<u>th</u> let Ω be an open set of $\mathbb{R}^n$, starshaped (at o) then Ω is C^∞-diffeomorph to $\mathbb{R}^n$.

<u>Proof</u> Let $F = \mathbb{R}^n \setminus \Omega$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^+$ be a C^∞ function such that $F = \varphi^{-1}(\{0\})$.

We set

$$f : \Omega \to \mathbb{R}^n, \quad x \mapsto \underbrace{\left[1 + \left(\int_0^1 \frac{dv}{\varphi(vx)}\right)^2 \|x\|_2^2\right]}_{\lambda(x)} . x = \left[1 + \left(\int_0^{\|x\|_2} \frac{dt}{\varphi(t\frac{x}{\|x\|_2})}\right)^2\right] . \|x\|_2 \times \frac{x}{\|x\|_2}$$

(where $\|x\|_2 = \left(\sum_i x_i^2\right)^{1/2}$)

f is smooth on $\mathbb{R}^n$. We set $A(x) = \operatorname{Sup}\{t > 0, \frac{tx}{\|x\|_2} \in \Omega\}$. f send injectively $[0, A(x)[\cdot \frac{x}{\|x\|_2}$ into $\mathbb{R}_+ \cdot \frac{x}{\|x\|_2}$ moreover, if we set $u = \frac{x}{\|x\|_2}$, then

$$\|f(0.u)\|_2 = 0 \text{ and } \lim_{t \to A(x)} \|f(t.u)\| = \left[1 + \left(\int_0^{A(x)} \frac{dt}{\varphi(tu)}\right)^2\right] \times A(x) = +\infty$$

indeed, if $A(x) = +\infty$ it is obvious

if $A(x) < \infty$ then $\begin{cases} \varphi(A(x)\, x) = 0 \\ \varphi \text{ smooth} \end{cases} \Rightarrow \varphi(tu) \underset{t \to A(x)}{=} O(t - A(x))$ and so $\int_0^{A(x)} \frac{ds}{\varphi(su)}$ diverges.

We infer that $\varphi\left([0, A(x)[\cdot \frac{x}{\|x\|_2}\right) = \mathbb{R}_+ \frac{x}{\|x\|_2}$ and so $\varphi(\Omega) = \mathbb{R}^n$.

To conclude, we have $d_x f(h) = \lambda(x) h + d_x \lambda(h) x$

so if $h \in \operatorname{Ker} d_x f \setminus \{0\}$ then there exists $\mu \in \mathbb{R}^*$ such that $h = \mu x$

and we get $\left[\lambda(x) + d_x \lambda(x)\right] = 0$ (note that $\lambda(0) = 1$ so $x \neq 0$).

but we have $\lambda(x) \geq 1$ and $g(t) = \lambda(tx)$ increasing so $g'(1) = d_x \lambda(x) x$

wich gives a contradiction.

<u>Nota bene</u>
- The withney Theorem is a classical result.
- in the case $n = 2$ the Riemann theorem implies that Ω is holomorphically diffeomorph to $\mathbb{R}^2 \simeq \mathbb{C}$.