}^{\mathscr{D}}$ is $\mathscr{V}$-enriched and $\mathscr{V}$ powered and copowered [Kel05, ch 2]. The enriched hom functor $(\mathscr{V}^{\mathscr{D}})^{\mathrm{op}} \times \mathscr{V}^{\mathscr{D}} \to \mathscr{V}$ is determined by a $\mathscr{V}$-powering $\mathscr{V}^{\mathrm{op}} \times \mathscr{V}^{\mathscr{D}} \to \mathscr{V}^{\mathscr{D}}$. This $\mathscr{V}$-powering on $\mathscr{V}^{\mathscr{D}}$ is defined as follows. Let $S : \mathscr{D} \to \mathscr{V}$ be a functor and let a be an object in $\mathscr{V}$. Then we define S^a to be the functor $\mathscr{D} \to \mathscr{V}$ defined on objects by $d \mapsto \underline{\mathscr{V}}(a, S(d))$ and extended naturally to morphisms.

Let $i : a \to b$ be a cofibration in $\mathscr{V}$. Let $f : S \to T$ be a fibration in $\mathscr{V}^{\mathscr{D}}$. For each d in $\mathscr{D}$, $(i^*, f_{d*}) : \underline{\mathscr{V}}(b, S(d)) \to \underline{\mathscr{V}}(a, S(d)) \times_{\underline{\mathscr{V}}(a,T(d))} \underline{\mathscr{V}}(b, T(d))$ is a fibration in $\mathscr{V}$, since the model category on $\mathscr{V}$ is monoidal. Thus $(i^*, f_*) : S^b \to S^a \times_{T^a} T^b$ is an objectwise fibration. So it is a fibration in the projective model structure on $\mathscr{V}^{\mathscr{D}}$. If either i or f is acyclic, then (i^*, f_*) is also acyclic. So by [Hov99, 4.2.2], $\mathscr{V}^{\mathscr{D}}$ is a $\mathscr{V}$-model category. □

Remark 4.2.2. We can of course get a projective algebraic model structure on $\mathbf{Pre}(\mathscr{D}, \mathscr{V})$ by applying the above theorem to $\mathbf{Fun}(\mathscr{D}^{\mathrm{op}}, \mathscr{V})$.

4.2.2 A Lifting of a Projective Algebraic Model Structure

Let $(\mathscr{V}, \otimes, I)$ be a bicomplete closed symmetric monoidal category. Suppose $(\mathcal{E}_{\mathscr{V}}, \mathcal{M}_{\mathscr{V}})$ is a well-copowered, left proper, orthogonal factorization systemon $\mathscr{V}$. Let $\mathscr{D}$ be a small $\mathscr{V}$-enriched category.

Suppose $\mathscr{C}$ is a bicomplete $\mathscr{V}$-enriched category that is powered and copowered over $\mathscr{V}$ and that is equipped with a $\mathscr{V}$-functor $\lambda : \mathscr{D} \to \mathscr{C}$. Let $\underline{\mathscr{C}}(-,-) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{V}$ be the enriched hom functor and let $\odot : \mathscr{V} \times \mathscr{C} \to \mathscr{C}$ be the copowering. In [GM11, §5.1], a $\mathscr{V}$-adjunction

$$\begin{tikzcd}
\mathbf{Pre}(\mathscr{D}, \mathscr{V}) \arrow[r, bend left, "\mathbb{T}"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "\mathbb{U}"]
\end{tikzcd}$$

is described. The functor $\mathbb{U}$ sends an object A in $\mathscr{C}$ to the presheaf $d \mapsto \underline{\mathscr{C}}(\lambda(d), A)$. The functor $\mathbb{T}$ sends a presheaf B to the object $B \odot_{\mathscr{D}} \lambda$, which is defined as the following coequalizer.

$$\begin{tikzcd} \coprod_{d,e} B(e) \otimes \underline{\mathscr{D}}(d,e) \odot \lambda(d) \arrow[r, shift left] \arrow[r, shift right] & \coprod_d B(d) \odot \lambda(d) \arrow[r, dashed] & B \odot_{\mathscr{D}} \lambda \end{tikzcd}$$

Let $\zeta : (\mathrm{C}_t, \mathrm{F}) \to (\mathrm{C}, \mathrm{F}_t)$ be an $\mathcal{E}_{\mathscr{V}}$-compact algebraic model category on $\mathscr{V}$ with weak equivalences $\mathcal{W}$. We assume $\mathbf{Pre}(\mathscr{D}, \mathscr{V})$ is equipped with the projective algebraic model structure in the following theorem. By 4.1.13, this algebraic model structure is $\mathscr{E}_{\mathscr{V}}^{\mathscr{D}}$-compact. We will further assume that $\mathscr{C}$ has a well-copowered, left proper, orthogonal factorization system $(\mathcal{E}_{\mathscr{C}}, \mathcal{M}_{\mathscr{C}})$.

Proposition 4.2.3. *If the adjunction* $\mathbb{T} \dashv \mathbb{U}$ *is* $(\mathcal{E}_{\mathscr{C}}; \mathcal{E}_{\mathscr{V}}^{\mathscr{D}})$*-compact and satisfies the acyclicity condition of proposition 4.1.9, then* $\mathscr{C}$ *has the structure of an* $\mathcal{E}_{\mathscr{C}}$*-compact algebraic model category. A morphism* f *in* $\mathscr{C}$ *is a fibration (respectively weak equivalence) if and only if* $\underline{\mathscr{C}}(\lambda(d), f)$ *is a fibration (weak equivalence) in* $\mathscr{V}$ *for each object* d *in* $\mathscr{D}$*.*

Proof. Consider the projective algebraic model structure on $\mathbf{Pre}(\mathscr{D}, \mathscr{V})$ described in proposition 4.1.12. Let $\mathcal{W}_{pre}$ and $\mathcal{F}_{pre}$ denote the weak equivalences and fibrations of this model category, respectively. Using the $\mathbb{T} \dashv \mathbb{U}$ adjunction and theorem 4.1.8, we have an $\mathcal{E}_{\mathscr{C}}$-compact algebraic model structure $\theta : (\mathrm{C}_t, \mathrm{F}) \to (\mathrm{C}, \mathrm{F}_t)$ on $\mathscr{C}$ with weak equivalences $\mathbb{U}^{-1}\mathcal{W}_{pre}$ and fibrations $|\mathbf{Alg}_{\mathrm{F}}| = \mathbb{U}^{-1}\mathcal{F}_{pre}$. □

This is a partial version of [GM11, 1.17]. We know that $\mathbb{U}$ is a right Quillen functor. So $\mathbb{T} \dashv \mathbb{U}$ is a Quillen adjunction. We also know by [GM11] that $\mathbb{T} \dashv \mathbb{U}$ is a $\mathscr{V}$-adjunction. So if we could show that the underlying model category is a $\mathscr{V}$-model category, then $\mathbb{T} \dashv \mathbb{U}$ would be a Quillen $\mathscr{V}$-adjunction. The author is not sure how to show this.

Chapter 5: Algebraic h-Model and m-Model Structures

5.1 Algebraic h-Model Structures

The h-model structure on topological spaces was first described in [Str72]. In [Col06a], it was shown that the h-model structure exists on topologically bicomplete categories when a minor condition is satisfied. In particular, this condition is satisfied in the category of k-spaces. There was a subtle mistake in Cole's proof, however, which was repeated in [MS06, §4]. This was rectified by the paper [BR13]. By proving one of the factorizations in the h-model structure is algebraic, Barthel and Riehl were able to prove that the h-model structure on a topologically bicomplete category exists when a condition, the monomorphism hypothesis, is satisfied. Riehl and Barthel did not prove that the second factorization is algebraic and so did not prove that the h-model structure is algebraic.

In this chapter, we prove that the h-model structure is algebraic. The monomorphism hypothesis is too restrictive to make this proof go through. We instead show that there are $\mathcal{E}$-compact factorizations for both of the WFSs of the h-model structure on k-spaces. So the h-model structure on k-spaces is an $\mathcal{E}$-compact algebraic model category. Not only does $\mathcal{E}$-compactness make this proof possible, but it also makes the proofs a lot easier. Verifying conditions (1)-(3) of 5.1.11 is very easy.

In the process of proving the result for the h-model structure on k-spaces, we get conditions under which the h-model structure on any topologically bicomplete category exists, is algebraic, and is $\mathcal{E}$-compact. In section 5.2, we use the $\mathcal{E}$-compactness of the algebraic h-model structure on k-spaces to show that the mixed model structure is an algebraic model structure on k-spaces.

It should be noted that it is shown in [Gau19, §4] that the q-, h-, and m-model structures on the locally presentable category of delta-generated spaces are accessible. Our results generalize Gaucher's results to more general categories of spaces.

5.1.1 Topologically Bicomplete Categories

We begin by summarizing some results about k-spaces and topologically bicomplete categories and fixing some notation. Most of these results are in [Rez18] and [MS06, §1.1, 1.2].

We begin by recalling colimits and limits in **Top**. The *final topology* on a set Y relative to a cocone $\{\theta_\alpha : X_\alpha \to Y\}_\alpha$ of spaces X_α is the finest topology that makes all of the maps θ_α continuous. The colimit of a diagram in **Top** is the space X whose underlying set is the colimit of the diagram in **Set** and whose topology is the final topology with respect to this colimiting cocone. The *initial topology* on a set Y relative to a cone $\{\theta_\alpha : Y \to X_\alpha\}_\alpha$ of spaces X_α is the coarsest topology that makes all of the maps θ_α continuous. The limit of a diagram in **Top** is the space X whose underlying set is the limit of the diagram in **Set** and whose topology is the initial topology with respect to this limiting cone.

A subset U of a topological space X is *k-open* if for every compact space K and every continuous map $f : K \to X$, $f^{-1}(U)$ is open. Of course every open subset of X is k-open. When every k-open subset of X is open, we say that X is a *k-space*. The collection of k-open subsets in a topological space X form a topology on the underlying set of X. We will use the notation kX for this new topological space. The identity map on sets is a continuous function $kX \to X$. The underlying map of sets of any continuous function $f : X \to Y$ is a continuous function $kX \to kY$, which we call kf.

Let **kTop** be the full subcategory of **Top** on the k-spaces. The subcategory inclusion functor $\mathbf{kTop} \hookrightarrow \mathbf{Top}$ is the inclusion of a coreflective subcategory. The right adjoint $k : \mathbf{Top} \to \mathbf{kTop}$ is the functor that sends spaces X to kX and maps $f : X \to Y$ to $kf : kX \to kY$.

$$\begin{tikzcd}
\mathbf{kTop} \arrow[r, hook, bend right] \arrow[r, phantom, "\top"] & \mathbf{Top} \arrow[l, bend right, "k"']
\end{tikzcd}$$

A consequence of this adjunction is that the category **kTop** is bicomplete. The colimit of a diagram in **kTop** is formed by taking the colimit of the diagram in **Set** and endowing this set with the final topology. The limit of a diagram in **kTop** is formed by taking the limit of the diagram in **Set**, endowing this set with the initial topology, and applying the functor k to this space.

We will use the notation $\times_k$ for the product of two spaces in **kTop** to avoid confusion with the product of the spaces in **Top**. If X and Y are k-spaces in **Top**, then $X \times_k Y = k(X \times Y)$. The category **kTop** is $\times_k$-cartesian closed. For a space X in **kTop**, the right adjoint to the functor $(-) \times_k X : \mathbf{kTop} \to \mathbf{kTop}$ is the functor $(-)^X : \mathbf{kTop} \to \mathbf{kTop}$ defined as follows. For spaces X and Y in **kTop**, let $\underline{\mathbf{Top}}(X, Y)$ be the set of maps $X \to Y$ in **Top** equipped with the compact-open topology. Since **kTop** is a full subcategory of **Top**, this is the same as the set of maps $X \to Y$ in **kTop**. The functor

$$(-)^X : \mathbf{kTop} \to \mathbf{kTop},$$

sends a space Y to the space $k\underline{\mathbf{Top}}(X, Y)$ and extends naturally to morphisms.

A category $\mathscr{C}$ is *topologically bicomplete* if $\mathscr{C}$ is bicomplete and is enriched, powered, and copowered over **kTop**. So we have bifunctors

$$\underline{\mathscr{C}}(-,-) : \mathscr{C}^{\text{op}} \times \mathscr{C} \to \mathbf{kTop}$$

$$(-) \otimes (-) : \mathscr{C} \times \mathbf{kTop} \to \mathscr{C}$$

$$(-)^{(-)} : \mathbf{kTop}^{\text{op}} \times \mathscr{C} \to \mathscr{C}$$

with natural bijections

$$\mathscr{C}\big(X \otimes K, Y\big) \cong \mathbf{kTop}\big(K, \underline{\mathscr{C}}(X,Y)\big) \cong \mathscr{C}\big(X, Y^K\big).$$

Some consequences of the natural bijections are that $X \otimes * \cong X$ and $Y^* \cong Y$ for the singleton space $*$ and that $\underline{\mathscr{C}}(\varnothing, X) \cong *$, $\underline{\mathscr{C}}(X, *) \cong *$, and $(*)^K \cong *$, where $\varnothing$ is the initial object of $\mathscr{C}$ and the $*$ in $\mathscr{C}$ is the terminal object of $\mathscr{C}$.

Every compact space in **Top** is a k-space. So the interval object I is a space in **kTop**. We call the objects $X \otimes I$ in a topologically bicomplete category *cylinder objects*. We will use the notation i_0 or $i_0(X)$ for the map $X \cong X \otimes * \to X \otimes I$ in $\mathscr{C}$ defined by the map $0 : * \to I$ in **kTop** whose image is 0. Similarly, the map $1 : * \to I$ whose image is 1 defines a map $i_1 : X \cong X \otimes * \to X \otimes I$ or $i_1(X)$ in $\mathscr{C}$ for each object X. Dually, we have a *cocylinder object* X^I for each object X in $\mathscr{C}$. The maps $0 : * \to I$ and $1 : * \to I$ define a restriction maps $p_0 = p_0(X) : X^I \to X^* \cong X$ and $p_1 = p_1(X) : X^I \to X^* \cong X$, respectively. Every space X has a map $\text{con} : X \to X^I$ that sends points to the constant map valued at that point and a projection map $\text{col} : X \otimes I \to X$. The adjunction $(-) \otimes I \dashv (-)^I$ has unit map $\text{coev}_X : X \to (X \otimes I)^I$ and counit map $\text{ev}_X : X^I \otimes I \to X$.

For each map $f : A \to B$ in $\mathscr{C}$, we define the *mapping cylinder* Mf by the following cocartesian square.

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "f"'] & A \otimes I \arrow[d, "s(f)"] \\
B \arrow[r, "t(f)"'] & Mf \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

For each map $f : X \to Y$, we will need two versions of the *mapping path space* Nf and $N_1 f$ defined by the cartesian squares below.

$$\begin{tikzcd}
Nf \arrow[r, "u(f)"] \arrow[d, "v(f)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y^I \arrow[d, "p_0"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
N_1 f \arrow[r, "u_1(f)"] \arrow[d, "v_1(f)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y^I \arrow[d, "p_1"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The mapping cylinder and mapping path space constructions are functorial. So M, N, and N_1 are functors $\mathscr{C}^{\mathbf{2}} \to \mathscr{C}$.

5.1.2 The h-Fibrations, h-Cofibrations, and h-Equivalences

Let $\mathscr{C}$ be a topologically bicomplete category equipped with a well-copowered, left proper, orthogonal factorization system $(\mathcal{E}, \mathcal{M})$. The *h-equivalences* in $\mathscr{C}$ are the homotopy equivalences relative to the cylinder objects $X \otimes I$. A map $f : X \to Y$ in $\mathscr{C}$ is an *h-fibration* if it has the right lifting property with respect to $i_0 : A \to A \otimes I$ for each object A in $\mathscr{C}$. A map $f : A \to B$ in $\mathscr{C}$ is a *h-cofibration* if it has the left lifting property with respect to $p_0 : X^I \to X$ for each object X in $\mathscr{C}$.

In general, the h-fibrations, h-cofibrations and h-equivalences do not have have the lifting properties required for the fibrations, cofibrations, and weak equivalences of a model category. To get the correct lifting properties, we need to use *strong cofibrations* in place of cofibrations. We can characterize these maps as follows. For each $g : X \to Y$, let $r(g) : X^I \to Ng$ be the map into the pullback defined by the maps $g^I : X^I \to X$ and $p_0(X) : X^I \to X$.

$$\begin{tikzcd}
X^I \arrow[r, dashed, "r(g)"] \arrow[rr, bend left, "g^I"] \arrow[dr, bend right, "p_0(X)"'] & Ng \arrow[r, "u(g)"] \arrow[d, "v(g)"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, "p_0(Y)"] \\
 & X \arrow[r, "g"] & Y
\end{tikzcd}$$

A *strong h-cofibration* is a map $f : A \to B$ that has the left lifting property with respect $r(g)$ for every h-fibration $g : X \to Y$.

Let $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$ be the collections of homotopy equivalences, strong h-cofibrations, and h-fibrations in $\mathscr{C}$, respectively.

Proposition 5.1.1 ([MS06, 4.3.3])**.** *$(\mathcal{C} \cap \mathcal{W})^{\square} = \mathcal{F}$, $\mathcal{C} \cap \mathcal{W} = {}^{\square}\mathcal{F}$, $\mathcal{C}^{\square} = \mathcal{F} \cap \mathcal{W}$, and $\mathcal{C} = {}^{\square}(\mathcal{F} \cap \mathcal{W})$.*

Proposition 5.1.2 ([MS06, 4.3.1])**.** *The class $\mathcal{W}$ of h-equivalences is closed under pushouts along maps in $\mathcal{C}$ and pullbacks along maps in $\mathcal{F}$.*

We define functorial factorizations (m, m_w) and (n_w, n) by the following diagrams.

$$\begin{tikzcd}
 & X \arrow[d, "i_1"] \arrow[dd, bend left, "m(f)"] & \\
X \arrow[r, "i_0"] \arrow[d, "f"] & X \otimes I \arrow[r, "\mathrm{col}"] \arrow[d, "s(f)"] & X \arrow[d, "f"] \\
Y \arrow[r, "t(f)"] \arrow[rr, bend right, "id"'] & Mf \arrow[r, dashed, "m_w(f)"] \arrow[ul, phantom, "\ulcorner", very near start] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, dashed, "n_w(f)"] \arrow[rr, bend left, "id"] \arrow[d, "f"] & N_1 f \arrow[r, "v_1(f)"] \arrow[d, "u_1(f)"] \arrow[dd, bend right, "n(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y \arrow[r, "\mathrm{con}"] & Y^I \arrow[r, "p_1"] \arrow[d, "p_0"] & Y \\
 & Y &
\end{tikzcd}$$

Proposition 5.1.3 ([MS06, 4.3.1])**.**

- *For each f, $m(f)$ is a strong h-cofibration and $m_w(f)$ is a homotopy equivalence.*
- *For each f, $n(f)$ is an h-fibration and $n_w(f)$ is a homotopy equivalence.*

5.1.3 The Construction of AWFSs

Let $(\mathrm{L}_{t1}, \mathrm{R}_1)$ be the functorial factorization defined by the following diagram.

$$\begin{tikzcd}
Y^I \arrow[d, "p_0"] & Nf \arrow[l, "u(f)"'] \arrow[r, "i_0"] \arrow[d, "v(f)"] \arrow[dl, phantom, very near start, "\llcorner"] & Nf \otimes I \arrow[r, "u(f)\otimes I"] \arrow[d, "s(v(f))"] & Y^I \otimes I \arrow[d, "\mathrm{ev}_Y"] \\
Y & X \arrow[l, "f"] \arrow[r, "\mathrm{L}_{t1} f"] \arrow[rr, bend right, "f"'] & E_{t1} f \arrow[r, dashed, "\mathrm{R}_1 f"] \arrow[ul, phantom, very near start, "\ulcorner"] & Y
\end{tikzcd}$$

Lemma 5.1.4. *The endofunctor* $\mathrm{L}_{t1} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ *is a comonad.*

Proof. Fix a space X and a map $f : Y \to Z$. There are bijective correspondences between the following sets, where the unlabeled maps are free to vary, but the diagrams must commute.

$$\left\{\text{maps } X \to Nf\right\} \cong \left\{\text{squares} \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ Z^I \arrow[r, "p_0(Z)"] & Z \end{tikzcd}\right\} \cong \left\{\text{squares} \begin{tikzcd} X \arrow[r] \arrow[d, "i_0(X)"] & Y \arrow[d, "f"] \\ X \otimes I \arrow[r] & Z \end{tikzcd}\right\}$$

In other words, there is a natural bijection

$$\mathscr{C}(X, Nf) \cong \mathscr{C}^{\mathbf{2}}(i_0(X), f).$$

So $N : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ is a right adjoint to $i_0 : \mathscr{C} \to \mathscr{C}^{\mathbf{2}}$. Thus $i_0 \circ N : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is a comonad. Since $\mathrm{L}_{t1} f$ is the pushout of $i_0(Nf)$ along $v(f)$, L_{t1} is a comonad. ☐

Lemma 5.1.5 ([BR13, 5.10])**.** $|\mathbf{Alg}_{\mathrm{R}_1}| = \mathcal{F}$

Proposition 5.1.6. *If the functorial factorization* $(\mathrm{L}_{t1}, \mathrm{R}_1)$ *is* $\mathcal{E}$*-compact, then there is a* $\mathcal{E}$*-compact AWFS* $(\mathrm{L}_t, \mathrm{R}) := \mathbb{F}_1(\mathrm{L}_{t1}, \mathrm{R}_1)$ *on* $\mathscr{C}$ *such that* $|\mathbf{Coalg}_{\mathrm{L}_t}| = \mathcal{C} \cap \mathcal{W}$ *and* $|\mathbf{Alg}_{\mathrm{R}}| = \mathcal{F}$*.*

Proof. A direct application of lemmas 5.1.4, 5.1.5, and 5.1.1 show that the conditions of proposition 3.2.19 are satisfied. By 3.3.3, the AWFS $(\mathrm{L}_t, \mathrm{R})$ is $\mathcal{E}$-compact. ☐

Let $(\mathrm{L}_1, \mathrm{R}_{t1})$ be the functorial factorization defined by the following diagram.

$$\begin{tikzcd}
N_1 f \arrow[r, "m(n(f))"] \arrow[d, "v_1(f)"] & M(n(f)) \arrow[d] \arrow[dr, bend left, "m_w(n(f))"] & \\
X \arrow[r, "\mathrm{L}_1 f"] \arrow[rr, bend right, "f"'] & E_1 f \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "\mathrm{R}_{t1} f"] & Y
\end{tikzcd} \tag{5.1}$$

Lemma 5.1.7. *The endofunctor* $\mathrm{L}_1 : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ *is a comonad on* $\mathscr{C}^{\mathbf{2}}$*.*

Proof. Fix maps $f : A \to B$ and $g : X \to Y$ in $\mathscr{C}$. There are bijections between the following sets, where the unlabeled maps are free to vary within the constraint that the diagrams commute.

$$\left\{\text{maps } m(f) \to g\right\} \cong \left\{\text{diagrams } \begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "f"] & A \otimes I \arrow[d] & A \arrow[l, "i_1"'] \arrow[d] \\
B \arrow[r] & Y & X \arrow[l, "g"']
\end{tikzcd}\right\} \cong$$

$$\left\{\text{diagrams } \begin{tikzcd}
B \arrow[d] & A \arrow[l, "f"'] \arrow[r] \arrow[d] & X \arrow[d, "g"] \\
Y & Y^I \arrow[l, "p_0"'] \arrow[r, "p_1"] & Y
\end{tikzcd}\right\} \cong \{\text{maps } f \to n(g)\}$$

So there is a natural bijection $\mathscr{C}^{\mathbf{2}}(m(f), g) \cong \mathscr{C}^{\mathbf{2}}(f, n(g))$. Thus m is a left adjoint to n. Therefore $m \circ n : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is a comonad. Since the pushout of a comonad is a comonad, L_1 is a comonad. ☐

Lemma 5.1.8. *For every map f in $\mathscr{C}$, $\mathrm{L}_1 f$ is a strong h-cofibration and $\mathrm{R}_{t1} f$ is a homotopy equivalence.*

Proof. By lemma 5.1.3, $m(n(f))$ is a strong h-cofibration. By lemma 5.1.1, $\mathcal{C}$ is closed under pushouts. Thus $\mathrm{L}_1 f$ is a strong h-cofibration.

Since $v_1(f)$ is a homotopy equivalence, the map $M(n(f)) \to E_1 f$ in diagram (5.1) is a homotopy equivalence by lemma 5.1.2. From lemma 5.1.3 we know $m_w(n(f))$ is a homotopy equivalence. Therefore $\mathrm{R}_{t1} f$ is a homotopy equivalence. ☐

Theorem 5.1.9. *If the factorizations* $(\mathrm{L}_1, \mathrm{R}_{t1})$ *and* $(\mathrm{L}_{t1}, \mathrm{R}_1)$ *are both* $\mathcal{E}$*-compact, then there is an* $\mathcal{E}$*-compact algebraic model category* $\zeta : (\mathrm{C}_t, \mathrm{F}) \to (\mathrm{C}, \mathrm{F}_t)$ *on* $\mathscr{C}$ *whose underlying model category has weak equivalences* $\mathcal{W}$*, cofibrations* $\mathcal{C}$*, and fibrations* $\mathcal{F}$*.*

Proof. We will use a similar approach to that of 3.3.4. Let $\mathrm{X} = (\mathrm{L}_t, \mathrm{R})$ be the $\mathcal{E}$-compact AWFS from proposition 5.1.6 and let $\mathrm{Y} = (\mathrm{L}_1, \mathrm{R}_{t1})$. By 3.2.1, $(\mathbf{LAWFS}(\mathscr{C}), \circledast, \mathtt{I})$ is a strict monoidal category. Since there is a unit map $\eta : \mathtt{I} \to \mathrm{Y}$ in $\mathbf{LAWFS}(\mathscr{C})$, $\mathrm{X} \circledast \eta : \mathrm{X} = \mathrm{X} \circledast \mathtt{I} \to \mathrm{X} \circledast \mathrm{Y}$ is a map in $\mathbf{LAWFS}(\mathscr{C})$. By 3.3.2, $\mathrm{X} \circledast \eta : \mathrm{X} \to \mathrm{X} \circledast \mathrm{Y}$ is a map of LAWFSs between $\mathcal{E}$-compact LAWFSs. Thus by 3.3.3, $\mathbb{F}_1(\mathrm{X} \circledast \eta) : \mathbb{F}_1(\mathrm{X}) \to \mathbb{F}_1(\mathrm{X} \circledast \mathrm{Y})$ is a map of AWFSs between $\mathcal{E}$-compact AWFSs.

Explicitly, on f, $X \circledast Y(f) = (\mathrm{L}_t \mathrm{R}_{t1} f \circ \mathrm{L}_1 f, \mathrm{R}\mathrm{R}_{t1} f)$. By lemma 5.1.8 and proposition 5.1.6, $\mathrm{L}_t \mathrm{R}_{t1} f \circ \mathrm{L}_1 f \in \mathcal{C}$ and $\mathrm{R}\mathrm{R}_{t1} f \in \mathcal{F} \cap \mathcal{W}$ for each f. Thus, by 3.1.4, $|\mathbf{Alg}_{\mathrm{R}\mathrm{R}_{t1}}| = \mathcal{F} \cap \mathcal{W}$. Let $(\mathrm{C}_t, \mathrm{F}) = \mathbb{F}_1(X)$, let $(\mathrm{C}, \mathrm{F}_t) = \mathbb{F}_1(X \circledast Y)$, and let $\zeta := \mathbb{F}_1(X \circledast \eta) : (\mathrm{C}_t, \mathrm{F}) \to (\mathrm{C}, \mathrm{F}_t)$. By 3.2.19, $(\mathrm{C}, \mathrm{F}_t)$ is an AWFS with $|\mathbf{Coalg}_{\mathrm{C}}| = \mathcal{C}$ and $|\mathbf{Alg}_{\mathrm{F}_t}| = \mathcal{F} \cap \mathcal{W}$ and $(\mathrm{C}_t, \mathrm{F})$ is an AWFS with $|\mathbf{Coalg}_{\mathrm{C}_t}| = \mathcal{C} \cap \mathcal{W}$ and $|\mathbf{Alg}_{\mathrm{F}}| = \mathcal{F}$. □

5.1.4 The Compactness Condition

Proposition 5.1.10. *If the comonad* $i_0 \circ N : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ *is* $\mathcal{E}$*-compact, then the factorizations* $(\mathrm{L}_1, \mathrm{R}_{t1})$ *and* $(\mathrm{L}_{t1}, \mathrm{R}_1)$ *are both* $\mathcal{E}$*-compact.*

Proof. Let λ be a regular cardinal such that $i_0 \circ N$ is $(\mathcal{E}, \lambda)$-compact. Let $\{(c_\alpha, d_\alpha) : f_\alpha \to g\}_\alpha$ be an $\mathcal{E}_{/\cong}$-tight $(\mathcal{V}_{/\cong}, \lambda)$-cocone. Let $(c, d) : \operatorname{colim}_\alpha f_\alpha \to g$ be the map defined by the cocone. The cocone $\{N(c_\alpha, d_\alpha) : N f_\alpha \to N g\}_\alpha$ defines a map $b : \operatorname{colim}_\alpha N f_\alpha \to N g$.

From the $(\mathcal{E}, \lambda)$-compactness of $i_0 \circ N$, we know each of the vertical maps in the following diagram are in $\mathcal{E}$.

$$\begin{tikzcd}
\operatorname{colim}_\alpha \operatorname{dom} f_\alpha \arrow[d, "c"] & \operatorname{colim}_\alpha N f_\alpha \arrow[l, "\operatorname{colim}_\alpha v(f_\alpha)"'] \arrow[r, "i_0(\operatorname{colim}_\alpha N f_\alpha)"] \arrow[d, "b"] & (\operatorname{colim}_\alpha N f_\alpha) \otimes I \arrow[d, "b \otimes I"] \\
\operatorname{dom} g & Ng \arrow[l, "v(g)"'] \arrow[r, "i_0(Ng)"] & Ng \otimes I
\end{tikzcd}$$

Since colimits commute, the colimit of the top row is $\operatorname{colim}_\alpha E_{t1} f_\alpha$. The colimit of the bottom row is $E_{t1} g$ and the map $\operatorname{colim}_\alpha E_{t1} f_\alpha \to E_{t1} g$ defined by the above diagram is in $\mathcal{E}$ by 2.4.3. So $\operatorname{colim}_\alpha \mathrm{L}_{t1} f_\alpha \to \mathrm{L}_{t1} g$ is in $\mathcal{E}^{\mathbf{2}}$. Thus $(\mathrm{L}_{t1}, \mathrm{R}_1)$ is $\mathcal{E}$-compact.

If we can show that $m \circ n : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is $(\mathcal{E}, \lambda)$-compact, then the same method we used above will prove $(\mathrm{L}_1, \mathrm{R}_{t1})$ is $(\mathcal{E}, \lambda)$-compact. It suffices to prove the following.

1. $n : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is strongly $(\mathcal{E}, \lambda)$-compact.

2. $m : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is $(\mathcal{E}, \lambda)$-compact.

Note that there is a homeomorphism $\varsigma : I \xrightarrow{\cong} I$ such that $p_0 \circ Y^\varsigma : Y^I \to Y$ is equal to $p_1 : Y^I \to Y$. So there is a natural isomorphism $N_1 \cong N$. Therefore $N_1 : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ sends $\mathcal{E}_{/\cong}$-tight $(\mathcal{V}_{/\cong}, \lambda)$-cocones to $\mathcal{E}$-tight cocones. But, since n is codomain-preserving, that is all that is required to show (1).

Let $*$ be the terminal object in $\mathscr{C}$. As we noted in section 5.1.1, $*^I \cong *$. So if $x : X \to *$ is the map to the terminal object, then $Nx \cong X$. Let $x_\alpha : \operatorname{dom} f_\alpha \to *$ and $y : \operatorname{dom} g \to *$ be the maps to the terminal object from the domains of the cocone $\{(c_\alpha, d_\alpha) : f_\alpha \to g\}_\alpha$. So $\{(c_\alpha, id_*) : x_\alpha \to y\}_\alpha$ is an $\mathcal{E}_{/\cong}$-tight $(\mathcal{V}_{/\cong}, \lambda)$-cocone. But applying $i_0 \circ N$ to this cocone yields the same cocone as applying $i_0 \circ \operatorname{dom}$ to the original cocone $\{(c_\alpha, d_\alpha) : f_\alpha \to g\}_\alpha$. In other words, $i_0 \circ \operatorname{dom} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is an $(\mathcal{E}, \lambda)$-compact functor. Applying the same method we used in the second paragraph of this proof to $i_0 \circ \operatorname{dom}$ proves that $M : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ sends

$\mathcal{E}_{/\cong}$-tight $(\mathcal{V}_{/\cong},\lambda)$-cocones to $\mathcal{E}$-tight cocones. Since $m : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is domain-preserving, (2) now follows. $\square$

Proposition 5.1.11. *Let λ be a regular cardinal. If the following conditions are satisfied, then $i_0 \circ N : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is an $(\mathcal{E},\lambda)$-compact comonad.*

1. *The functor $(-) \otimes I : \mathscr{C} \to \mathscr{C}$ sends every map in $\mathcal{E}$ to a map in $\mathcal{E}$.*
2. *If $f \in \mathcal{E}$ and g and h are isomorphisms, then the limit in $\mathscr{C}^{\mathbf{2}}$ of any diagram $f \to h \leftarrow g$ is in $\mathscr{E}$.*
3. *For every diagram $D : \mathscr{A} \times \mathbf{in} \to \mathscr{C}$ on a λ-filtered category $\mathscr{A}$, the canonical map*
$$\operatorname*{colim}_{\alpha} \lim_{\beta} D(\alpha,\beta) \longrightarrow \lim_{\beta} \operatorname*{colim}_{\alpha} D(\alpha,\beta)$$
is in $\mathcal{E}$.

Remark 5.1.12. By proposition 4.1.2, condition (1) is equivalent to the requirement that the functor $(-)^I : \mathscr{C} \to \mathscr{C}$ sends every map in $\mathcal{M}$ to a map in $\mathcal{M}$. If $\mathcal{E} = \mathcal{E}^{\downarrow}$, the collection of epimorphisms, then (1) holds by lemma 4.1.4. If in addition, $\mathscr{C}$ is a concrete category with a realization functor $\mathscr{C} \to \mathbf{Set}$ that preserves limits, then (2) also holds.

Proof. Let $\{(c_\alpha, d_\alpha) : f_\alpha \to g\}_\alpha$ be an $\mathcal{E}_{/\cong}$-tight $(\mathcal{V}_{/\cong},\lambda)$-cocone. Let $(c,d) : \mathrm{colim}_\alpha f_\alpha \to g$ be the map defined by the cocone. Since $d_\alpha : \mathrm{cod}\ f_\alpha \to \mathrm{cod}\ g$ is an isomorphism for each α, each map $d_\alpha^I : (\mathrm{cod}\ f_\alpha)^I \to (\mathrm{cod}\ g)^I$ is an isomorphism. Therefore the map $b : \mathrm{colim}_\alpha (\mathrm{cod}\ f_\alpha)^I \to (\mathrm{cod}\ g)^I$ defined by the cocone $\{d_\alpha^I : (\mathrm{cod}\ f_\alpha)^I \to (\mathrm{cod}\ g)^I\}_\alpha$ is an isomorphism. So, in the following diagram, $c \in \mathcal{E}$ and d and b are isomorphisms.

$$\begin{tikzcd}
\mathrm{colim}_\alpha (\mathrm{cod}\ f_\alpha)^I \arrow[r, "\mathrm{colim}_\alpha p_0"] \arrow[d, "b", "\cong"'] & \mathrm{colim}_\alpha \mathrm{cod}\ f_\alpha \arrow[d, "d", "\cong"'] & \mathrm{colim}_\alpha \mathrm{dom}\ f_\alpha \arrow[l, "\mathrm{colim}_\alpha f_\alpha"'] \arrow[d, "c"] \\
(\mathrm{cod}\ g)^I \arrow[r, "p_0"] & \mathrm{cod}\ g & \mathrm{dom}\ g \arrow[l, "g"']
\end{tikzcd}$$

Let P be the limit of the top row. By (2), the map $P \to Ng$ induced by the above diagram is in $\mathcal{E}$. By (3), the map $\operatorname{colim}_\alpha Nf_\alpha \to P$ is in $\mathcal{E}$. Therefore the map $\operatorname{colim}_\alpha Nf_\alpha \to Ng$ defined by the cocone $\{N(c_\alpha, d_\alpha) : Nf_\alpha \to Ng\}_\alpha$ is in $\mathcal{E}$. Since $(-)\otimes I$ preserves colimits and maps in $\mathcal{E}$, the map $\operatorname{colim}_\alpha(Nf_\alpha \otimes I) \to Ng \otimes I$ defined by the cocone $\{N(c_\alpha, d_\alpha) \otimes I : Nf_\alpha \otimes I \to Ng \otimes I\}$ is in $\mathcal{E}$. $\square$

5.1.5 An Algebraic h-Model Structure on k-Spaces

The epimorphisms in **kTop** are exactly the (continuous) surjective maps. Indeed, if f is an epimorphism, one only needs to consider the cocartesian square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, "g"] \\
Y \arrow[r, "h"] \arrow[ur, phantom, "\ulcorner", very near end] & Z.
\end{tikzcd}$$

If $y \in Y$ is not in the image of f, then $g(y) \neq h(y)$, so $g \neq h$ and thus $gf \neq hf$, a contradiction. Since the spaces themselves are sets and the possible topologies on a set are limited by its cardinality, there can only be a set's worth of isomorphism classes of epimorphism quotients for each object. So by 2.4.9, $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail})$ is a well-copowered, left-proper, orthogonal factorization system on **kTop**.

Definition 5.1.13. An injective map $f : X \to Y$ in the category **kTop** is a *k-inclusion* if $X = k(f(X))$, where $f(X)$ has the same underlying set as X, but has the subspace topology from Y.

We can of course factor each map $f : X \to Y$ as a map $\hat{f} : X \to kf(X)$ in $\mathcal{E}$ followed by a k-inclusion $i : kf(X) \to Y$.

Proposition 5.1.14. *The collection $\mathcal{M}^{s\downarrowtail}$ of strong monomorphisms in* **kTop** *is the collection of k-inclusions.*

Proof. We will show that the class of k-inclusions is equal to $(\mathcal{E}^{\twoheaddownarrow})^{\square}$. Let $f : X \to Y$ be a k-inclusion, let $p : A \to B$ be a map in $\mathcal{E}^{\twoheaddownarrow}$, and let $(u, v) : p \to f$ be a map in $\mathbf{kTop^2}$. Since the image of v is contained in the image of f, v lifts to a map $v' : B \to f(X)$. For every k-open set U in $f(X)$ and every continuous map $l : L \to B$ on a compact space L, $l^{-1}(v'^{-1}(U))$ is open. So $v'^{-1}(U)$ is a k-open set in B. Thus the map of sets defined by v' is continuous as a map to $kf(X)$. So a solution to the lifting problem $(u, v) : p \to f$ exists.

Conversely, suppose $f : X \to Y$ is a map in $(\mathcal{E}^{\twoheaddownarrow})^{\square}$. Then a lift l exists in the following diagram, where $\hat{f} \in \mathcal{E}^{\twoheaddownarrow}$ and i is a k-inclusion.

$$\begin{tikzcd}
X \arrow[r, "id"] \arrow[d, "\hat{f}"] & X \arrow[d, "f"] \\
kf(X) \arrow[r, "i"] \arrow[ur, dashed, "l"] & Y
\end{tikzcd}$$

So $\hat{f}$ is a bijection and $l \circ \hat{f} = id$. Therefore $\hat{f} \circ l = id$. Since f is isomorphic to i, f must be a k-inclusion. □

Theorem 5.1.15 (The Algebraic h-Model Structure on k-Spaces)**.** *There is an $\mathcal{E}$-compact algebraic model category $q : (\mathrm{C}_t, \mathrm{F}) \to (\mathrm{C}, \mathrm{F}_t)$ on* **kTop** *whose underlying model category has weak equivalences $\mathcal{W}$, cofibrations $\mathcal{C}$, and fibrations $\mathcal{F}$.*

Proof. We will prove conditions (1)-(3) of 5.1.11 hold. Since $(-) \otimes I$ is a left adjoint, it preserves epimorphisms (4.1.4). So (1) holds.

