Recent Work in Homotopy Type Theory: Modal, Algebraic, Synthetic, and Cubical

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Overview and Acknowledgements

- I will give an overview of some of our current work, emphasizing the main themes: **Modal** type theory and **Cubical** models, and some closely related **Algebraic** and **Synthetic** developments.
- Last year I told you about our progress toward a **realizability** ∞-**topos** RT_∞. I will also give an update on that.
- I'm surveying the work of people in the CMU Philosophy Group: Jeremy Avigad, Steve Awodey, Jonas Frey, Pieter Hofstra (2017-18), Clive Newstead, Egbert Rijke, Sam Speight (2017-18), Floris van Doorn, Felix Wellen (2018-19), and Colin Zwanziger.
- But we are building on the work of external MURI team members Thierry Coquand, Nicola Gambino, and their collaborators, as well as that of other MURI team members, particularly the modal work of Dan Licata and Mike Shulman.

1. Modal type theory

- In logic, a **modality** is a unary operator $\diamond p$ on propositions that satifies certain laws, such as

$$p \Rightarrow \diamond p, \qquad \diamond\diamond p \Rightarrow \diamond p, \qquad (p \Rightarrow q) \Rightarrow (\diamond p \Rightarrow \diamond q)$$

- Under Propositions as Types this becomes a **monad** $\diamond : \mathbb{T} \longrightarrow \mathbb{T}$ on the category $\mathbb{T}$ of types and terms.
- In **simple type theory**, it's relatively easy to encode such a modality:

$$\frac{\Gamma \vdash A \ \mathtt{type}}{\Gamma \vdash \diamond A \ \mathtt{type}} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash a^{\diamond} : \diamond A}$$

  and so on.
- In **dependent type theory** there are many subtleties, as well as many possible applications, all of which are worked out in the recent paper by MURI researchers Rijke, Shulman, and Spitters.

Comodal type theory

- Another common logical modality $\Box p$ satifies the **dual laws**,

$$\Box p \Rightarrow p, \qquad \Box p \Rightarrow \Box\Box p, \qquad (p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$$

- This becomes a **comonad** $\Box : \mathbb{T} \longrightarrow \mathbb{T}$ on the types and terms.
- Encoding such a **comonadic modality** ("comodality"?) is much trickier, especially in dependent type theory. This is the focus of the current thesis research of Colin Zwanziger, drawing on recent work of MURI team members Shulman and Licata and past work of CMU researcher Frank Pfenning.
- Shulman has recently developed an elaborate system with several interlocking modalities and comodalities for describing **cohesion** in HoTT. New MURI team member Felix Wellen uses such modal systems to describe **differential cohesion** for synthetic differential and algebraic geometry.

Implementing modal type theory

- I will let Jeremy Avigad tell you about the **recent developments in Lean** this afternoon, but one short-range goal is to implement such modal operators.
- The monadic case seems to be doable without too much difficulty, but the **comonadic** one is more challenging.
- An experimental implementation of comonads in Agda by Vezzosi (called **Agda-flat**) was recently used by Licata, Orton, Pitts, and Spitters (2018) for a **surprising and beautiful** new construction of a universe in cubical type theory. There should be many other useful applications of such an implementation of modal type theory.

2. Algebraic type theory

- A **natural model of type theory** (Awodey 2015) is a natural transformation $p : \dot{U} \longrightarrow U$ of presheaves with a right adjoint to its functor of elements $\int_{\mathbb{C}} p$. This is an algebraic formulation of the notion of a **category with families**, a presentation of dependent type theory due to Dybjer.
- Every map $p : \dot{U} \longrightarrow U$ in presheaves $\widehat{\mathbb{C}}$ determines a **polynomial endofunctor** $P : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$. The map p is a natural model iff P preserves all colimits. P is a **monad** iff $p : \dot{U} \longrightarrow U$, regarded as a type-theoretic universe in $\widehat{\mathbb{C}}$, has unit $\mathbf{1}$ and Σ types, and it is a P-**algebra** iff p has Π types.
- A system of dependent type theory $\mathbb{T}[\sigma]$ over a finite signature σ generates a **classifying natural model** $U[\sigma]$, which is initial among all models interpreting σ. This is an instance of Voevodsky's "initiality conjecture".
- Clive Newstead's current thesis research includes all this and more about such algebraic models of type theory.

Functorial semantics for type theory

One way to formulate the **initiality of syntax** is as follows:

1. We specify a notion of "semantic category" $\mathcal{C}$ (e.g. CwF), along with the morphisms $f : \mathcal{C} \longrightarrow \mathcal{D}$ of such things.
2. We say what a "signature" σ for a dependently typed theory is, and what an interpretation of σ in a semantic category $\mathcal{C}$ is.
3. Let $\mathsf{Int}_\sigma(\mathcal{C})$ be the set of all interpretations of σ in $\mathcal{C}$. Morphisms $f : \mathcal{C} \longrightarrow \mathcal{D}$ are shown to take interpretations in $\mathcal{C}$ to interpretations in $\mathcal{D}$, in a functorial way.
4. The functor Int_σ is representable: there is a semantic category $\mathcal{C}[\sigma]$ and for each $\mathcal{C}$ an isomorphism, natural in $\mathcal{C}$,
$$\mathsf{Int}_\sigma(\mathcal{C}) \;\cong\; \mathsf{Hom}(\mathcal{C}[\sigma], \mathcal{C})\,.$$
5. Finally, $\mathcal{C}[\sigma]$ can be constructed syntactically as the category of types and terms, or similar, over the signature σ.