Let $f \in \mathcal{E}$, let g and h be isomorphisms, and let $(x, y) : f \to h$ and $(u, v) : g \to h$ be maps in $\mathbf{kTop^2}$. The functor $k : \mathbf{Top} \to \mathbf{kTop}$ only changes the topology of each space, but does not change the underlying set or map of sets. Since the map $f \times_h g : \operatorname{dom} f \times_{\operatorname{dom} h} \operatorname{dom} g \to \operatorname{cod} f \times_{\operatorname{cod} h} \operatorname{cod} g$ is surjective, the map $k(f \times_h g)$ must also be surjective. So condition (2) holds.

Finally, let $D : \mathscr{A} \times \mathbf{in} \to \mathscr{C}$ be a diagram of the shape described in (3). Since finite limits commute with filtered colimits in **Set**, the map

$$\operatorname*{colim}_{\alpha} \lim_{\beta} D(\alpha, \beta) \longrightarrow \lim_{\beta} \operatorname*{colim}_{\alpha} D(\alpha, \beta)$$

in **kTop** is bijective. So in particular, it is in $\mathcal{E}$. The above map is in general not an isomorphism in **kTop**. □

The above proof is more elementary than [Col06a, §4] and [Lew78, 9.5], showing the benefits of using $\mathcal{E}$-compactness.

5.2 Mixed Algebraic Model Structures

It was shown in [Col06b, 2.1] that when a category has two model structures, one of which has larger classes of both weak equivalences and fibrations, then there is a third model structure on the category whose weak equivalences are from the larger class and whose fibrations are from the smaller class. The third model structure is called the *mixed model structure.*

When we mix the h-model structure on k-spaces with the classical Quillen model structure (or q-model structure), we get a new model structure which we call the *m-model structure.* The weak equivalences of the m-model structure are the weak homotopy equivalences and the fibrations are the h-fibrations. We will show that the m-model structure is an algebraic model structure. Unfortunately, we are not able to show it is $\mathcal{E}$-compact.

5.2.1 Mixing Model Structures

Let $\mathscr{C}$ be a bicomplete category with well-copowered proper orthogonal factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$. Suppose $\zeta : (\mathrm{C}_t, \mathrm{F}) \to (\mathrm{C}, \mathrm{F}_t)$ is an $\mathcal{E}$-compact algebraic model category on $\mathscr{C}$ with weak equivalences $\mathcal{W}_1$. We will use the notation $\mathcal{C}_1 = |\mathbf{Coalg}_{\mathrm{C}}|$ and

$\mathcal{F}_1 = |\mathbf{Alg}_{\mathrm{F}}|$. Suppose there is a second model category on $\mathscr{C}$ with weak equivalences $\mathcal{W}_2$, fibrations $\mathcal{F}_2$, and cofibrations $\mathcal{C}_2$ such that $\mathcal{W}_1 \subseteq \mathcal{W}_2$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

Theorem 5.2.1 (Mixed Model Structure)**.** *If there is an* $(\mathcal{E}, \mathcal{M}')$*-compact LAWFS* $(\mathrm{L}, \mathrm{R}_t)$ *on* $\mathscr{C}$ *such that* $|\mathbf{Coalg}_{\mathrm{L}}| \subseteq \mathcal{C}_2$ *and* $|\mathbf{Alg}_{\mathrm{R}_t}| \subseteq \mathcal{F}_2 \cap \mathcal{W}_2$*, then there is an algebraic model category* $\rho : (\mathrm{X}_t, \mathrm{Y}) \to (\mathrm{X}, \mathrm{Y}_t)$ *on* $\mathscr{C}$ *with weak equivalences* $\mathcal{W}_2$ *such that* $|\mathbf{Alg}_{\mathrm{Y}}| = \mathcal{F}_1$*.*

Proof. Let $\mathcal{C}_m = {}^{\square}(\mathcal{W}_2 \cap \mathcal{F}_1)$. If $f \in \mathcal{C}_m \cap \mathcal{W}_2$, then $\mathrm{F}f \in \mathcal{W}_2 \cap \mathcal{F}_1$. So a solution to the lifting problem $(\mathrm{C}_t f, id) : f \to \mathrm{F}f$ exists. Thus $f \in |\mathbf{Coalg}_{\mathrm{C}_t}| = \mathcal{C}_1 \cap \mathcal{W}_1$. Conversely, if $f \in \mathcal{C}_1 \cap \mathcal{W}_1 = |\mathbf{Coalg}_{\mathrm{C}_t}|$, then $f \in \mathcal{W}_2$ and a solution to any lifting problem $(u, v) : f \to g$ with $g \in \mathcal{F}_1$ exists. So $f \in \mathcal{C}_m \cap \mathcal{W}_2$. Since $(\mathcal{C}_1 \cap \mathcal{W}_1, \mathcal{F}_1)$ is a weak factorization system, $(\mathcal{C}_m \cap \mathcal{W}_2, \mathcal{F}_1)$ is a weak factorization system with an associated $\mathcal{E}$-compact AWFS $(\mathrm{C}_t, \mathrm{F})$.

Let $(\mathrm{X}_1, \mathrm{Y}_{t1}) = (\mathrm{C}_t, \mathrm{F}) \circledast (\mathrm{L}, \mathrm{R}_t)$. So $(\mathrm{X}_1, \mathrm{Y}_{t1})$ is functorial factorization with $\mathrm{X}_1 f = \mathrm{C}_t \mathrm{R}_t f \circ \mathrm{L} f$ on each object f in $\mathscr{C}^{\mathbf{2}}$. By 3.3.1, $(\mathrm{X}_1, \mathrm{Y}_{t1})$ is an $(\mathcal{E}, \mathcal{M}')$-compact LAWFS. Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$, we know $|\mathbf{Coalg}_{\mathrm{L}}| \subseteq \mathcal{C}_2 = {}^{\square}(\mathcal{F}_2 \cap \mathcal{W}_2) \subseteq {}^{\square}(\mathcal{F}_1 \cap \mathcal{W}_2) = \mathcal{C}_m$. Therefore $\mathrm{X}_1 f \in \mathcal{C}_m$ and thus $|\mathbf{Coalg}_{\mathrm{X}_1}| \subseteq \mathcal{C}_m$. Since $\mathrm{Y}_{t1} f = \mathrm{F}\mathrm{R}_t f$ and the unit map for $\mathrm{F}\mathrm{R}_t$ is $F\vec{\eta}^{\mathrm{R}_t} \circ \vec{\eta}^{\mathrm{F}} = \vec{\eta}^{F} \mathrm{R}_t \circ \vec{\eta}^{\mathrm{R}_t}$, every Y_{t1}-algebra is both an F-algebra and an R_t-algebra. So $|\mathbf{Alg}_{\mathrm{Y}_{t1}}| \subseteq |\mathbf{Alg}_{\mathrm{F}}| \cap |\mathbf{Alg}_{\mathrm{R}_t}| = \mathcal{F}_1 \cap \mathcal{W}_2$.

Since the unit map $\eta : \mathtt{I} \to (\mathrm{L}, \mathrm{R}_t)$ is a map in $\mathbf{LAWFS}(\mathscr{C})$, by 3.2.1, the map

$$(\mathrm{C}_t, \mathrm{F}) \circledast \eta : (\mathrm{C}_t, \mathrm{F}) = (\mathrm{C}_t, \mathrm{F}) \circledast \mathtt{I} \to (\mathrm{C}_t, \mathrm{F}) \circledast (\mathrm{L}, \mathrm{R}_t) = (\mathrm{X}_1, \mathrm{Y}_{t1})$$

is a map in $\mathbf{LAWFS}(\mathscr{C})$. Since $(\mathrm{C}_t, \mathrm{F})$ is $\mathcal{E}$-compact and $(\mathrm{X}_1, \mathrm{Y}_{t1})$ is $(\mathcal{E}, \mathcal{M}')$-compact, both LAWFSs have reflections in $\mathbf{AWFS}(\mathscr{C})$. Let $(\mathrm{X}_t, \mathrm{Y}) = \mathbb{F}_1((\mathrm{C}_t, \mathrm{F}))$ and let $(\mathrm{X}, \mathrm{Y}_t) = \mathbb{F}_1((\mathrm{X}_1, \mathrm{Y}_{t1}))$. It follows that the reflection of the map $(\mathrm{C}_t, \mathrm{F}) \circledast \eta$ in $\mathbf{AWFS}(\mathscr{C})$ exists. So $\rho := \mathbb{F}_1((\mathrm{C}_t, \mathrm{F}) \circledast \eta) : (\mathrm{X}_t, \mathrm{Y}) \to (\mathrm{X}, \mathrm{Y}_t)$ is a map in $\mathbf{AWFS}(\mathscr{C})$.

Since $(\mathrm{C}_t, \mathrm{F})$ is a LAWFS associated to the weak factorization system $(\mathcal{C}_m \cap \mathcal{W}_2, \mathcal{F}_1)$, by 3.2.19 (3.1.4), $(\mathrm{X}_t, \mathrm{Y})$ is an AWFS associated to the weak factorization system $(\mathcal{C}_m \cap \mathcal{W}_2, \mathcal{F}_1)$.

Since $(\mathrm{X}_1, \mathrm{Y}_{t1})$ is a LAWFS with $|\mathbf{Coalg}_{\mathrm{X}_1}| \subseteq \mathcal{C}_m$ and $|\mathbf{Alg}_{\mathrm{Y}_{t1}} \subseteq \mathcal{F}_1 \cap \mathcal{W}_2$, by 3.1.4 and 3.2.19, $(\mathrm{X}, \mathrm{Y}_t)$ is an AWFS associated to the weak factorization system $(\mathcal{C}_m, \mathcal{F}_1 \cap \mathcal{W}_2)$. □

Corollary 5.2.2. *If the LAWFS* $(\mathrm{L}, \mathrm{R}_t)$ *of theorem 5.2.1 is an* $\mathcal{E}$*-compact, then the algebraic model category* $\rho : (\mathrm{X}_t, \mathrm{Y}) \to (\mathrm{X}, \mathrm{Y}_t)$ *is* $\mathcal{E}$*-compact.*

5.2.2 An Algebraic m-Model Structure on k-Spaces

We now return to the category **kTop** of k-spaces. As we saw, **kTop** is a bicomplete closed monoidal category with a well-copowered proper orthogonal factorization system $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail})$. We will now use the notation $\mathcal{W}_h$, $\mathcal{F}_h$, and $\mathcal{C}_h$ for the homotopy equivalences, h-fibrations, and strong h-cofibrations in **kTop**, respectively.

Let $\mathcal{W}_q$ be the collection of weak homotopy equivalences in **kTop**, let $\mathcal{F}_q$ be the collection of Serre fibrations in **kTop**, and let $\mathcal{C}_q$ be the collection of retracts of inclusions of cell complexes in **kTop**. By [Hov99, 2.4.23], **kTop** is a model category with weak equivalences $\mathcal{W}_q$, fibrations $\mathcal{F}_q$, and cofibrations $\mathcal{C}_q$. We will refer to this model structure as the *q-model structure* on **kTop**.

Since $\mathcal{W}_h \subseteq \mathcal{W}_q$ and $\mathcal{F}_h \subseteq \mathcal{F}_q$, the h- and q-model structures on **kTop** mix to produce a model structure on **kTop** with weak equivalences $\mathcal{W}_q$, fibrations $\mathcal{F}_h$, and cofibrations $\mathcal{C}_m := {}^{\square}(\mathcal{F}_h \cap \mathcal{W}_q)$ ([Col06b, 2.1]). We will show that this model structure has an associated algebraic model structure. First we need the following lemma.

Lemma 5.2.3. *For each space* X *in* **kTop***, there is a regular cardinal* λ *such that the functor*

$$\mathbf{kTop}(X, -) : \mathbf{kTop} \to \mathbf{Set}$$

preserves $\mathcal{E}^{\twoheaddownarrow}$*-tightness of* $(\mathcal{M}^{s\downarrowtail}, \lambda)$*-cocones.*

Proof. Let X be a space in **kTop** and let λ be a regular cardinal greater than $|X|$. Let $\{i_\alpha : Y_\alpha \to Y\}$ be an $\mathcal{E}^{\twoheaddownarrow}$-tight $(\mathcal{M}^{s\downarrowtail}, \lambda)$-cocone in **kTop**. Let f be a map $X \to Y$. Every point in X is in the image of an i_α. So there is a collection of fewer than λ distinct Y_α whose images under the i_α's collectively contain the image of f. Since $\{i_\alpha : Y_\alpha \to Y\}$ is λ-filtered, there is a single Y_β such that the image of f is contained in the image of i_β. Because $i_\beta(Y_\beta)$ has the subspace topology, $f : X \to Y$ lifts to a continuous map $f' : X \to i_\beta(Y_\beta)$. For every k-open set U in $i_\beta(Y_\beta)$, the set $f'^{-1}(U)$ is k-open in X. So f' lifts to a map $X \to Y_\beta$. Thus the induced map

$$\operatorname*{colim}_{\alpha} \mathbf{kTop}(X, Y_\alpha) \to \mathbf{kTop}(X, Y)$$

is a surjection. $\square$

Theorem 5.2.4 (Algebraic m-Model Structure)**.** *There is an algebraic model category* $\theta : (\mathrm{X}_t, \mathrm{Y}) \to (\mathrm{X}, \mathrm{Y}_t)$ *on* $\mathbf{kTop}$ *with weak equivalences* $\mathcal{W}_q$ *such that* $|\mathbf{Coalg}_{\mathrm{X}}| = \mathcal{C}_m$ *and* $|\mathbf{Alg}_{\mathrm{Y}}| = \mathcal{F}_h$.

Proof. By 5.2.1, it suffices to show that there is an $(\mathcal{E}^{⤈}, \mathcal{M}^{s↧})$-compact LAWFS $(\mathrm{L}_1^I, \mathrm{R}_1^I)$ such that $|\mathbf{Coalg}_{\mathrm{L}_1^I}| \subseteq \mathcal{C}_q$ and $|\mathbf{Alg}_{\mathrm{R}_1^I}| \subseteq \mathcal{F}_q \cap \mathcal{W}_q$.

Let $\mathcal{I}$ be the collection of boundary inclusions $S^{n-1} \to D^n$ for all n, where D^n is the n-disk and S^n is the n-sphere with $S^{-1} = \varnothing$. Then $\mathcal{I}^{\square} = \mathcal{F}_q \cap \mathcal{W}_q$. Let $\mathscr{I} = \mathbf{Disc}(\mathcal{I})$, viewed as a discrete subcategory of $\mathbf{kTop^2}$, and let $I : \mathscr{I} \to \mathbf{kTop^2}$ be the subcategory inclusion functor. By 5.2.3 and 3.2.27, the reflection $\mathrm{L}_0^I = \mathbb{F}_3(I)$ of J and I in $\mathbf{Cmd(kTop^2)}$ exists and is $(\mathcal{E}^{⤈}, \mathcal{M}^{s↧})$-compact. By 3.2.22, the reflection $(\mathrm{L}_1^I, \mathrm{R}_1^I) = \mathbb{F}_2(\mathrm{L}_0^I)$ in $\mathbf{LAWFS(kTop)}$ exists and is $(\mathcal{E}^{⤈}, \mathcal{M}^{s↧})$-compact.

By 3.2.26, 3.2.23, and 3.1.15, $I^{\boxslash} \cong \mathbf{Coalg}_{\mathrm{L}_0^I}^{\mathrm{EM}\boxslash} \cong \mathbf{Coalg}_{\mathrm{L}_1^I}^{\mathrm{EM}\boxslash} \cong \mathbf{Alg}_{\mathrm{R}_1^I}$. But we also know $|I^{\boxslash}| = \mathcal{I}^{\square} = \mathcal{F}_q \cap \mathcal{W}_q$. So $|\mathbf{Alg}_{\mathrm{R}_1^I}| = |I^{\boxslash}| = \mathcal{F}_q \cap \mathcal{W}_q$. Since $\mathbf{Coalg}_{\mathrm{L}_1^I}^{\mathrm{EM}}$ is a subcategory of ${}^{\boxslash}(\mathbf{Coalg}_{\mathrm{L}_1^I}^{\mathrm{EM}\boxslash})$, $|\mathbf{Coalg}_{\mathrm{L}_1^I}^{\mathrm{EM}}| \subseteq |{}^{\boxslash}(J^{\boxslash})| \subseteq {}^{\square}|J^{\boxslash}| = {}^{\square}(\mathcal{F}_q \cap \mathcal{W}_q) = \mathcal{C}_q$. Since $\mathcal{C}_q$ is retract closed, $|\mathbf{Coalg}_{\mathrm{L}_1^I}| \subseteq \mathcal{C}_q$. $\square$

Remark 5.2.5. If for each X, there is a regular cardinal λ such that the functor $\mathbf{kTop}(X, -) : \mathbf{kTop} \to \mathbf{Set}$ preserves $\mathcal{E}^{\downarrow}$-tightness of λ-filtered cocones then the m-model structure will be $\mathcal{E}^{\downarrow}$-compact.

Chapter 6: Quasiaccessible Categories

Quasiaccessible categories both generalize locally presentable categories and include the category of topological spaces and the category of k-spaces. An impressive amount of the theory of accessible categories still applies to quasiaccessible categories.

Showing that when L is a “small” copointed endofunctor on **Top**, the forgetful functor $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathbf{Top}$ has a right adjoint was one of the motivating problems for developing the theory of quasiaccessible categories. By the special adjoint functor theorem, the only obstruction to showing the right adjoint to U_{L} exists is showing that the category $\mathbf{Coalg}_{\mathrm{L}}$ has a family of generators. This result would be true if $\mathbf{Coalg}_{\mathrm{L}}$ were in some sense accessible “up to epimorphisms”.

Being accessible “up to epimorphisms” is roughly the requirement for a category to be quasiaccessible. Even though **Set** is accessible and colimits in **Top** are found by topologizing the colimit in **Set**, the topologies on the spaces in **Top** prevent the category from being accessible. It is, however, easy to show that **Top** is accessibility “up to epimorphisms”.

Once we have the sought after result 6.2.20 for quasiaccessible copointed endofunctors, we are able to prove that every quasiaccessible functorial factorization associated to a WFS $(\mathcal{L}, \mathcal{R})$ can be replaced by an associated compact LAWFS. So $(\mathcal{L}, \mathcal{R})$ has an associated AWFS. Thus every quasiaccessible model category can be given the structure of an algebraic model category.

Since, under mild assumptions, the Bousfield-Friedlander theorem outputs a localized quasiaccessible model category when the input is a quasiaccessible model category, one application of our results is a proof that the localized model category output by the Bousfield-Friedlander theorem can be given the structure of an algebraic model category.

In section 6.3.3, we prove that the h-model structure on **Top** is quasiaccessible. We have not yet been able to show that the h-model structure on k-spaces is quasiaccessible. The author believes it may be possible to modify the definition of quasiaccessible functors so that all the proofs go through, but so that it is also possible to prove that the h-model structure on k-spaces is quasiaccessible.

Many of the theorems in this chapter are analogs of results in [AR94] for locally presentable and accessible categories. This material bears an even stronger resemblance to the equivalent locally generated categories of [GU71] which are summarized in [AR94, §1.E].

6.1 Quasiaccessible Categories

6.1.1 Presentable Objects

Recall, in a given category, $\mathcal{E}^{\twoheaddownarrow}$, $\mathcal{E}^{s\twoheaddownarrow}$, $\mathcal{M}^{\downarrowtail}$, and $\mathcal{M}^{s\downarrowtail}$ are the collections of epimorphisms, strong epimorphisms, monomorphisms, and strong monomorphisms, respectively.

Definition 6.1.1. Let $\mathscr{C}$ be a category with a proper orthogonal factorization system $(\mathcal{E}, \mathcal{M})$. An object K in $\mathscr{C}$ is $(\mathcal{E}, \mathcal{M}, \lambda)$-*presentable* for a regular cardinal λ if the functor $\mathscr{C}(K, -) : \mathscr{C} \to \mathbf{Set}$ sends $\mathcal{E} \cap \mathcal{M}^{\downarrowtail}$-tight $(\mathcal{M}, \lambda)$-cocones to colimiting cocones.

Proposition 6.1.2. *Let $\mathscr{C}$ be a category with a proper orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ and let λ be a regular cardinal. Let X be an object in $\mathscr{C}$ with an $\mathcal{E} \cap \mathcal{M}^{\downarrowtail}$-tight $(\mathcal{M}, \lambda)$-cocone $\{x_\alpha : X_\alpha \to X\}_\alpha$.*

1. *If $f : L \to X$ is map on an $(\mathcal{E}, \mathcal{M}, \lambda)$ presentable object L, then there is an α and a map $f' : L \to X_\alpha$ such that $x_\alpha \circ f' = f$.*

2. *If $\{f_\beta : L_\beta \to X\}_\beta$ is a λ-small cocone of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects over X, then there is an α and a cocone $\{f'_\beta : L_\beta \to X_\alpha\}_\beta$ such that $x_\alpha \circ f'_\beta = f_\beta$.*

Proof. (1) For every $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object L in $\mathscr{C}$, the map of sets $\operatorname{colim}_\alpha \mathscr{C}(L, X_\alpha) \to \mathscr{C}(L, X)$ induced by the cocone maps $x_{\alpha *} : \mathscr{C}(L, X_\alpha) \to \mathscr{C}(L, X)$ is an isomorphism. So $f : L \to X$ lifts to a map $f' : L \to X_\alpha$.

(2) By (1), for every map $f_\beta : L_\beta \to X$, there is an α_β and a map $g_\beta : L_\beta \to X_{\alpha_\beta}$ such that $x_{\alpha_\beta} \circ g_\beta = f_\beta$. Since $\{x_{\alpha_\beta} : X_{\alpha_\beta} \to X\}_\beta$ is a λ-small set of maps, there is an α' such that every map x_{α_β} factors through $x_{\alpha'} : X_{\alpha'} \to X$ via a connecting map $x_{\alpha_\beta}^{\alpha'} : X_{\alpha_\beta} \to X_{\alpha'}$ in the cocone $\{x_\alpha\}_\alpha$. Let $f'_\beta = x_{\alpha_\beta}^{\alpha'} \circ g_\beta : L_\beta \to X_{\alpha'}$ for each β. Then $x_{\alpha'} \circ f'_\beta = f_\beta$ for each β. Furthermore, if $l : L_{\beta_1} \to L_{\beta_2}$ is a connecting map of the cocone $\{f_\beta\}_\beta$, then $x_{\alpha'} \circ f'_{\beta_2} \circ l = f_{\beta_2} \circ l = f_{\beta_1} = x_{\alpha'} \circ f'_{\beta_1}$. Since $x_{\alpha'}$ is an $\mathcal{M}$-map, it is a monomorphism. Therefore $f'_{\beta_2} \circ l = f'_{\beta_1}$. So $\{f'_\beta : L_\beta \to X_{\alpha'}\}_\beta$ is a cocone over $X_{\alpha'}$. ☐

Proposition 6.1.3. *Let $\mathscr{C}$ be a category with a proper orthogonal factorization system $(\mathcal{E}, \mathcal{M})$. Let λ and κ be regular cardinals with $\lambda \le \kappa$.*

1. *Every $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object is $(\mathcal{E}, \mathcal{M}, \kappa)$-presentable.*

2. *An $\mathcal{E}$-quotient of an $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object is $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable.*

3. *A retract of an $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object is $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable.*

4. *A κ-small colimit of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects, when it exists, is an $(\mathcal{E}, \mathcal{M}, \kappa)$-presentable object.*

5. *If* $\{K_\alpha \to X\}$ *is a* κ*-small* $\mathcal{E}$*-tight cocone of* $(\mathcal{E},\mathcal{M},\lambda)$*-presentable objects over* X*, then* X *is an* $(\mathcal{E},\mathcal{M},\kappa)$*-presentable object.*

Proof. (1) is immediate.

(2) Let K be an $(\mathcal{E},\mathcal{M},\lambda)$-presentable object and let $g : K \to L$ be a map in $\mathcal{E}$. Let $\{x_\alpha : X_\alpha \to X\}_\alpha$ be an $\mathcal{E} \cap \mathcal{M}^{\downfishtail}$-tight $(\mathcal{M},\lambda)$-cocone and let $h : L \to X$ be a map. Then by 6.1.2, there is an α and a map $k : K \to X_\alpha$ such that $x_\alpha \circ k = h \circ g$. Since $x_\alpha \in \mathcal{M}$ and $g \in \mathcal{E}$, a solution to the lifting problem $(k,h) : g \to x_\alpha$ exists. So the map $\operatorname{colim}_\alpha \mathscr{C}(L, X_\alpha) \to \mathscr{C}(L,X)$ defined by the cocone $\{x_{\alpha *} : \mathscr{C}(L,X_\alpha) \to \mathscr{C}(L,X)\}_\alpha$ is surjective. Since each map x_α is in $\mathcal{M}$, each map $x_{\alpha *}$ is injective. It follows that the map $\operatorname{colim}_\alpha \mathscr{C}(L,X_\alpha) \to \mathscr{C}(L,X)$ is injective.

(3) Let K be an $(\mathcal{E},\mathcal{M},\lambda)$-presentable object and let L be a retract of K. Let $\{x_\alpha : X_\alpha \to X\}_\alpha$ be an $\mathcal{E} \cap \mathcal{M}^{\downfishtail}$-tight $(\mathcal{M},\lambda)$-cocone. Since the map $\operatorname{colim}_\alpha x_{\alpha *} : \operatorname{colim}_\alpha \mathscr{C}(L,X_\alpha) \to \mathscr{C}(L,X)$ is a retract in $\mathbf{Set}^{\mathbf{2}}$ of the map $\operatorname{colim}_\alpha x_{\alpha *} : \operatorname{colim}_\alpha \mathscr{C}(K,X_\alpha) \to \mathscr{C}(K,X)$ and the latter map is a bijection, the former map is bijective.

(4) Let $\operatorname{colim}_i K_i$ be the colimit of a κ-small diagram of $(\mathcal{E},\mathcal{M},\lambda)$-presentable objects in $\mathscr{C}$. Let $\{x_\alpha : X_\alpha \to X\}_\alpha$ be an $\mathcal{E} \cap \mathcal{M}^{\downfishtail}$-tight $(\mathcal{M},\kappa)$-cocone in $\mathscr{C}$. This cocone is in particular λ-filtered. Since κ-small limits commute with κ-filtered colimits in $\mathbf{Set}$,

$$\begin{aligned}
\operatorname*{colim}_\alpha \mathscr{C}(\operatorname*{colim}_i K_i, X_\alpha) &\cong \operatorname*{colim}_\alpha \lim_i \mathscr{C}(K_i, X_\alpha) \\
&\cong \lim_i \operatorname*{colim}_\alpha \mathscr{C}(K_i, X_\alpha) \\
&\cong \lim_i \mathscr{C}(K_i, X) \\
&\cong \mathscr{C}(\operatorname*{colim}_i K_i, X)
\end{aligned}$$

and this isomorphism is the map determined by the cocone maps $x_\alpha : X_\alpha \to X$.

(5) is an immediate consequence of (4) and (2). □

6.1.2 Quasiaccessible Categories

Definition 6.1.4. A category $\mathscr{C}$ is $(\mathcal{E}, \mathcal{M}, \lambda)$-*quasiaccessible* if the following conditions are satisfied.

1. $(\mathcal{E}, \mathcal{M})$ is a proper orthogonal factorization system on $\mathscr{C}$ and λ is a regular cardinal.
2. $\mathscr{C}$ is $\mathcal{E}$-well-copowered.
3. $\mathscr{C}$ is closed under λ-filtered colimits.
4. There is a small set $\mathcal{S}$ of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects such that every object X in $\mathscr{C}$ has an $\mathcal{E} \cap \mathcal{M}^{\downarrowtail}$-tight $(\mathcal{M}, \lambda)$-cocone $\{S_\alpha \to X\}_\alpha$ of objects in $\mathcal{S}$ over X.

The category $\mathscr{C}$ is $(\mathcal{E}, \mathcal{M})$-*quasiaccessible* if it is $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible for some regular cardinal λ. A category $\mathscr{C}$ is *quasiaccessible* if it is $(\mathcal{E}, \mathcal{M})$-quasiaccessible for some proper orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ in $\mathscr{C}$.

Before developing the theory of quasiaccessible categories, we will look at some examples. The category of topological spaces is our motivating example.

The only thing preventing **Top** from being an accessible category is the topologies on its sets. Indeed, **Set** is an $\aleph_0$-accessible category and the colimit of a diagram in **Top** is the colimit of the underlying diagram of sets equipped with the final topology. So in particular, every discrete space is the colimit of a finitely-filtered diagram of discrete spaces. Furthermore, each topological space X has a bijective map $X_0 \to X$ from a discrete space X_0. So **Top** is "accessible up to bijective maps".

For every topological space X, there is only a small collection of isomorphism classes of topological spaces of cardinality at most $|X|$. So **Top** is well-copowered. Since **Top** is also cocomplete, by 2.4.9, $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail})$ is a proper orthogonal factorization system on **Top**. For the same reason we saw in section 5.1.5, the epimorphisms in **Top** are exactly the (continuous) surjections. A similar argument to the one in 5.1.14, shows that $\mathcal{E}^{\text{⤈}}$ is the collection of surjective continuous maps and $\mathcal{M}^{s\text{↧}}$ is the collection of subspace inclusions. Condition (4) of definition 6.1.4 is all that remains to be checked.

Remark 6.1.5. If $\{x_\alpha : X_\alpha \to X\}_\alpha$ is an $\mathcal{M}^{s\text{↧}}$-cocone in **Top**, then, as a set, $\operatorname{colim}_\alpha X_\alpha$ is the union $\bigcup_\alpha X_\alpha$ in X. In general, the space $\operatorname{colim}_\alpha X_\alpha$ will have more open subsets than the set $\bigcup_\alpha X_\alpha$ with the subspace topology from X. So the map $\operatorname{colim}_\alpha X_\alpha \to X$ defined by the cocone $\{x_\alpha\}_\alpha$ is not necessarily a subspace inclusion.

A good example to keep in mind is $\mathbb{R}$. The canonical cocone $\{y_\alpha : Y_\alpha \to \mathbb{R}\}_\alpha$ of discrete subspaces of $\mathbb{R}$ is a finitely filtered $\mathcal{M}^{s\text{↧}}$-cocone. As a set, $\mathbb{R}$ is the colimit of $\{y_\alpha\}_\alpha$. However, any colimit in **Top** of discrete spaces must be a discrete space. So the colimit $\operatorname{colim}_\alpha Y_\alpha$ does not have the subspace topology from $\mathbb{R}$.

If all of the x_α's in an $\mathcal{M}^{s\text{↧}}$-cocone $\{x_\alpha : X_\alpha \to X\}_\alpha$ in **Top** are open maps, then the map $\operatorname{colim}_\alpha X_\alpha \to X$ defined by the cocone $\{x_\alpha\}_\alpha$ *is* a subspace inclusion. But the requirement that the subspaces are open places some lower bound on their cardinalities. Open subsets of $\mathbb{R}$ have to have the same cardinality as $\mathbb{R}$.

Proposition 6.1.6. *Let λ be a regular cardinal. The $(\mathcal{E}^{\text{⤈}}, \mathcal{M}^{s\text{↧}}, \lambda)$-presentable objects in* **Top** *are exactly the spaces with cardinality less than λ.*

Proof. Let $\{f_\alpha : X_\alpha \to X\}_\alpha$ be an $\mathcal{E}^{\text{⤈}} \cap \mathcal{M}^{\text{↧}}$-tight $(\mathcal{M}^{s\text{↧}}, \lambda)$-cocone in **Top**. Let $g : K \to X$ be a map on a space K with $|K| < \lambda$. Because the cocone $\{f_\alpha\}_\alpha$ is $\mathcal{E}^{\text{⤈}}$-tight, for every point $x \in X$, there is an α' and $x' \in X_{\alpha'}$ such that $f_{\alpha'}(x') = x$. So every point in the image of g is in the image of an f_α. Because $|K| < \lambda$, there is a λ small collection of X_α's that cover K. Since the cocone $\{f_\alpha\}_\alpha$ is λ-filtered, there is an index α_g such that the image of g is contained in the image of f_{α_g}. So g lifts to a map $g' : K \to X_{\alpha_g}$. Thus $\mathbf{Top}(K, -) : \mathbf{Top} \to \mathbf{Set}$ sends

$\mathcal{E}^{\not\downarrow} \cap \mathcal{M}^{\not\uparrow}$-tight $(\mathcal{M}^{s\not\uparrow}, \lambda)$-cocones to $\mathcal{E}^{\not\downarrow}$-tight $(\mathcal{M}^{\not\uparrow}, \lambda)$-cocones. Since a λ-filtered colimit of injections is **Set** is an injection, K is $(\mathcal{E}^{\not\downarrow}, \mathcal{M}^{s\not\uparrow}, \lambda)$-presentable.

Conversely, suppose K is an $(\mathcal{E}^{\not\downarrow}, \mathcal{M}^{s\not\uparrow}, \lambda)$-presentable object in **Top**. Let $\{f_\alpha : K_\alpha \to K\}_\alpha$ be the canonical $\mathcal{M}^{s\not\uparrow}$-cocone of K with respect to the K_α with $|K_\alpha| < \lambda$. For every λ-small subcocone $\{f_\alpha\}_{\alpha \in \mathcal{A}_0}$, the underlying set of $\operatorname{colim}_{\alpha_\alpha \in \mathcal{A}_0} K_\alpha$ has cardinality less than λ, since it is a λ-small union of sets of cardinality less than λ. Let $r : \operatorname{colim}_{\alpha \in \mathcal{A}_0} K_\alpha \to K$ be the map defined by the λ-small cocone $\{f_\alpha\}_{\alpha \in \mathcal{A}_0}$. We can factor r as $r = p \circ q$, where $q \in \mathcal{E}^{\not\downarrow}$ and $p \in \mathcal{M}^{s\not\uparrow}$. Since an $\mathcal{E}^{\not\downarrow}$-quotient of a space with cardinality less than λ is a space with cardinality less than λ, p is equal to f_α for some α. Thus $\{f_\alpha\}_\alpha$ is a λ-filtered cocone. For every point $k \in K$, $\{k\}$ is a set of cardinality less than λ and $\{k\} \to K$ is a subspace inclusion. Thus on the level of sets, $\operatorname{colim}_\alpha f_\alpha = \cup_\alpha K_\alpha = K$. So $\{f_\alpha\}_\alpha$ is an $\mathcal{E}^{\not\downarrow} \cap \mathcal{M}^{\not\uparrow}$-tight $(\mathcal{M}^{s\not\uparrow}, \lambda)$-cocone. Thus the identity map $id : K \to K$ lifts to a map $i : K \to K_{\alpha'}$ for some α' such that $f_{\alpha'} \circ i = id$. Since $K_{\alpha'}$ has cardinality less than λ, K must have cardinality less than λ. □

Theorem 6.1.7. *The category* **Top** *is* $(\mathcal{E}^{\not\downarrow}, \mathcal{M}^{s\not\uparrow}, \lambda)$*-quasiaccessible for every regular cardinal* λ.

Proof. Condition (4) of definition 6.1.4 is the only condition that needs to be checked. If X is a space in **Top** and if $\{f_\alpha : K_\alpha \to X\}_\alpha$ is the canonical $\mathcal{M}^{s\not\uparrow}$-cocone of X with respect to the K_α with $|K_\alpha| < \lambda$, then our proof of 6.1.6 shows that $\{f_\alpha\}_\alpha$ is λ-filtered and $\mathcal{E}^{\not\downarrow}$-tight. Since a λ-filtered colimit of monomorphisms in **Top** is a monomorphism, $\{f_\alpha\}_\alpha$ is $\mathcal{E}^{\not\downarrow} \cap \mathcal{M}^{\not\uparrow}$-tight. □

We can also prove that the category **kTop** of k-spaces is a quasiaccessible category. Just like **Top**, **kTop** is cocomplete and well-copowered and has a proper orthogonal factorization system $(\mathcal{E}^{\not\downarrow}, \mathcal{M}^{s\not\uparrow})$. As we saw in section 5.1.5, $\mathcal{E}^{\not\downarrow}$ is the collection of surjective maps and $\mathcal{M}^{s\not\uparrow}$ is the collection of k-inclusions (see definition 5.1.13).

Proposition 6.1.8. *Let λ be a regular cardinal. The $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail}, \lambda)$-presentable objects in* **kTop** *are exactly the spaces with cardinality less than λ.*

Proof. This proof is very similar to 6.1.6. We let $\{f_\alpha : X_\alpha \to X\}_\alpha$ be an $\mathcal{E}^{\twoheaddownarrow} \cap \mathcal{M}^{\downarrowtail}$-tight $(\mathcal{M}^{s\downarrowtail}, \lambda)$-cocone in **kTop** and let $g : K \to X$ be a map on a space K with $|K| < \lambda$. Then every point in the image of g is in the image of an f_α and because $|K| < \lambda$, there is a λ small collection of X_α that cover K. Since the cocone $\{f_\alpha\}_\alpha$ is λ-filtered, there is an index α_g such that the image of g is contained in the image of f_{α_g}. So g lifts to a map $g' : K \to f_{\alpha_g}(X_{\alpha_g})$ in **Top**, where $f_{\alpha_g}(X_{\alpha_g})$ has the subspace topology from X. Since, for every k-open set U in $f_{\alpha_g}(X_{\alpha_g})$, $g'^{-1}(U)$ is k-open in K, g' lifts to a map $g' : K \to X_{\alpha_g}$. It follows that K is $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail}, \lambda)$-presentable.

Conversely, suppose K is an $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail}, \lambda)$-presentable object in **kTop**. Let $\{f_\alpha : K_\alpha \to K\}_\alpha$ be the canonical $\mathcal{M}^{s\downarrowtail}$-cocone of K with respect to the K_α with $|K_\alpha| < \lambda$. Then $\{f_\alpha\}_\alpha$ is a λ-filtered cocone. For every point $k \in K$, $\{k\}$ is a set of cardinality less than λ and $\{k\} \to K$ is a k-inclusion. So as sets, $\operatorname{colim}_\alpha f_\alpha = \cup_\alpha K_\alpha = K$. Thus $\{f_\alpha\}_\alpha$ is an $\mathcal{E}^{\twoheaddownarrow} \cap \mathcal{M}^{\downarrowtail}$-tight $(\mathcal{M}^{s\downarrowtail}, \lambda)$-cocone. So the identity map $id : K \to K$ lifts to a map $i : K \to K_{\alpha'}$ for some α' such that $f_\alpha \circ i = id$. Since $K_{\alpha'}$ has cardinality less than λ, K must have cardinality less than λ. □

Theorem 6.1.9. *The category* **kTop** *is $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail}, \lambda)$-quasiaccessible for every regular cardinal λ.*

Proof. Just as in 6.1.7, we only need to check condition (4) of the definition and this follows from the proof of 6.1.8. □

Quasiaccessible categories are a generalization of locally presentable categories.

Proposition 6.1.10. *A cocomplete category is accessible if and only if it is $(\mathcal{E}^{s\twoheaddownarrow}, \mathcal{M}^{\downarrowtail})$-quasiaccessible.*

Proof. Let $\mathscr{C}$ be an accessible category. By [AR94, 1.56], $\mathscr{C}$ is complete, well-powered, and well-copowered. So by 2.4.9, $(\mathcal{E}^{s\not\downarrow}, \mathcal{M}^{\curlyvee\!\!\!\!\downarrow})$ is a proper orthogonal factorization system on $\mathscr{C}$. Since every $\mathcal{E}^{s\not\downarrow} \cap \mathcal{M}^{\curlyvee\!\!\!\!\downarrow}$-map is an isomorphism, every λ-presentable object in $\mathscr{C}$ is an $(\mathcal{E}^{s\not\downarrow}, \mathcal{M}^{\curlyvee\!\!\!\!\downarrow}, \lambda)$-presentable object. By [AR94, 2.29, 2.34], there is a regular cardinal λ such that $\mathscr{C}$ is closed under λ-filtered colimits and every object in $\mathscr{C}$ is the colimit of an $(\mathcal{M}^{\curlyvee\!\!\!\!\downarrow}, \lambda)$-cocone of λ-presentable objects. So $\mathscr{C}$ satisfies conditions 6.1.4 (3) and (4).

Conversely, suppose $\mathscr{C}$ is an $(\mathcal{E}^{s\not\downarrow}, \mathcal{M}^{\curlyvee\!\!\!\!\downarrow}, \lambda)$-quasiaccessible category. Since every $\mathcal{E}^{s\not\downarrow} \cap \mathcal{M}^{\curlyvee\!\!\!\!\downarrow}$-map is an isomorphism, the $(\mathcal{E}^{s\not\downarrow}, \mathcal{M}^{\curlyvee\!\!\!\!\downarrow}, \lambda)$-presentable objects in $\mathscr{C}$ are exactly the λ-*generated objects* [AR94, 1.67]. Furthermore, there is a regular cardinal λ such that every object in $\mathscr{C}$ is the colimit of an $(\mathcal{M}^{\curlyvee\!\!\!\!\downarrow}, \lambda)$-cocone of λ-generated objects. Therefore there is a regular cardinal κ such that $\mathscr{C}$ is κ-accessible [AR94, 1.70]. □

Categories of presheaves are quasiaccessible in two different ways. By the above proposition they are $(\mathcal{E}^{s\not\downarrow}, \mathcal{M}^{\curlyvee\!\!\!\!\downarrow})$-quasiaccessible and by the following proposition, they are also $(\mathcal{E}^{\not\downarrow}, \mathcal{M}^{s\curlyvee\!\!\!\!\downarrow})$-quasiaccessible.

Proposition 6.1.11. *Every category of presheaves is* $(\mathcal{E}^{\not\downarrow}, \mathcal{M}^{s\curlyvee\!\!\!\!\downarrow})$*-quasiaccessible.*

Proof. Let $\mathscr{A}$ be a small category. Since **Set** is well-copowered, the category of presheaves $\mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}$ is well-copowered. The category $\mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}$ is also accessible [AR94, 1.46]. By 2.4.9, $(\mathcal{E}^{\not\downarrow}, \mathcal{M}^{s\curlyvee\!\!\!\!\downarrow})$ is a proper orthogonal factorization system on $\mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}$. Since every monomorphism in a category of presheaves is an objectwise injection and every epimorphism in a category of presheaves is an objectwise surjection, every $\mathcal{E}^{\not\downarrow} \cap \mathcal{M}^{\curlyvee\!\!\!\!\downarrow}$-map in $\mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}$ is an isomorphism. Therefore every λ-presentable object in $\mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}$ is an $(\mathcal{E}^{\not\downarrow}, \mathcal{M}^{s\curlyvee\!\!\!\!\downarrow}, \lambda)$-presentable object. By [AR94, 2.31, 2.34], there is a regular cardinal λ such that $\mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}$ is closed under λ-filtered colimits and every object in $\mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}$ is the colimit of an $(\mathcal{M}^{s\curlyvee\!\!\!\!\downarrow}, \lambda)$-cocone of λ-presentable objects. So every object has an $\mathcal{E}^{\not\downarrow} \cap \mathcal{M}^{s\mapsdown}$-tight $(\mathcal{M}^{s\mapsdown}, \lambda)$-presentable cocone of $(\mathcal{E}^{\not\downarrow}, \mathcal{M}^{s\mapsdown}, \lambda)$-presentable objects. ☐

6.1.3 Basic Properties of Quasiaccessible Categories

Proposition 6.1.12. *Let $\mathscr{C}$ be an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible category.*

1. *If $g \circ f \in \mathcal{M}$, then $f \in \mathcal{M}$. If $g \circ f \in \mathcal{E}$, then $g \in \mathcal{E}$.*

2. *A map $f : X \to Y$ is a monomorphism in $\mathscr{C}$ if and only if $a = b$ for each $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object K and each pair of maps $a, b : K \to X$ with $f \circ a = f \circ b$.*

3. *If $\{x_\alpha : f_\alpha \to f\}_\alpha$ is an $\mathcal{E} \cap \mathcal{M}^{\mapsdown}$-tight $(\mathcal{M}, \lambda)$-cocone in $\mathscr{C} \downarrow Y$ of monomorphisms $f_\alpha : X_\alpha \to Y$, then $f : X \to Y$ is a monomorphism.*

4. *Every $(\mathcal{M}, \lambda)$-cocone $D \overset{\cdot}{\longrightarrow} Y$ factors as an $\mathcal{E} \cap \mathcal{M}^{\mapsdown}$-tight $(\mathcal{M}, \lambda)$-cocone $D \overset{\cdot}{\longrightarrow} X$ followed by an $\mathcal{M}$-map $X \to Y$.*

5. *Every $\mathcal{E}$-tight $(\mathcal{M}, \lambda)$-cocone in $\mathscr{C}$ is an $\mathcal{E} \cap \mathcal{M}^{\mapsdown}$-tight cocone.*

Proof. (1) This is a restatement of 2.4.5.

(2) Suppose f is a monomorphism relative to $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects. Let Z be an object in $\mathscr{C}$ and let $u : Z \to \operatorname{dom} f$ and $v : Z \to \operatorname{dom} f$ be two maps such that $f \circ u = f \circ v$. There is an $\mathcal{E}$-tight $(\mathcal{M}, \lambda)$-cocone $\{k_\alpha : K_\alpha \to Z\}_\alpha$ of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects over Z. So the map $h : \operatorname{colim}_\alpha K_\alpha \to Z$ defined by the cocone $\{k_\alpha\}_\alpha$ is an $\mathcal{E}$-map. For each α, since $f \circ u \circ k_\alpha = f \circ v \circ k_\alpha$, $u \circ k_\alpha = v \circ k_\alpha$. Therefore $u \circ h = v \circ h$. Since h is an epimorphism, $u = v$.

(3) Now suppose $a : K \to X$ and $b : K \to X$ are maps on an $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object K such that $f \circ a = f \circ b$. There are indices α_1 and α_2 such that a lifts to a map $a' : K \to X_{\alpha_1}$ and b lifts to a map $b' : K \to X_{\alpha_2}$. Since the cocone $\{x_\alpha\}_\alpha$ is λ-filtered, there is an index α_3

such that a and b lift to maps $a'' : K \to X_{\alpha_3}$ and $b'' : K \to X_{\alpha_3}$. Since

$$f_{\alpha_3} \circ a'' = f \circ x_{\alpha_3} \circ a'' = f \circ a = f \circ b = f \circ x_{\alpha_3} \circ b'' = f_{\alpha_3} \circ b'',$$

$a'' = b''$. But this means $a = b$.

(4) By (1), the cocone $\theta : D \xrightarrow{\centerdot} \operatorname{colim} D$ is an $(\mathcal{M}, \lambda)$-cocone. Of course this cocone is $\mathcal{E} \cap \mathcal{M}^{\downzigzagarrow}$-tight, since it is colimiting. So by (3), the induced map $f : \operatorname{colim} D \to Y$ is a monomorphism. By applying the $(\mathcal{E}, \mathcal{M})$-factorization to f, we get $f = p \circ q$, where $q : \operatorname{colim} D \to X$ is a map in $\mathcal{E} \cap \mathcal{M}^{\downzigzagarrow}$ and $p : X \to Y$ is a map in $\mathcal{M}$. Another application of (1) shows that $q \circ \theta : D \xrightarrow{\centerdot} X$ is an $\mathcal{M}$-cocone.

(5) Let $\{x_\alpha : X_\alpha \to X\}_\alpha$ be an $\mathcal{E}$-tight $(\mathcal{M}, \lambda)$-cocone. Let $f : \operatorname{colim}_\alpha X_\alpha \to X$ be the map defined by the cocone $\{x_\alpha\}_\alpha$. Taking $f_\alpha = x_\alpha$ in part (3), we get that f is a monomorphism. □

Proposition 6.1.13. *Every object X in an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible category $\mathscr{C}$ is $(\mathcal{E}, \mathcal{M}, \kappa)$-presentable for some regular cardinal κ.*

Proof. This is an immediate consequence of 6.1.4 (4) and 6.1.3 (5). □

Proposition 6.1.14. *The collection of isomorphism classes of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects in an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible category $\mathscr{C}$ is a set.*

Proof. Let K be an $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object in $\mathscr{C}$. Let $\mathcal{S}$ be a set of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects in $\mathscr{C}$ that satisfies 6.1.4 (4). There is an $\mathcal{E} \cap \mathcal{M}^{\downzigzagarrow}$-tight $(\mathcal{M}, \lambda)$-cocone $\{k_\alpha : K_\alpha \to K\}_\alpha$ of $\mathcal{S}$-objects over K. By remark 6.1.2, the identity map $id : K \to K$ factors through some $k_\alpha : K_\alpha \to K$. Since k_α is a split monomorphism, it is an $\mathcal{E}$-map. So K is an $\mathcal{E}$-quotient of K_α. Since $\mathscr{C}$ is $\mathcal{E}$-well-copowered, there can only be a set of isomorphism classes of such objects. □

Definition 6.1.15. Let $\mathscr{C}$ be an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible category. Let $\mathbf{Pres}^{\mathcal{E}}_{\mathcal{M},\lambda}(\mathscr{C})$ be the full subcategory of $\mathscr{C}$ on a set of representatives for the isomorphism classes of the $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects in $\mathscr{C}$.

In the following proposition, we make use of definition 2.1.12.

Proposition 6.1.16. *Let $\mathscr{C}$ be an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible category. Every canonical $\mathcal{M}$-cocone with respect to $\mathbf{Pres}^{\mathcal{E}}_{\mathcal{M},\lambda}(\mathscr{C})$ is λ-filtered and $\mathcal{E} \cap \mathcal{M}^{\between}$-tight.*

Proof. Let X be an object in $\mathscr{C}$. By 6.1.4 (4), there is an $\mathcal{E} \cap \mathcal{M}^{\between}$-tight $(\mathcal{M}, \lambda)$-cocone $\{k_\alpha : K_\alpha \to X\}_\alpha$ of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects over X.

Let $\{l_\beta : L_\beta \to X\}_\beta$ be a λ-small $\mathcal{M}$-cocone of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects over X. By 6.1.2 (2), there is an α_0 such that $\{l_\beta\}_\beta$ lifts to a cocone $\{l'_\beta : L_\beta \to K_{\alpha_0}\}_\beta$. Since $k_{\alpha_0} : K_{\alpha_0} \to X$ is an $\mathcal{M}$-map on an $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object K_{α_0}, the category $\mathbf{Pres}^{\mathcal{E}}_{\mathcal{M},\lambda}(\mathscr{C}) \downarrow_{\mathcal{M}} X$ is λ-filtered.

Now let $\{l_\beta : L_\beta \to X\}_\beta$ be the canonical $\mathcal{M}$-cocone for X relative to $\mathbf{Pres}^{\mathcal{E}}_{\mathcal{M},\lambda}(\mathscr{C})$. Let $f : \operatorname{colim}_\alpha K_\alpha \to X$ and $g : \operatorname{colim}_\beta L_\beta \to X$ be the maps defined by the cocones $\{k_\alpha\}_\alpha$ and $\{l_\beta\}_\beta$, respectively. Since $\{k_\alpha\}_\alpha$ is a subcocone of $\{l_\beta\}_\beta$, there is a map $h : \operatorname{colim}_\alpha K_\alpha \to \operatorname{colim}_\beta L_\beta$ such that $g \circ h = f$. By 6.1.12 (1), $g \in \mathcal{E}$. We also know by 6.1.12 (1) that each map in the colimiting cocone $\{L_\beta \to \operatorname{colim}_\beta L_\beta\}_\beta$ is an $\mathcal{M}$-map. Since the colimiting cocone $\{L_\beta \to \operatorname{colim}_\beta L_\beta\}_\beta$ is an $\mathcal{E} \cap \mathcal{M}^{\between}$-tight $(\mathcal{M}, \lambda)$-cocone, we know by 6.1.12 (3) that g is a monomorphism. So $\{l_\beta : L_\beta \to X\}_\beta$ is an $\mathcal{E} \cap \mathcal{M}^{\between}$-tight cocone. □

Proposition 6.1.17. *If $\mathscr{C}$ is an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible category and κ is a regular cardinal such that $\lambda \lhd \kappa$, then $\mathscr{C}$ is $(\mathcal{E}, \mathcal{M}, \kappa)$-quasiaccessible.*

Proof. Since every κ-filtered diagram is in particular λ-filtered, we only need to prove condition (4) of 6.1.4. Let X be an object in $\mathscr{C}$. Let $\mathcal{T}_X$ be the set of all objects in $\mathscr{C}$ with an

$\mathcal{M}$-map to X that are $\mathcal{E} \cap \mathcal{M}^{\Ydown}$-quotients of colimits of κ-small $(\mathcal{M}, \lambda)$-cocones of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects over X. In other words, an object T is in $\mathcal{T}_X$ if the following conditions hold.

- There is a map $t : T \to X$ which is in $\mathcal{M}$.
- There is a κ-small, $\mathcal{E} \cap \mathcal{M}^{\Ydown}$-tight, $(\mathcal{M}, \lambda)$-cocone $\theta : D \xrightarrow{\cdot} T$ of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects.
- The cocone $t \circ \theta : D \xrightarrow{\cdot} X$ is an $\mathcal{M}$-cocone.

By 6.1.12 (4), every κ-small $(\mathcal{M}, \lambda)$-cocone of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects over X factors through an object in $\mathcal{T}_X$. The set $\mathcal{T}_X$ is small and every object in $\mathcal{T}_X$ is $(\mathcal{E}, \mathcal{M}, \kappa)$-presentable by 6.1.3 (5).

Let $\{k_\alpha : K_\alpha \to X\}_{\alpha \in \mathcal{A}}$ be an $\mathcal{E} \cap \mathcal{M}^{\Ydown}$-tight λ-directed $\mathcal{M}$-cocone of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects. Let $\mathcal{D}_\lambda(\mathcal{A})$ be the poset of λ-directed subsets of $\mathcal{A}$, ordered by inclusion, and let $\widehat{\mathcal{A}}$ be the subposet of $\mathcal{D}_\lambda(\mathcal{A})$ on the λ-directed subsets of $\mathcal{A}$ that are κ-small. A κ-small union $\mathcal{U}$ of elements of $\widehat{\mathcal{A}}$ is a κ-small subset of $\mathcal{A}$. Since $\lambda \lhd \kappa$, $\mathcal{U}$ is contained in a κ-small λ-directed subset $\mathcal{V}$ of $\mathcal{A}$. So $\mathcal{V}$ is an element of $\widehat{\mathcal{A}}$ and thus $\widehat{\mathcal{A}}$ is κ-directed. Let $F : \widehat{\mathcal{A}} \to \mathscr{C}$ be the diagram that sends each λ-filtered poset $\mathcal{X}$ in $\widehat{\mathcal{A}}$ to the object $\operatorname{colim}_{\alpha \in \mathcal{X}} K_\alpha$ and sends an inclusion $\mathcal{X} \subseteq \mathcal{Y}$ of posets in $\widehat{\mathcal{A}}$ to the induced map of colimits $\operatorname{colim}_{\alpha \in \mathcal{X}} K_\alpha \to \operatorname{colim}_{\alpha \in \mathcal{Y}} K_\alpha$. There are of course also compatible maps $\varphi_{\mathcal{X}} : \operatorname{colim}_{\alpha \in \mathcal{X}} K_\alpha \to X$ that define a κ-filtered cocone $\varphi : F \xrightarrow{\cdot} X$. By the proof of 6.1.12 (4), this cocone factors as a natural transformation $\rho : F \to F'$ of $\mathcal{E} \cap \mathcal{M}^{\Ydown}$-maps followed by an $\mathcal{M}$-cocone $\varphi' : F' \xrightarrow{\cdot} X$ and the objects of F' are in $\mathcal{T}_X$.

Since colimits commute, $\operatorname{colim}_\alpha K_\alpha \cong \operatorname{colim} F$. So $\varphi : F \xrightarrow{\cdot} X$ is $\mathcal{E} \cap \mathcal{M}^{\Ydown}$-tight and the map $f : \operatorname{colim} F \to X$ defined by φ factors through the map $f' : \operatorname{colim} F' \to X$ defined by φ'.

Therefore by 6.1.12 (1), f' is in $\mathcal{E}$. The colimiting cocone $F' \overset{\cdot}{\longrightarrow} \operatorname{colim} F'$ is an $\mathcal{M}$-cocone by 6.1.12 (1). So 6.1.12 (3) tells us that, $f' : \operatorname{colim} F' \to X$ is in $\mathcal{M}^{\lightning}$. So $\varphi' : F' \overset{\cdot}{\longrightarrow} X$ is an $\mathcal{E} \cap \mathcal{M}^{\lightning}$-tight $(\mathcal{M}, \kappa)$-cocone of objects in $\mathcal{T}_X$. ☐

Corollary 6.1.18. *Let $\mathscr{C}$ be an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible category. Let κ be a regular cardinal with $\lambda \lhd \kappa$. Every $(\mathcal{E}, \mathcal{M}, \kappa)$-presentable object K in $\mathscr{C}$ is a retract of an object T with a κ-small $\mathcal{E} \cap \mathcal{M}^{\lightning}$-tight $(\mathcal{M}, \lambda)$-cocone $\{K_\alpha \to T\}_\alpha$ of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects.*

Proof. The proof of proposition 6.1.17 shows that there is an $\mathcal{E} \cap \mathcal{M}^{\lightning}$-tight $(\mathcal{M}, \kappa)$-cocone $\{t_\beta : T_\beta \to K\}_\beta$ of objects T_β in $\mathcal{T}_K$. By 6.1.2, there is a β such that the identity map $id : K \to K$ factors through $t_\beta : T_\beta \to K$. ☐

6.2 Quasiaccessible Functors

6.2.1 Quasiaccessible and Weakly Quasiaccessible Functors

Definition 6.2.1. Let $\mathscr{A}$ and $\mathscr{B}$ be $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-quasiaccessible and $(\mathcal{E}_2, \mathcal{M}_2, \lambda)$-quasiaccessible categories, respectively. A functor $F : \mathscr{A} \to \mathscr{B}$ is $(\mathcal{E}_1, \mathcal{E}_2; \mathcal{M}_1, \mathcal{M}_2; \lambda)$-*quasiaccessible* if it sends $\mathcal{E}_1 \cap \mathcal{M}^{\lightning}$-tight $(\mathcal{M}_1, \lambda)$-cocones to $\mathcal{E}_2$-tight $(\mathcal{M}_2, \lambda)$-cocones.

Usually, once we specify that $\mathscr{A}$ is $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-quasiaccessible and that $\mathscr{B}$ is $(\mathcal{E}_2, \mathcal{M}_2, \lambda)$-quasiaccessible, then we can refer to an $(\mathcal{E}_1, \mathcal{E}_2; \mathcal{M}_1, \mathcal{M}_2; \lambda)$-quasiaccessible functor $F : \mathscr{A} \to \mathscr{B}$ as a just a λ-quasiaccessible functor without confusion. We will say F is quasiaccessible if it is λ-quasiaccessible for some regular cardinal λ.

By 6.1.12 (5), the functor $F : \mathscr{A} \to \mathscr{B}$ in the definition is λ-quasiaccessible if and only if it sends $\mathcal{E}_1 \cap \mathcal{M}^{\lightning}$-tight $(\mathcal{M}_1, \lambda)$-cocones to $\mathcal{E}_2 \cap \mathcal{M}^{\lightning}$-tight $(\mathcal{M}_2, \lambda)$-cocones.

Remark 6.2.2. Let $\mathscr{A}$ and $\mathscr{B}$ be $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-quasiaccessible and $(\mathcal{E}_2, \mathcal{M}_2, \lambda)$-quasiaccessible categories, respectively. A λ-quasiaccessible functor $F : \mathscr{A} \to \mathscr{B}$ in particular sends maps in

$\mathcal{M}_1$ to maps in $\mathcal{M}_2$. Indeed, if $f : X \to Y$ is an $\mathcal{M}_1$-map, then the cocone

$$\begin{tikzcd} X \arrow[dr, "f"] \arrow[dd, "f"'] & \\ & Y \\ Y \arrow[ur, "id"'] & \end{tikzcd}$$

over Y is a colimiting $(\mathcal{M}_1, \lambda)$-cocone. So $Ff \in \mathcal{M}_2$.

Remark 6.2.3. If $F : \mathscr{A} \to \mathscr{B}$ is a λ-quasiaccessible functor between quasiaccessible categories, and $\lambda \lhd \kappa$, then, by 6.1.17, F is κ-quasiaccessible.

Theorem 6.2.4. *Let $\mathscr{A}$ be an $(\mathcal{E}_1, \mathcal{M}_1)$-quasiaccessible category and let $\mathscr{B}$ be an $(\mathcal{E}_2, \mathcal{M}_2)$-quasiaccessible category. For every quasiaccessible functor $F : \mathscr{A} \to \mathscr{B}$, there is a regular cardinal λ such that for every regular cardinal κ with $\lambda \lhd \kappa$, F is κ-quasiaccessible and sends $(\mathcal{E}_1, \mathcal{M}_1, \kappa)$-presentable objects to $(\mathcal{E}_2, \mathcal{M}_2, \kappa)$-presentable objects.*

Proof. Let λ_1 be a regular cardinal such that F is λ_1-quasiaccessible. By 6.1.13, 6.1.3 (1), and 6.1.14, there is a regular cardinal λ_2 such that FK is $(\mathcal{E}_2, \mathcal{M}_2, \lambda_2)$-presentable for every $(\mathcal{E}_1, \mathcal{M}_1, \lambda_1)$-presentable object K in $\mathscr{A}$. We can further arrange for $\lambda_1 \lhd \lambda_2$. Let κ be a regular cardinal with $\lambda_2 \lhd \kappa$ and let K be an $(\mathcal{E}_1, \mathcal{M}_1, \kappa)$-presentable object in $\mathscr{A}$. By 6.1.18, K is a retract of an object T, which has an $\mathcal{E}_1 \cap \mathcal{M}^{\pitchfork}$-tight κ-small $(\mathcal{M}_1, \lambda_1)$-cocone $\{K_\alpha \to T\}_\alpha$ of $(\mathcal{E}_1, \mathcal{M}_1, \lambda_1)$-presentable objects. Then $\{FK_\alpha \to FT\}_\alpha$ is a $\mathcal{E}_2 \cap \mathcal{M}^{\pitchfork}$-tight κ-small $(\mathcal{M}_2, \lambda_1)$-cocone of $(\mathcal{E}, \mathcal{M}, \lambda_2)$-presentable objects. So in particular, by 6.1.3 (5), FT is a κ-presentable object. Since FK is a retract of FT, FK is κ-presentable by 6.1.3 (3). $\square$

Proposition 6.2.5. *Let $\mathscr{C}$ be a $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible category and let $f : X \to Y$ be a map in $\mathscr{C}$. If the map of sets $f_* : \mathscr{C}(K, X) \to \mathscr{C}(K, Y)$ is a bijection for every $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object K, then f is an $\mathcal{E} \cap \mathcal{M}^{\pitchfork}$-map.*

Proof. By 6.1.12 (2), a map in $\mathscr{C}$ is a monomorphism if and only if it is a monomorphism relative to each $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object K. So f is a monomorphism.

Let $\{k_\alpha : K_\alpha \to Y\}_\alpha$ be an $\mathcal{E} \cap \mathcal{M}^{\curlyveedownarrow}$-tight $(\mathcal{M}, \lambda)$-cocone of $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects. Every map $k_\alpha : K_\alpha \to Y$ has a unique lift $l_\alpha : K_\alpha \to X$. If $k_\alpha^\beta : K_\alpha \to K_\beta$ is a connecting map of the $\{k_\alpha\}_\alpha$-cocone, then the uniqueness of the lifts l_α means that $l_\beta \circ k_\alpha^\beta = l_\alpha$. The maps l_α therefore define a cocone $\{l_\alpha : K_\alpha \to X\}_\alpha$ which lifts the cocone $\{k_\alpha\}$. So if $h : \operatorname{colim}_\alpha K_\alpha \to Y$ is the map defined by the cocone $\{k_\alpha\}_\alpha$ and if $g : \operatorname{colim}_\alpha K_\alpha \to X$ is the map defined by the cocone $\{l_\alpha\}_\alpha$, then $f \circ g = h$. Since $h \in \mathcal{E}$, $f \in \mathcal{E}$ by 6.1.12 (1). □

Proposition 6.2.6. *Let $\mathscr{A}$ be an $(\mathcal{E}_1, \mathcal{M}_1)$-quasiaccessible category and let $\mathscr{B}$ be an $(\mathcal{E}_2, \mathcal{M}_2)$-quasiaccessible category. Let $F : \mathscr{A} \to \mathscr{B}$ be a quasiaccessible functor with a right adjoint $G : \mathscr{B} \to \mathscr{A}$. If $F(\mathcal{E}_1) \subseteq \mathcal{E}_2$ or if $G(\mathcal{M}_2) \subseteq \mathcal{M}_1$, then G is quasiaccessible.*

Remark 6.2.7. The conditions $F(\mathcal{E}_1) \subseteq \mathcal{E}_2$ and $G(\mathcal{M}_2) \subseteq \mathcal{M}_1$ are equivalent by 4.1.2. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are both the collection of epimorphisms or if they are both the collection of strong epimorphisms in their respective categories, then the conditions hold by 4.1.4.

Proof. By 6.2.4, there is a regular cardinal λ such that F is λ-quasiaccessible and sends $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-presentable objects to $(\mathcal{E}_2, \mathcal{M}_2, \lambda)$-presentable objects. Let $\{x_\alpha : X_\alpha \to X\}_\alpha$ be an $\mathcal{E}_2 \cap \mathcal{M}^{\curlyveedownarrow}$-tight $(\mathcal{M}_2, \lambda)$-cocone in $\mathscr{B}$. Then the colimiting cocone $\{GX_\alpha \to \operatorname{colim}_\alpha GX_\alpha\}_\alpha$ is an $\mathcal{E}_1 \cap \mathcal{M}^{\curlyveedownarrow}$-tight $(\mathcal{M}_1, \lambda)$-cocone by 6.1.12 (1). So, for each $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-presentable object K in $\mathscr{A}$,

$$\begin{aligned}
\mathscr{A}(K, \operatorname*{colim}_\alpha GX_\alpha) &\cong \operatorname*{colim}_\alpha \mathscr{A}(K, GX_\alpha) \\
&\cong \operatorname*{colim}_\alpha \mathscr{B}(FK, X_\alpha) \\
&\cong \mathscr{B}(FK, X) \\
&\cong \mathscr{A}(K, GX)
\end{aligned}$$

By 6.2.5, the map $\mathrm{colim}_\alpha GX_\alpha \to X$ defined by the cocone $\{Gx_\alpha\}_\alpha$ is in $\mathcal{E}_1 \cap \mathcal{M}^{\pitchfork}$. So the cocone $\{Gx_\alpha\}_\alpha$ is an $\mathcal{E}_1 \cap \mathcal{M}^{\pitchfork}$-tight $(\mathcal{M}_1, \lambda)$-cocone. □

Definition 6.2.8. A functor $A : \mathscr{A} \to \mathscr{C}$ is *weakly* $(\mathcal{E}, \mathcal{M}, \lambda)$-*quasiaccessible* relative to a set $\mathcal{S}$ of objects in $\mathscr{A}$ if the following conditions are satisfied.

1. $\mathscr{C}$ is an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible category.
2. A sends objects in $\mathcal{S}$ to $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects in $\mathscr{C}$.
3. For every object X in $\mathscr{A}$, the canonical $A^{-1}(\mathcal{M})$-cocone $\{s_\alpha : S_\alpha \to X\}_\alpha$ of X relative to $\mathcal{S}$ is sent by A to an $\mathcal{E} \cap \mathcal{M}^{\pitchfork}$-tight $(\mathcal{M}, \lambda)$-cocone $\{As_\alpha : AS_\alpha \to AX\}_\alpha$ in $\mathscr{C}$.

We will say that $A : \mathscr{A} \to \mathscr{C}$ is weakly $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible when a set $\mathcal{S}$ of objects in $\mathscr{A}$ exists such that A is weakly $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible relative to $\mathcal{S}$.

Proposition 6.2.9. *Let $A : \mathscr{A} \to \mathscr{C}$ be a weakly $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible functor relative to a set $\mathcal{S}$. If $\mathscr{B}$ is a full subcategory of $\mathscr{A}$ that is closed under $A^{-1}(\mathcal{M})$-subobjects, then the restriction functor $A|_{\mathscr{B}} : \mathscr{B} \to \mathscr{C}$ is weakly $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible relative to the subset of $\mathcal{S}$ on the objects in $\mathscr{B}$.*

Proof. Let $\mathcal{S}$ be a set of objects in $\mathscr{A}$ such that A is weakly $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible relative to $\mathcal{S}$, let X be an object in $\mathscr{B}$, and let $\{s_\alpha : S_\alpha \to X\}_\alpha$ be the canonical $A^{-1}(\mathcal{M})$-cocone of X with respect to $\mathcal{S}$. By definition, $\{As_\alpha : AS_\alpha \to AX\}_\alpha$ is an $\mathcal{E} \cap \mathcal{M}^{\pitchfork}$-tight $(\mathcal{M}, \lambda)$-cocone in $\mathscr{C}$. Since the maps s_α are all in $A^{-1}(\mathcal{M})$, every S_α is an object in $\mathscr{B}$. So the subset $\mathcal{S}_{\mathscr{B}}$ of $\mathcal{S}$ on the objects in $\mathscr{B}$ satisfies conditions (2) and (3) of 6.2.8 with respect to $A|_{\mathscr{B}}$. □

Proposition 6.2.10. *Let $A : \mathscr{A} \to \mathscr{C}$ be a weakly $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible functor with respect to a set $\mathcal{S}$ of objects in $\mathscr{A}$ and let $f : X \to Y$ be a map in $\mathscr{A}$. If the map of sets $f_* : \mathscr{A}(K, X) \to \mathscr{A}(K, Y)$ is a surjection for each $K \in \mathcal{S}$, then Af is an $\mathcal{E}$-map.*

Proof. Let $\{x_\alpha : K_\alpha \to X\}_\alpha$ be the canonical $A^{-1}(\mathcal{M})$-cocone for X with respect to $\mathcal{S}$ and let $\{y_\beta : L_\beta \to Y\}_\beta$ be the canonical $A^{-1}(\mathcal{M})$-cocone for Y with respect to $\mathcal{S}$. Let $x : \operatorname{colim}_\alpha AK_\alpha \to AX$ and $y : \operatorname{colim}_\beta AL_\beta \to AY$ be the maps defined by the cocones $\{Ax_\alpha\}_\alpha$ and $\{Ay_\beta\}_\beta$, respectively. Then x and y are $\mathcal{E} \cap \mathcal{M}^{\pitchfork}$-maps. Every map $Af \circ Ax_\alpha : AK_\alpha \to AY$ factors through some $Ay_\beta : AL_\beta \to AY$. So it factors through $y : \operatorname{colim}_\beta AL_\beta \to AY$ in particular. Because y is a monomorphism, we get a cocone $\{AK_\alpha \to \operatorname{colim}_\beta AL_\beta\}_\beta$ that lifts the cocone $\{Af \circ Ax_\alpha\}_\alpha$. Therefore, there is a map $d : \operatorname{colim}_\alpha AK_\alpha \to \operatorname{colim}_\beta AL_\beta$ such that $y \circ d = Af \circ x$.

We will show d is an $\mathcal{E}$-map. Let g be an $\mathcal{M}$-map and let $(u, v) : d \to g$ be a map in $\mathscr{C}^{\mathbf{2}}$. Because g is a monomorphism, to get a lift of the cocone $\{v \circ Ay_\beta : AL_\beta \to \operatorname{cod} g\}_\alpha$ along g, it is sufficient to lift each map $v \circ Ay_\beta$ individually. These individual lifts are defined by using the fact that $f_* : \mathscr{A}(K, X) \to \mathscr{A}(K, Y)$ is a surjection on $K \in \mathcal{S}$. Indeed, there is a map $m_\beta : L_\beta \to X$ such that $f \circ m_\beta = y_\beta$. So there is an α_β and an $m'_\beta : AL_\beta \to AK_{\alpha_\beta}$ such that $Af \circ Ax_{\alpha_\beta} \circ m'_\beta = Af \circ Am_\beta = Ay_\beta$. If $k_{\alpha_\beta} : AK_{\alpha_\beta} \to \operatorname{colim}_\alpha AK_\alpha$ and $l_\beta : AL_\beta \to \operatorname{colim}_\beta AL_\beta$ are the maps in the colimiting cocone, then

$$y \circ d \circ k_{\alpha_\beta} \circ m'_\beta = Af \circ Ax_{\alpha_\beta} \circ m'_\beta = Ay_\beta = y \circ l_\beta.$$

$$\begin{tikzcd}
& AK_{\alpha_\beta} \arrow[r, "k_{\alpha_\beta}"] \arrow[rr, bend left, "Ax_{\alpha_\beta}"] & \operatorname{colim}_\alpha AK_\alpha \arrow[r, "x"] \arrow[d, "d"] & AX \arrow[d, "Af"] \\
AL_\beta \arrow[ur, dashed, "m'_\beta"] \arrow[rr, "l_\beta"] \arrow[rrr, bend right, "Ay_\beta"'] & & \operatorname{colim}_\beta AL_\beta \arrow[r, "y"] & AY
\end{tikzcd}$$

Since y is a monomorphism, $d \circ k_{\alpha_\beta} \circ m'_\beta = l_\beta$. Therefore $u \circ k_{\alpha_\beta} \circ m'_\beta$ is a map $AL_\beta \to \operatorname{dom} g$ such that $g \circ u \circ k_{\alpha_\beta} \circ m'_\beta = v \circ l_\beta$. So the cocone $\{v \circ Ay_\beta\}_\beta$ lifts along g. Therefore, there is a map $s : \operatorname{colim}_\beta AL_\beta \to \operatorname{dom} g$ such that $g \circ s = v$. Since $g \circ s \circ d = v \circ d = g \circ u$ and g is a monomorphism, s is a solution to the lifting problem $(u,v) : d \to g$. So d is in ${}^{\square}\mathcal{M} = \mathcal{E}$. Since $d \in \mathcal{E}$ and $y \in \mathcal{E}$, $Af \in \mathcal{E}$ by 6.1.12 (1). $\square$

Corollary 6.2.11. *If $f : X \to Y$ is a map in an $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible category $\mathscr{C}$ such that the map of sets $f_* : \mathscr{C}(K,X) \to \mathscr{C}(K,Y)$ is a surjection for each $(\mathcal{E},\mathcal{M},\lambda)$-presentable object K, then f is an $\mathcal{E}$-map.*

Proof. Since $\mathrm{Id} : \mathscr{C} \to \mathscr{C}$ is weakly $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible relative to $\mathbf{Pres}_\lambda(\mathscr{C})$, this is an immediate consequence of 6.2.10. $\square$

Proposition 6.2.12. *Let $\mathscr{C}$ be an $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible category. Let $\mathscr{D}$ be a small λ-filtered category, let $\mathscr{E}$ be a λ-small category, and let $F : \mathscr{D} \times \mathscr{E} \to \mathscr{C}$ be a diagram in $\mathscr{C}$ such that $\lim_e F(d,e)$ and $\lim_e \operatorname{colim}_d F(d,e)$ exist. If the colimiting cocone $\{f_{d,e} : F(d,e) \to \operatorname{colim}_{d'} F(d',e)\}_d$ is an $\mathcal{M}$-cocone for each object e, then the canonical map*

$$\operatorname*{colim}_d \lim_e F(d,e) \to \lim_e \operatorname*{colim}_d F(d,e)$$

is an $\mathcal{E}$-map.

Proof. The $\mathscr{E}$-limit of the maps $f_{d,e} : F(d,e) \to \operatorname{colim}_{d'} F(d',e)$ is a map $h_d = \lim_e f_{d,e} : \lim_e F(d,e) \to \lim_e \operatorname{colim}_{d'} F(d',e)$. The maps $\{h_d\}_d$ define the canonical map $h : \operatorname{colim}_d \lim_e F(d,e) \to \lim_e \operatorname{colim}_d F(d,e)$. By taking the $\mathscr{E}$-limit first, we also get maps $g_d : \lim_e F(d,e) \to \operatorname{colim}_{d'} \lim_e F(d',e)$. For each $(\mathcal{E},\mathcal{M},\lambda)$-presentable object K in $\mathscr{C}$, the maps $h_{d*} : \mathscr{C}(K, \lim_e F(d,e)) \to \mathscr{C}(K, \lim_e \operatorname{colim}_{d'} F(d',e$ define an isomorphism

$$\begin{aligned}
\operatorname*{colim}_d \mathscr{C}(K, \lim_e F(d,e)) &\cong \operatorname*{colim}_d \lim_e \mathscr{C}(K, F(d,e)) \\
&\cong \lim_e \operatorname*{colim}_d \mathscr{C}(K, F(d,e)) \\
&\cong \lim_e \mathscr{C}(K, \operatorname*{colim}_d F(d,e)) \\
&\cong \mathscr{C}(K, \lim_e \operatorname*{colim}_d F(d,e)).
\end{aligned}$$

Since the following diagram commutes, h_* is a surjection.

$$\begin{tikzcd}
\operatorname{colim}_d \mathscr{C}(K, \lim_e F(d,e)) \arrow[r, "\operatorname{colim}_d g_{d*}"] \arrow[dr, "\cong", "\operatorname{colim}_d h_{d*}"'] & \mathscr{C}(K, \operatorname{colim}_{d'} \lim_e F(d',e)) \arrow[d, "h_*"] \\
 & \mathscr{C}(K, \lim_e \operatorname{colim}_{d'} F(d',e))
\end{tikzcd}$$

By 6.2.11, h is an $\mathcal{E}$-map. $\square$

6.2.2 Comma Categories

Let $F : \mathscr{A} \to \mathscr{C}$ and $G : \mathscr{B} \to \mathscr{C}$ be functors. Recall, we use the notation (A, f, B) for an object and $(u, v) : (A, f, B) \to (A', g, B')$ for a morphism in the comma category $F \downarrow G$, where $f : FA \to GB$ and $g : FA' \to GB'$ are maps in $\mathscr{C}$ and $u : A \to A'$ is a map in $\mathscr{A}$ and $v : B \to B'$ is a map in $\mathscr{B}$. We often refer to $f : FA \to GB$ itself as an object in $F \downarrow G$. We will use the notation $P_1 : F \downarrow G \to \mathscr{A}$ for the projection of the comma category onto $\mathscr{A}$. In other words, $P_1(A, f, B) = A$ and $P_1(u, v) = u$. Similarly, $P_2 : F \downarrow G \to \mathscr{B}$ will denote the projection onto $\mathscr{B}$. The projection functors P_1 and P_2 define a functor $P_{1\times 2} : F \downarrow G \to \mathscr{A} \times \mathscr{B}$ which sends object (A, f, B) to (A, B) and sends morphisms $(u, v) : (A, f, B) \to (A', g, B')$ to morphisms $(u, v) : (A, B) \to (A', B')$. We will use the notation $P_{\mathscr{C}^{\mathbf{2}}} : F \downarrow G \to \mathscr{C}^{\mathbf{2}}$ for the forgetful functor that sends objects (A, f, B) to f and sends morphisms $(u, v) : (A, f, B) \to (A', g, B')$ to $(u, v) : f \to g$.

Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be $(\mathcal{E}_1, \mathcal{M}_1)$-quasiaccessible, $(\mathcal{E}_2, \mathcal{M}_2)$-quasiaccessible, and $(\mathcal{E}_3, \mathcal{M}_3)$-quasiaccessible categories, respectively. Let $F : \mathscr{A} \to \mathscr{C}$ and $G : \mathscr{B} \to \mathscr{C}$ be functors. The following collection of objects in $F \downarrow G$

$$\left\{ f : FK \to GL \;\middle|\; \begin{array}{l} K \text{ is } (\mathcal{E}_1, \mathcal{M}_1, \lambda)\text{-presentable in } \mathscr{A} \text{ and} \\ L \text{ is } (\mathcal{E}_2, \mathcal{M}_2, \lambda)\text{-presentable in } \mathscr{B} \end{array} \right\}$$

has only a set of isomorphism classes in $F \downarrow G$ by 6.1.14. We will use the notation $\mathcal{S}_\lambda^{F\downarrow G}$ for a set of representatives for the isomorphism classes of the objects in the above collection. We note that the pair $(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{M}_1 \times \mathcal{M}_2)$ is a proper orthogonal factorization system on $\mathscr{A} \times \mathscr{B}$.

Theorem 6.2.13. *If $F : \mathscr{A} \to \mathscr{C}$ and $G : \mathscr{B} \to \mathscr{C}$ are quasiaccessible functors, then there is a regular cardinal λ_0 such that for every regular cardinal λ with $\lambda_0 \lhd \lambda$, the functor $P_{1\times 2} : F \downarrow G \to \mathscr{A} \times \mathscr{B}$ is weakly $(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{M}_1 \times \mathcal{M}_2, \lambda)$-quasiaccessible with respect to $\mathcal{S}_\lambda^{F\downarrow G}$. Furthermore, the functor $P_{\mathscr{C}^{\mathbf{2}}} : F \downarrow G \to \mathscr{C}^{\mathbf{2}}$ satisfies conditions (2) and (3) of 6.2.8 with respect to $\mathcal{S}_\lambda^{F\downarrow G}$.*

Proof. By 6.1.17 and 6.2.4, there is a regular cardinal λ_0 such that for every regular cardinal λ with $\lambda_0 \lhd \lambda$, the following conditions hold.

- $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are, respectively, $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-quasiaccessible, $(\mathcal{E}_2, \mathcal{M}_2, \lambda)$-quasiaccessible, and $(\mathcal{E}_3, \mathcal{M}_3, \lambda)$-quasiaccessible categories.

- F is a λ-quasiaccessible functor that sends $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-presentable objects to $(\mathcal{E}_3, \mathcal{M}_3, \lambda)$-presentable objects.

- G is a λ-quasiaccessible functor that sends $(\mathcal{E}_2, \mathcal{M}_2, \lambda)$-presentable objects to $(\mathcal{E}_3, \mathcal{M}_3, \lambda)$-presentable objects.

(I) Clearly, the $(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{M}_1 \times \mathcal{M}_2, \lambda)$-presentable objects in $\mathscr{A} \times \mathscr{B}$ are exactly the pairs (A, B) such that A is $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-presentable and B is $(\mathcal{E}_2, \mathcal{M}_2, \lambda)$-presentable. So $P_{1\times 2}$ sends every object in $\mathcal{S}_\lambda^{F\downarrow G}$ is to a λ-presentable object in $\mathscr{A} \times \mathscr{B}$.

Let $f : FX \to GY$ be any object in $F \downarrow G$. Let $\{(u_\gamma, v_\gamma) : f_\gamma \to f\}_{\gamma \in \mathcal{A}}$ be the canonical $P_{1\times 2}^{-1}((\mathcal{M}_1 \times \mathcal{M}_2)^{\mathbf{2}})$-cocone of f relative to $\mathbf{Full}(\mathcal{S}_\lambda^{F\downarrow G})$ in the category $F \downarrow G$. So $(u_\gamma, v_\gamma) : f_\gamma \to f$ is a map in the cocone if $f_\gamma \in \mathcal{S}_\lambda^{F\downarrow G}$, $u_\gamma \in \mathcal{M}_1$, and $v_\gamma \in \mathcal{M}_2$. We will start by showing that $\{(u_\gamma, v_\gamma)\}_{\gamma \in \mathcal{A}}$ is λ-filtered. To show that $\{P_{1\times 2}(u_\gamma, v_\gamma)\}$ is an $\mathcal{E}_1 \times \mathcal{E}_2 \cap \mathcal{M}^{\between}$-tight cocone in $\mathscr{A} \times \mathscr{B}$, it suffices to show that $\{P_1(u_\gamma, v_\gamma)\}_{\gamma \in \mathcal{A}}$ is an $\mathcal{E}_1 \cap \mathcal{M}^{\between}$-tight cocone in $\mathscr{A}$ and that $\{P_2(u_\gamma, v_\gamma)\}_{\gamma \in \mathcal{A}}$ is an $\mathcal{E}_2 \cap \mathcal{M}^{\between}$-tight cocone in $\mathscr{B}$. For each γ, let K_γ and L_γ be objects in $\mathscr{A}$ and $\mathscr{B}$, respectively, such that f_γ is a map $FK_\gamma \to GL_\gamma$.

Let $\{x_\alpha : X_\alpha \to X\}_\alpha$ be the canonical $\mathcal{M}_1$-cocone of X relative to $\mathbf{Pres}^{\mathcal{E}_1}_{\mathcal{M}_1,\lambda}(\mathscr{A})$ and let $\{y_\beta : Y_\beta \to Y\}_\beta$ be the canonical $\mathcal{M}_2$-cocone of Y relative to $\mathbf{Pres}^{\mathcal{E}_2}_{\mathcal{M}_2,\lambda}(\mathscr{B})$. We will use these cocones in our proof. We will also make use of the observation that $\{Fx_\alpha : FX_\alpha \to FX\}_\alpha$ and $\{Gy_\beta : GY_\beta \to GY\}_\beta$ are $\mathcal{E}_3 \cap \mathcal{M}^{\lightning}$-tight $(\mathcal{M}_3, \lambda)$-cocones on $(\mathcal{E}_3, \mathcal{M}_3, \lambda)$-presentable objects in $\mathscr{C}$.

(I a) λ-*Filtered.* Let $\mathcal{A}_0$ be a λ-small subset of $\mathcal{A}$. By 6.1.2 (2), there is an α_1 such that the cocone $\{u_\gamma : K_\gamma \to X\}_{\gamma \in \mathcal{A}_0}$ factors through the map $x_{\alpha_1} : X_{\alpha_1} \to X$. So there is a cocone $\{u'_\gamma : K_\gamma \to X_{\alpha_1}\}_{\gamma \in \mathcal{A}_0}$ such that $x_{\alpha_1} \circ u'_\gamma = u_\gamma$. Similarly, the cocone $\{v_\gamma : L_\gamma \to Y\}_{\gamma \in \mathcal{A}_0}$ factors through $y_{\beta_1} : Y_{\beta_1} \to Y$ for some β_1. So there is a cocone $\{v'_\gamma : L_\gamma \to Y_{\beta_1}\}_{\gamma \in \mathcal{A}_0}$ such that $y_{\beta_1} \circ v'_\gamma = v_\gamma$. Since $f \circ Fx_\alpha : FX_{\alpha_1} \to GY$ is a map on an $(\mathcal{E}_3, \mathcal{M}_3, \lambda)$-presentable object FX_{α_1} it factors through $Gy_{\beta_2} : GY_{\beta_2} \to GY$ for some β_2. Therefore, there is an index β_3 such that $f \circ Fx_{\alpha_1} : FX_{\alpha_1} \to GY$ factors through $Gy_{\beta_3} : GY_{\beta_3} \to GY$ and $y_{\beta_1} : Y_{\beta_1} \to Y$ factors through $y_{\beta_3} : Y_{\beta_3} \to Y$. So in particular, there is a map $h : FX_{\alpha_1} \to GY_{\beta_3}$ in $\mathcal{S}^{F\downarrow G}_\lambda$ such that $Gy_{\beta_3} \circ h = f \circ Fx_{\alpha_1}$ and there is a map $y^{\beta_3}_{\beta_1} : Y_{\beta_1} \to Y_{\beta_3}$ which is a connecting map of the cocone $\{y_\beta\}_\beta$. Note that for each $\gamma \in \mathcal{A}_0$,

$$Gy_{\beta_3} \circ Gy^{\beta_3}_{\beta_1} \circ Gv'_\gamma \circ f_\gamma = Gv_\gamma \circ f_\gamma = f \circ Fu_\gamma = f \circ Fx_{\alpha_1} \circ Fu'_\gamma = Gy_{\beta_3} \circ h \circ Fu'_\gamma.$$

$$\begin{tikzcd}
FK_\gamma \arrow[rr, "Fu'_\gamma"] \arrow[rrr, bend left, "Fu_\gamma"] \arrow[dd, "f_\gamma"] & & FX_{\alpha_1} \arrow[r, "Fx_{\alpha_1}"] \arrow[d] \arrow[dd, bend right, "h"'] & FX \arrow[dd, "f"] \\
 & & GY_{\beta_2} \arrow[d] & \\
GL_\gamma \arrow[r, "Gv'_\gamma"] \arrow[rrr, bend right, "Gv_\gamma"'] & GY_{\beta_1} \arrow[r, "Gy^{\beta_3}_{\beta_1}"] & GY_{\beta_3} \arrow[r, "Gy_{\beta_3}"] & GY
\end{tikzcd}$$

Since Gy_{β_3} is in $\mathcal{M}_3$, it is a monomorphism. So $G(y^{\beta_3}_{\beta_1} \circ v'_\gamma) \circ f_\gamma = h \circ Fu'_\gamma$ for each $\gamma \in \mathcal{A}_0$. Since $h \in \mathcal{S}^{F\downarrow G}_\lambda$, $Fx_{\alpha_1} \in \mathcal{M}_3$, and $Gy_{\beta_3} \in \mathcal{M}_3$, the cocone $\{(u_\gamma, v_\gamma)\}_\gamma$ is λ-filtered.

(I b) $(\mathcal{E}_1 \times \mathcal{E}_2) \cap \mathcal{M}^{\mapsdown}$-*Tight.* To show that $\{(u_\gamma, v_\gamma)\}_\gamma$ is $((\mathcal{E}_1 \times \mathcal{E}_2) \cap \mathcal{M}^{\mapsdown})^{\mathbf{2}}$-tight, it suffices to show that the functor P_1, when restricted to $\{(u_\gamma, v_\gamma)\}_\gamma$, defines a final functor from the indexing category of $\{(u_\gamma, v_\gamma)\}_\gamma$ to the indexing category of $\{x_\alpha\}_\alpha$ and that the functor P_2 restricted to $\{(u_\gamma, v_\gamma)\}_\gamma$ defines a final functor from the indexing category of $\{(u_\gamma, v_\gamma)\}_\gamma$ to the indexing category of $\{y_\beta\}_\beta$. We will use proposition A.2.4 for this purpose.

We will first prove the result for the functor P_1. Note that since $\{x_\alpha : X_\alpha \to X\}_\alpha$ is an $\mathcal{M}_1$-cocone, it is a cocone of monomorphisms. Fix an index α and consider the map $x_\alpha : X_\alpha \to X$. Since $f \circ Fx_\alpha : FX_\alpha \to GY$ is a map on an $(\mathcal{E}_3, \mathcal{M}_3, \lambda)$-presentable object, it factors through $Gy_\beta : GY_\beta \to GY$ for some β. So there is a an object $h \in \mathcal{S}_\lambda^{F\downarrow G}$ and a map $(x_\alpha, y_\beta) : h \to f$ in $F \downarrow G$ such that $P_1(x_\alpha, y_\beta) = x_\alpha : X_\alpha \to X$. Finality now follows from A.2.4. So $\{P_1(u_\gamma, v_\gamma)\}_\gamma$ is an $\mathcal{E}_1 \cap \mathcal{M}^{\mapsdown}$-tight $(\mathcal{M}_1, \lambda)$-cocone in $\mathscr{A}$.

It remains to prove the result for P_2. Fix indices β_1 and γ_1. Since $v_{\gamma_1} : L_{\gamma_1} \to Y$ is equal to $y_{\beta_2} : Y_{\beta_2} \to Y$ for some β_2, there is an index β_3 such that $y_{\beta_1} : Y_{\beta_1} \to Y$ and $y_{\beta_2} = v_{\gamma_1} : L_{\gamma_1} \to Y$ both factor through $y_{\beta_3} : Y_{\beta_3} \to Y$. Let $y_{\beta_2}^{\beta_3} : Y_{\beta_2} \to Y_{\beta_3}$ be a connecting map of the cocone $\{y_\beta\}_\beta$. Then $Gy_{\beta_2}^{\beta_3} \circ f_{\gamma_1} : FK_{\gamma_1} \to GY_{\beta_3}$ is an object in $\mathcal{S}_\lambda^{F\downarrow G}$ and $(u_{\gamma_1}, y_{\beta_3}) : Gy_{\beta_2}^{\beta_3} \circ f_{\gamma_1} \to f$ is a map in $F \downarrow G$.

$$\begin{tikzcd}
L_{\gamma_1} = Y_{\beta_2} \arrow[dr, "y_{\beta_2}^{\beta_3}"'] \arrow[drr, bend left, "v_{\gamma_1} = y_{\beta_2}"] & & \\
& Y_{\beta_3} \arrow[r, "y_{\beta_3}"] & Y \\
Y_{\beta_1} \arrow[ur] \arrow[urr, bend right, "y_{\beta_1}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
FK_{\gamma_1} \arrow[rr, "Fu_{\gamma_1}"] \arrow[d, "f_{\gamma_1}"'] & & FX \arrow[dd, "f"] \\
GL_{\gamma_1} \arrow[d, "Gy_{\beta_2}^{\beta_3}"'] \arrow[drr, "Gv_{\gamma_1}"] & & \\
GY_{\beta_3} \arrow[rr, "Gy_{\beta_3}"] & & GY
\end{tikzcd}$$

So we've shown that $y_{\beta_1} : Y_{\beta_1} \to Y$ factors through the map $P_2(u_{\gamma_1}, y_{\beta_3})$, where $(u_{\gamma_1}, y_{\beta_3})$ is a map in the cocone $\{(u_\gamma, v_\gamma)\}_\gamma$. Finality now follows from A.2.4. So $\{P_2(u_\gamma, v_\gamma)\}_\gamma$ is an $\mathcal{E}_1 \cap \mathcal{M}^{\mapsdown}$-tight $(\mathcal{M}_1, \lambda)$-cocone in $\mathscr{A}$.

(II a) Let $\{(a_\tau, b_\tau) : h_\tau \to f\}_\tau$ be the canonical $P^{-1}_{\mathscr{C}^{\mathbf{2}}}(\mathcal{M}^{\mathbf{2}}_3)$-cocone of f relative to $\mathbf{Full}(\mathcal{S}^{F\downarrow G}_\lambda)$ in the category $F \downarrow G$. So $(a_\tau, b_\tau) : h_\tau \to f$ is a map in the cocone if $h_\tau \in \mathcal{S}^{F\downarrow G}_\lambda$, $Fa_\gamma \in \mathcal{M}_3$, and $Gb_\gamma \in \mathcal{M}_3$. An identical proof to the one in part (I a) shows that this sequence is λ-filtered. An identical proof to the one in part (I b) shows that $\{P_1(a_\tau, b_\tau)\}_\tau$ is an $\mathcal{E}_1 \cap \mathcal{M}^{\lightning}$-tight cocone in $\mathscr{A}$ and that $\{P_2(a_\tau, b_\tau)\}_\tau$ is an $\mathcal{E}_2 \cap \mathcal{M}^{\lightning}$-tight cocone in $\mathscr{B}$. So $\{Fa_\tau\}_\tau$ and $\{Gb_\tau\}_\tau$ are $\mathcal{E}_3 \cap \mathcal{M}^{\lightning}$-tight cocones in $\mathscr{C}$. Thus

$$\left\{P_{\mathscr{C}^{\mathbf{2}}}(a_\tau, b_\tau) = (Fa_\tau, Gb_\tau) : h_\tau \to f\right\}_\tau$$

is $(\mathcal{E}_3 \cap \mathcal{M}^{\lightning})^{\mathbf{2}}$-tight cocone in $\mathscr{C}^{\mathbf{2}}$.

(II b) We will conclude by showing that every $f : FK \to GL$ in $\mathcal{S}^{F\downarrow G}_\lambda$ is an $(\mathcal{E}^{\mathbf{2}}_3, \mathcal{M}^{\mathbf{2}}_3, \lambda)$-presentable object in $\mathscr{C}^{\mathbf{2}}$. Let $\{(u_\alpha, v_\alpha) : g_\alpha \to g\}_\alpha$ be an $(\mathcal{E}_3 \cap \mathcal{M}^{\lightning})^{\mathbf{2}}$-tight $(\mathcal{M}^{\mathbf{2}}_3, \lambda)$-cocone and let $(s, t) : f \to g$ be a map. Since FK and GL are $(\mathcal{E}_3, \mathcal{M}_3, \lambda)$-presentable, there are lifts $s' : FK \to \operatorname{dom} g_{\alpha_1}$ and $t' : GL \to \operatorname{cod} g_{\alpha_2}$ such that $s = u_{\alpha_1} \circ s'$ and $t = v_{\alpha_2} \circ t'$. There is an α_3 such that $(u_{\alpha_1}, v_{\alpha_1}) : g_{\alpha_1} \to g$ and $(u_{\alpha_2}, v_{\alpha_2}) : g_{\alpha_2} \to g$ factor through $(u_{\alpha_3}, v_{\alpha_3}) : g_{\alpha_3} \to g$. So there are maps $s'' : FK \to \operatorname{dom} g_{\alpha_3}$ and $t'' : GL \to \operatorname{cod} g_{\alpha_3}$ such that $s = u_{\alpha_3} \circ s''$ and $t = v_{\alpha_3} \circ t''$. Since v_{α_3} is a monomorphism and

$$v_{\alpha_3} \circ g_{\alpha_3} \circ s'' = g \circ u_{\alpha_3} \circ s'' = g \circ s = t \circ f = v_{\alpha_3} \circ t'' \circ f,$$

$(s'', t'') : f \to g_{\alpha_3}$ is a map in $\mathscr{C}^{\mathbf{2}}$. It follows that f is $(\mathcal{E}^{\mathbf{2}}_3, \mathcal{M}^{\mathbf{2}}_3, \lambda)$-presentable. ☐

Corollary 6.2.14. *If $\mathscr{C}$ is an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible category, then $\mathscr{C}^{\mathbf{2}}$ is an $(\mathcal{E}^{\mathbf{2}}, \mathcal{M}^{\mathbf{2}}, \lambda)$-quasiaccessible category.*

Proof. Since $\mathscr{C}$ has λ-filtered colimits, the category $\mathscr{C}^{\mathbf{2}}$ will also have λ-filtered colimits. The category $\mathscr{C}^{\mathbf{2}}$ will also inherit well-copoweredness from $\mathscr{C}$. The proper orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ defines a proper orthogonal factorization system $(\mathcal{E}^{\mathbf{2}}, \mathcal{M}^{\mathbf{2}})$ on $\mathscr{C}^{\mathbf{2}}$. So the only thing left to verify is condition (4) of definition 6.1.4. Let F and G in theorem 6.2.13 both be the identity functor on $\mathscr{C}$. Then $\mathcal{S}_\lambda^{F\downarrow G}$ is just a set of maps $f : K \to L$ between $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable objects in $\mathscr{C}$. We know from the theorem that the objects of $\mathcal{S}_\lambda^{F\downarrow G}$ are $(\mathcal{E}^{\mathbf{2}}, \mathcal{M}^{\mathbf{2}}, \lambda)$-presentable objects in $\mathscr{C}^{\mathbf{2}}$ and that for every object g in $\mathscr{C}^{\mathbf{2}}$ there is an $(\mathcal{E} \cap \mathcal{M}^{\updownarrow})$-tight $(\mathcal{M}, \lambda)$-cocone of objects in $\mathcal{S}_\lambda^{F\downarrow G}$ over g. □

Corollary 6.2.15. *If $F : \mathscr{A} \to \mathscr{C}$ and $G : \mathscr{B} \to \mathscr{C}$ are quasiaccessible functors, then there is a regular cardinal λ_0 such that for every regular cardinal λ with $\lambda_0 \lhd \lambda$, $P_{\mathscr{C}^{\mathbf{2}}} : F \downarrow G \to \mathscr{C}^{\mathbf{2}}$ is weakly $(\mathcal{E}_3^{\mathbf{2}}, \mathcal{M}_3^{\mathbf{2}}, \lambda)$-quasiaccessible relative to $\mathcal{S}_\lambda^{F\downarrow G}$.*

6.2.3 Inserter Categories

Given functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{A} \to \mathscr{B}$, the *inserter category* for F and G, $\mathbf{Ins}(F, G)$, is the subcategory of the comma category $F \downarrow G$ whose objects are the objects are triples (A, f, A) and whose morphisms are the maps $(u, u) : (A, f, A) \to (A', g, A')$.

$$\begin{tikzcd}
FA \arrow[r, "Fu"] \arrow[d, "f"] & FA' \arrow[d, "g"] \\
GA \arrow[r, "Gu"] & GA'
\end{tikzcd}$$

We will use the notation $P_{\mathscr{A}} : \mathbf{Ins}(F, G) \to \mathscr{A}$ for the projection functor to $\mathscr{A}$ that sends objects (A, f, A) to A and sends morphisms $(u, v) : (A, f, A) \to (A', g, A')$ to $u : A \to A'$. Let $\mathcal{S}_\lambda^{\mathbf{Ins}(F,G)}$ be the subset of $\mathcal{S}_\lambda^{F\downarrow G}$ on the objects in $\mathbf{Ins}(F, G)$.

Theorem 6.2.16. *Let $\mathscr{A}$ be an $(\mathcal{E}_1, \mathcal{M}_1)$-quasiaccessible category and let $\mathscr{B}$ be an $(\mathcal{E}_2, \mathcal{M}_2)$-quasiaccessible category. If $F, G : \mathscr{A} \to \mathscr{B}$ are quasiaccessible functors, then there is a regular cardinal λ such that the functor $P_{\mathscr{A}} : \mathbf{Ins}(F, G) \to \mathscr{A}$ is weakly $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-quasiaccessible relative to $\mathcal{S}_\lambda^{\mathbf{Ins}(F,G)}$.*

Proof. By 6.2.4 and 6.2.15, there are a regular cardinals λ_0 and λ with $\lambda_0 \lhd \lambda$ such that the following hold.

1. $\mathscr{A}$ is $(\mathcal{E}_1, \mathcal{M}_1, \lambda_0)$-quasiaccessible and $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-quasiaccessible.

2. $\mathscr{B}$ is $(\mathcal{E}_2, \mathcal{M}_2, \lambda_0)$-quasiaccessible and $(\mathcal{E}_2, \mathcal{M}_2, \lambda)$-quasiaccessible.

3. F and G are both λ_0-quasiaccessible and λ-quasiaccessible functors that send $(\mathcal{E}_1, \mathcal{M}_1, \lambda_0)$-presentable objects to $(\mathcal{E}_2, \mathcal{M}_2, \lambda_0)$-presentable objects and that send $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-presentable objects to $(\mathcal{E}_2, \mathcal{M}_2, \lambda)$-presentable objects.

4. $P_{1\times 2} : F \downarrow G \to \mathscr{A} \times \mathscr{A}$ is weakly $(\mathcal{E}_1 \times \mathcal{E}_1, \mathcal{M}_1 \times \mathcal{M}_1, \lambda)$-quasiaccessible relative to $\mathcal{S}_\lambda^{F\downarrow G}$.

Let $f : FX \to GX$ be an object in $\mathbf{Ins}(F, G)$.

(I) Let $g : FA \to GB$ be an object in $\mathcal{S}_\lambda^{F\downarrow G}$, and let $(u, v) : g \to f$ be a map in $F \downarrow G$. We will show that, in the category $F \downarrow G$, (u, v) factors through a map $(q, q) : h \to f$ on an object $h \in \mathcal{S}_\lambda^{\mathbf{Ins}(F,G)}$. We do this by first selecting an $\mathcal{E}_1 \cap \mathcal{M}^{↧}$-tight λ-directed $\mathcal{M}_1$-cocone $\{k_\beta : K_\beta \to X\}_\beta$ of $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-presentable objects in $\mathscr{A}$ and then constructing a λ-small λ_0-filtered cocone $\{(k_{\beta_\iota}, k_{\beta_{\iota+1}}) : h_\iota \to f\}_\iota$ in $F \downarrow G$ such that $\{k_{\beta_\iota}\}_\iota$ is a subcocone of $\{k_\beta\}_\beta$. The desired object h will then be a particular $\mathcal{E}_2^{\mathbf{2}}$-quotient of the colimit of the cocone $\{(Fk_{\beta_\iota}, Gk_{\beta_{\iota+1}}) : h_\iota \to f\}_\iota$ in $\mathscr{B}^{\mathbf{2}}$.

(I a) *Initial Step.* There is a β_0 such that $u : A \to X$ factors through $k_{\beta_0} : K_{\beta_0} \to X$. Let $u' : A \to K_{\beta_0}$ be the lift of u. Since $\{Gk_\beta : GK_\beta \to GX\}_\beta$ is an $\mathcal{E}_2 \cap \mathcal{M}^{↧}$-tight $(\mathcal{M}_2, \lambda)$-cocone, there is a β_0' such that $f \circ Fk_{\beta_0} : FK_{\beta_0} \to GX$ factors through $Gk_{\beta_0'} : GK_{\beta_0'} \to GX$ in $\mathscr{B}$. It follows that there is an β_1, $\beta_0 \leq \beta_1$, $\beta_0' \leq \beta_1$ such that $B \to X$ factors through $k_{\beta_1} : K_{\beta_1} \to X$. So there is a map $h_0 : FK_{\beta_0} \to GK_{\beta_1}$ in $\mathscr{B}$ and $v' : B \to K_{\beta_1}$ in $\mathscr{A}$ such that $f \circ Fk_{\beta_0} = Gk_{\beta_1} \circ h_0$ and $k_{\beta_1} \circ v' = v$. Thus

$$Gk_{\beta_1} \circ Gv' \circ g = Gv \circ g = f \circ Fu = f \circ Fk_{\beta_0} \circ Fu' = Gk_{\beta_1} \circ h_0 \circ Fu'.$$

Since Gk_{β_1} is in $\mathcal{M}_2$, it is a monomorphism. So $(u', v') : g \to h_0$ is a map in $F \downarrow G$.

(I b) *Successor Step.* Suppose the following hold for an ordinal ι.

- There is a map $h_\iota : FK_{\beta_\iota} \to GK_{\beta_{\iota+1}}$ in $\mathscr{B}$.
- $(k_{\beta_\iota}, k_{\beta_{\iota+1}}) : h_\iota \to f$ is a map in $F \downarrow G$.
- There is a connecting map $k_{\beta_\iota}^{\beta_{\iota+1}} : K_{\beta_\iota} \to K_{\beta_{\iota+1}}$ in the cocone $\{k_\beta\}_\beta$.

There is a $\beta_{\iota+2}$ such that $f \circ Fk_{\beta_{\iota+1}} : FK_{\beta_{\iota+1}} \to GX$ factors through $Gk_{\beta_{\iota+2}} : GK_{\beta_{\iota+2}} \to GX$ via a map $h_{\iota+1} : FK_{\beta_{\iota+1}} \to GK_{\beta_{\iota+2}}$ in $\mathscr{B}$. In other words, $(k_{\beta_{\iota+1}}, k_{\beta_{\iota+2}}) : h_{\iota+1} \to f$ is a map in $F \downarrow G$. Without loss of generality, we can assume $\beta_{\iota+2} \geq \beta_{\iota+1}$. So there is a connecting map $k_{\beta_{\iota+1}}^{\beta_{\iota+2}} : K_{\beta_{\iota+1}} \to K_{\beta_{\iota+2}}$ in the cocone $\{k_\beta\}_\beta$. Since

$$Gk_{\beta_{\iota+2}} \circ h_{\iota+1} \circ Fk_{\beta_\iota}^{\beta_{\iota+1}} = f \circ Fk_{\beta_\iota} = Gk_{\beta_{\iota+1}} \circ h_\iota = Gk_{\beta_{\iota+2}} \circ Gk_{\beta_{\iota+1}}^{\beta_{\iota+2}} \circ h_\iota$$

and since $Gk_{\beta_{\iota+2}} : GK_{\beta_{\iota+2}} \to GX$ is a monomorphism, the following diagram commutes.

$$\begin{tikzcd}
FK_{\beta_\iota} \arrow[r, "Fk_{\beta_\iota}^{\beta_{\iota+1}}"] \arrow[d, "h_\iota"] \arrow[rr, bend left, "Fk_{\beta_\iota}"] & FK_{\beta_{\iota+1}} \arrow[r, "Fk_{\beta_{\iota+1}}"] \arrow[d, "h_{\iota+1}"] & FX \arrow[d, "f"] \\
GK_{\beta_{\iota+1}} \arrow[r, "Gk_{\beta_{\iota+1}}^{\beta_{\iota+2}}"] \arrow[rr, bend right, "Gk_{\beta_{\iota+1}}"'] & GK_{\beta_{\iota+2}} \arrow[r, "Gk_{\beta_{\iota+2}}"] & GX
\end{tikzcd}$$

So $(k_{\beta_\iota}^{\beta_{\iota+1}}, k_{\beta_{\iota+1}}^{\beta_{\iota+2}}) : h_\iota \to h_{\iota+1}$ is a map in $F \downarrow G$.

(I c) *Limit Step.* Suppose κ is a limit ordinal with $|\kappa| \leq \lambda$ and the following hold for every ordinal $\iota < \kappa$.

- $h_\iota : FK_{\beta_\iota} \to GK_{\beta_{\iota+1}}$ is a map in $\mathscr{B}$.
- $(k_{\beta_\iota}, k_{\beta_{\iota+1}}) : h_\iota \to f$ is a map in $F \downarrow G$.
- $k_{\beta_\iota}^{\beta_{\iota+1}} : K_{\beta_\iota} \to K_{\beta_{\iota+1}}$ is a connecting map in the cocone $\{k_\beta\}_\beta$.

- $(k^{\beta_\iota}_{\beta_{\iota-1}}, k^{\beta_{\iota+1}}_{\beta_\iota}) : h_{\iota-1} \to h_\iota$ is a map in $F \downarrow G$.

Since $\{(k_{\beta_\iota}, k_{\beta_{\iota+1}}) : h_\iota \to f\}_{\iota<\kappa}$ is a λ-small subcocone of the λ-filtered cocone $\{k_\beta\}_\beta$, there is an index β_κ with $\beta_\kappa > \beta_\iota$ for every $\iota < \kappa$. There is then an index $\beta_{\kappa+1}$, $\beta_{\kappa+1} \geq \beta_\kappa$, and a map $h_\kappa : FK_{\beta_\kappa} \to GK_{\beta_{\kappa+1}}$ in $\mathscr{B}$ such that $f \circ Fk_{\beta_\kappa} = Gk_{\beta_{\kappa+1}} \circ h_\kappa$. So $(k_{\beta_\kappa}, k_{\beta_{\kappa+1}}) : h_\kappa \to f$ is a map in $F \downarrow G$ and there is a connecting map $k^{\beta_{\kappa+1}}_{\beta_\kappa} : K_{\beta_\kappa} \to K_{\beta_{\kappa+1}}$ in the cocone $\{k_\beta\}_\beta$. For each $\iota < \kappa$,

$$Gk_{\beta_{\kappa+1}} \circ h_\kappa \circ Fk^{\beta_\kappa}_{\beta_\iota} = f \circ Fk_{\beta_\iota} = Gk_{\beta_{\iota+1}} \circ h_\iota = Gk_{\beta_{\kappa+1}} \circ Gk^{\beta_{\kappa+1}}_{\beta_{\iota+1}} \circ h_\iota,$$

Since $Gk_{\beta_{\kappa+1}} : GK_{\beta_{\kappa+1}} \to GX$ is a monomorphism, $(k^{\beta_\kappa}_{\beta_\iota}, k^{\beta_{\kappa+1}}_{\beta_{\iota+1}}) : h_\iota \to h_\kappa$ is a map in $F \downarrow G$ for each $\iota < \kappa$.

(I d) *Final Step.* Let κ be the initial ordinal for the cardinal λ. Since λ is a regular cardinal, κ is a regular ordinal. So κ is, in particular, a limit ordinal. Since $|\kappa| = \lambda$, the cocone $\{(k_{\beta_\iota}, k_{\beta_{\iota+1}}) : h_\iota \to f\}_{\iota<\kappa}$ is defined. The fact that λ is a regular cardinal means that no λ_0-small subposet of κ is final. So, viewed as a totally ordered set, κ is λ_0-filtered. Thus $\{(k_{\beta_\iota}, k_{\beta_{\iota+1}}) : h_\iota \to f\}_{\iota<\kappa}$ is a λ-small λ_0-filtered cocone.

Because the cocone $\{k_{\beta_\iota}\}_{\iota<\kappa}$ is a subcocone of $\{k_\beta\}_\beta$, $k_{\beta_\iota} : K_{\beta_\iota} \to X$ is an $\mathcal{M}_1$-map for each ι. By 6.1.12 (4), there is an object T with an $\mathcal{M}_1$-map $q : T \to X$ and an $\mathcal{E}_1 \cap \mathcal{M}^{\lightning}$-tight $(\mathcal{M}_1, \lambda_0)$-cocone $\{t_\iota : K_{\beta_\iota} \to T\}_{\iota<\kappa}$ such that $q \circ t_\iota = k_{\beta_\iota}$ for each $\iota < \kappa$. By 6.1.3 (5), T is an $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-presentable object.

By our assumptions on F and G, the cocones $\{Fk_{\beta_\iota} : FK_{\beta_\iota} \to FT\}_{\iota<\kappa}$ and $\{Gk_{\beta_\iota} : GK_{\beta_\iota} \to GT\}_{\iota<\kappa}$ are $\mathcal{E}_2 \cap \mathcal{M}^{\lightning}$-tight cocones in $\mathscr{B}$. Furthermore, by 6.2.2, $Fq : FT \to FX$ and $Gq : GT \to GX$ are $\mathcal{M}_2$-maps. So we have the following commutative diagram, where r and

s are the maps defined by the cocones $\{Ft_\iota\}_{\iota<\kappa}$ and $\{Gt_\iota\}_{\iota<\kappa}$, respectively.

$$\begin{tikzcd}
\operatorname{colim}_{\iota<\kappa} FK_{\beta_\iota} \arrow[r, "r"] \arrow[d, "\operatorname{colim} h_\iota"] & FT \arrow[r, "Fq"] \arrow[d, dashed, "h"] & FX \arrow[d, "f"] \\
\operatorname{colim}_{\iota<\kappa} GK_{\beta_{\iota+1}} \arrow[r, "s"] & GT \arrow[r, "Gq"] & GX
\end{tikzcd}$$

Since $r \in \mathcal{E}_2$ and $Gq \in \mathcal{M}_2$, there is a unique map $h : FT \to GT$ making the above diagram commute.

(II) Let $\{(x_\alpha, x_\alpha) : f_\alpha \to f\}_{\alpha\in\mathcal{A}}$ be the canonical $P^{-1}_{1\times2}(\mathcal{M}_1 \times \mathcal{M}_1)$-cocone of f with respect to $\mathcal{S}^{\mathbf{Ins}(F,G)}_\lambda$ in $\mathbf{Ins}(F,G)$. We will denote the objects $P_{\mathscr{A}} f_\alpha$ in $\mathscr{A}$ by X_α. So f_α is a map $f_\alpha : FX_\alpha \to GX_\alpha$. Showing that the cocone $\{P_{1\times2}(x_\alpha, x_\alpha)\}_{\alpha\in\mathcal{A}}$ is an $(\mathcal{E}_1 \times \mathcal{E}_1) \cap \mathcal{M}^{\lightning}$-tight λ-filtered cocone in $\mathscr{A} \times \mathscr{A}$ is equivalent to showing that $\{x_\alpha : X_\alpha \to X\}_{\alpha\in\mathcal{A}}$ is an $\mathcal{E}_1$-tight λ-filtered cocone.

(II a) λ-*Filtered.* Let $\{x_\alpha\}_{\alpha\in\mathcal{A}_0}$ be a λ-small subcocone of $\{x_\alpha\}_{\alpha\in\mathcal{A}}$. We know that there is an $\mathcal{E}_1 \cap \mathcal{M}^{\lightning}$-tight $(\mathcal{M}_1, \lambda)$-cocone $\{k_\beta : K_\beta \to X\}_\beta$ of $(\mathcal{E}_1, \mathcal{M}_1, \lambda)$-presentable objects over X. Therefore, by 6.1.2 (2), the cocone $\{x_\alpha : X_\alpha \to X\}_{\alpha\in\mathcal{A}_0}$ lifts to a cocone $\{y_\alpha : X_\alpha \to K_{\beta'}\}_{\alpha\in\mathcal{A}_0}$ for some β'. Since $\{Gk_\beta : GK_\beta \to GX\}_\beta$ is an $\mathcal{E}_2 \cap \mathcal{M}^{\lightning}$-tight $(\mathcal{M}_2, \lambda)$-cocone, the map $f \circ Fk_{\beta'} : FK_{\beta'} \to GX$ factors through a map $Gk_{\beta''} : GK_{\beta''} \to GX$. We can choose $\beta'' > \beta'$ so that there is a map $k^{\beta''}_{\beta'} \circ y_\alpha : X_\alpha \to K_{\beta''}$ with $x_\alpha = k^{\beta''}_{\beta'} \circ y_\alpha$ for each $\alpha \in \mathcal{A}_0$. So there is a map $g : FK_{\beta'} \to FK_{\beta''}$ in $\mathscr{B}$ and a map $(k_{\beta'}, k_{\beta''}) : g \to f$ in $F \downarrow G$ and

$$Gk_{\beta''} \circ g \circ Fy_\alpha = f \circ Fx_\alpha = Gx_\alpha \circ f_\alpha = Gk_{\beta''} \circ G(k^{\beta''}_{\beta'} \circ y_\alpha) \circ f_\alpha.$$

Since $Gk_{\beta''}$ is a monomorphism the following diagram commutes for every $\alpha \in \mathcal{A}_0$.

$$\begin{tikzcd}
FX_\alpha \arrow[r, "Fy_\alpha"] \arrow[rr, bend left, "Fx_\alpha"] \arrow[d, "f_\alpha"'] & FK_{\beta'} \arrow[r, "Fk_{\beta'}"] \arrow[d, "g"] & FX \arrow[d, "f"] \\
GX_\alpha \arrow[r, "G(k^{\beta''}_{\beta'} \circ y_\alpha)"] \arrow[rr, bend right, "Gx_\alpha"'] & GK_{\beta''} \arrow[r, "Gk_{\beta''}"] & GX
\end{tikzcd}$$

But by our work in (I), we know that, in $F \downarrow G$, the map $(k_{\beta'}, k_{\beta''}) : g \to f$ factors through a map $(q,q) : h \to f$, where $h \in \mathcal{S}_\lambda^{\mathbf{Ins}(F,G)}$ and $q \in \mathcal{M}_1$. Let $(a,b) : g \to h$ be the map such that $(q,q) \circ (a,b) = (k_{\beta'}, k_{\beta''})$. We now know that the following diagram in $\mathscr{A}$ commutes for each $\alpha \in \mathcal{A}_0$.

$$\begin{tikzcd}
K_{\beta'} \arrow[dr, "q\circ a=k_{\beta'}"'] & X_\alpha \arrow[l, "y_\alpha"'] \arrow[r, "k^{\beta''}_{\beta'}\circ y_\alpha"] \arrow[d, "x_\alpha"] & K_{\beta''} \arrow[dl, "q\circ b=k_{\beta''}"] \\
 & X &
\end{tikzcd}$$

Since q is a monomorphism, $a \circ y_\alpha = b \circ k^{\beta''}_{\beta'} \circ y_\alpha$. So $(a \circ y_\alpha, b \circ k^{\beta''}_{\beta'} \circ y_\alpha) : f_\alpha \to h$ is a map in $\mathbf{Ins}(F,G)$. Since $h \in \mathcal{S}_\lambda^{\mathbf{Ins}(F,G)}$, $(q,q) : h \to f$ is equal to $(x_{\alpha'}, x_{\alpha'}) : f_{\alpha'} \to f$ for some α'. Thus $\{(x_\alpha, x_\alpha)\}_{\alpha \in \mathcal{A}}$ is λ-filtered.

(II b) $\mathcal{E}_1 \cap \mathcal{M}^{\mapsdown}$-*Tight.* In the category $F \downarrow G$, let $\{(u_\gamma, v_\gamma) : g_\gamma \to f\}_\gamma$ be the canonical $P^{-1}_{1\times2}((\mathcal{M}_1 \times \mathcal{M}_1)^{\mathbf{2}})$-cocone of f relative to maps $g_\gamma : FA_\gamma \to GB_\gamma$ in $\mathcal{S}_\lambda^{F\downarrow G}$. By (4), $\{u_\gamma\}_\gamma$ and $\{v_\gamma\}_\gamma$ are $\mathcal{E}_1 \cap \mathcal{M}^{\mapsdown}$-tight $(\mathcal{M}_1, \lambda)$-cocones in $\mathscr{A}$. We will apply A.2.4 to show that $\{(x_\alpha, x_\alpha) : f_\alpha \to f\}_\alpha$ is a final subcocone of $\{(u_\gamma, v_\gamma) : g_\gamma \to f\}_\gamma$ in $\mathscr{B}^{\mathbf{2}}$. By (I), each map $(u_\gamma, v_\gamma) : g_\gamma \to f$ factors through some $(x_\alpha, x_\alpha) : f_\alpha \to f$ in $F \downarrow G$. Since the maps $(u_\gamma, v_\gamma) : g_\gamma \to f$ are all monomorphisms in $F \downarrow G$, A.2.4 applies. Thus $\{x_\alpha : X_\alpha \to X\}_\alpha$ is an $\mathcal{E}_1 \cap \mathcal{M}^{\mapsdown}$-tight $(\mathcal{M}_1, \lambda)$-cocone in $\mathscr{A}$.

$\square$

6.2.4 Categories of Coalgebras

Let $\varphi : F \to G$ and $\psi : F \to G$ be natural transformations between functors $F, G : \mathscr{A} \to \mathscr{B}$. The *equifier category*, $\mathbf{Eq}(\varphi, \psi)$, of φ and ψ is the full subcategory of $\mathscr{A}$ on the objects A such that $\varphi_A = \psi_A : FA \to GA$.

Proposition 6.2.17. *Let $F : \mathscr{A} \to \mathscr{B}$ be a weakly $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible functor relative to a set $\mathcal{S}_\lambda$ of objects in $\mathscr{A}$. Let $G : \mathscr{A} \to \mathscr{B}$ be a functor such that $F^{-1}(\mathcal{M}) \subseteq G^{-1}(\mathcal{M})$. If φ :*

$F \to G$ and $\psi : F \to G$ are natural transformations, then the functor $F|_{\mathbf{Eq}(\varphi,\psi)} : \mathbf{Eq}(\varphi,\psi) \to \mathscr{B}$ we get by restricting F to $\mathbf{Eq}(\varphi,\psi)$ is a weakly $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible functor relative to the subset $\mathcal{S}_\lambda^{\mathbf{Eq}(\varphi,\psi)}$ of $\mathcal{S}_\lambda$ on the objects in $\mathbf{Eq}(\varphi,\psi)$.

Proof. By 6.2.9, it is sufficient to show that $\mathbf{Eq}(\varphi,\psi)$ is closed under $F^{-1}(\mathcal{M})$-subobjects. Let X be an object in $\mathbf{Eq}(\varphi,\psi)$ and let $f : K \to X$ be a map $\mathscr{A}$ such that $Ff \in \mathcal{M}$. Then, since $F^{-1}(\mathcal{M}) \subseteq G^{-1}(\mathcal{M})$, $Gf \in \mathcal{M}$. Because $Gf \circ \varphi_K = \varphi_X \circ Ff = \psi_X \circ Ff = Gf \circ \psi_K$ and Gf is a monomorphism, $\varphi_K = \psi_K$. □

Theorem 6.2.18. *Let $\mathscr{A}$ be a $(\mathcal{E},\mathcal{M})$-quasiaccessible category. If (L,ε) is a quasiaccessible pointed endofunctor on $\mathscr{A}$, then the forgetful functor $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathscr{A}$ is weakly $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible for some regular cardinal λ.*

Proof. By 6.2.16, there is a regular cardinal λ such that the functor $P_{\mathscr{A}} : \mathbf{Ins}(Id,\mathrm{L}) \to \mathscr{A}$ is $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible relative to $\mathcal{S}_\lambda^{\mathbf{Ins}(Id,\mathrm{L})}$ and the functor $\mathrm{L} : \mathscr{A} \to \mathscr{A}$ is λ-quasiaccessible. There is a natural transformation $\zeta : P_{\mathscr{A}} \to \mathrm{L}P_{\mathscr{A}}$ with $\zeta_{(X,f,X)} = f : X \to \mathrm{L}X$ on each object (X,f,X) in $\mathbf{Ins}(Id,\mathrm{L})$. Let $\psi = \varepsilon P_{\mathscr{A}} \circ \zeta : P_{\mathscr{A}} \to P_{\mathscr{A}}$ and let $id : P_{\mathscr{A}} \to P_{\mathscr{A}}$ be the identity natural transformation. Then $\mathbf{Coalg}_{\mathrm{L}}$ is the category $\mathbf{Eq}(id,\psi)$. Since L is λ-quasiaccessible, $\mathrm{L}(\mathcal{M}) \subseteq \mathcal{M}$. So $P_{\mathscr{A}}^{-1}(\mathcal{M}) \subseteq P_{\mathscr{A}}^{-1}(\mathrm{L}^{-1}(\mathcal{M}))$. Thus, by 6.2.17, the forgetful functor $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathscr{A}$ is weakly $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible relative to a set of representatives for the isomorphism classes of $(\mathcal{E},\mathcal{M},\lambda)$-presentable objects in $\mathscr{A}$ that are L-coalgebras. □

The following result uses definition A.3.5.

Corollary 6.2.19. *If $\mathscr{A}$ is a quasiaccessible category and (L,ε) is a quasiaccessible pointed endofunctor on $\mathscr{A}$, then the category $\mathbf{Coalg}_{\mathrm{L}}$ has a $U_{\mathrm{L}}^{-1}(\mathcal{E})$-strong family of generators.*

Proof. By 6.2.18, there is a set $\mathcal{S}$ of objects in $\mathbf{Coalg}_{\mathrm{L}}$ such that for every object $\langle X, k\rangle$ in $\mathbf{Coalg}_{\mathrm{L}}$, there is a cocone $\{s_\alpha : \langle S_\alpha, l_\alpha\rangle \to \langle X, k\rangle\}_\alpha$ of objects $\langle S_\alpha, l_\alpha\rangle \in \mathcal{S}$ such that $\{s_\alpha : S_\alpha \to X\}_\alpha$ is an $\mathcal{E} \cap \mathcal{M}^{\lightning}$-tight $(\mathcal{M}, \lambda)$-cocone. Let

$$s : \coprod_{\alpha} \langle S_\alpha, l_\alpha\rangle \to \langle X, k\rangle$$

be the map defined by the maps $s_\alpha : \langle S_\alpha, l_\alpha\rangle \to \langle X, k\rangle$. We need to show s is in $U_{\mathrm{L}}^{-1}(\mathcal{E})$. By proposition 6.2.11, to show $s : \coprod_\alpha S_\alpha \to X$ is an $\mathcal{E}$-map, it suffices to show $s_* : \mathscr{C}(K, \coprod_\alpha S_\alpha) \to \mathscr{C}(K, X)$ is an epimorphism for each $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object K.

Let $k : K \to X$ be a map on a $(\mathcal{E}, \mathcal{M}, \lambda)$-presentable object K. Then there is an α' such that k factors through $s_{\alpha'}$. Let $in_{S'_\alpha} \to \coprod_\alpha S_\alpha$ be the inclusion of $S_{\alpha'}$ into the α' copy of the coproduct. Then $s \circ in_{S'_\alpha} = s_{\alpha'}$. So k factors through s. Thus $s_* : \mathscr{C}(K, \coprod_\alpha S_\alpha) \to \mathscr{C}(K, X)$ is an epimorphism. $\square$

Proposition 6.2.20. *If $\mathscr{A}$ is a cocomplete $(\mathcal{E}, \mathcal{M})$-quasiaccessible category and $(\mathrm{L}, \varepsilon)$ is an $(\mathcal{E}, \mathcal{M})$-quasiaccessible pointed endofunctor on $\mathscr{A}$, then the forgetful functor $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathscr{A}$ has a right adjoint.*

Proof. We will verify that $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathscr{A}$ satisfies the conditions of the generalized special adjoint functor theorem in appendix A.3. Since $\mathscr{A}$ is cocomplete, the category $\mathbf{Coalg}_{\mathrm{L}}$ is cocomplete and the forgetful functor $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathscr{A}$ preserves colimits. Since $\mathscr{A}$ is $(\mathcal{E}, \mathcal{M})$-quasiaccessible, it is $\mathcal{E}$-well-copowered. There can only be a set of lifts $\langle X, k\rangle$ in $\mathbf{Coalg}_{\mathrm{L}}$ for each object X in $\mathscr{A}$, so $\mathbf{Coalg}_{\mathrm{L}}$ is $U_{\mathrm{L}}^{-1}(\mathcal{E})$-well-copowered. By 6.2.19, $\mathbf{Coalg}_{\mathrm{L}}$ has a $U_{\mathrm{L}}^{-1}(\mathcal{E})$-strong family of generators. Because $\mathcal{E}$ is stable under pushouts and U_{L} preserves colimits, the collection $U_{\mathrm{L}}^{-1}(\mathcal{E})$ is stable under pushouts. The existence of the right adjoint now follows from A.3.6. $\square$

Theorem 6.2.21. *Let $\mathscr{A}$ be a cocomplete $(\mathcal{E},\mathcal{M})$-quasiaccessible category and let (L,ε) be an $(\mathcal{E},\mathcal{M})$-quasiaccessible pointed endofunctor on $\mathscr{A}$. There is a comonad $(\mathrm{L}',\varepsilon,\delta)$ on $\mathscr{A}$ such that $|\mathbf{Coalg}_{\mathrm{L}'}| = |\mathbf{Coalg}_{\mathrm{L}}|$ and $\mathrm{L}' : \mathscr{A} \to \mathscr{A}$ preserves $\mathcal{E}$-tightness of $(\mathcal{M},\lambda)$-cocones.*

Proof. Let $G : \mathscr{A} \to \mathbf{Coalg}_{\mathrm{L}}$ be the right adjoint to the forgetful functor $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathscr{A}$. Let $\nu : I \to GU_{\mathrm{L}}$ and $\xi : U_{\mathrm{L}}G \to I$ be the unit and counit maps for the adjunction, respectively. Let $\mathrm{L}' : \mathscr{A} \to \mathscr{A}$ be the endofunctor $\mathrm{L}' = U_{\mathrm{L}}G$. Then L' is a comonad with counit map $\varepsilon = \xi : U_{\mathrm{L}}G \to I$ and comultiplication map $\delta = U_{\mathrm{L}}\nu G : U_{\mathrm{L}}G \to U_{\mathrm{L}}GU_{\mathrm{L}}G$. Since $\mathrm{L}'X \in |\mathbf{Coalg}_{\mathrm{L}}|$ for every object X in $\mathscr{A}$, $|\mathbf{Coalg}_{\mathrm{L}'}| \subseteq |\mathbf{Coalg}_{\mathrm{L}}|$. Now let X be an object in $|\mathbf{Coalg}_{\mathrm{L}}|$. So X has a lift $\langle X,k\rangle$ in $\mathbf{Coalg}_{\mathrm{L}}$. The commutativity of the following diagram shows that $X \in |\mathbf{Coalg}_{\mathrm{L}'}|$.

$$\begin{tikzcd}
U_{\mathrm{L}}\langle X,k\rangle \arrow[r, "U_{\mathrm{L}}\nu_{\langle X,k\rangle}"] \arrow[rr, bend left, "id"] & U_{\mathrm{L}}GU_{\mathrm{L}}\langle X,k\rangle \arrow[r, "\xi_{U_{\mathrm{L}}\langle X,k\rangle}"] & U_{\mathrm{L}}\langle X,k\rangle
\end{tikzcd}$$

By 6.2.18, there is a regular cardinal λ and a set $\mathcal{S}_\lambda$ of objects in $\mathbf{Coalg}_{\mathrm{L}}$ such that $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathscr{A}$ is an $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible category relative to $\mathcal{S}_\lambda$. Specifically, by 6.2.16 and 6.2.17, $\mathcal{S}_\lambda$ is the collection of objects in $\mathbf{Coalg}_{\mathrm{L}}$ that forget to $(\mathcal{E},\mathcal{M},\lambda)$-presentable objects in $\mathscr{A}$. Let X be an object in $\mathscr{A}$ and let $\{x_\alpha : X_\alpha \to X\}_\alpha$ be an $\mathcal{E}$-tight $(\mathcal{M},\lambda)$-cocone. By 6.1.12 (5), $\{x_\alpha\}_\alpha$ is $\mathcal{E}\cap\mathcal{M}^{\pitchfork}$-tight. The cocone $\{Gx_\alpha\}_\alpha$ defines a map $g : \operatorname{colim}_\alpha GX_\alpha \to GX$. Let $\{c_\alpha : GX_\alpha \to \operatorname{colim}_\alpha GX_\alpha\}_\alpha$ be the colimiting cocone. Then the following diagram commutes for each $\langle K,k\rangle$ in $\mathbf{Coalg}_{\mathrm{L}}$.

$$\begin{tikzcd}
\operatorname{colim}_\alpha \mathbf{Coalg}_{\mathrm{L}}\big(\langle K,k\rangle, GX_\alpha\big) \arrow[r, "\operatorname{colim} c_{\alpha *}"] \arrow[dr, "\operatorname{colim} Gx_{\alpha *}"'] & \mathbf{Coalg}_{\mathrm{L}}\big(\langle K,k\rangle, \operatorname{colim}_\alpha GX_\alpha\big) \arrow[d, "g_*"] \\
 & \mathbf{Coalg}_{\mathrm{L}}\big(\langle K,k\rangle, GX\big)
\end{tikzcd}$$

For each $\langle K,k\rangle$ in $\mathcal{S}_\lambda$,

$$\begin{aligned}\operatorname*{colim}_\alpha \mathbf{Coalg}_{\mathrm{L}}(\langle K,k\rangle, GX_\alpha) &\cong \operatorname*{colim}_\alpha \mathscr{A}(U_{\mathrm{L}}\langle K,k\rangle, X_\alpha)\\ &\cong \mathscr{A}(U_{\mathrm{L}}\langle K,k\rangle, X)\\ &\cong \mathbf{Coalg}_{\mathrm{L}}(\langle K,k\rangle, GX),\end{aligned}$$

and this bijection is the map $\operatorname{colim}_\alpha Gx_{\alpha *}$ defined by the cocone $\{Gx_{\alpha *}\}_\alpha$. So, in particular, the map g_* is a surjection on the objects $\langle K,k\rangle$ in $\mathcal{S}_\lambda$. By 6.2.10, the map $U_{\mathrm{L}}g$ is in $\mathcal{E}$. Since U_{L} commutes with colimits, $U_{\mathrm{L}}g$ is the map $\operatorname{colim}_\alpha U_{\mathrm{L}}GX_\alpha \to U_{\mathrm{L}}GX$ defined by the cocone $\{U_{\mathrm{L}}Gx_\alpha : U_{\mathrm{L}}GX_\alpha \to U_{\mathrm{L}}GX\}_\alpha$. ☐

Proposition 6.2.22. *If $\alpha : \mathrm{L} \to \mathrm{C}$ is a map of $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible copointed endofunctors, then there is a map $\alpha' : \mathrm{L}' \to \mathrm{C}'$ of comonads, where L' and C' are the comonads constructed in theorem 6.2.21.*

Proof. There is a functor $\alpha_* : \mathbf{Coalg}_{\mathrm{L}} \to \mathbf{Coalg}_{\mathrm{C}}$ such that $U_{\mathrm{C}}\alpha_* = U_{\mathrm{L}}$, where $U_{\mathrm{L}} : \mathbf{Coalg}_{\mathrm{L}} \to \mathscr{C}$ and $U_{\mathrm{C}} : \mathbf{Coalg}_{\mathrm{C}} \to \mathscr{C}$ are the forgetful functors. By 6.2.20, the forgetful functors U_{L} and U_{C} both have right adjoints. Therefore, by the dual of 2.1.34, U_{L} and U_{C} are comonadic functors. The result now follows from A.3.2. ☐

6.3 Quasiaccessible Model Categories

6.3.1 Quasiaccessible Model Categories are Algebraic Model Categories

Definition 6.3.1. Let $\mathscr{C}$ be an $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible category.

- A functorial factorization (L,R) on $\mathscr{C}$ is $(\mathcal{E},\mathcal{M},\lambda)$-*quasiaccessible* if $\mathrm{L} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is an $(\mathcal{E}^{\mathbf{2}},\mathcal{M}^{\mathbf{2}},\lambda)$-quasiaccessible functor.

- A weak factorization system (L, R) is $(\mathcal{E}, \mathcal{M}, \lambda)$-*quasiaccessible* if it has an associated functorial factorization that is $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible.
- A model category on $\mathscr{C}$ is $(\mathcal{E}, \mathcal{M}, \lambda)$-*quasiaccessible* if both of its weak factorization systems are $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible.
- An algebraic model category $\zeta : (\mathrm{C}_t, \mathrm{F}) \to (\mathrm{C}, \mathrm{F}_t)$ on $\mathscr{C}$ with weak equivalences $\mathcal{W}$ is $(\mathcal{E}, \mathcal{M}, \lambda)$-*quasiaccessible* if $(\mathrm{C}_t, \mathrm{F})$ and $(\mathrm{C}, \mathrm{F}_t)$ are both $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible functorial factorizations.

A functorial factorization, weak factorization system, model category, or algebraic model category is $(\mathcal{E}, \mathcal{M})$-*quasiaccessible* if it is $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible for some regular cardinal λ.

Remark 6.3.2. A functorial factorization (L, R) is $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible if and only if $\mathrm{R} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is $(\mathcal{E}^{\mathbf{2}}, \mathcal{M}^{\mathbf{2}}, \lambda)$-quasiaccessible.

Theorem 6.3.3. *Every $(\mathcal{E}, \mathcal{M})$-quasiaccessible weak factorization system on an $(\mathcal{E}, \mathcal{M})$-quasiaccessible category $\mathscr{C}$ has an associated AWFS.*

Proof. Let $(\mathcal{L}, \mathcal{R})$ be an $(\mathcal{E}, \mathcal{M})$-quasiaccessible weak factorization system on $\mathscr{C}$. There is a λ regular cardinal such that $\mathscr{C}$ and $(\mathcal{L}, \mathcal{R})$ are $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible. Let $(\mathrm{L}_1, \mathrm{R}_1)$ be an $(\mathcal{E}, \mathcal{M})$-quasiaccessible functorial factorization associated to $(\mathcal{L}, \mathcal{R})$. By our remarks following proposition 3.1.2 in section 3.1.1, $|\mathbf{Coalg}_{\mathrm{L}_1}| = \mathcal{L}$. By 6.2.21, there is a comonad $\mathrm{L}'_1 : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ such that $|\mathbf{Coalg}_{\mathrm{L}'_1}| = \mathcal{L}$ and L'_1 preserves $\mathcal{E}^{\mathbf{2}}$-tightness of $(\mathcal{M}^{\mathbf{2}}, \lambda)$-cocones. By 3.1.13 and 3.1.16, $\mathcal{L} = |\mathbf{Coalg}_{\mathrm{L}'_1}| = |{}^{\boxslash}\mathbf{Alg}_{\mathrm{R}'_1}| = {}^{\square}|\mathbf{Alg}_{\mathrm{R}'_1}|$. Thus $|\mathbf{Alg}_{\mathrm{R}'_1}| \subseteq ({}^{\square}|\mathbf{Alg}_{\mathrm{R}'_1}|)^{\square} = \mathcal{L}^{\square} = \mathcal{R}$. By 3.1.4, $|\mathbf{Alg}_{\mathrm{R}'_1}| = \mathcal{R}$. So, by 3.2.19, $(\mathrm{L}', \mathrm{R}') \coloneqq \mathbb{F}_1((\mathrm{L}'_1, \mathrm{R}'_1))$ is an AWFS such that $|\mathbf{Coalg}_{\mathrm{L}'}| = \mathcal{L}$ and $|\mathbf{Alg}_{\mathrm{R}'}| = \mathcal{R}$. Of course the monad structure on R' and the comonad structure on L' tell us that for each object f in $\mathscr{C}^{\mathbf{2}}$, $\mathrm{L}'f \in \mathcal{L}$ and $\mathrm{R}'f \in \mathcal{R}$. □

Theorem 6.3.4. *Every* $(\mathcal{E}, \mathcal{M}, \lambda)$*-quasiaccessible model category is the underlying model category of an algebraic model category.*

Proof. Let $\mathscr{C}$ be a bicomplete $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible category. Let $\mathcal{W}$, $\mathcal{C}$, and $\mathcal{F}$ be the collections of weak equivalences, cofibrations, and fibrations, respectively, for an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible model category on $\mathscr{C}$. There are $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible functorial factorizations $(\mathrm{C}_t, \mathrm{F})$ and $(\mathrm{C}, \mathrm{F}_t)$ for the weak factorization systems $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$, respectively.

Let $(\mathrm{L}_t, \mathrm{R})$ be the factorization $(\mathrm{C}, \mathrm{F}_t) \odot (\mathrm{C}_t, \mathrm{F})$. Explicitly, $\mathrm{L}_t f = \mathrm{C}\mathrm{C}_t f$ and $\mathrm{R}f = \mathrm{F}f \circ \mathrm{F}_t \mathrm{C}_t f$. So $(\mathrm{L}_t, \mathrm{R})$ is an associated factorization for $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$. Since the composition of $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible functors is $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible, the functorial factorization $(\mathrm{L}_t, \mathrm{R})$ is $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible. The counit map $\varepsilon : \mathrm{C}_t \to \mathrm{Id}$ defines a map of copointed endofunctors $\mathrm{C}\varepsilon : \mathrm{L}_t = \mathrm{C}\mathrm{C}_t \to \mathrm{C}$.

By the remarks following proposition 3.1.2, $|\mathbf{Coalg}_{\mathrm{L}_t}| = \mathcal{C} \cap \mathcal{W}$ and $|\mathbf{Coalg}_{\mathrm{C}}| = \mathcal{C}$. So by 6.2.21, there are comonads L'_t and C' that preserve $\mathcal{E}^{\mathbf{2}}$-tightness of $(\mathcal{M}^{\mathbf{2}}, \lambda)$-cocones such that $|\mathbf{Coalg}_{\mathrm{L}'_t}| = \mathcal{C} \cap \mathcal{W}$, $|\mathbf{Coalg}_{\mathrm{C}'}| = \mathcal{C}$. By 6.2.22, there is a map of these comonads $\alpha : \mathrm{L}'_t \to \mathrm{C}'$. By 3.2.22, $\mathbb{F}_2\alpha : \mathbb{F}_2\mathrm{L}'_t \to \mathbb{F}_2\mathrm{C}'$ is a map of $(\mathcal{E}, \mathcal{M})$-compact LAWFSs. By 3.2.16, $\mathbb{F}_1\mathbb{F}_2\alpha : \mathbb{F}_1\mathbb{F}_2\mathrm{L}'_t \to \mathbb{F}_1\mathbb{F}_2\mathrm{C}'$ is a map of AWFSs. Let $(\mathrm{L}''_t, \mathrm{R}'') = \mathbb{F}_1\mathbb{F}_2\mathrm{L}'_t$ and let $(\mathrm{L}'', \mathrm{R}''_t) = \mathbb{F}_1\mathbb{F}_2\mathrm{C}'$. Using the remarks following proposition 3.1.2 and the facts that $\mathcal{C} \cap \mathcal{W} = {}^{\square}\mathcal{F}$ and $\mathcal{C} = {}^{\square}(\mathcal{F} \cap \mathcal{W})$, we can show that $\mathbb{F}_2(\mathrm{L}'_t)$ and $\mathbb{F}_2(\mathrm{C}')$ are functorial factorizations associated to $(\mathcal{C} \cap \mathcal{W})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$, respectively. Then, by 3.2.19 and 3.1.4, $|\mathbf{Coalg}_{\mathrm{L}''_t}| = \mathcal{C} \cap \mathcal{W}$, $|\mathbf{Coalg}_{\mathrm{L}''}| = \mathcal{C}$, $|\mathbf{Alg}_{\mathrm{R}''}| = \mathcal{F}$, and $|\mathbf{Alg}_{\mathrm{R}''_t}| = \mathcal{F} \cap \mathcal{W}$. So $\mathbb{F}_1\mathbb{F}_2\alpha : (\mathrm{L}''_t, \mathrm{R}'') \to (\mathrm{L}'', \mathrm{R}''_t)$ is the desired algebraic model category. $\square$

6.3.2 The Bousfield-Friedlander Theorem

To get an algebraic model category after applying the Bousfield-Friedlander theorem to an $(\mathcal{E},\mathcal{M})$-quasiaccessible model category we will need to place the following restriction on the collection $\mathcal{E}$.

$$\text{If } f \in \mathcal{E} \text{ and } g,h \in \mathcal{E} \cap \mathcal{M}^{\natural}\text{, then the limit of any diagram } f \to h \leftarrow g \text{ in } \mathscr{C}^{\mathbf{2}} \text{ is in } \mathcal{E}. \qquad (6.1)$$

Let $\mathscr{C}$ be a bicomplete $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible category such that $\mathcal{E}$ satisfies condition (6.1). Suppose there is an $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible model structure on $\mathscr{C}$ with cofibrations $\mathcal{C}$, fibrations $\mathcal{F}$, and weak equivalences $\mathcal{W}$. Let $(\mathrm{C}_t,\mathrm{F})$ and $(\mathrm{C},\mathrm{F}_t)$ be $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible functorial factorizations associated to the weak factorization systems $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ and $(\mathcal{C},\mathcal{F}\cap\mathcal{W})$, respectively. Let (Q,α) be a pointed endofunctor on $\mathscr{C}$.

Definition 6.3.5. An object f in $\mathscr{C}^{\mathbf{2}}$ is a *Q-equivalence* if $Q(f)$ is a weak equivalence. An object f in $\mathscr{C}^{\mathbf{2}}$ is a *Q-fibration* if it has the right lifting property with respect to cofibrations which are Q-equivalences. An object X in $\mathscr{C}$ is *Q-fibrant* if the map $X \to *$ to the terminal object is a Q-fibration.

We will use the notation $\mathcal{W}_Q = \{f \mid Q(f) \in \mathcal{W}\}$ for the Q-equivalences and $\mathcal{F}_Q = (\mathcal{C}\cap\mathcal{W}_Q)^{\square}$ for the Q-fibrations.

The endofunctor Q on $\mathscr{C}$ defines an endofunctor $Q^{\mathbf{2}} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ which sends objects $f : A \to B$ to $Qf : QA \to QB$ and morphisms $(u,v) : f \to g$ to $(Qu,Qv) : Qf \to Qg$. The functor is pointed by $\vec{\alpha} = (\alpha,\alpha) : Id \to Q^{\mathbf{2}}$.

Theorem 6.3.6. *If the following conditions on $Q^{\mathbf{2}}$ are satisfied, then there is an $(\mathcal{E},\mathcal{M},\lambda)$-quasiaccessible model category on $\mathscr{C}$ with weak equivalences $\mathcal{W}_Q$ and fibrations $\mathcal{F}_Q$.*

1. *$Q^{\mathbf{2}} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is an $(\mathcal{E}^{\mathbf{2}},\mathcal{M}^{\mathbf{2}},\lambda)$-quasiaccessible endofunctor.*

2. $Q^{\mathbf{2}}$ preserves weak equivalence objects.

3. The maps $\vec{\alpha}_{Q^{\mathbf{2}}} : Q^{\mathbf{2}} \to Q^{\mathbf{2}}Q^{\mathbf{2}}$ and $Q^{\mathbf{2}}\vec{\alpha} : Q^{\mathbf{2}} \to Q^{\mathbf{2}}Q^{\mathbf{2}}$ are natural $\mathcal{W}^{\mathbf{2}}$-maps.

4. The collection $\mathcal{W}_Q$ is stable under pullbacks along the fibrations $f : X \to Y$ between fibrant objects for which $\vec{\alpha}_f : f \to Q^{\mathbf{2}}f$ is a $\mathcal{W}^{\mathbf{2}}$-map.

Proposition 6.3.7 ([Sta08])**.** *Assuming conditions (2) - (4) of theorem 6.3.6, the following hold.*

1. A map in $\mathscr{C}$ is an $\mathcal{F} \cap \mathcal{W}$-map if and only if it is an $\mathcal{F}_Q \cap \mathcal{W}_Q$-map.

2. A map $f : X \to *$ is an $\mathcal{F}_Q$-map if $\vec{\alpha}_f : f \to Q^{\mathbf{2}}f$ is a $\mathcal{W}^{\mathbf{2}}$-map.

3. A map between Q-fibrant objects is an $\mathcal{F}_Q$-map if and only if it is an $\mathcal{F}$-map.

Proof of Theorem 6.3.6. We will construct an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible functorial factorization (L^Q, R^Q) such that $L^Q f \in \mathcal{C} \cap \mathcal{W}_Q$ and $R^Q f \in \mathcal{F}_Q$ for each map f in $\mathscr{C}$. Let E_t be the middomain functor of $(\mathrm{C}_t, \mathrm{F})$. Let $f : X \to Y$ be a map in $\mathscr{C}$ and let $x' : QX \to *$ and $y' : QY \to *$ be maps to the terminal object. Consider the following diagram.

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[d, "f"] & QX \arrow[r, "\mathrm{C}_t x'"] \arrow[d, "Qf"] & E_t x' \arrow[r, "\mathrm{C}_t E_t Qf"] \arrow[d, "E_t Qf"] & E_t E_t Qf \arrow[d, "\mathrm{F} E_t Qf"] \\
Y \arrow[r, "\alpha_Y"] & QY \arrow[r, "\mathrm{C}_t y'"] & E_t y' \arrow[r, "id"] & E_t y'
\end{tikzcd}$$

Let $R_1 f : E_1 f \to Y$ be the pullback of $\mathrm{F}E_t Qf$ along $\mathrm{C}_t y' \circ \alpha_Y$. Let $L_1 f : X \to E_1 f$ be the map into the pullback defined by the maps $f : X \to Y$ and $\mathrm{C}_t E_t Qf \circ \mathrm{C}_t x' \circ \alpha_X : X \to E_t E_t Qf$. Let $(L^Q, R^Q) = (\mathrm{C}, \mathrm{F}_t) \odot (L_1, R_1)$. So $R^Q f = R_1 f \circ \mathrm{F}_t L_1 f$ and $L^Q f = \mathrm{C} L_1 f$.

By 6.3.6 (3), $\vec{\alpha}_{Qf}$ is a $\mathcal{W}^{\mathbf{2}}$-map. So α_{QX} and α_{QY} are in $\mathcal{W}^{\mathbf{2}}$. Since $\mathrm{C}_t y'$ is in $\mathcal{W}$, $Q(\mathrm{C}_t y') : QQY \to QE_t y'$ is in $\mathcal{W}$. Thus $\alpha_{E_t y'} : E_t y' \to QE_t y'$ is in $\mathcal{W}$. Similarly, since $\mathrm{C}_t E_t Qf \circ \mathrm{C}_t x'$ is in $\mathcal{W}$, $Q(\mathrm{C}_t E_t Qf \circ \mathrm{C}_t x')$ is in $\mathcal{W}$ and thus $\alpha_{E_t E_t Qf}$ is in $\mathcal{W}$. Therefore, by 6.3.7 (2), E_tE_tQf and E_ty' are Q-fibrant objects. So by 6.3.7 (3), $\mathrm{F}E_tQf$ is an $\mathcal{F}_Q$-map. Therefore R_1f is an $\mathcal{F}_Q$-map. By 6.3.7 (1), $R^Qf \in \mathcal{F}_Q$. Since the maps $\mathrm{C}_ty' \circ \alpha_Y$ and $\mathrm{C}_tE_tQf \circ \mathrm{C}_tx' \circ \alpha_X$ are Q-weak equivalences, the 2-out-of-3 property of $\mathcal{W}_Q$ and 6.3.6 (4) imply that L_1f is an $\mathcal{C} \cap \mathcal{W}_Q$-map. So $\mathrm{C}L_1f \in \mathcal{C} \cap \mathcal{W}_Q$.

Let $\{(u_\alpha, v_\alpha) : f_\alpha \to f\}_\alpha$ be an $\mathcal{E}^{\mathbf{2}}$-tight $(\mathcal{M}^{\mathbf{2}}, \lambda)$-cocone in $\mathscr{C}^{\mathbf{2}}$. By 6.3.6 (1), $\{(Qu_\alpha, Qv_\alpha) : Qf_\alpha \to Qf\}_\alpha$ is an $\mathcal{E}^{\mathbf{2}}$-tight $(\mathcal{M}^{\mathbf{2}}, \lambda)$-cocone. Let $X_\alpha = \operatorname{dom} f_\alpha$ and $Y_\alpha = \operatorname{cod} f_\alpha$. So $\{(Qu_\alpha, id_*) : x' \circ Qu_\alpha \to x'\}_\alpha$ and $\{(Qv_\alpha, id_*) : y' \circ Qv_\alpha \to y'\}_\alpha$ are $\mathcal{E}^{\mathbf{2}}$-tight $(\mathcal{M}^{\mathbf{2}}, \lambda)$-cocones. Since $\{\mathrm{C}_t(Qu_\alpha, id_*) : \mathrm{C}_t(x' \circ Qu_\alpha) \to \mathrm{C}_tx'\}_\alpha$ is an $\mathcal{E}^{\mathbf{2}}$-tight $(\mathcal{M}^{\mathbf{2}}, \lambda)$-cocone, it follows that $\{E_t(Qu_\alpha, Qv_\alpha) : E_tQf_\alpha \to E_tQf\}_\alpha$ is an $\mathcal{E}^{\mathbf{2}}$-tight $(\mathcal{M}^{\mathbf{2}}, \lambda)$-cocone. Thus $\{\mathrm{F}E_t(Qu_\alpha, Qv_\alpha) : \mathrm{F}E_tQf_\alpha \to \mathrm{F}E_tQf\}_\alpha$ is an $\mathcal{E}^{\mathbf{2}}$-tight $(\mathcal{M}^{\mathbf{2}}, \lambda)$-cocone. Because $\mathcal{M}$ is stable under pullbacks, $\{R_1(u_\alpha, v_\alpha) : R_1f_\alpha \to R_1f\}_\alpha$ is an $(\mathcal{M}^{\mathbf{2}}, \lambda)$-cocone. By 6.1.12 (5), the cocones $\{(u_\alpha, v_\alpha)\}_\alpha$ and $\{\mathrm{F}E_tQ(u_\alpha, v_\alpha)\}_\alpha$ are $(\mathcal{E} \cap \mathcal{M}^{\mapsdown})^{\mathbf{2}}$-tight. It follows from condition (6.1) and proposition 6.2.12 that the cocone $\{L_1(u_\alpha, v_\alpha)\}_\alpha$ is $\mathcal{E}^{\mathbf{2}}$-tight. Therefore $\{\mathrm{C}L_1(u_\alpha, v_\alpha) : \mathrm{C}L_1f_\alpha \to \mathrm{C}L_1f\}_\alpha$ is an $\mathcal{E}^{\mathbf{2}}$-tight $(\mathcal{M}^{\mathbf{2}}, \lambda)$-cocone. So (L^Q, R^Q) is an $(\mathcal{E}, \mathcal{M}, \lambda)$-quasiaccessible functorial factorization. ☐

Corollary 6.3.8. *If the conditions of 6.3.6 are satisfied, then there is an algebraic model category on $\mathscr{C}$ with weak equivalences $\mathcal{W}_Q$ and fibrations $\mathcal{F}_Q$.*

Proof. This is just an application of theorem 6.3.4 to theorem 6.3.6. ☐

6.3.3 A Quasiaccessible h-Model Structure

Let $\mathcal{W}$, $\mathcal{C}$, and $\mathcal{F}$ be the collections of homotopy equivalences, closed h-cofibrations, and h-fibrations in **Top**, respectively. By [Str72], **Top** is a model category with weak equivalences $\mathcal{W}$, cofibrations $\mathcal{C}$, and fibrations $\mathcal{F}$. In fact, the only results we need from [Str72] are that

$(\mathcal{C}\cap\mathcal{W})^{\square} = \mathcal{F}$, $\mathcal{C}\cap\mathcal{W} = {}^{\square}\mathcal{F}$, $\mathcal{C}^{\square} = \mathcal{F}\cap\mathcal{W}$, and $\mathcal{C} = {}^{\square}(\mathcal{F}\cap\mathcal{W})$. We will use different factorizations than those used by Strøm.

Following [BR13], we will use the Moore path space factorization $(\mathrm{L}_t, \mathrm{R})$ to factor a map $f : X \to Y$ into a map $\mathrm{L}_t f \in \mathcal{C} \cap \mathcal{W}$ followed by a map in $\mathrm{R}f \in \mathcal{F}$. The space ΠY of Moore paths on Y is given by the pullback square, where $\mathbb{R}_{\geq 0} = [0, \infty)$ and $\mathrm{shift} : Y^{\mathbb{R}_{\geq 0}} \times \mathbb{R}_{\geq 0} \to Y^{\mathbb{R}_{\geq 0}}$ is the map that sends (p, t) to the path $x \mapsto p(x + t)$.

$$\begin{tikzcd}
\Pi Y \arrow[r] \arrow[d, "\mathrm{ev}_{\mathrm{end}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\mathbb{R}_{\geq 0}} \times \mathbb{R}_{\geq 0} \arrow[d, "\mathrm{shift}"] \\
Y \arrow[r, "\mathrm{const}"] & Y^{\mathbb{R}_{\geq 0}}
\end{tikzcd}$$

On the point-set level, ΠY is the set of triples $(p, t, y) \in Y^{\mathbb{R}_{\geq 0}} \times \mathbb{R}_{\geq 0} \times Y$ such that $p(x) = y$ for all $x \geq t$. This set can be identified with triples (p, t, y) of paths $p : [0, t] \to Y$ such that $p(t) = y$. The map $\mathrm{ev}_{\mathrm{end}}$ sends a triple (p, t, y) to the point $p(t) = y$.

The factorization $f \mapsto \mathrm{R}f \circ \mathrm{L}_t f$ is given by the following pullback diagram. The space Γf consists of the triples (p, t, x) such that p is a path $p : [0, t] \to Y$ and $p(0) = f(x)$. The map $\mathrm{L}_t f$ sends a point x to the point (p, t, x) consisting of the constant path $p : [0, t] \to Y$ valued at $f(x)$ and $\mathrm{R}f$ sends a triple (p, t, x) to the point $p(t)$ in Y.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{L}_t f"] \arrow[dr, "id"'] & \Gamma f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "\mathrm{R}f"] & \Pi Y \arrow[r, "\mathrm{ev}_{\mathrm{end}}"] \arrow[d, "\mathrm{ev}_0"] & Y \\
& X \arrow[r, "f"] & Y &
\end{tikzcd}$$

It is shown in [BR13] that $\mathrm{L}_t f \in \mathcal{C} \cap \mathcal{W}$ and that $\mathrm{R}f \in \mathcal{F}$. Although [BR13] worked in a convenient category of spaces, the proof transfers seamlessly to **Top**. The proof that $\mathrm{R}f$ is an h-fibration relies on the fact that composition of paths in the Moore path space is strictly associative. This fact makes it possible to put a monad structure on $\mathrm{R} : \mathbf{Top^2} \to \mathbf{Top^2}$, which implies that $\mathrm{R}f \in \mathcal{F}$. Beyond establishing that $(\mathrm{L}_t, \mathrm{R})$ is a factorization associated to $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, we will not use the monad structure on R.

We will now describe our factorization of a map $f : X \to Y$ into a map in $\mathcal{C}$ followed by a map in $\mathcal{F} \cap \mathcal{W}$. We will start with the functorial factorization (m, m_w) defined by the following diagram. The maps i_0 and i_1 send x to $(x, 0)$ and $(x, 1)$, respectively, $s(f)$ and $t(f)$ are the colimiting cocone maps, and $m(f) = s(f) \circ i_1$. The collapse map $\mathrm{col} : X \times I \to X$ is just the projection onto X. The map $m_w(f)$ is the map out of the colimit defined by the cocone maps $id : Y \to Y$ and $f \circ \mathrm{col} : X \times I \to Y$.

$$\begin{tikzcd}
& X \arrow[d, "i_1"] \arrow[dd, bend left=60, "m(f)"] & \\
X \arrow[r, "i_0"] \arrow[d, "f"] & X \times I \arrow[r, "\mathrm{col}"] \arrow[d, "s(f)"'] & X \arrow[d, "f"] \\
Y \arrow[r, "t(f)"] \arrow[rr, bend right, "id"'] & Mf \arrow[r, dashed, "m_w(f)"] \arrow[ul, phantom, "\ulcorner", very near start] & Y
\end{tikzcd}$$

Since i_0 is in $\mathcal{C} \cap \mathcal{W}$, $t(f) \in \mathcal{C} \cap \mathcal{W}$. Therefore $m_w(f) \in \mathcal{W}$. Standard arguments show that $m(f)$ is a closed h-cofibration. Now, the functorial factorization $(\mathrm{L}, \mathrm{R}_t) := (\mathrm{L}_t, \mathrm{R}) \circledast (m, m_w)$ given by $f \mapsto (\mathrm{R}m_w(f), \mathrm{L}_t m_w(f) \circ m(f))$ factors f into a map in $\mathcal{C}$ followed by a map in $\mathcal{F} \cap \mathcal{W}$.

The remainder of this section is devoted to the proof of the following result.

Theorem 6.3.9. *The functorial factorizations* $(\mathrm{L}_t, \mathrm{R})$ *and* $(\mathrm{L}, \mathrm{R}_t)$ *are* $(\mathcal{E}^{⤈}, \mathcal{M}^{s↧}, \lambda)$*-quasiaccessible for every regular cardinal* $\lambda > 2^{\aleph_0}$.

For spaces A and X, let X^A be the set $\mathbf{Top}(A, X)$ equipped with the compact-open topology. For a compact subset K of A and an open subset U of X, we will use the notation

$$V_X(K, U) = \{f : A \to X \mid f(K) \subseteq U\}.$$

The collection of all such $V_X(K, U)$ is a subbase for the compact-open topology on $\mathbf{Top}(A, X)$. Even though $\mathbf{Top}$ is not cartesian closed, we can still prove the following result.

Lemma 6.3.10. *If $f : X \to Y$ is an $\mathcal{M}^{s\text{↧}}$-map, then for every A, $f^A : X^A \to Y^A$ is an $\mathcal{M}^{s\text{↧}}$-map.*

Proof. Since f is a monomorphism, $f^A : X^A \to Y^A$ is a monomorphism. Let K be a compact subset of A and let U be an open subset of X. Let U' be an open subset of Y such that $U' \cap X = U$. Then $V_Y(K, U') \cap X^A = V_X(K, U)$. It follows that f^A is a subspace inclusion. □

Lemma 6.3.11. *If $(u, v) : f \to g$ is an $(\mathcal{M}^{s\text{↧}})^{\mathbf{2}}$-map in $\mathbf{Top^2}$, then $\Gamma(u, v) : \Gamma f \to \Gamma g$ is an $\mathcal{M}^{s\text{↧}}$-map.*

Proof. By 6.3.10, the map $v^{\mathbb{R}_{\geq 0}} : (\operatorname{cod} f)^{\mathbb{R}_{\geq 0}} \to (\operatorname{cod} g)^{\mathbb{R}_{\geq 0}}$ is a subspace inclusion. Since $(\mathcal{E}^{\text{⤈}}, \mathcal{M}^{s\text{↧}})$ is a orthogonal factorization system, the limit of a diagram in $\mathbf{Top^2}$ of objects in $\mathcal{M}^{s\text{↧}}$ is an object in $\mathcal{M}^{s\text{↧}}$. So the map

$$v^{\mathbb{R}_{\geq 0}} \times \mathbb{R}_{\geq 0} : (\operatorname{cod} f)^{\mathbb{R}_{\geq 0}} \times \mathbb{R}_{\geq 0} \to (\operatorname{cod} g)^{\mathbb{R}_{\geq 0}} \times \mathbb{R}_{\geq 0}$$

is a subspace inclusion and the map $\Pi v : \Pi(\operatorname{cod} f) \to \Pi(\operatorname{cod} g)$ is a subspace inclusion. Therefore $\Gamma(u, v) : \Gamma f \to \Gamma g$ is a subspace inclusion. □

Proposition 6.3.12. *The functorial factorization* $(\mathrm{L}_t, \mathrm{R})$ *is* $(\mathcal{E}^{\text{⤈}}, \mathcal{M}^{s\text{↧}}, \lambda)$*-quasiaccessible for every regular cardinal* $\lambda > 2^{\aleph_0}$.

Proof. We know from 6.3.11 that L_t sends $(\mathcal{M}^{s\text{↧}})^{\mathbf{2}}$-cocones to $(\mathcal{M}^{s\text{↧}})^{\mathbf{2}}$-cocones. Let λ be a regular cardinal with $\lambda > |\mathbb{R}_{\geq 0}| = 2^{\aleph_0}$ and let $\{(u_\alpha, v_\alpha) : f_\alpha \to f\}_\alpha$ be an $(\mathcal{E}^{\text{⤈}})^{\mathbf{2}}$-tight $((\mathcal{M}^{s\text{↧}})^{\mathbf{2}}, \lambda)$-cocone in $\mathbf{Top^2}$. By 6.1.6, $\mathbb{R}_{\geq 0}$ is an $(\mathcal{E}^{\text{⤈}}, \mathcal{M}^{s\text{↧}}, \lambda)$-presentable object in $\mathbf{Top}$. So by 6.3.10, $\{v_\alpha^{\mathbb{R}_{\geq 0}} : (\operatorname{cod} f_\alpha)^{\mathbb{R}_{\geq 0}} \to (\operatorname{cod} f)^{\mathbb{R}_{\geq 0}}\}_\alpha$ is an $\mathcal{E}^{\text{⤈}}$-tight $(\mathcal{M}^{s\text{↧}}, \lambda)$-cocone. By 6.1.12 (5), $\{v_\alpha^{\mathbb{R}_{\geq 0}}\}_\alpha$ is in particular $\mathcal{M}^{\text{↧}}$-tight. From proposition 6.2.12 and the fact that $\mathcal{E}^{\text{⤈}}$ satisfies condition (6.1), we can conclude in order that each of the following cocones are $\mathcal{E}^{\text{⤈}}$-tight.

$$\left\{v_\alpha^{\mathbb{R}_{\geq 0}} \times \mathbb{R}_{\geq 0} : (\operatorname{cod} f_\alpha)^{\mathbb{R}_{\geq 0}} \times \mathbb{R}_{\geq 0} \to (\operatorname{cod} f)^{\mathbb{R}_{\geq 0}} \times \mathbb{R}_{\geq 0}\right\}_\alpha$$

$$\{\Pi v_\alpha : \Pi(\operatorname{cod} f_\alpha) \to \Pi(\operatorname{cod} f)\}_\alpha$$

$$\{\Gamma(u_\alpha, v_\alpha) : \Gamma f_\alpha \to \Gamma f\}_\alpha$$

Therefore L_t preserves $(\mathcal{E}^{\not\downarrow})^{\mathbf{2}}$-tightness of $((\mathcal{M}^{s\mapsdown})^{\mathbf{2}}, \lambda)$-cocones. ☐

Lemma 6.3.13. *Consider the following pushout square and cocone in* **Top**.

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "f'"'] \arrow[ddr, bend left, "h"] & \\
B \arrow[r, "g'"] \arrow[drr, bend right, "k"] & B \coprod_A C \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "l"] & \\
& & D
\end{tikzcd}$$

If f, g, h, and k are subspace inclusions, then l is a subspace inclusion.

Proof. The set $B \coprod_A C$ is just the union $B \cup C$ in D. So l is an injection. Let L be a closed set in $B \coprod_A C$. Then $L \cap B$ is a closed set in B and $L \cap C$ is a closed set in C. But that means $\overline{l(L)} \cap B = L \cap B$ and $\overline{l(L)} \cap C = L \cap C$. So $\overline{l(L)} \cap (B \coprod_A C) = \big(\overline{l(L)} \cap B\big) \cup \big(\overline{l(L)} \cap C\big) = L$. ☐

Lemma 6.3.14. *Consider the following map of pushout squares in* **Top**.

$$\begin{tikzcd}
A_1 \arrow[rr, "g_1"] \arrow[dr, "a"] \arrow[dd, "f_1"] & & C_1 \arrow[dr, "c"] \arrow[dd, "f_1'" near end] & \\
& A_2 \arrow[rr, crossing over, "g_2" near start] & & C_2 \arrow[dd, "f_2'"] \\
B_1 \arrow[rr, "g_1'" near start] \arrow[dr, "b"] & & B_1 \coprod_{A_1} C_1 \arrow[dr, "d"] & \\
& B_2 \arrow[rr, "g_2'"] \arrow[from=uu, crossing over, "f_2" near end] & & B_2 \coprod_{A_2} C_2
\end{tikzcd}$$

Suppose g_1 and g_2 are subspace inclusions such that $c(C_1 \smallsetminus g_1(A_1)) \subseteq C_2 \smallsetminus g_2(A_2)$.

1. *If a, b, and c are injections, then d is an injection.*

2. *If a, b, and c are subspace inclusions, then d is a subspace inclusion.*

Proof. (1) It is easy to check that g_2' is an injection and that the restriction of f_2' to $C_2 \smallsetminus g_2(A_2)$ is an injection. Every point in $B_1 \coprod_{A_1} C_1$ either lifts to a point in $C_1 \smallsetminus g_1(A_1)$ or to a point in B_1. Let x and y be distinct points in $B_1 \coprod_{A_1} C_1$. If both x and y have lifts in B_1, then the injectivity of b and g_2' imply that $d(x) \neq d(y)$. If both x and y have lifts in $C_1 \smallsetminus g_1(A_1)$, then the fact that $c(C_1 \smallsetminus g_1(A_1)) \subseteq C_2 \smallsetminus g_2(A_2)$, the injectivity of c, and the injectivity of the restriction of f_2' to $C_2 \smallsetminus g_2(A_2)$ imply that $d(x) \neq d(y)$. Suppose x has a lift $\tilde{x} \in C_1 \smallsetminus g_1(A_1)$ and y has a lift $\tilde{y} \in B_1$. Since $c(\tilde{x}) \in C_2 \smallsetminus g_2(A_2)$, $f_2'(c(\tilde{x})) \neq g_2'(b(\tilde{y}))$. So d is an injection.

(2) Recall that $B_2 \coprod_{A_2} C_2$ has the final topology. So a subset V is open if and only if $V \cap B_2$ and $f_2'^{-1}(V)$ are open. Let U be an open set in $B_1 \coprod_{A_1} C_1$. Then $U \cap B_1$, $f_1'^{-1}(U)$, and $f_1^{-1}(U \cap B_1)$ are open. There is an open set V_B in B_2 such that $V_B \cap B_1 = U \cap B_1$. Let $V_A = f_2^{-1}(V_B)$. Then $V_A \cap A_1 = f_1^{-1}(U \cap B_1)$. Since $f_1'^{-1}(U) \cap A_1 = f_1^{-1}(U \cap B_1) = V_A \cap A_1$, the set $V_D := f_1'^{-1}(U) \cup V_A$ in $A_2 \coprod_{A_1} C_1$ is open. By 6.3.13, there is an open set V_C in C_2 such that $V_C \cap (A_2 \coprod_{A_1} C_1) = V_D$. So in particular, $V_C \cap C_1 = f_1'^{-1}(U)$ and $V_C \cap A_2 = V_A$. By (1), $V_B \coprod_{V_A} V_C$ is a subset of $B_2 \coprod_{A_2} C_2$. Let $V = V_B \coprod_{V_A} V_C$. Since $V \cap B_2 = V_B$ and $f_2'^{-1}(V) = V_C$, V is an open subset of $B_2 \coprod_{A_2} C_2$. Since $V_B \cap B_1 = U \cap B_1$, $V_A \cap A_1 = f^{-1}(U \cap B_1)$, and $V_C \cap C_1 = f_1'^{-1}(U)$, we must have that $d^{-1}(V) = U$. □

Proposition 6.3.15. *The functorial factorization* (m, m_w) *is* $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail}, \lambda)$*-quasiaccessible for every regular cardinal* λ*.*

Proof. Let λ be a regular cardinal. Let $\{(u_\alpha, v_\alpha) : f_\alpha \to f\}_\alpha$ be an $(\mathcal{E}^{\twoheaddownarrow})^{\mathbf{2}}$-tight $((\mathcal{M}^{s\downarrowtail})^{\mathbf{2}}, \lambda)$-cocone in $\mathbf{Top^2}$, where f and the f_α are maps $f : X \to Y$ and $f_\alpha : X_\alpha \to Y_\alpha$. It suffices to show the cocone $\{M(u_\alpha, v_\alpha) : Mf_\alpha \to Mf\}_\alpha$ in $\mathbf{Top}$ is an $\mathcal{E}^{\twoheaddownarrow}$-tight $\mathcal{M}^{s\downarrowtail}$-cocone. The cocone $\{M(u_\alpha, v_\alpha)\}_\alpha$ is $\mathcal{E}^{\twoheaddownarrow}$-tight because colimits commute and the colimit of any diagram in $\mathbf{Top^2}$ whose objects are in $\mathcal{E}^{\twoheaddownarrow}$ is in $\mathcal{E}^{\twoheaddownarrow}$.

Of course, the functor $(-) \times I : \mathbf{Top} \to \mathbf{Top}$ preserves subspace inclusions. So the vertical arrows in the following diagram are all subspace inclusions.

$$\begin{tikzcd}
Y_\alpha \arrow[d, "v_\alpha"] & X_\alpha \arrow[l, "f_\alpha"'] \arrow[r, "i_0"] \arrow[d, "u_\alpha"] & X_\alpha \times I \arrow[d, "u_\alpha \times I"] \\
Y & X \arrow[l, "f"'] \arrow[r, "i_0"] & X \times I
\end{tikzcd}$$

Both of the maps $i_0 : X_\alpha \to X_\alpha \times I$ and $i_0 : X \to X \times I$ are subspace inclusions, and $X_\alpha \times (0,1] \subseteq X \times (0,1]$. Thus, by 6.3.14, $M(u_\alpha, v_\alpha) : Mf_\alpha \to Mf$ is a subspace inclusion. So $m(u_\alpha, v_\alpha)$ is an $(\mathcal{M}^{s\downarrowtail})^{\mathbf{2}}$-map for each α. $\square$

We can now prove the following restatement of theorem 6.3.9.

Theorem 6.3.16. *The h-model structure on* **Top** *is* $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail})$*-quasiaccessible.*

Proof. Let λ be a regular cardinal larger than $2^{\aleph_0}$. The functorial factorization $(\mathrm{L}_t, \mathrm{R})$ is an $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail}, \lambda)$-quasiaccessible functorial factorization for the weak factorization system $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$. By 6.3.15 and 6.3.2, $m_w : \mathbf{Top^2} \to \mathbf{Top^2}$ is an $((\mathcal{E}^{\twoheaddownarrow})^{\mathbf{2}}, (\mathcal{M}^{s\downarrowtail})^{\mathbf{2}}, \lambda)$-quasiaccessible functor. Since the composition of functors preserves $((\mathcal{E}^{\twoheaddownarrow})^{\mathbf{2}}, (\mathcal{M}^{s\downarrowtail})^{\mathbf{2}}, \lambda)$-quasiaccessibility, $\mathrm{R}_t = \mathrm{R} m_w : \mathbf{Top^2} \to \mathbf{Top^2}$ is $((\mathcal{E}^{\twoheaddownarrow})^{\mathbf{2}}, (\mathcal{M}^{s\downarrowtail})^{\mathbf{2}}, \lambda)$-quasiaccessible. So $(\mathrm{L}, \mathrm{R}_t)$ is an $(\mathcal{E}^{\twoheaddownarrow}, \mathcal{M}^{s\downarrowtail}, \lambda)$-quasiaccessible functorial factorization for the weak factorization system $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$. $\square$

Chapter 7: A Characterization of Accessible Model Categories with Accessible Weak Equivalences

A model structure on a category $\mathscr{C}$ is *combinatorial* if it is cofibrantly generated and if $\mathscr{C}$ is a locally presentable category. Smith's theorem for combinatorial model categories provides a complete characterization of combinatorial model categories. It in particular produces an (acyclic cofibration, fibration) factorization for the model category when only given the data of the (cofibration, acyclic fibration) WFS. There are many references for this material. Beke's paper [Bek00] is the earliest source that contains many of the relevant ideas. Summaries can also be found in [Ros09] and [Lur09, §A.2.6].

We attempt to prove a version of Smith's theorem for accessible model categories. Unfortunately, it does not seem to be the case that the weak equivalences in an accessible model category have to be accessible and accessibly embedded. Without this assumption, any characterization of accessible model categories seems hopeless. We instead only attempt to classify the accessible model categories whose weak equivalences are accessible and accessibly embedded. The classification we get is much harder to work with than Smith's theorem and likely of limited utility.

The papers [BG19] and [Bou20] outline alternative incomplete, but potentially productive, approaches towards Bousfield localizations, which could provide an alternative to Smith's theorem for accessible categories.

7.1 Accessible Model Categories

7.1.1 Properties of Accessible Categories

We will assume the reader is familiar with the definitions of accessible categories and accessible functors. A good summary can be found in [AR94]. If $\mathscr{C}$ is a λ-accessible category for a regular cardinal λ, then the λ-presentable objects in $\mathscr{C}$ only have a set of isomorphism classes. We will use the notation $\mathbf{Pres}_\lambda(\mathscr{C})$ the full subcategory of $\mathscr{C}$ on such a set of representatives.

We will state a few of the important results and definitions in [AR94].

Proposition 7.1.1. *Let λ and κ be regular cardinals with $\lambda \lhd \kappa$.*

1. *[AR94, 2.11] Every λ-accessible category is κ-accessible.*
2. *[AR94, 2.18] Every λ-accessible functor is κ-accessible.*
3. *[AR94, 2.20] If a functor preserves λ-presentable objects, then it preserves κ-presentable objects.*

Definition 7.1.2. A subcategory $\mathscr{B}$ of an accessible category $\mathscr{C}$ is *accessibly embedded* in $\mathscr{C}$ if $\mathscr{B}$ is a full subcategory of $\mathscr{C}$ and there is a regular cardinal λ such that $\mathscr{B}$ is closed under λ-filtered colimits in $\mathscr{C}$.

Definition 7.1.3. Let λ be a regular cardinal. A map $f : X \to Y$ in a category $\mathscr{A}$ is λ-*pure* if for every map $a : A_1 \to A_1$ between λ-presentable objects and every commutative square of the following form

$$\begin{tikzcd} A_1 \arrow[r, "u"] \arrow[d, "a"'] & X \arrow[d, "f"] \\ A_2 \arrow[r, "v"] & Y, \end{tikzcd}$$

there is a map $s : A_2 \to X$ such that $s \circ a = u$.

Proposition 7.1.4. *Let λ and κ be a regular cardinals with $\kappa \geq \lambda$.*

1. *If $f : X \to Y$ and $g : Y \to Z$ are λ-pure then $g \circ f$ is λ-pure.*

2. *If $g \circ f$ is λ-pure, then f is λ-pure.*

3. *If f is κ-pure, then f is λ-pure.*

4. *If $D : \mathscr{D} \to \mathscr{A}^{\mathbf{2}}$ is a κ-filtered diagram whose objects are λ-pure morphisms in $\mathscr{A}$, then $\operatorname{colim} D$ is a λ-pure morphism in $\mathscr{A}$.*

Proof. (1) and (2) are immediate. Since κ-filtered diagrams are λ-filtered, every λ-presentable object is κ-presentable. So (3) holds. (4) Every map $a : A_1 \to A_2$ between λ-presentable objects in $\mathscr{A}$ is λ-presentable. Every map $a \to \operatorname{colim} D$ from a λ-presentable object a in $\mathscr{A}^{\mathbf{2}}$ factors through Dd for some d. Since Dd is a λ-pure map in $\mathscr{A}$, the result holds. $\square$

Proposition 7.1.5. *[AR94, 2.31] Let λ be a regular cardinal and let $\mathscr{A}$ be a cocomplete λ-accessible category. If f is a λ-pure map in $\mathscr{A}$, then it is a regular monomorphism.*

Definition 7.1.6. Let $\mathscr{A}$ be a category and let λ be a regular cardinal. We define $\mathbf{Pure}_\lambda(\mathscr{A})$ to be the subcategory $\mathscr{A}$ whose objects are the objects of $\mathscr{A}$ and whose morphisms are the λ-pure morphisms in $\mathscr{A}$.

Remark 7.1.7. To check that a subcategory $\mathscr{B}$ of a κ-accessible category $\mathscr{A}$ is κ-accessible, it is sufficient to show the following.

1. $\mathscr{B}$ is closed under κ-filtered colimits.

2. The inclusion $\mathscr{B} \hookrightarrow \mathscr{A}$ preserves κ-filtered colimits.

3. There is a set $\mathcal{B}_0$ of κ-presentable objects in $\mathscr{A}$ that are also objects in $\mathscr{B}$ such that every object in $\mathscr{B}$ is a κ-filtered colimit in $\mathscr{B}$ of the objects in $\mathcal{B}_0$.

In this case, the κ-presentable objects of $\mathscr{B}$ are the κ-presentable objects of $\mathscr{A}$ that are in $\mathscr{B}$.

Proposition 7.1.8. *Let λ be a regular cardinal and let $\mathscr{A}$ be a λ-accessible category. The category $\mathbf{Pure}_\lambda(\mathscr{A})$ is accessible and the subcategory inclusion functor $\mathbf{Pure}_\lambda(\mathscr{A}) \hookrightarrow \mathscr{A}$ is accessible.*

Proof. By [AR94, 2.34], there is a regular cardinal κ, $\lambda \lhd \kappa$, such that $\mathbf{Pure}_\lambda(\mathscr{A})$ is κ-accessible. Since λ is strictly less than κ, $\mathscr{A}$ is also κ-accessible. Let $D : \mathscr{D} \to \mathbf{Pure}_\lambda(\mathscr{A})$ be a κ-filtered diagram. Let C be the colimit of D in $\mathscr{A}$ and let $\alpha : D \overset{\centerdot}{\longrightarrow} C$ be the colimiting cocone. For each object d and each map $f : d \to d'$ in $\mathscr{D}$, the map $Df : Dd \to Dd'$ in $\mathscr{A}$ is λ-pure. Furthermore, for a fixed object d, the colimit of the κ-filtered diagram $\{Dd \to Dd'\}_{d'}$ indexed by $d \downarrow \mathscr{D}$ is the map $\alpha_d : Dd \to C$. So by proposition 7.1.4 (4), α_d is λ-pure. So the cocone α is a cocone in $\mathbf{Pure}_\lambda(\mathscr{A})$. Suppose $\beta : D \overset{\centerdot}{\longrightarrow} C'$ is another cocone in $\mathbf{Pure}_\lambda(\mathscr{A})$. Let $k : C \to C'$ be the map in $\mathscr{A}$ out of the colimit defined by this cocone. Then k is the colimit in $\mathscr{A}^{\mathbf{2}}$ of the κ-filtered diagram $\{\beta_d\}_d$. So another application of proposition 7.1.4 (4) tells us that k is λ-pure. ☐

Proposition 7.1.9. *Let λ be a regular cardinal. Let $F : \mathscr{A} \to \mathscr{B}$ be a λ-accessible functor that preserves λ-presentable objects. Then F restricts to an accessible functor $F_\lambda : \mathbf{Pure}_\lambda(\mathscr{A}) \to \mathbf{Pure}_\lambda(\mathscr{B})$.*

Proof. By proposition 7.1.8 and uniformization, there is a regular cardinal κ, $\lambda \lhd \kappa$, such that $\mathscr{A}$, $\mathscr{B}$, $\mathbf{Pure}_\lambda(\mathscr{A})$, and $\mathbf{Pure}_\lambda(\mathscr{B})$ are κ-accessible categories, F is a κ-accessible functor, and the subcategory inclusion functors $\mathbf{Pure}_\lambda(\mathscr{A}) \hookrightarrow \mathscr{A}$ and $\mathbf{Pure}_\lambda(\mathscr{B}) \hookrightarrow \mathscr{B}$ are κ-accessible functors. By theorem [AR94, 2.38], F preserves λ-pure morphisms. A κ-filtered colimit in

$\mathbf{Pure}_\lambda(\mathscr{A})$ is a κ-filtered colimits in $\mathscr{A}$. So F preserves this colimit, the colimiting cocone is a cocone in $\mathbf{Pure}_\lambda(\mathscr{B})$, and this cocone must be a colimiting cocone in $\mathbf{Pure}_\lambda(\mathscr{B})$. ☐

7.1.2 Accessible Model Categories

Definition 7.1.10. Let λ be a regular cardinal.

1. A functorial factorization (L, R) on $\mathscr{C}$ is λ-*accessible* if $\mathrm{L} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ is a λ-accessible functor.
2. A weak factorization system $(\mathcal{L}, \mathcal{R})$ is λ-*accessible* if it has an associated functorial factorization (L, R) that is λ-accessible.
3. A model category is λ-*accessible* if both of its weak factorization systems are λ-accessible.
4. An algebraic model category is λ-*accessible* if both of its AWFSs are λ-accessible functorial factorizations.

Note that for a functorial factorization (L, R), L is λ-accessible if and only if R is λ-accessible.

A functorial factorization, weak factorization system, model category, or algebraic model category is *accessible* if it is λ-accessible for some regular cardinal λ.

Accessible comonads, LAWFSs, and AWFSs are in particular $\mathcal{E}$-compact comonads, LAWFSs, and AWFSs, where $(\mathcal{E}, \mathcal{M})$ is the (isomorphism, any map) left proper orthogonal factorization system. So the results of section 3.3 apply to accessible objects.

In particular, when $\mathscr{C}$ is a cocomplete λ-accessible category for some regular cardinal λ, then there are adjunctions

$$\begin{tikzcd}
\mathbf{AWFS}_\lambda(\mathscr{C}) \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathbb{G}_1"'] & \mathbf{LAWFS}_\lambda(\mathscr{C}) \arrow[l, bend right, "\mathbb{F}_1"'] \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathbb{G}_2"'] & \mathbf{Cmd}_\lambda(\mathscr{C}^{\mathbf{2}}) \arrow[l, bend right, "\mathbb{F}_2"'] \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathbb{G}_3"'] & \mathbf{CAT}/_{LK_\lambda}\mathscr{C}^{\mathbf{2}}, \arrow[l, bend right, "\mathbb{F}_3"']
\end{tikzcd}$$

where $\mathbf{AWFS}_\lambda(\mathscr{C})$, $\mathbf{LAWFS}_\lambda(\mathscr{C})$, and $\mathbf{Cmd}_\lambda(\mathscr{C}^{\mathbf{2}})$ are the categories of λ-accessible AWFSs, λ-accessible LAWFSs, and λ-accessible comonads, respectively, and where $\mathbf{CAT}/_{LK_\lambda}\mathscr{C}^{\mathbf{2}}$ is the metacategory of functors $A : \mathscr{A} \to \mathscr{C}^{\mathbf{2}}$ for which the left Kan extension of A along itself exists and is a λ-accessible functor on $\mathscr{C}^{\mathbf{2}}$. Every cocomplete λ-accessible category $\mathscr{C}$ permits the algebraic small object argument, so every small category over $\mathscr{C}^{\mathbf{2}}$ is an object in $\mathbf{CAT}/_{LK_\lambda}\mathscr{C}^{\mathbf{2}}$. In other words, we have the following specializations of 3.3.3, 3.2.22 and 3.2.27.

Proposition 7.1.11.

- *A free AWFS on an accessible LAWFS is accessible.*
- *Every cofibrantly generated AWFS on an accessible category is accessible.*

Proposition 7.1.12 ([Ros17])**.** *A weak factorization system is accessible if and only if it has an associated functorial factorization which is a cofibrantly generated AWFS.*

Although a cofibrantly generated AWFS is accessible, an accessible functorial factorization associated to a weak factorization system is not guaranteed to be a cofibrantly generated AWFS by the proposition. We may have to make a different choice of associated functorial factorization.

Remark 7.1.13. As the above proposition makes clear, for every accessible AWFS (L, R), there is a cofibrantly generated AWFS $(\mathrm{L}^I, \mathrm{R}^I)$ such that $|\mathbf{Coalg}_{\mathrm{L}^I}| = |\mathbf{Coalg}_{\mathrm{L}}|$ and $|\mathbf{Alg}_{\mathrm{R}^I}| =$

$|\mathbf{Alg}_{\mathrm{R}}|$. However, the AWFS (L, R) does not have to be cofibrantly generated. So the term "accessible AWFS" is more general than "cofibrantly generated AWFS".

The definition of an accessible model category differs from the definition of an algebraic model category in two ways. First of all, passing to an algebraic model category requires making a choice of AWFS associated to each accessible weak factorization system. As we just discussed, an arbitrary choice of associated functorial factorization will not do. We have to choose one that is both accessible and algebraic. Secondly, an algebraic model category comes with the structure of a map of AWFSs. While it is not clear a priori that the AWFSs associated to the accessible weak factorization systems can be chosen in such a way that there is a map of AWFSs between them, proposition 3.3.4 tells us that this is indeed possible.

Proposition 7.1.14. *Every accessible model category is the underlying model category of a cofibrantly generated algebraic model category.*

The following technical result can be proven for more general cofibrantly generated LAWFSs, but the hypotheses rarely holds for nonaccessible categories. It will be useful in our characterization of accessible model categories with accessible weak equivalences.

Proposition 7.1.15. *Let $\mathscr{C}$ be an accessible cocomplete category. Let $\mathcal{L}_0$ be a collection of objects in $\mathscr{C}^{\mathbf{2}}$ and let $\mathcal{L}$ be the retract closure of $\mathcal{L}_0$. Suppose $I : \mathscr{I} \to \mathscr{C}^{\mathbf{2}}$ is a small category over $\mathscr{C}^{\mathbf{2}}$ satisfying the following conditions.*

1. *Every object in $\mathcal{L}_0$ is the colimit of a diagram that factors through I*

2. *For every object f in $\mathscr{C}^{\mathbf{2}}$, the colimit of the diagram $(I \downarrow f) \to \mathscr{C}^{\mathbf{2}}$ is an object in $\mathcal{L}$*

Then $|\mathbf{Coalg}_{\mathrm{L}_0^I}| = \mathcal{L}$, where $\mathrm{L}_0^I = \mathbb{F}_3(I)$. If in addition $\mathcal{L} = {}^{\square}(\mathcal{L}^{\square})$, then $|\mathbf{Coalg}_{\mathrm{L}^I}| = \mathcal{L}$, where $(\mathrm{L}^I, \mathrm{R}^I) = \mathbb{F}_1\mathbb{F}_2\mathbb{F}_3(I)$.

Proof. Since $\mathrm{L}_0^I : \mathscr{C} \to \mathscr{C}$ is the left Kan extension of I along I, there is a natural transformation $\alpha : I \to \mathrm{L}_0^I I$. By definition, $\vec{\varepsilon} : \mathrm{L}_0^I \to id$ is the unique map such that $\vec{\varepsilon}_I \circ \alpha = id : I \to I$. Therefore, there is a map of categories $N : \mathscr{I} \to \mathbf{Coalg}_{\mathrm{L}_0^I}$ over $\mathscr{C}$. So, if $U_I : \mathbf{Coalg}_{\mathrm{L}_0^I} \to \mathscr{C}$ is the forgetful functor, then $U_I N = I$.

Let f be an object in $\mathcal{L}$. Then f is a retract of a $g \in \mathcal{L}_0$. By (1), there is a diagram $D : \mathscr{D} \to \mathscr{I}$ such that $g = \operatorname{colim} ID$. Since $\operatorname{colim} ID = \operatorname{colim} U_I N D = U_I(\operatorname{colim} ND)$, g is in the image of U_I. Since $|\mathbf{Coalg}_{\mathrm{L}_0^I}|$ is closed under retracts, f must be in the collection $|\mathbf{Coalg}_{\mathrm{L}_0^I}|$. Now, let f be an object in $|\mathbf{Coalg}_{\mathrm{L}_0^I}|$. Since $\mathrm{L}_0^I f$ is defined as the colimit of the diagram $I \downarrow f \to \mathscr{C}^{\mathbf{2}}$, by (2), $\mathrm{L}_0^I f \in \mathcal{L}$. But f is a retract of $\mathrm{L}_0^I f$. So $f \in \mathcal{L}$. Thus $|\mathbf{Coalg}_{\mathrm{L}_0^I}| = \mathcal{L}$.

Now suppose in addition to (1) and (2), we also know $\mathcal{L} = {}^{\square}(\mathcal{L}^{\square})$. Then $\mathcal{L}$ is closed under cobase change. So $\mathrm{L}_1^I f \in \mathcal{L}$ for each f in $\mathscr{C}^{\mathbf{2}}$. So $|\mathbf{Coalg}_{\mathrm{L}_1^I}| \subseteq \mathcal{L}$. Since there is a map of comonads $\mathrm{L}_0^I \to \mathrm{L}_1^I$, there is a functor $\mathbf{Coalg}_{\mathrm{L}_0^I} \to \mathbf{Coalg}_{\mathrm{L}_1^I}$ over $\mathscr{C}^{\mathbf{2}}$ and thus $\mathcal{L} = |\mathbf{Coalg}_{\mathrm{L}_0^I}| \subseteq |\mathbf{Coalg}_{\mathrm{L}_1^I}| \subseteq \mathcal{L}$. So $|\mathbf{Coalg}_{\mathrm{L}_1^I}| = \mathcal{L}$. Now, by propositions 3.1.13 and 3.1.16, $\mathcal{L}^{\square} = |\mathbf{Coalg}_{\mathrm{L}_1^I}|^{\square} = |\mathbf{Coalg}_{\mathrm{L}_1^I}^{\boxslash}| = |\mathbf{Alg}_{\mathrm{R}_1^I}|$. But, by the construction of $\mathbb{F}_1((\mathrm{L}_1^I, \mathrm{R}_1^I)) = (\mathrm{L}^I, \mathrm{R}^I)$, $|\mathbf{Alg}_{\mathrm{R}_1^I}| = |\mathbf{Alg}_{\mathrm{R}^I}^{\mathrm{EM}}|$. So $\mathcal{L} = {}^{\square}|\mathbf{Alg}_{\mathrm{R}_1^I}| = {}^{\square}|\mathbf{Alg}_{\mathrm{R}^I}^{\mathrm{EM}}| = |{}^{\boxslash}\mathbf{Alg}_{\mathrm{R}^I}^{\mathrm{EM}}| = |\mathbf{Coalg}_{\mathrm{L}^I}|$. □

Remark 7.1.16. An easy modification of the above proof shows that when $\mathcal{L}_0$ in the above proposition is retract closed, so that $\mathcal{L} = \mathcal{L}_0$, then $|\mathbf{Coalg}_{\mathrm{L}_0^I}^{\mathrm{EM}}| = |\mathbf{Coalg}_{\mathrm{L}_0^I}| = \mathcal{L}$. It can then easily be shown that $|\mathbf{Coalg}_{\mathrm{L}_1^I}^{\mathrm{EM}}| = |\mathbf{Coalg}_{\mathrm{L}_1^I}|$ too. However, it does not follow that $|\mathbf{Coalg}_{\mathrm{L}^I}^{\mathrm{EM}}| = |\mathbf{Coalg}_{\mathrm{L}^I}|$.

7.2 Characterizing Accessible Model Categories

7.2.1 Constructing Accessible Weak Factorization Systems

We will need the following nonstandard definition.

Definition 7.2.1. Let λ be a regular cardinal. A collection of objects $\mathcal{X}$ in a λ-accessible category $\mathscr{C}$ is *λ-preaccessible* if for every regular cardinal κ with $\kappa = \lambda$ or $\kappa \rhd \lambda$, every object in $\mathcal{X}$ is a κ-filtered colimit of κ-presentable objects in $\mathcal{X}$.

A collection of objects $\mathcal{X}$ in an accessible category is *preaccessible* if it is λ-preaccessible for some regular cardinal λ. The following result is immediate.

Proposition 7.2.2. *The collection of objects in the image of an accessible functor $F : \mathscr{A} \to \mathscr{B}$ is a preaccessible collection in $\mathscr{B}$.*

Let $\mathscr{C}$ be a bicomplete accessible category. Suppose $\mathcal{W}$, $\mathcal{C}$, and $\mathcal{F}$ are collections of maps in $\mathscr{C}$ that satisfy the following conditions.

1. $\mathbf{Full}(\mathcal{W})$ is accessible and accessibly embedded in $\mathscr{C}^{\mathbf{2}}$
2. $\mathcal{C} \cap \mathcal{W} = {}^{\square}\mathcal{F}$ and $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\square}$
3. $\mathcal{F}$ is preaccessible

We will describe for arbitrarily large regular cardinals λ the construction of an accessible algebraic weak factorization system $(\mathrm{L}^{J_\lambda}, \mathrm{R}^{J_\lambda})$ on $\mathscr{C}$ such that $\mathbf{Coalg}_{\mathrm{L}^{J_\lambda}} \subseteq \mathcal{C} \cap \mathcal{W}$. If we were able to prove the equality $\mathbf{Coalg}_{\mathrm{L}^{J_\lambda}} = \mathcal{C} \cap \mathcal{W}$, then we would have that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is an accessible weak factorization system. However, with the above assumptions, it does not seem possible to prove equality. Although not useful as a condition for constructing accessible weak factorization systems, we do get $\mathbf{Coalg}_{\mathrm{L}^{J_\lambda}} = \mathcal{C} \cap \mathcal{W}$ for arbitrarily large regular cardinals λ when we know that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is an accessible weak factorization system. Since our construction of L^{J_λ} did not rely on this assumption, this fact at least serves as a reassurance that our construction is correct. We only need to find an additional, useful condition to add to those above that will guarantee that $\mathbf{Coalg}_{\mathrm{L}^{J_\lambda}} = \mathcal{C} \cap \mathcal{W}$. We will describe such a condition in the following section and we will prove in section 7.2.3 that our conditions fully characterize the $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ weak factorization system of an accessible model categories with accessible weak equivalences.

Let λ be a regular cardinal. Let $\mathcal{F}^\lambda$ be the collection of objects that are in $\mathbf{Pres}_\lambda(\mathscr{C})$ and in $\mathcal{F}$. Let $K_\lambda : \mathbf{Disc}(\mathcal{F}^\lambda) \hookrightarrow \mathscr{C}^{\mathbf{2}}$ be the inclusion of the discrete subcategory of $\mathscr{C}^{\mathbf{2}}$ on the objects of $\mathcal{F}^\lambda$. Let $V : {}^{\boxslash}K_\lambda \to \mathscr{C}^{\mathbf{2}}$ be the forgetful functor. Let $\mathscr{J}_\lambda$ be the full subcategory of ${}^{\boxslash}K_\lambda$ on the objects in $V^{-1}(\mathbf{Pres}_\lambda(\mathscr{C}^{\mathbf{2}}))$ and let $N : \mathscr{J}_\lambda \hookrightarrow {}^{\boxslash}K_\lambda$ be the subcategory inclusion functor. Then $\mathscr{J}_\lambda$ is a small category. Let $J_\lambda : \mathscr{J}_\lambda \to \mathscr{C}^{\mathbf{2}}$ be the restriction of V to $\mathscr{J}_\lambda$.

$$\begin{tikzcd}
\mathscr{J}_\lambda \arrow[r, hook, "N"] \arrow[dr, "J_\lambda"'] & {}^{\boxslash}K_\lambda \arrow[d, "V"] \\
 & \mathscr{C}^{\mathbf{2}}
\end{tikzcd}$$

Lemma 7.2.3. *The functor V is surjective on objects in $\mathcal{C} \cap \mathcal{W}$.*

Proof. Indeed, if f is an object in $\mathcal{C} \cap \mathcal{W}$, then a solution exists to every lifting problem $(u, v) : f \to p$ when $p \in \mathcal{F}^\lambda$. By choosing a single solution to each such problem, we get a lift $\langle f, \varphi \rangle$ of f in ${}^{\boxslash}K_\lambda$. We are able to make these choices independently because the discrete category $\mathbf{Disc}(\mathcal{F}^\lambda)$ does not impose any coherence conditions on the lifts. ☐

To prove more about $\mathscr{J}_\lambda$, we will need to be more specific about the cardinal λ. Specifically, let λ be a regular cardinal that satisfies the following conditions.

- The collection $\mathcal{F}$ is λ-preaccessible.
- The subcategory inclusion functor $\mathbf{Full}(\mathcal{W}) \hookrightarrow \mathscr{C}^{\mathbf{2}}$ is a λ-accessible functor.

Note that whenever κ is a regular cardinal with $\kappa \rhd \lambda$, the above conditions hold for κ.

Lemma 7.2.4. *Every object in $\mathscr{J}_\lambda$ forgets to an object in $\mathcal{C} \cap \mathcal{W}$.*

Proof. Let $\langle f, \varphi\rangle$ be an object in $\mathscr{J}_\lambda$. Let $p \in \mathcal{F}$ and let $(u, v) : f \to p$ be a map in $\mathscr{C}^{\mathbf{2}}$. By (3), p is the colimit of a λ-filtered cocone $\{(s_\alpha, t_\alpha) : p_\alpha \to p\}$ of objects p_α in $\mathcal{F}^\lambda$. Since f is λ-presentable, there is an index α and a map $(x, y) : f \to p_\alpha$ such that $(u, v) = (s_\alpha, t_\alpha) \circ (x, y)$. So the map $\varphi(x, y, p_\alpha) : \operatorname{cod} f \to \operatorname{dom} p_\alpha$ is a solution to the lifting problem (x, y). So $s_\alpha \circ \varphi(x, y, p_\alpha) : \operatorname{cod} f \to \operatorname{dom} p$ is a solution to the lifting problem (u, v). Thus f has the left lifting property with respect to every object in $\mathcal{F}$. ☐

We will show condition (2) of proposition 7.1.15 holds with respect to the collection $\mathcal{C} \cap \mathcal{W}$.

Lemma 7.2.5. *The category $\mathscr{J}_\lambda$ has λ-small colimits.*

Proof. Let $D : \mathscr{D} \to \mathscr{J}_\lambda$ be a λ-small diagram. Since ${}^{\boxslash}K_\lambda$ is cocomplete, $\operatorname{colim} ND$ exists. Since V preserves colimits, $V(\operatorname{colim} ND) = \operatorname{colim} VND = \operatorname{colim} J_\lambda D$. Because $J_\lambda D$ is a λ-small diagram of λ-presentable objects in $\mathscr{C}^{\mathbf{2}}$, $\operatorname{colim} J_\lambda D$ is a λ-presentable object. So $\operatorname{colim} ND$ is an object in $\mathscr{J}_\lambda$. Since $\mathscr{J}_\lambda$ is a full subcategory of ${}^{\boxslash}K_\lambda$, $\operatorname{colim} D$ exists in $\mathscr{J}_\lambda$ and $N(\operatorname{colim} D) = \operatorname{colim} ND$. ☐

Lemma 7.2.6. *For every object f in $\mathscr{C}^{\mathbf{2}}$, the colimit of the canonical cocone of f with respect to J_λ is in $\mathcal{C} \cap \mathcal{W}$.*

Proof. Let $\{(u_\alpha, v_\alpha) : J_\lambda\langle f_\alpha, \varphi_\alpha\rangle \to f\}_\alpha$ be the canonical cocone of f with respect to J_λ. By 7.2.5, every λ-small subdiagram factors through a (u_α, v_α). So the cocone $\{(u_\alpha, v_\alpha)\}_\alpha$ is λ-filtered. By condition (1) and the fact that $\operatorname{colim}_\alpha f_\alpha$ is the colimit of a λ-filtered diagram of objects in $\mathcal{C} \cap \mathcal{W}$, $\operatorname{colim}_\alpha f_\alpha$ must be in $\mathcal{W}$. ☐

Lemma 7.2.7. *The collection $|\mathbf{Coalg}_{\mathrm{L}^{J_\lambda}}|$ is a subset of $\mathcal{C} \cap \mathcal{W}$.*

Proof. By 7.2.6, $\mathrm{L}_0^{J_\lambda} f \in \mathcal{C} \cap \mathcal{W}$ for every object f in $\mathscr{C}^{\mathbf{2}}$. Since $\mathcal{C} \cap \mathcal{W}$ is stable under pushouts, $\mathrm{L}_1^{J_\lambda} f \in \mathcal{C} \cap \mathcal{W}$. So $|\mathbf{Coalg}_{\mathrm{L}_1^{J_\lambda}}| \subseteq \mathcal{C} \cap \mathcal{W}$. By propositions 3.1.13 and 3.1.16, $(\mathcal{C} \cap$

$\mathcal{W})^{\square} \subseteq |\mathbf{Coalg}_{\mathrm{L}_1^{J_\lambda}}|^{\square} = |\mathbf{Coalg}^{\boxslash}_{\mathrm{L}_1^{J_\lambda}}| = |\mathbf{Alg}_{\mathrm{R}_1^{J_\lambda}}|$. By the freeness of $\mathbb{F}_1((\mathrm{L}_1^{J_\lambda}, \mathrm{R}_1^{J_\lambda})) = (\mathrm{L}^{J_\lambda}, \mathrm{R}^{J_\lambda})$, $|\mathbf{Alg}_{\mathrm{R}_1^{J_\lambda}}| = |\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}^{J_\lambda}}|$. So $|\mathbf{Coalg}_{\mathrm{L}^{J_\lambda}}| = |{}^{\boxslash}\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}^{J_\lambda}}| = {}^{\square}|\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}^{J_\lambda}}| = {}^{\square}|\mathbf{Alg}_{\mathrm{R}_1^{J_\lambda}}| \subseteq \mathcal{C} \cap \mathcal{W}$. □

Since L^{J_λ} is a comonad, $\mathrm{L}^{J_\lambda} f \in \mathcal{C} \cap \mathcal{W}$ for every object f in $\mathscr{C}^{\mathbf{2}}$. Also, we know the AWFS $(\mathrm{L}^{J_\lambda}, \mathrm{R}^{J_\lambda}) = \mathbb{F}_1\mathbb{F}_2\mathbb{F}_3(J_\lambda)$ is accessible by 7.1.11.

Lemma 7.2.8. *If $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is an accessible weak factorization system, then there is a regular cardinal κ such that every object $f \in \mathcal{C} \cap \mathcal{W}$ is the colimit of a diagram that factors through J_κ.*

Proof. By 7.1.11, there is an accessible algebraic weak factorization system $(\mathrm{C}_t, \mathrm{F})$ such that $|\mathbf{Coalg}_{\mathrm{C}_t}| = \mathcal{C} \cap \mathcal{W}$ and $|\mathbf{Alg}_{\mathrm{F}}| = \mathcal{F}$. Let κ be a regular cardinal such that the forgetful functor $U_{\mathrm{C}_t} : \mathbf{Coalg}_{\mathrm{C}_t} \to \mathscr{C}^{\mathbf{2}}$ is κ-accessible and preserves κ-presentable objects. We will choose $\mathbf{Pres}_\kappa(\mathbf{Coalg}_{\mathrm{C}_t})$ and $\mathbf{Pres}_\kappa(\mathscr{C}^{\mathbf{2}})$ so that U_{C_t} restricts to a functor $\mathbf{Pres}_\kappa(\mathbf{Coalg}_{\mathrm{C}_t}) \to \mathbf{Pres}_\kappa(\mathscr{C}^{\mathbf{2}})$. Since every object in $\mathcal{F}^\kappa$ is in the image of $U_{\mathrm{F}} : \mathbf{Alg}_{\mathrm{F}} \to \mathscr{C}^{\mathbf{2}}$ and there are no nonidentity morphisms in $\mathbf{Disc}(\mathcal{F}^\kappa)$, the subcategory inclusion functor $K_\kappa : \mathbf{Disc}(\mathcal{F}^\kappa) \hookrightarrow \mathscr{C}^{\mathbf{2}}$ has a lift $\widetilde{K}_\kappa : \mathbf{Disc}(\mathcal{F}^\kappa) \to \mathbf{Alg}_{\mathrm{F}}$ such that $U_{\mathrm{F}}\widetilde{K}_\kappa = K_\kappa$. By 3.1.13, there is a functor $F : \mathbf{Coalg}_{\mathrm{C}_t} \to {}^{\boxslash}\mathbf{Alg}_{\mathrm{F}}$ over $\mathscr{C}^{\mathbf{2}}$. Let G be the following composite functor over $\mathscr{C}^{\mathbf{2}}$

$$\begin{tikzcd}
\mathbf{Coalg}_{\mathrm{C}_t} \arrow[r, "F"] \arrow[rr, bend left, "G"] & {}^{\boxslash}\mathbf{Alg}_{\mathrm{F}} \arrow[r, "{}^{\boxslash}\widetilde{K}_\kappa"] & {}^{\boxslash}K_\kappa.
\end{tikzcd}$$

Then $VG = U_{\mathrm{C}_t}$, where $V : {}^{\boxslash}K_\kappa \to \mathscr{C}^{\mathbf{2}}$ is the forgetful functor.

Let f be an object in $\mathcal{C} \cap \mathcal{W}$ and let $\langle f, \vec{k} \rangle$ be a lift in $\mathbf{Coalg}_{\mathrm{C}_t}$. Since $\mathbf{Coalg}_{\mathrm{C}_t}$ is κ-accessible, there is a colimiting κ-filtered cocone $\{(u_\alpha, v_\alpha) : \langle f_\alpha, \vec{k}_\alpha \rangle \to \langle f, \vec{k} \rangle\}_\alpha$ of κ-presentable objects $\langle f_\alpha, \vec{k}_\alpha \rangle$ in $\mathbf{Coalg}_{\mathrm{C}_t}$. Since U_{C_t} preserves colimits and V reflects colimits, $\{G(u_\alpha, v_\alpha) : G\langle f_\alpha, \vec{k}_\alpha \rangle \to G\langle f, \vec{k} \rangle\}_\alpha$ is a colimiting cocone in ${}^{\boxslash}K_\kappa$. Since U_{C} preserves κ-presentable objects, $\{G(u_\alpha, v_\alpha)\}_\alpha$ is a κ-filtered cocone of objects $G\langle f_\alpha, \vec{k}_\alpha \rangle$ in $\mathscr{J}_\kappa$. Of course $VG\langle f,\vec{k}\rangle = U_{\mathrm{C}_t}\langle f,\vec{k}\rangle = f$. Along with the facts that V preserves colimits and $\mathscr{J}_\kappa$ is a full subcategory of ${}^{\boxslash}K_\kappa$, this means f is the colimit of a diagram that factors through J_κ. $\square$

If $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ is an accessible weak factorization system, then whenever κ is a regular cardinal that satisfies all of the following conditions, results 7.2.3 through 7.2.8 hold.

- The collection $\mathcal{F}$ is κ-preaccessible.
- The subcategory inclusion functor $\mathbf{Full}(\mathcal{W}) \hookrightarrow \mathscr{C}^{\mathbf{2}}$ is a κ-accessible functor.
- The forgetful functor $U_{\mathrm{C}_t} : \mathbf{Coalg}_{\mathrm{C}_t} \to \mathscr{C}^{\mathbf{2}}$ is κ-accessible and preserves κ-presentable objects.

From 7.1.15, it then follows that $(\mathrm{L}^{J_\kappa},\mathrm{R}^{J_\kappa})$ is a κ-accessible algebraic weak factorization system such that $|\mathbf{Coalg}_{\mathrm{L}^{J_\kappa}}| = \mathcal{C}\cap\mathcal{W}$ and $|\mathbf{Alg}_{\mathrm{R}^{J_\kappa}}| = \mathcal{F}$.

7.2.2 Dense Pairs

The codensity monad on $K_\lambda : \mathbf{Disc}(\mathcal{F}^\lambda) \hookrightarrow \mathscr{C}^{\mathbf{2}}$ provides an alternative description of the category ${}^{\boxslash}K_\lambda$. Unfortunately, unlike density comonads, a codensity monad on an accessible category does not have to be accessible.

Let $\mathscr{C}$ be a complete category. Let $K : \mathscr{K} \to \mathscr{C}^{\mathbf{2}}$ be a small category. We define an endofunctor ${}^K\mathrm{R}_0 : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ as the right Kan extension of K along itself. So on an object f in $\mathscr{C}^{\mathbf{2}}$,

$$ {}^K\mathrm{R}_0 f = \lim_{f\to f_\alpha} f_\alpha, $$

where the limit is indexed by the comma category $f \downarrow K$. By definition, there is a natural transformation $\vec{\alpha} : {}^K\mathrm{R}_0 K \to K$ such that for any functor $S : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ and any natural transformation $\vec{\beta} : SK \to K$, there is a unique natural transformation $\vec{\gamma} : S \to R$ such that

$\vec{\alpha}_K \vec{\gamma}_K = \vec{\beta}_K$. The universal property defines a unit map $\vec{\eta} : Id \to {}^K\mathrm{R}_0$ and a multiplication map $\vec{\mu} : {}^K\mathrm{R}_0 {}^K\mathrm{R}_0 \to {}^K\mathrm{R}_0$ that make ${}^K\mathrm{R}_0$ a monad.

Let $\vec{\eta} = (\eta^0, \eta^1)$ be the components of the unit map. We define a functorial factorization $({}^K\mathrm{L}_1, {}^K\mathrm{R}_1)$ objectwise by taking the pullback of ${}^K\mathrm{R}_0 f$ against η^0_f.

$$\begin{tikzcd}
\cdot \arrow[r, dashed, "{}^K\mathrm{L}_1 f"] \arrow[rr, bend left, "\eta^0_f"] \arrow[d, "f"'] & \cdot \arrow[r, "\sigma_f"] \arrow[d, "{}^K\mathrm{R}_1 f"] \arrow[dr, phantom, very near start, "\lrcorner"] & \cdot \arrow[d, "{}^K\mathrm{R}_0 f"] \\
\cdot \arrow[r, "id"'] & \cdot \arrow[r, "\eta^1_f"'] & \cdot
\end{tikzcd}$$

${}^K\mathrm{L}_1 f$ is the map into the limit defined by the cocone maps η^0_f and f. Dualizing the our discussion of the reflection $\mathbb{G}_2$ in section 3.2.5, we get that $({}^K\mathrm{L}_1, {}^K\mathrm{R}_1)$ is a RAWFS.

Proposition 7.2.9. *Let $K : \mathscr{K} \to \mathscr{C}^{\mathbf{2}}$ be a small category over $\mathscr{C}^{\mathbf{2}}$. Then ${}^{\boxslash}K \cong \mathbf{Coalg}_{{}^K\mathrm{L}_1}$.*

Proof. This is dual to 3.2.26. $\square$

If $\mathcal{K}$ is a set of objects, then we will use the notation ${}^{\mathcal{K}}\mathrm{L}$ for ${}^K\mathrm{L}$ when $K : \mathbf{Disc}(\mathcal{K}) \to \mathscr{C}^{\mathbf{2}}$ is the subcategory inclusion functor.

Definition 7.2.10. Let $\mathcal{L}$ be a collection of morphisms in a category $\mathscr{C}$. We will say a pair $(\mathcal{J}, \mathcal{K})$ of subsets $\mathcal{J} \subseteq \mathcal{L}$ and $\mathcal{K} \subseteq \mathcal{L}^{\square}$ is a *dense pair* if every object f in $\mathcal{L}$ is the colimit of a diagram of objects in $\mathcal{J}$ that factors through the forgetful functor $U_{{}^{\mathcal{K}}\mathrm{L}_1} : \mathbf{Coalg}_{{}^{\mathcal{K}}\mathrm{L}_1} \to \mathscr{C}^{\mathbf{2}}$.

7.2.3 Smith's Theorem for a Class of Accessible Model Categories

The following theorem is our attempt at a version of Smith's theorem for accessible model categories with accessible and accessibly embedded weak equivalences. While the weak equivalences in combinatorial model categories are always accessible and accessibly embedded, this is not the case for accessible model categories. We only attempt to characterize the accessible model categories whose weak equivalences are accessible and accessibly embedded. The author does not know whether this is an assumption that holds in all reasonable, well-behaved accessible model categories, or if it is a very restrictive assumption that doesn't often apply beyond combinatorial model categories.

Theorem 7.2.11. *Let $\mathscr{C}$ be a λ-accessible bicomplete category and let $\mathcal{W}$ and $\mathcal{C}$ be collections of morphisms in $\mathscr{C}^{\mathbf{2}}$ that satisfy the following conditions.*

- *$\mathcal{W}$ has the 2-out-of-3 property.*
- *$\mathbf{Full}(\mathcal{W})$ is accessible and accessibly embedded in $\mathscr{C}^{\mathbf{2}}$.*
- *$(\mathcal{C}, \mathcal{C}^{\square})$ is an accessible weak factorization system on $\mathscr{C}$.*
- *$\mathcal{C}^{\square} \subseteq \mathcal{W}$.*

Then there is an accessible model category on $\mathscr{C}$ with weak equivalences $\mathcal{W}$ and cofibrations $\mathcal{C}$ if and only if the following conditions are satisfied.

1. *$\mathcal{C} \cap \mathcal{W} = {}^{\square}\big((\mathcal{C} \cap \mathcal{W})^{\square}\big)$*
2. *$(\mathcal{C} \cap \mathcal{W})^{\square}$ is preaccessible*
3. *There is a regular cardinal λ_0 such that for every regular cardinal $\lambda \rhd \lambda_0$, the sets $\mathcal{C}_t^{\lambda} = |\mathbf{Pres}_{\lambda}(\mathscr{C}^{\mathbf{2}})| \cap \mathcal{C} \cap \mathcal{W}$ and $\mathcal{F}^{\lambda} = |\mathbf{Pres}_{\lambda}(\mathscr{C}^{\mathbf{2}})| \cap (\mathcal{C} \cap \mathcal{W})^{\square}$ form a dense pair $(\mathcal{C}_t^{\lambda}, \mathcal{F}^{\lambda})$ for $(\mathcal{C} \cap \mathcal{W}, (\mathcal{C} \cap \mathcal{W})^{\square})$*

Proof. Suppose $\mathscr{C}$ is an accessible model category with weak equivalences $\mathcal{W}$ and cofibrations $\mathcal{C}$. Then $(\mathcal{C} \cap \mathcal{W}, (\mathcal{C} \cap \mathcal{W})^{\square})$ is an accessible weak factorization system. So condition (1) holds. By 7.1.12, there is an accessible algebraic weak factorization system $(\mathrm{C}_t, \mathrm{F})$ such that $|\mathbf{Coalg}_{\mathrm{C}_t}| = \mathcal{C} \cap \mathcal{W}$ and $|\mathbf{Alg}_{\mathrm{F}}| = (\mathcal{C} \cap \mathcal{W})^{\square}$. The category $\mathbf{Alg}_{\mathrm{F}}$ and the forgetful functor

$\mathbf{Alg}_{\mathrm{F}} \to \mathscr{C}^{\mathbf{2}}$ are accessible [AR94, 2.76]. Since the full image of an accessible functor is preaccessible 7.2.2, condition (2) holds. Condition 3 holds by 7.2.8

Conversely, suppose conditions (1) - (3) hold. Let λ be a regular cardinal that satisfies the following conditions.

- The collection $(\mathcal{C} \cap \mathcal{W})^{\square}$ is λ-preaccessible.
- The subcategory inclusion functor $\mathbf{Full}(\mathcal{W}) \hookrightarrow \mathscr{C}^{\mathbf{2}}$ is λ-accessible.
- $\lambda \rhd \lambda_0$ for some regular cardinal λ_0 that satisfies (3).

Using the construction of section 7.2.1, we know from 7.2.6 that $J_\lambda : \mathscr{J}_\lambda \to \mathscr{C}^{\mathbf{2}}$ satisfies condition (2) of 7.1.15 for the collection $\mathcal{C} \cap \mathcal{W}$. Condition 3 is the equivalent to every $f \in \mathcal{C} \cap \mathcal{W}$ being a colimit of a diagram that factors through J_λ. So condition (1) of 7.1.15 for the collection $\mathcal{C} \cap \mathcal{W}$ is satisfied. Thus $(\mathrm{L}^{J_\lambda}, \mathrm{R}^{J_\lambda})$ is an accessible AWFS such that $|\mathbf{Coalg}_{\mathrm{L}^{J_\lambda}}| = \mathcal{C} \cap \mathcal{W}$ and, necessarily, $|\mathbf{Alg}_{\mathrm{R}^{J_\lambda}}| = (\mathcal{C} \cap \mathcal{W})^{\square}$. So $(\mathcal{C} \cap \mathcal{W}, (\mathcal{C} \cap \mathcal{W})^{\square})$ is an accessible weak factorization system. Since $(\mathcal{C}, \mathcal{C}^{\square})$ is an accessible weak factorization system by assumption, it remains to show that $\mathcal{C}^{\square} = (\mathcal{C} \cap \mathcal{W})^{\square} \cap \mathcal{W}$. Let $f \in (\mathcal{C} \cap \mathcal{W})^{\square} \cap \mathcal{W}$ and let $(\mathrm{C}, \mathrm{F}_t)$ be a functorial factorization associated to $(\mathcal{C}, \mathcal{C}^{\square})$. Since $\mathcal{W}$ has the 2-out-of-3 property and $\mathrm{F}_t f \in \mathcal{C}^{\square} \subset \mathcal{W}$, $\mathrm{C}f \in \mathcal{C} \cap \mathcal{W}$. So a solution exists to the lifting problem $(id, \mathrm{F}_t f) : \mathrm{C}f \to f$ and $f \in \mathcal{C}^{\square}$. Thus $(\mathcal{C} \cap \mathcal{W})^{\square} \cap \mathcal{W} \subseteq \mathcal{C}^{\square}$. The reverse inclusion is immediate. $\square$

We conclude with some facts we do know about the weak equivalences in an accessible model category.

Proposition 7.2.12. *Suppose $\mathscr{C}$ is an accessible model category with weak equivalences $\mathcal{W}$, cofibrations $\mathcal{C}$, and fibrations $\mathcal{F}$. The following are equivalent.*

1. *The category $\mathbf{Full}(\mathcal{F} \cap \mathcal{W})$ is accessible and accessibly embedded in $\mathscr{C}^{\mathbf{2}}$.*

2. *The category* $\mathbf{Full}(\mathcal{F} \cap \mathcal{W})$ *is a small injectivity class in* $\mathscr{C}^{\mathbf{2}}$*.*

3. *The weak factorization system* $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ *is cofibrantly generated by a set.*

Proof. (1) ⇔ (2) is [AR94, 4.8]. (2) ⇔ (3) is easy. See the proof of [Ros09, 3.3]. □

Proposition 7.2.13. *Suppose* $\mathscr{C}$ *is an accessible model category with weak equivalences* $\mathcal{W}$*, cofibrations* $\mathcal{C}$*, and fibrations* $\mathcal{F}$*. The following are equivalent.*

1. *The category* $\mathbf{Full}(\mathcal{W})$ *is accessible and accessibly embedded in* $\mathscr{C}^{\mathbf{2}}$*.*

2. *There is an accessible copointed endofunctor* C *with* $|\mathbf{Coalg}_{\mathrm{C}}| = \mathcal{C}$ *such that the category* $\mathbf{Coalg}_{\mathrm{C}_{\mathcal{W}}} = \mathbf{Full}(U_{\mathrm{C}}^{-1}(\mathcal{W}))$ *is accessible and accessibly embedded in* $\mathbf{Coalg}_{\mathrm{C}}$*.*

3. *There is an accessible pointed endofunctor* F *with* $|\mathbf{Alg}_{\mathrm{F}}| = \mathcal{F}$ *such that the category* $\mathbf{Alg}_{\mathrm{F}_{\mathcal{W}}} = \mathbf{Full}(U_{\mathrm{F}}^{-1}(\mathcal{W}))$ *is accessible and accessibly embedded in* $\mathbf{Alg}_{\mathrm{F}}$*.*

Proof. These all follow from [AR94, 2.50]. □

Lemma 7.2.14. *If a map of monads* $(\alpha, id) : \mathrm{R} \to \mathrm{R}'$ *is an epimorphism on objects in the image of* $V_{\mathrm{R}'} : \mathbf{Alg}_{\mathrm{R}'}^{\mathrm{EM}} \to \mathscr{M}^{\mathbf{2}}$*, then the functor* $\alpha^* : \mathbf{Alg}_{\mathrm{R}'}^{\mathrm{EM}} \to \mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}}$ *is the inclusion of a full subcategory.*

Proof. Suppose $\langle f, \vec{k} \rangle$ and $\langle f, \vec{l} \rangle$ are objects in $\mathbf{Alg}_{\mathrm{R}'}^{\mathrm{EM}}$ such that $\alpha^*\langle f, \vec{k} \rangle = \alpha^*\langle f, \vec{l} \rangle$. Then $\vec{k} \circ (\alpha_f, id) = \vec{l} \circ (\alpha_f, id)$. Since $f = U\langle f, \vec{k} \rangle$, (α_f, id) is an epimorphism. So $\vec{k} = \vec{l}$. Thus α^* is injective on objects.

We already know α^* is a faithful functor, so it remains to show that it is full. Let $\langle f, \vec{k} \rangle$ and $\langle g, \vec{m} \rangle$ be objects in $\mathbf{Alg}_{\mathrm{R}'}^{\mathrm{EM}}$ and let $\vec{u} : \alpha^*\langle f, \vec{k} \rangle \to \alpha^*\langle g, \vec{m} \rangle$ be a map in $\mathbf{Alg}_{\mathrm{R}}^{\mathrm{EM}}$. This means the outer rectangle in the following diagram must commute.

$$\begin{tikzcd}
\mathrm{R}f \arrow[r, "{(\alpha_f, id)}"] \arrow[d, "\mathrm{R}\vec{u}"] & \mathrm{R}'f \arrow[r, "\vec{k}"] \arrow[d, "\mathrm{R}'\vec{u}"] & f \arrow[d, "\vec{u}"] \\
\mathrm{R}f \arrow[r, "{(\alpha_g, id)}"] & \mathrm{R}'g \arrow[r, "\vec{m}"] & g
\end{tikzcd}$$

Since f is in the image of U, (α_f, id) is an epimorphism. Thus the right square in the above diagram commutes. So $\vec{u}$ is a map $\langle f, \vec{k}\rangle \to \langle g, \vec{m}\rangle$ in $\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}'}$. $\square$

Proposition 7.2.15. *Let* $\zeta : (\mathrm{C}_t, \mathrm{F}) \to (\mathrm{C}, \mathrm{F}_t)$ *be an accessible algebraic model category on an accessible bicomplete category* $\mathscr{M}$ *with weak equivalences* $\mathcal{W}$*. There is an AWFS* $(\mathrm{C}', \mathrm{F}'_t)$ *on* $\mathscr{M}$ *and a map of AWFSs* $\zeta' : (\mathrm{C}_t, \mathrm{F}) \to (\mathrm{C}', \mathrm{F}'_t)$ *such that* $\zeta'^* : \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{F}'_t} \to \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{F}}$ *is an accessible embedding of an accessible category.*

Proof. Let $\mathcal{C} = |\mathbf{Coalg}_{\mathrm{C}}|$ and let $\mathcal{F} = |\mathbf{Alg}_{\mathrm{F}}|$. Let X be the AWFS $(\mathrm{C}, \mathrm{F}_t)$. Let X_t be the AWFS $(\mathrm{C}_t, \mathrm{F})$. We define a new functorial factorization $X_t \odot X$ on $\mathscr{M}$ by sending an object f in $\mathscr{M}^{\mathbf{2}}$ to the object $(\mathrm{C}_t\mathrm{C}f, \mathrm{F}_t f \circ \mathrm{FC}f)$ in $\mathscr{M}^{\mathbf{3}}$. We will also use the notation $(\mathrm{L}f, \mathrm{R}f) \coloneqq (\mathrm{C}_t\mathrm{C}f, \mathrm{F}_t f \circ \mathrm{FC}f) = X_t \odot X f$. The operation $(X_t, X) \mapsto X_t \odot X$ preserves monad structure. So $X_t \odot X$ is a RAWFS. The functorial factorization $\perp$ defined by $\perp f = (f, id)$ is a unit for the operation $\odot$. There is a counit map $X_t \to \perp$ defined by the horizontal arrows in the following commutative diagram.

$$\begin{tikzcd}
{} \arrow[r, "id"] \arrow[d, "\mathrm{C}_t f"'] & {} \arrow[d, "f"] \\
{} \arrow[r, "\mathrm{F}f"] \arrow[d, "\mathrm{F}f"'] & {} \arrow[d, "id"] \\
{} \arrow[r, "id"] & {}
\end{tikzcd}$$

The counit map is a map of AWFSs. We therefore get that $X_t \odot X \to \perp \odot X \cong X$ is a map of RAWFSs.

Let $f \in \mathcal{F} \cap \mathcal{W}$ be given. Then $\mathrm{C}f \in \mathcal{C} \cap \mathcal{W}$. So the map $\vec{\varepsilon}_{\mathrm{C}f} = (id, \mathrm{FC}f) : \mathrm{C}_t\mathrm{C}f \to \mathrm{C}f$ is an epimorphism. But $\mathrm{FC}f$ is the counit map $X_t \odot X \to \perp \odot X \cong X$. In other words, the map of monads $(\mathrm{FC}f, id) : \mathrm{R}f \to \mathrm{F}_t f$ is an epimorphism. Since this holds for every $f \in \mathcal{F} \cap \mathcal{W} = |\mathbf{Alg}_{\mathrm{F}_t}|$, the functor

$$(\mathrm{FC}f)^* : \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{F}_t} \to \mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}}$$

is the inclusion of a full subcategory. Since $(\mathrm{FC}f)^*$ is an accessible functor and $\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{F}_t}$ is an accessible category, $\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{F}_t}$ is accessibly embedded in $\mathbf{Alg}^{\mathrm{EM}}_{\mathrm{R}}$. □

Proposition 7.2.16. *Let $\mathscr{M}$ be an accessible model category with weak equivalences $\mathcal{W}$, cofibrations $\mathcal{C}$, and fibrations $\mathcal{F}$. There is an algebraic model category $\zeta : (\mathrm{C}_t, \mathrm{F}) \to (\mathrm{C}, \mathrm{F}_t)$ on $\mathscr{M}$ whose underlying model category has weak equivalences $\mathcal{W}$ and cofibrations $\mathcal{C}$ such that $\zeta_* : \mathbf{Coalg}^{\mathrm{EM}}_{\mathrm{C}_t} \to \mathbf{Coalg}^{\mathrm{EM}}_{\mathrm{C}'}$ is an accessible embedding of an accessible category.*

Proof. This is dual to 7.2.15. □

An attempt at a version of Smith's theorem was made in [Ros17, 5.3]. The claim in that paper that the weak equivalences are accessible and accessibly embedded is a major error, which was recently corrected in [Ros20]. Some other issues with [Ros17, 5.3] are that it is not a complete characterizations and that condition (5) is likely almost never true. In [Ros20] it is shown that the weak equivalences in an accessible model category are preaccessible. The author is not sure if the definition of preaccessible collections there agrees with the one in this paper.

Appendix A: Appendix 1

A.1 Well-Pointed Endofunctor Construction

Let $G : \mathscr{C} \to \mathscr{B}$ be a functor with a left adjoint $F : \mathscr{B} \to \mathscr{C}$ and let $\xi : FG \to \mathrm{Id}$ and $\nu : \mathrm{Id} \to GF$ be the counit and unit maps of the adjunction. Let (R, ρ) be a pointed endofunctor on $\mathscr{B}$ and suppose $\mathscr{C}$ is cocomplete. Then $\mathbf{End}(\mathscr{C})$ is cocomplete and we can define an endofunctor $\mathrm{S} : \mathscr{C} \to \mathscr{C}$ by the following pushout diagram in $\mathbf{End}(\mathscr{C})$.

$$\begin{tikzcd}
FG \arrow[r, "F\rho G"] \arrow[d, "\xi"] & F\mathrm{R}G \arrow[d, "\zeta"] \\
\mathrm{Id} \arrow[r, "\sigma"] & \mathrm{S} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Proposition A.1.1 ([Kel80, 9.2]). *If* (R, ρ) *is a well-pointed endofunctor, then* (S, σ) *is well-pointed.*

Proof. We know $\mathrm{S}\sigma \circ \sigma = \sigma\mathrm{S} \circ \sigma$, so it suffices to show $\mathrm{S}\sigma \circ \zeta = \sigma\mathrm{S} \circ \zeta$. Let $\varphi = G\zeta \circ \nu\mathrm{R}G : \mathrm{R}G \to G\mathrm{S}$. The commutativity of the left diagram below shows that the equality $G\sigma = \varphi \circ \rho G$ holds. Therefore the right diagram commutes.

$$\begin{tikzcd}
G \arrow[r, "\rho G"] \arrow[d, "\nu G"] \arrow[dd, bend right=60, "id"'] & \mathrm{R}G \arrow[d, "\nu\mathrm{R}G"'] \arrow[dd, bend left=60, "\varphi"] \\
GFG \arrow[r, "GF\rho G"] \arrow[d, "G\xi"] & GF\mathrm{R}G \arrow[d, "G\zeta"'] \\
G \arrow[r, "G\sigma"] & G\mathrm{S}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{R}G \arrow[r, "\rho\mathrm{R}G"] \arrow[d, "\varphi"] & \mathrm{R}\mathrm{R}G \arrow[d, "\mathrm{R}\varphi"] & \mathrm{R}G \arrow[l, "\mathrm{R}\rho G"'] \arrow[dl, "\mathrm{R}G\sigma"] \arrow[d, "\varphi"] \\
G\mathrm{S} \arrow[r, "\rho G\mathrm{S}"] \arrow[dr, "G\sigma\mathrm{S}"'] & \mathrm{R}G\mathrm{S} \arrow[d, "\varphi\mathrm{S}"] & G\mathrm{S} \arrow[dl, "G\mathrm{S}\sigma"] \\
& G\mathrm{S}\mathrm{S} &
\end{tikzcd}$$

Since (R,ρ) is well-pointed, $\rho\mathrm{R}G = \mathrm{R}\rho G$. Thus $G\sigma\mathrm{S}\circ\varphi = G\mathrm{S}\sigma\circ\varphi$. Applying F to this equality and composing with $\xi\mathrm{SS} : FG\mathrm{SS} \to \mathrm{SS}$ gives the desired equality.

$$\mathrm{S}\sigma\circ\zeta = \mathrm{S}\sigma\circ\xi\mathrm{S}\circ F\varphi = \xi\mathrm{SS}\circ FG\mathrm{S}\sigma\circ F\varphi = \xi\mathrm{SS}\circ FG\sigma\mathrm{S}\circ F\varphi$$
$$= \sigma\mathrm{S}\circ\xi\mathrm{S}\circ F\varphi = \sigma\mathrm{S}\circ\zeta.$$

□

Proposition A.1.2 ([Kel80, 9.2]). *An object X in $\mathscr{C}$ is an* S*-algebra if and only if GX is an* R*-algebra.*

Proof. We have bijective correspondences between the following classes.

- Maps $m : \mathrm{S}X \to X$ such that $m\circ\tau_X = id_X$.
- Maps $n : F\mathrm{R}GX \to X$ such that $n\circ F\rho_{GX} = \xi_X$.
- Maps $p : \mathrm{R}GX \to GX$ such that $p\circ\rho_{GX} = id_{GX}$.

□

A.2 Final Functors

Definition A.2.1. A functor $F : \mathscr{E} \to \mathscr{D}$ is *final* if for every object d in $\mathscr{D}$, $d \downarrow F$ is nonempty and connected.

If $E : \mathscr{E} \to \mathscr{D}$ and $D : \mathscr{D} \to \mathscr{C}$ are diagrams, then any cocone $\alpha : D \to \Delta^{\mathscr{D}}_X$ determines a cocone $\alpha E : DE \to \Delta^{\mathscr{D}}_X E = \Delta^{\mathscr{E}}_X$. So we get a map $\operatorname{colim} DE \to X$. In particular, when $\alpha : D \xrightarrow{\cdot} \operatorname{colim} D$ is the colimiting cocone of D, then we get a cocone $\alpha E : DE \xrightarrow{\cdot} \operatorname{colim} D$ and a map $\operatorname{colim} DE \to \operatorname{colim} D$. The following result is well-known.

Proposition A.2.2 ([Mac71, IX. §3]). *If* $E : \mathscr{E} \to \mathscr{D}$ *and* $D : \mathscr{D} \to \mathscr{C}$ *are diagrams and* E *is a final functor, then the cocone* $DE \overset{\cdot}{\longrightarrow} \operatorname{colim} D$ *defines an isomorphism* $\operatorname{colim} DE \cong \operatorname{colim} D$.

It will be useful to have a couple of shortcuts for checking that a functor is final. In certain situations we only need to check the nonemptiness condition to guarantee finality.

Proposition A.2.3. *Let* $\mathscr{D}$ *be a finitely filtered category and let* $F : \mathscr{E} \hookrightarrow \mathscr{D}$ *be the inclusion of a full subcategory. If* $d \downarrow F$ *is nonempty for each object* d *in* $\mathscr{D}$*, then* F *is a final functor.*

Proof. Let d be an object in $\mathscr{D}$. We only need to show $d \downarrow F$ is connected. Let $u_1 : d \to e_1$ and $u_2 : d \to e_2$ be two maps in $\mathscr{D}$ to objects e_1 and e_2 in $\mathscr{E}$. Since $\mathscr{D}$ is finitely-filtered, there are maps $v_1 : e_1 \to d_1$ and $v_2 : e_2 \to d_1$ in $\mathscr{D}$ such that $v_1 \circ u_1 = v_2 \circ u_2$. Since $d_1 \downarrow D$ is nonempty, there is a map $u_3 : d_1 \to e_3$. Because $\mathscr{E}$ is a full subcategory of $\mathscr{D}$, the maps $u_3 \circ v_1$ and $u_3 \circ v_2$ are maps in $\mathscr{E}$. ☐

Proposition A.2.4. *Let* $D : \mathscr{D} \to \mathscr{C}$ *be a diagram in* $\mathscr{C}$*, let* $\mathcal{M}$ *be some collection of monomorphisms in* $\mathscr{C}$*, and let* X *be an object in* $\mathscr{C}$*. A finitely filtered diagram* $E : \mathscr{E} \to D \downarrow_{\mathcal{M}} X$ *is final if for every object* (d, f) *in* $D \downarrow_{\mathcal{M}} X$*, the category* $(d, f) \downarrow E$ *is nonempty.*

Proof. Let $f : Dd \to X$ be an object in $D \downarrow_{\mathcal{M}} X$. It suffices to check that $(d, f) \downarrow E$ is connected. Suppose $u_1 : (d, f) \to Ee_1$ and $u_2 : (d, f) \to Ee_2$ are maps in $D \downarrow_{\mathcal{M}} X$. Since $\mathscr{E}$ is finitely filtered, there are maps $x_1 : e_1 \to e_3$ and $x_2 : e_2 \to e_3$ in $\mathscr{E}$. So $Ex_1 : Ee_1 \to Ee_3$ and $Ex_2 : Ee_2 \to Ee_3$ are maps in $D \downarrow_{\mathcal{M}} X$. Thus, as maps in $\mathscr{C}$,

$$Ee_3 \circ Ex_1 \circ u_1 = Ee_1 \circ u_1 = f = Ee_2 \circ u_2 = Ee_3 \circ Ex_2 \circ u_2.$$

But Ee_3 is a monomorphism in $\mathscr{C}$. So $Ex_1 \circ u_1 = Ex_2 \circ u_2$. ☐

A.3 Adjunctions

Given an adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\top"] & \mathscr{C} \arrow[l, bend right, "G"'] \end{tikzcd}$$

with unit $\nu : I \to GF$ and counit $\xi : FG \to I$, there is a comparison functor $H_{FG} : \mathscr{A} \to \mathbf{Coalg}^{\mathrm{EM}}_{FG}$ defined by sending an object A to the object $\langle FA, F\nu_A \rangle$ and sending a map $f : A \to B$ to the map $Ff : \langle FA, F\nu_A \rangle \to \langle FB, F\nu_B \rangle$. Then $U_{FG} H_{FG} = F$, where $U_{FG} : \mathbf{Coalg}^{\mathrm{EM}}_{FG} \to \mathscr{C}$ is the forgetful functor.

Definition A.3.1. An adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\top"] & \mathscr{C} \arrow[l, bend right, "G"'] \end{tikzcd}$$

is *comonadic* if the comparison functor $H_{FG} : \mathscr{A} \to \mathbf{Coalg}^{\mathrm{EM}}_{FG}$ is an equivalence of categories. A functor $F : \mathscr{A} \to \mathscr{C}$ is *comonadic* if it is the left adjoint in a comonadic adjunction.

Proposition A.3.2. *Let*

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend right, "F_1"'] \arrow[r, phantom, "\top"] & \mathscr{C} \arrow[l, bend right, "G_1"'] & \text{and} & \mathscr{B} \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\top"] & \mathscr{C} \arrow[l, bend right, "G_2"'] \end{tikzcd}$$

be comonadic adjunctions. Every map of comonads $\alpha : F_1G_1 \to F_2G_2$ defines a functor $K : \mathscr{A} \to \mathscr{B}$ with a natural isomorphism $F_2K \to F_1$. Conversely, every functor $K : \mathscr{A} \to \mathscr{B}$ with a natural isomorphism $F_2K \to F_1$ defines a map of comonads $\alpha : F_1G_1 \to F_2G_2$.

Proof. Suppose $\alpha : F_1G_1 \to F_2G_2$ is a map of comonads. There is a functor $\alpha_* : \mathbf{Coalg}^{\mathrm{EM}}_{F_1G_1} \to \mathbf{Coalg}^{\mathrm{EM}}_{F_2G_2}$ defined by sending an object $\langle X, k \rangle$ in $\mathbf{Coalg}^{\mathrm{EM}}_{F_1G_1}$ to the object $\langle X, \alpha_X \circ k \rangle$ in $\mathbf{Coalg}^{\mathrm{EM}}_{F_2G_2}$ and sending a map $f : \langle X, k \rangle \to \langle Y, l \rangle$ to the map $f : \langle X, \alpha_X \circ k \rangle \to \langle Y, \alpha_Y \circ l \rangle$. So in particular, $U_{F_2G_2}\alpha_* = U_{F_1G_1}$, where $U_{F_1G_1} : \mathbf{Coalg}^{\mathrm{EM}}_{F_1G_1} \to \mathscr{C}$ and $U_{F_2G_2} : \mathbf{Coalg}^{\mathrm{EM}}_{F_2G_2} \to \mathscr{C}$ are the forgetful functors. Let $\widehat{H}_{F_2G_2} : \mathbf{Coalg}^{\mathrm{EM}}_{F_2G_2} \to \mathscr{B}$ be an up-to-natural-isomorphism inverse for the comparison functor $H_{F_2G_2} : \mathscr{B} \to \mathbf{Coalg}^{\mathrm{EM}}_{F_2G_2}$. We define K to be the composition of the functors in the top row of the following diagram.

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "H_{F_1G_1}"] \arrow[dr, "F_1"'] & \mathbf{Coalg}^{\mathrm{EM}}_{F_1G_1} \arrow[r, "\alpha_*"] \arrow[d, "U_{F_1G_1}"] & \mathbf{Coalg}^{\mathrm{EM}}_{F_2G_2} \arrow[r, "\widehat{H}_{F_2G_2}"] \arrow[d, "U_{F_2G_2}"'] & \mathscr{B} \arrow[d, "F_2"] \\
 & \mathscr{C} \arrow[r, "\mathrm{Id}"] & \mathscr{C} \arrow[r, "\cong"] & \mathscr{C}
\end{tikzcd}$$

Then

$$F_2K = F_2\widehat{H}_{F_2G_2}\alpha_*H_{F_1G_1} = U_{F_2G_2}H_{F_2G_2}\widehat{H}_{F_2G_2}\alpha_*H_{F_1G_1}$$

$$\cong U_{F_2G_2}\alpha_*H_{F_1G_1} = U_{F_1G_1}H_{F_1G_1} = F_1.$$

Conversely, suppose $K : \mathscr{A} \to \mathscr{B}$ is a functor with a natural isomorphism $\beta : F_2K \to F_1$. We will use ν for both of the adjunction unit maps $\nu : \mathrm{Id} \to G_1F_1$ and $\nu : \mathrm{Id} \to G_2F_2$ and ξ for both of the counit maps $\xi : F_1G_1 \to \mathrm{Id}$ and $\xi : F_2G_2 \to \mathrm{Id}$. Which unit and counit maps we are using should be clear from context. Let α be the composite natural transformation

$$F_1G_1 \xrightarrow[\cong]{\beta^{-1}G_1} F_2KG_1 \xrightarrow{F_2\nu KG_1} F_2G_2F_2KG_1 \xrightarrow[\cong]{F_2G_2\beta G_1} F_2G_2F_1G_1 \xrightarrow{F_2G_2\xi} F_2G_2.$$

It is easy to check that $\xi \circ \alpha = \xi : F_1G_1 \to \mathrm{Id}$. A diagram chase shows that the following diagram commutes.

$$\begin{tikzcd}
F_1G_1 \arrow[r, "\beta^{-1}G_1", "\cong"'] \arrow[d, "F_1\nu G_1"] & F_2KG_1 \arrow[r, "F_2\nu KG_1"] & F_2G_2F_2KG_1 \arrow[r, "F_2G_2\beta G_1", "\cong"'] & F_2G_2F_1G_1 \\
F_1G_1F_1G_1 \arrow[urrr, bend right, "\alpha F_1G_1"] & & &
\end{tikzcd}$$

We can also show that

$$\begin{tikzcd}
F_2KG_1 \arrow[r, "F_2\nu KG_1"] \arrow[d, "F_2\nu KG_1"] & F_2G_2F_2KG_1 \arrow[r, "F_2G_2\beta G_1"] & F_2G_2F_1G_1 \arrow[r, "F_2G_2\xi"] & F_2G_2 \arrow[d, "F_2\nu G_2"] \\
F_2G_2F_2KG_1 \arrow[r, "F_2G_2\beta G_1"] & F_2G_2F_1G_1 \arrow[rr, "F_2G_2\alpha"] & & F_2G_2F_2G_2
\end{tikzcd}$$

commutes. Therefore $F_2\nu G_2 \circ \alpha = F_2G_2\alpha \circ \alpha F_1G_1 \circ F_1\nu G_1$. So α is a map of comonads. $\square$

Definition A.3.3 (Cosolution Set Condition)**.** A functor $F : \mathscr{A} \to \mathscr{B}$ between locally small categories satisfies the *cosolution set condition* relative to an object B in $\mathscr{B}$ if there is a set $\mathcal{S}_B$ of objects in $\mathscr{A}$ such that for every object A in $\mathscr{A}$ and every map $f : FA \to B$ in $\mathscr{B}$, there is an object A' in $\mathcal{S}_B$ and maps $a : A \to A'$ and $f' : FA' \to B$ such that $f' \circ Fa = f$.

$$\begin{tikzcd}
A \arrow[d, dashed, "a"'] & FA \arrow[r, "f"] \arrow[d, "Fa"'] & B \\
A' & FA' \arrow[ur, "f'"'] &
\end{tikzcd}$$

Proposition A.3.4. *If $\mathscr{A}$ is cocomplete, then a colimit-preserving functor $F : \mathscr{A} \to \mathscr{B}$ between locally small categories has a right adjoint if and only if it satisfies the cosolution set condition with respect to each object B in $\mathscr{B}$.*

Proof. This is dual to [Bor94b, 3.3.3]. □

Definition A.3.5. Let $\mathcal{E}$ be some collection of epimorphisms in a category $\mathscr{A}$. An $\mathcal{E}$*-strong family of generators* for $\mathscr{A}$ is a set of objects $\{G_i\}_{i\in\mathcal{I}}$ such that for each object X, the map $\coprod_{i\in\mathcal{I}} \coprod_{\mathscr{A}(G_i,X)} G_i \to X$ is an $\mathcal{E}$-map.

Proposition A.3.6. *Let $\mathscr{A}$ be a category and let $\mathcal{E}$ be some collection of epimorphisms in $\mathscr{A}$ that is stable under pushouts. A functor $F : \mathscr{A} \to \mathscr{B}$ between locally small categories has a right adjoint if the following conditions are satisfied.*

1. *$\mathscr{A}$ is cocomplete*

2. *F preserves colimits*

3. *$\mathscr{A}$ is $\mathcal{E}$-well-copowered*

4. *$\mathscr{A}$ has an $\mathcal{E}$-strong family of generators*

Proof. We will closely follow the proof of [Bor94b, 3.3.4]. Let B be an object in $\mathscr{B}$. It suffices to show that F satisfies the cosolution set condition with respect to B. Let $\{G_i\}_{i\in I}$ be an $\mathcal{E}$-strong family of generators for $\mathscr{A}$. Since $\mathscr{A}$ is $\mathcal{E}$-well-copowered, the collection of $\mathcal{E}$-quotients of the object $\coprod_{i\in I}\coprod_{\mathscr{B}(FG_i,B)} G_i$ in $\mathscr{A}$ has only a set of isomorphism classes. Recall from definition 2.4.8 that two $\mathcal{E}$-quotients $p : \coprod_{i\in I}\coprod_{\mathscr{B}(FG_i,B)} G_i \to X$ and $q : \coprod_{i\in I}\coprod_{\mathscr{B}(FG_i,B)} G_i \to X$ are in the same isomorphism class if there is an isomorphism $g : X \to Y$ such that $g \circ p = q$. Let $\mathcal{S}_B^{\mathrm{mor}}$ be a set of representatives for the isomorphism classes of $\mathcal{E}$-quotients of $\coprod_{i\in I}\coprod_{\mathscr{B}(FG_i,B)} G_i$. Let $\mathcal{S}_B = \{X \mid X = \operatorname{cod} p,\ p \in \mathcal{S}_B^{\mathrm{mor}}\}$.

Let A be an object in $\mathscr{A}$ and let $f : FA \to B$ be a map in $\mathscr{B}$. We will refer to the below diagrams. Let in_g be the inclusion map into the term of the coproduct indexed by g. The map τ is determined by the inclusion maps. Let q be the map defined by the canonical cocone $\{G_i \to A\}_{i\in\mathcal{I}}$ of A with respect to $\mathbf{Disc}(\{G_i\}_{i\in\mathcal{I}})$. The maps p and a are the cocone maps of the pushout square. Since $\mathcal{E}$ is closed under pushouts, $p \in \mathcal{E}$. Thus $A' \in \mathcal{S}_B$.

$$\begin{tikzcd}
G_i \arrow[rr, bend left, "g"] \arrow[dr, "in_g"] \arrow[ddr, bend right, "in_{f\circ Fg}"'] & & \\
& \coprod_{i\in I}\coprod_{\mathscr{A}(G_i,A)} G_i \arrow[r, "q"] \arrow[d, "\tau"] & A \arrow[d, "a"] \\
& \coprod_{i\in I}\coprod_{\mathscr{B}(FG_i,B)} G_i \arrow[r, "p"'] & A' \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
\coprod_{i\in I}\coprod_{\mathscr{A}(G_i,A)} FG_i \arrow[r, "Fq"] \arrow[d, "F\tau"] & FA \arrow[d, "Fa"] \arrow[ddr, bend left, "f"] & \\
\coprod_{i\in I}\coprod_{\mathscr{B}(FG_i,B)} FG_i \arrow[r, "Fp"'] \arrow[drr, "u"] & FA' \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "f'"] & \\
FG_i \arrow[u, "in_g"] \arrow[rr, "g"'] & & B
\end{tikzcd}$$

Because F preserves colimits, the top rectangle of the diagram on the right is cocartesian. The map u is defined by the discrete canonical cocone $\{FG_i \to B\}_{i\in\mathcal{I}}$. So there is a map $f' : FA' \to B$ making the right diagram commute. □

A.4 Density Comonad

The *density comonad* is obtained from a left Kan extension. Let $K : \mathscr{I} \to \mathscr{C}$ and $L : \mathscr{I} \to \mathscr{C}$ be functors between locally small categories.

Definition A.4.1. The left Kan extension of L along K is a functor $\mathrm{Lan}_K(L) : \mathscr{C} \to \mathscr{C}$ with a natural transformation $\alpha : L \to \mathrm{Lan}_K(L)K$ such that for any other functor $F : \mathscr{C} \to \mathscr{C}$ with a natural transformation $\beta : L \to FK$, there is a unique natural transformation $\gamma : \mathrm{Lan}_K(L) \to F$ such that $\gamma_K \circ \alpha = \beta$.

$$\begin{tikzcd}
\mathscr{I} \arrow[r, "L"] \arrow[d, "K"'] \arrow[dr, phantom, "\circlearrowright" near start] & \mathscr{C} \\
\mathscr{C} \arrow[ur, dashed, "\mathrm{Lan}_K(L)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathrm{Lan}_K(L)K \arrow[d, "\gamma K"] & \mathrm{Lan}_K(L) \arrow[d, dashed, "\exists!"', "\gamma"] \\
& FK & F
\end{tikzcd}$$

When it exists, the left Kan extension of L along K is unique up to unique natural isomorphism.

Proposition A.4.2 ([Bor94b, 3.7.2])**.** *When $\mathscr{C}$ is a locally small cocomplete category and $\mathscr{I}$ is a small category, the left Kan extension of L along K exists.*

Suppose $\mathscr{C}$ is a cocomplete locally small category and $\mathscr{I}$ is a small category. Recall that $\mathscr{C}^{\mathscr{I}}$ is the locally small category whose objects are functors from $\mathscr{I}$ to $\mathscr{C}$ and whose morphisms are natural transformations. Fix a functor $K : \mathscr{I} \to \mathscr{C}$. There is a functor $\mathrm{ev}_K(-) : \mathbf{End}(\mathscr{C}) \to \mathscr{C}^{\mathscr{I}}$ that sends an endofunctor $F : \mathscr{C} \to \mathscr{C}$ to the functor $FK : \mathscr{I} \to \mathscr{C}$. If $\iota : J \to I$ is a natural transformation of functors in $\mathscr{C}^{\mathscr{I}}$, then the universal property of the left Kan extension of J along K implies the existence of a natural transformation $\mathrm{Lan}_K(J) \to \mathrm{Lan}_K(I)$ of endofunctors on $\mathscr{C}$.

$$\begin{tikzcd}
J \arrow[rr, "\alpha^J"] \arrow[dr, "\iota"'] & & \mathrm{Lan}_K(J)K \arrow[d, "\gamma K"] & \mathrm{Lan}_K(J) \arrow[d, dashed, "\exists!"', "\gamma"] \\
& I \arrow[r, "\alpha^I"] & \mathrm{Lan}_K(I)K & \mathrm{Lan}_K(I)
\end{tikzcd}$$

So $\mathrm{Lan}_K(-) : \mathscr{C}^{\mathscr{I}} \to \mathbf{End}(\mathscr{C})$ is a functor. Furthermore, the universal property of $\mathrm{Lan}_K(J)$ implies that there is a bijection

$$\mathbf{End}(\mathscr{C})\big(\mathrm{Lan}_K(J), F\big) \cong \mathscr{C}^{\mathscr{I}}\big(J, \mathrm{ev}_K(F)\big) \tag{A.1}$$

which is natural over F in $\mathbf{End}(\mathscr{C})$ and J in $\mathscr{C}^{\mathscr{I}}$. So $\mathrm{Lan}_K(-)$ is a left adjoint to ev_K, which proves the following result.

Proposition A.4.3. *The endofunctor $Lan_K(K) : \mathscr{C} \to \mathscr{C}$ is a comonad on $\mathscr{C}$.*

We call $\mathrm{Lan}_K(K)$ the *density comonad* on K.

Lemma A.4.4. *Let $\alpha : K \to Lan_K(K)K$ be the universal natural transformation of the left Kan extension.*

- *If $(\mathrm{L}, \varepsilon)$ is a copointed endofunctor on $\mathscr{C}$, then for every natural transformation $\beta : K \to \mathrm{L}K$ such that $\varepsilon K \circ \beta = id$, there is a unique map of copointed endofunctors $\gamma : Lan_K(K) \to \mathrm{L}$ such that $\gamma K \circ \alpha = \beta$.*

- *If $(\mathrm{L}, \varepsilon, \delta)$ is a comonad on $\mathscr{C}$, then for every natural transformation $\beta : K \to \mathrm{L}K$ such that $\varepsilon K \circ \beta = id$ and $\delta K \circ \beta = \mathrm{L}\beta \circ \beta$, there is a unique map of comonads $\gamma : Lan_K(K) \to \mathrm{L}$ such that $\gamma K \circ \alpha = \beta$.*

Proof. We will prove the second result. By the universal property of the left Kan extension, there is a unique natural transformation $\gamma : \mathrm{Lan}_K(K) \to \mathrm{L}$ such that $\gamma K \circ \alpha = \beta$. We only need to show that γ is a map of comonads. Let $\varepsilon^0 : \mathrm{Lan}_K(K) \to Id$ and $\delta^0 : \mathrm{Lan}_K(K) \to \mathrm{Lan}_K(K)\mathrm{Lan}_K(K)$ be the counit and comultiplication maps of the comonad $\mathrm{Lan}_K(K)$. Then $\varepsilon^0 K \circ \alpha = id$ and $\delta K \circ \alpha = \mathrm{Lan}_K(K)\alpha \circ \alpha$.

By definition, $\varepsilon^0 : \mathrm{Lan}_K(K) \to \mathit{Id}$ is the unique natural transformation such that $\varepsilon^0 K \circ \alpha = \mathit{id}$. But $\varepsilon \circ \gamma : \mathrm{Lan}_K(K) \to \mathit{Id}$ is a natural transformation such that $\varepsilon K \circ \gamma K \circ \alpha = \varepsilon^0 K \circ \alpha = \mathit{id}$. So $\varepsilon \circ \gamma = \varepsilon^0$. So γ is a map of copointed endofunctors.

By the universal property of the left Kan extension, there is a unique map $\theta : \mathrm{Lan}_K(K) \to \mathrm{LL}$ such that $\mathrm{L}\beta \circ \beta = \theta K \circ \alpha$. Since $\delta \circ \gamma : \mathrm{Lan}_K(K) \to \mathrm{LL}$ is a natural transformation such that $\delta K \circ \gamma K \circ \alpha = \delta K \circ \beta = \mathrm{L}\beta \circ \beta$, $\delta \circ \gamma = \theta$. Since $\gamma \mathrm{L} \circ \mathrm{Lan}_K(K)\gamma \circ \delta^0 : \mathrm{Lan}_K(K) \to \mathrm{LL}$ is a natural transformation that makes the following diagram commute, $\gamma \mathrm{L} \circ \mathrm{Lan}_K(K)\gamma \circ \delta^0 = \theta$.

$$\begin{tikzcd}
\mathrm{Lan}_K(K)K \arrow[d, "\delta^0 K"] & K \arrow[l, "\alpha"'] \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & \\
\mathrm{Lan}_K(K)\mathrm{Lan}_K(K)K \arrow[dr, "\mathrm{Lan}_K(K)\gamma K"'] & \mathrm{Lan}_K(K)K \arrow[l, "\mathrm{Lan}_K(K)\alpha"'] \arrow[r, "\gamma K"] \arrow[d, "\mathrm{Lan}_K(K)\beta"] & \mathrm{L}K \arrow[d, "\mathrm{L}\beta"] \\
 & \mathrm{Lan}_K(K)\mathrm{L}K \arrow[r, "\gamma \mathrm{L}K"] & \mathrm{LL}K
\end{tikzcd}$$

Therefore $\gamma : \mathrm{Lan}_K(K) \to \mathrm{L}$ is a map of comonads. ☐

A.5 Maps of AWFSs are AWFSs

We prove the folklore result that there is a bijective correspondence between maps of AWFSs on $\mathscr{C}$ and maps of AWFSs on $\mathscr{C}^{\mathbf{2}}$. In fact, we can show that this correspondence is functorial. We do not use this result anywhere else in this thesis.

Proposition A.5.1. *There is an isomorphism of categories* $\Phi : \mathbf{AWFS}(\mathscr{C})^{\mathbf{2}} \to \mathbf{AWFS}(\mathscr{C}^{\mathbf{2}})$.

Proof. We will begin with an object α in $\mathbf{AWFS}(\mathscr{C})^{\mathbf{2}}$. This is a map of algebraic weak factorization systems $\alpha : (\mathrm{C}^l, \mathrm{F}^l) \to (\mathrm{C}^r, \mathrm{F}^r)$ on $\mathscr{C}$. Let $E : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ be the middomain functor of $(\mathrm{C}^l, \mathrm{F}^l)$ and let $E' : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ be the middomain functor of $(\mathrm{C}^r, \mathrm{F}^r)$. We will define an algebraic weak factorization system $\Phi(\alpha) = (\mathrm{L}_\alpha, \mathrm{R}_\alpha)$ on $\mathscr{C}^{\mathbf{2}}$. Let $(\mathrm{L}_\alpha, \mathrm{R}_\alpha) : (\mathscr{C}^{\mathbf{2}})^{\mathbf{2}} \to (\mathscr{C}^{\mathbf{2}})^{\mathbf{3}}$ be the functorial factorization given by

$$f \xrightarrow{(u,v)} g \qquad \longmapsto \qquad f \xrightarrow{(\mathrm{C}^l u, \mathrm{C}^r v)} \alpha_v \circ E(f,g) \xrightarrow{(\mathrm{F}^l u, \mathrm{F}^r v)} g$$

on the objects of $(\mathscr{C}^{\mathbf{2}})^{\mathbf{2}}$ and by

$$\begin{tikzcd}
f \arrow[r, "{(u,v)}"] \arrow[d, "{(a,b)}"'] & g \arrow[d, "{(c,d)}"] \\
h \arrow[r, "{(s,t)}"] & k
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
f \arrow[r, "{(\mathrm{C}^l u, \mathrm{C}^r v)}"] \arrow[d, "{(a,b)}"'] & \alpha_v \circ E(f,g) \arrow[r, "{(\mathrm{F}^l u, \mathrm{F}^r v)}"] \arrow[d, "{(E(a,c),E'(b,d))}"] & g \arrow[d, "{(c,d)}"] \\
h \arrow[r, "{(\mathrm{C}^l s, \mathrm{C}^r t)}"] & \alpha_t \circ E(h,k) \arrow[r, "{(\mathrm{F}^l s, \mathrm{F}^r t)}"] & k
\end{tikzcd}$$

on morphisms of $(\mathscr{C}^{\mathbf{2}})^{\mathbf{2}}$. Let $E^*_\alpha : (\mathscr{C}^{\mathbf{2}})^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ be the middomain functor of $(\mathrm{L}_\alpha, \mathrm{R}_\alpha)$.

We can define a counit map $\vec{\varepsilon}^{\,\mathrm{L}_\alpha}_{(u,v)}$ and a comultiplication map $\vec{\delta}^{\,\mathrm{L}_\alpha}_{(u,v)}$ for the endofunctor L_α by

$$\begin{tikzcd}
f \arrow[r, "{(\mathrm{C}^l u, \mathrm{C}^r v)}"] \arrow[d, "{(id,id)}"'] & \alpha_v \circ E(f,g) \arrow[d, "{\varepsilon^{\mathrm{L}_\alpha}_{(u,v)}}"] \\
f \arrow[r, "{(u,v)}"] & g
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
f \arrow[r, "{(\mathrm{C}^l u, \mathrm{C}^r v)}"] \arrow[d, "{(id,id)}"'] & \alpha_v \circ E(f,g) \arrow[d, "{\delta^{\mathrm{L}_\alpha}_{(u,v)}}"] \\
f \arrow[r, "{(\mathrm{C}^l\mathrm{C}^l u, \mathrm{C}^r\mathrm{C}^r v)}"] & \alpha_{\mathrm{C}^r v} \circ E(f, \alpha_v \circ E(f,g))
\end{tikzcd}$$

respectively, where $\varepsilon^{\mathrm{L}_\alpha}_{(u,v)} = (\varepsilon^{\mathrm{C}^l}_u, \varepsilon^{\mathrm{C}^r}_v) = (\mathrm{F}^l u, \mathrm{F}^r v)$ and where $\delta^{\mathrm{L}_\alpha}_{(u,v)} = (\delta^{\mathrm{C}^l}_u, \delta^{\mathrm{C}^r}_v)$. It is easy to check that $\mathrm{L}_\alpha \vec{\varepsilon}^{\,\mathrm{L}_\alpha} \circ \vec{\delta}^{\,\mathrm{L}_\alpha} = id = \vec{\varepsilon}^{\,\mathrm{L}_\alpha} \mathrm{L}_\alpha \circ \vec{\delta}^{\,\mathrm{L}_\alpha}$ and $\vec{\delta}^{\,\mathrm{L}_\alpha} \mathrm{L}_\alpha \circ \vec{\delta}^{\,\mathrm{L}_\alpha} = \mathrm{L}_\alpha \vec{\delta}^{\,\mathrm{L}_\alpha} \circ \vec{\delta}^{\,\mathrm{L}_\alpha}$, since these coherence conditions hold for C^l and C^r. Thus L_α is a comonad. Similarly, the endofunctor R_α is a monad with unit map $\vec{\eta}^{\,\mathrm{R}_\alpha}_{(u,v)}$ and multiplication map $\vec{\mu}^{\,\mathrm{R}_\alpha}_{(u,v)}$ given by

$$\begin{tikzcd}
f \arrow[r, "{(u,v)}"] \arrow[d, "{\eta^{\mathrm{R}_\alpha}_{(u,v)}}"'] & g \arrow[d, "{(id,id)}"] \\
\alpha_v \circ E(f,g) \arrow[r, "{(\mathrm{F}^l u, \mathrm{F}^r v)}"] & g
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\alpha_{\mathrm{F}^r v} \circ E(\alpha_v \circ E(f,g), g) \arrow[r, "{(\mathrm{F}^l\mathrm{F}^l u, \mathrm{F}^r\mathrm{F}^r v)}"] \arrow[d, "{\mu^{\mathrm{R}_\alpha}_{(u,v)}}"'] & g \arrow[d, "{(id,id)}"] \\
\alpha_v \circ E(f,g) \arrow[r, "{(\mathrm{F}^l u, \mathrm{F}^r v)}"] & g
\end{tikzcd}$$

respectively, where $\eta^{\mathrm{R}_\alpha}_{(u,v)} = (\eta^{\mathrm{F}^l}_u, \eta^{\mathrm{F}^r}_v) = (\mathrm{C}^l u, \mathrm{C}^r v)$ and where $\mu^{\mathrm{R}_\alpha}_{(u,v)} = (\mu^{\mathrm{F}^l}_u, \mu^{\mathrm{F}^r}_v)$. The map $(\delta^{\mathrm{L}_\alpha}, \mu^{\mathrm{R}_\alpha}) : \mathrm{L}_\alpha \mathrm{R}_\alpha \to \mathrm{R}_\alpha \mathrm{L}_\alpha$ satisfies the distributivity axioms, since the component maps $(\delta^{\mathrm{C}^l}, \mu^{\mathrm{F}^l}) : \mathrm{C}^l \mathrm{F}^l \to \mathrm{F}^l \mathrm{C}^l$ and $(\delta^{\mathrm{C}^r}, \mu^{\mathrm{F}^r}) : \mathrm{C}^r \mathrm{F}^r \to \mathrm{F}^r \mathrm{C}^r$ do. Therefore $(\mathrm{L}_\alpha, \mathrm{R}_\alpha)$ is an algebraic weak factorization system on $\mathscr{C}^{\mathbf{2}}$.

Consider the following morphism in $\mathbf{AWFS}(\mathscr{C})^{\mathbf{2}}$ from α to β.

$$\begin{tikzcd}
(\mathrm{C}^l, \mathrm{F}^l) \arrow[r, "\alpha"] \arrow[d, "\zeta"] & (\mathrm{C}^r, \mathrm{F}^r) \arrow[d, "\theta"] \\
(\mathrm{X}^l, \mathrm{Y}^l) \arrow[r, "\beta"] & (\mathrm{X}^r, \mathrm{Y}^r)
\end{tikzcd}$$

Let E'' and E''' be the functors $\mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ associated to the factorizations $(\mathrm{X}^l, \mathrm{Y}^l)$ and $(\mathrm{X}^r, \mathrm{Y}^r)$, respectively. Define $\Phi((\zeta, \theta))$ as the natural transformation $E^*_\alpha \to E^*_\beta$ given by $(\zeta_u, \theta_v) : \alpha_v \circ E(f,g) \to \beta_v \circ E''(f,g)$ on the objects $(u,v) : f \to g$ of $(\mathscr{C}^{\mathbf{2}})^{\mathbf{2}}$. Since $(id, \zeta) : \mathrm{C}^l \to \mathrm{X}^l$ and $(id, \theta) : \mathrm{C}^r \to \mathrm{X}^r$ are maps of comonads, the map $\mathrm{L}_\alpha \to \mathrm{L}_\beta$ given by $(id_f, (\zeta_u, \theta_v)) : \mathrm{L}_\alpha(u,v) \to \mathrm{L}_\beta(u,v)$ on $(u,v) : f \to g$ is a map of comonads. Similarly, the natural transformation $\mathrm{R}_\alpha \to \mathrm{R}_\beta$ given by $(u,v) \mapsto ((\zeta_u, \theta_v), id_g)$ is a map of monads on $\mathscr{C}^{\mathbf{2}}$. Since Φ commutes with composition of morphisms and Φ of the identity morphism in $\mathbf{AWFS}(\mathscr{C})^{\mathbf{2}}$ is the identity morphism in $\mathbf{AWFS}(\mathscr{C}^{\mathbf{2}})$, Φ is a functor $\mathbf{AWFS}(\mathscr{C})^{\mathbf{2}} \to \mathbf{AWFS}(\mathscr{C}^{\mathbf{2}})$.

Now suppose (L, R) is an object in $\mathbf{AWFS}(\mathscr{C}^{\mathbf{2}})$. We will produce a map of algebraic weak factorization systems $\Psi((\mathrm{L}, \mathrm{R}))$ on $\mathscr{C}$. Let $E^* : (\mathscr{C}^{\mathbf{2}})^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$ be the middomain functor of (L, R). Consider the objects $(u,u) : id_A \to id_B$, $(v,v) : id_C \to id_D$ and the morphism $\big((f,f),(g,g)\big) : (u,u) \to (v,v)$ in $(\mathscr{C}^{\mathbf{2}})^{\mathbf{2}}$. We will define two endofunctors on $\mathscr{C}^{\mathbf{2}}$, L^l and L^r. Since $\mathrm{L}(u,u)$ is an object in $(\mathscr{C}^{\mathbf{2}})^{\mathbf{2}}$, we define $(\mathrm{L}^l u, \mathrm{L}^r u) := \mathrm{L}(u,u)$ on the objects u of $\mathscr{C}^{\mathbf{2}}$. Since $\mathrm{L}\big((f,f),(g,g)\big) = \Big((f,f), E^*\big((f,f),(g,g)\big)\Big)$ is a morphism in $(\mathscr{C}^{\mathbf{2}})^{\mathbf{2}}$, there are functors $\mathscr{C}^{\mathbf{2}} \to \mathscr{C}$, E^l and E^r, such that $E^*\big((f,f),(g,g)\big) = \big(E^l(f,g), E^r(f,g)\big)$. So we define $\mathrm{L}^l(f,g) := \big(f, E^l(f,g)\big)$ and $\mathrm{L}^r(f,g) := \big(f, E^r(f,g)\big)$ on the morphisms of $\mathscr{C}^{\mathbf{2}}$. Similarly, we define endofunctors R^l and R^r on $\mathscr{C}^{\mathbf{2}}$. On objects u in $\mathscr{C}^{\mathbf{2}}$, $(\mathrm{R}^l u, \mathrm{R}^r u) := \mathrm{R}(u,u)$. On morphisms, $\mathrm{R}^l(f,g) := \big(E^l(f,g), g\big)$ and $\mathrm{R}^r(f,g) := \big(E^r(f,g), g\big)$. This gives us two functorial factorizations $(\mathrm{L}^l, \mathrm{R}^l)$ and $(\mathrm{L}^r, \mathrm{R}^r)$ on $\mathscr{C}$ with associated functors E^l and E^r, respectively.

We get counit and comultiplication maps for L^l and L^r from the counit and comultiplication maps of L. Namely, $(\varepsilon^{\mathrm{L}^l}_u, \varepsilon^{\mathrm{L}^r}_u) := \varepsilon^{\mathrm{L}}_{(u,u)}$ and $(\delta^{\mathrm{L}^l}_u, \delta^{\mathrm{L}^r}_u) := \delta^{\mathrm{L}}_{(u,u)}$, where $\vec{\varepsilon}^{\,\mathrm{L}}_{(u,u)} = \big(id_{id_A}, \varepsilon^{\mathrm{L}}_{(u,u)}\big)$ and $\vec{\delta}^{\,\mathrm{L}}_{(u,u)} = \big(id_{id_A}, \delta^{\mathrm{L}}_{(u,u)}\big)$. Similarly, $(\eta^{\mathrm{R}^l}_u, \eta^{\mathrm{R}^r}_u) := \eta^{\mathrm{R}}_{(u,u)}$ and $(\mu^{\mathrm{R}^l}_u, \mu^{\mathrm{R}^r}_u) := \mu^{\mathrm{R}}_{(u,u)}$, where $\vec{\eta}^{\,\mathrm{R}}_{(u,u)} = \big(\eta^{\mathrm{R}}_{(u,u)}, id_{id_B}\big)$ and $\vec{\mu}^{\,\mathrm{R}}_{(u,u)} = \big(\mu^{\mathrm{R}}_{(u,u)}, id_{id_B}\big)$. With these maps, L^l and L^r are comonads, while R^l and R^r are monads. Because $(\delta^{\mathrm{L}}, \mu^{\mathrm{R}}) : \mathrm{LR} \to \mathrm{RL}$ satisfies the distributivity axioms, the component maps $(\delta^{\mathrm{L}^l}, \mu^{\mathrm{R}^l}) : \mathrm{L}^l\mathrm{R}^l \to \mathrm{R}^l\mathrm{L}^l$ and $(\delta^{\mathrm{L}^r}, \mu^{\mathrm{R}^r}) : \mathrm{L}^r\mathrm{R}^r \to \mathrm{R}^r\mathrm{L}^r$ each satisfy distributivity of the comonad over the monad.

Define $\xi_f^{(\mathrm{L},\mathrm{R})}$ as $E^*(f,f)$ on each object f in $\mathscr{C}^{\mathbf{2}}$. Let (u,v) be a map $f \to g$ in $\mathscr{C}^{\mathbf{2}}$. Consider the following map from (u,v) to (v,v) in $(\mathscr{C}^{\mathbf{2}})^{\mathbf{2}}$.

$$\begin{tikzcd}
f \arrow[r, "{(u,v)}"] \arrow[d, "{(f,id_B)}"'] & g \arrow[d, "{(g,id_D)}"] \\
id_B \arrow[r, "{(v,v)}"] & id_D
\end{tikzcd}$$

Applying E^* to this map gives us the map $E^*\big((f,id_B),(g,id_D)\big) = \big(E^l(f,g), E^r(id_B,id_D)\big) : E^*(u,v) \to E^*(v,v)$ in $\mathscr{C}^{\mathbf{2}}$. Since $E^r(id_B,id_D) = id_{E^r v}$, this map expresses the relation $E^*(u,v) = E^*(v,v) \circ E^l(f,g) = \xi_v^{(\mathrm{L},\mathrm{R})} \circ E^l(f,g)$. Now the existence of $\vec{\delta}^{\mathrm{L}}_{(f,f)}$ tells us that the following square commutes.

$$\begin{tikzcd}
E^l f \arrow[r, "\delta_f^{\mathrm{L}^l}"] \arrow[d, "{E^*(f,f)}"'] & E^l\mathrm{L}^l f \arrow[d, "{E^*(\mathrm{L}^l f,\mathrm{L}^r f)}"] \\
E^r f \arrow[r, "\delta_f^{\mathrm{L}^r}"] & E^r\mathrm{L}^r f
\end{tikzcd}$$

But this means that $\delta_f^{\mathrm{L}^r} \circ \xi_f^{(\mathrm{L},\mathrm{R})} = \xi_{\mathrm{L}^r f}^{(\mathrm{L},\mathrm{R})} \circ E^l\big(id_A, \xi_f^{(\mathrm{L},\mathrm{R})}\big) \circ \delta_f^{\mathrm{L}^l}$. Thus $\big(id, \xi^{(\mathrm{L},\mathrm{R})}\big) : \mathrm{L}^l \to \mathrm{L}^r$ is a map of comonads. Similarly, the existence of $\vec{\mu}^{\mathrm{R}}_{(f,f)}$ tells us that $\mu_f^{\mathrm{R}^r} \circ E^*(\mathrm{R}^r f, \mathrm{R}^l f) = E^*(f,f) \circ \mu_f^{\mathrm{R}^l}$. So $\mu_f^{\mathrm{R}^r} \circ \xi_{\mathrm{R}^r f}^{(\mathrm{L},\mathrm{R})} \circ E^l\big(\xi_f^{(\mathrm{L},\mathrm{R})}, id_B\big) = \xi_f^{(\mathrm{L},\mathrm{R})} \circ \mu_f^{\mathrm{R}^l}$ and thus $\big(id, \xi^{(\mathrm{L},\mathrm{R})}\big) : \mathrm{R}^l \to \mathrm{R}^r$ is a map of monads. So $\xi^{(\mathrm{L},\mathrm{R})}$ defines a map $(\mathrm{L}^l, \mathrm{R}^l) \to (\mathrm{L}^r, \mathrm{R}^r)$ of algebraic weak factorization systems on $\mathscr{C}$. We therefore take $\Psi((\mathrm{L},\mathrm{R}))$ to be $\xi^{(\mathrm{L},\mathrm{R})}$.

Let $\rho : (\mathrm{L},\mathrm{R}) \to (\mathrm{S},\mathrm{T})$ be a map of algebraic weak factorization systems on $(\mathscr{C}^{\mathbf{2}})^{\mathbf{2}}$. Let F^* be the middomain functor of (S,T). Since E^* and F^* are functors $(\mathscr{C}^{\mathbf{2}})^{\mathbf{2}} \to \mathscr{C}^{\mathbf{2}}$, they can be decomposed as pairs of functors $\mathscr{C}^{\mathbf{2}} \to \mathscr{C}$. So $E^* = (E^l, E^r)$ and $F^* = (F^l, F^r)$. The natural transformation $\rho : E^* \to F^*$ also decomposes as a pair of natural transformations

$\rho = (\rho^l, \rho^r)$, where ρ^l is a natural transformation $E^l \to F^l$ and ρ^r is a natural transformation $E^r \to F^r$. Since $(id, \rho) : \mathrm{L} \to \mathrm{S}$ is a map of comonads, $(id, \rho^l) : \mathrm{L}^l \to \mathrm{S}^l$ and $(id, \rho^r) : \mathrm{L}^r \to \mathrm{S}^r$ are both maps of comonads. Similarly, $(\rho^l, id) : \mathrm{R}^l \to \mathrm{T}^l$ and $(\rho^r, id) : \mathrm{L}^r \to \mathrm{T}^r$ are maps of monads. Thus $\rho^l : (\mathrm{L}^l, \mathrm{R}^l) \to (\mathrm{S}^l, \mathrm{T}^l)$ and $\rho^r : (\mathrm{L}^r, \mathrm{R}^r) \to (\mathrm{S}^r, \mathrm{T}^r)$ are maps of algebraic weak factorization systems on $\mathscr{C}$. Define $\Psi(\rho)$ to be the morphism

$$\begin{tikzcd}
(\mathrm{L}^l, \mathrm{R}^l) \arrow[r, "\xi^{(\mathrm{L},\mathrm{R})}"] \arrow[d, "\rho^l"] & (\mathrm{L}^r, \mathrm{R}^r) \arrow[d, "\rho^r"] \\
(\mathrm{S}^l, \mathrm{T}^l) \arrow[r, "\xi^{(\mathrm{S},\mathrm{T})}"] & (\mathrm{S}^r, \mathrm{T}^r)
\end{tikzcd}$$

in $\mathbf{AWFS}(\mathscr{C})^{\mathbf{2}}$ from $\xi^{(\mathrm{L},\mathrm{R})}$ to $\xi^{(\mathrm{S},\mathrm{T})}$. We then have $\Psi(\rho \circ \sigma) = \Psi(\rho) \circ \Psi(\sigma)$ and $\Psi(id) = id$. So Ψ is a functor $\mathbf{AWFS}(\mathscr{C}^{\mathbf{2}}) \to \mathbf{AWFS}(\mathscr{C})^{\mathbf{2}}$. It is easy to see that Ψ is an inverse functor for Φ. $\square$

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