This framework, due to Lawvere, works for algebraic theories, lambda-calculus, higher-order logic, ... It works just as well for type theory since it is also **essentially algebraic**.

3. Synthetic methods

- A milestone formalization was completed this year by Floris van Doorn in his thesis project on the **Serre spectral sequence**. It uses synthetic methods proposed by Shulman, and had substantial input from MURI researcher Ulrik Buchholtz and others.
- Egbert Rijke's thesis research on **Classifying types** has focussed on synthetic homotopy theory. One result has been to show how much can be done with a limited supply of basic HITs, permitting the construction of many others. The new theory of ∞-**equivalence relations** is a big step toward solving the important problem of higher coherence laws in HoTT. This work can be seen as exploring the extent to which HoTT can serve as a **formal calculus for abstract homotopy theory**.

Synthetic methods

- Buchholtz, Rijke, and van Doorn have also developed a synthetic theory of **Higher groups**, reported in a LICS 2018 paper. It includes a partial solution to a problem first proposed in 2006 of using HoTT to prove the "**Baez-Dolan stabilization hypothesis**".
- Felix Wellen's work on **differential cohesion** uses a modality in HoTT to represent structures such as fiber bundles in differential and algebraic geometry in a uniform synthetic way. This work opens up the possibility of **formalization** of areas of mathematics that would be impossible using current conventional methods – as well as new mathematical results, and new proofs of known results, arising from the synthetic reformulation of these subjects.

4. Update on the Realizability ∞-Topos

- Last year, I reported on on-going work by Awodey, Frey, and Hofstra toward a **realizability model** RT_∞ using cubical assemblies, with $\Sigma, \Pi, \mathtt{Id}_X$ and an impredicative univalent universe of sets. We had not yet succeeded in building an untrucated univalent universe.
- As an application, Awodey, Frey, and Speight gave **Impredicative encodings of (higher) inductive types**. (reported in a 2018 LICS paper and Speight's 2017 MS thesis). Frey also uses related ideas to **construct coproducts** in an ∞-topos with a subobject classifier.
- The **cubical realizability model** can in fact now be completed to one with an untrucated universe using the new results of Licata, Orton, Pitts, and Spitters (2018) mentioned earlier. This uses a cubical AWFS, tinyness of the interval $\mathbb{I}$, and a "0-skeleton" **comodality**. A paper doing this was just posted by Taichi Uemura, a PhD student in Amsterdam.

5. Cubical models of HoTT

- In late 2013, Bezem, Coquand, and Huber produced the first **cubical** model of univalence, giving a constructive version of Voevodsky's **simplicial** model. This used a symmetric version of the "classical" (i.e. monoidal) category of cubes.
- In early 2014, I proposed to build a model in presheaves on the symmetric cubes with diagonals, which I named the *Cartesian cubes* and defined formally as the **free finite product category on an "interval"** $1 \to \mathbb{I} \leftarrow 1$.
- There are many different cube categories, some of which have recently been studied by Buchholtz and Morehouse.

Table : Some cube categories

maps \ structure	0,1	$\wedge, \vee$	$\neg$
faces & degen.s	monoidal	classical	
+ symmetries	symmetric monoidal		
+ diagonals	Cartesian	Dedekind	deMorgan

Cartesian cubes

A concrete description of the **Cartesian cubes** is as the dual

$$\mathbb{C} = \mathbb{B}^{op}$$

of the category $\mathbb{B}$ of finite, strictly bipointed sets. (2015 MS thesis of Jason Parker)

In 2014 CMU lectures I developed the **basic properties** of the Cartesian cubical sets (see github/awodey/math/cubical):

- the presheaf topos $\widehat{\mathbb{C}}$ classifies **strictly bipointed objects**,
- the **geometric realization** to Top preserves finite products,
- the **nerve** functor from Cat is full and faithful,
- the Id-type can be taken to be the **pathspace** $X^{\mathbb{I}}$,
- the pathspace $X^{\mathbb{I}}$ is a shift and therefore has a right adjoint, the **root** $X_{\mathbb{I}}$, thus the interval $\mathbb{I}$ is **tiny**,
- **calculation** of the root $X_{\mathbb{I}}$ of a cubical set X.

Id-types and Path-types

- Coquand's cubical type theory using the deMorgan cubes **does not interpret Identity types as pathspaces**. Instead, for each type X there is both an exponential "path type" $X^{\mathbb{I}}$ and a non-isomorphic Identity type $\mathtt{Id}_X$. This may be necessary for some computational purposes, but I consider it unsatisfactory as a model of HoTT.
- In summer 2016, I gave an AWFS model in the Cartesian cubical sets that interprets the identity type as an exponential pathobject,
$$\mathtt{Id}_X = X^{\mathbb{I}}.$$
- This is the set-up that we used for the **realizability model** that I presented last year. However, we had difficulty building a univalent universe in this setting.

Quillen model categories

- I now think that one needs a full **Quillen model structure** in order to properly model HoTT with a univalent universe.
- Moreover, many people are now working in **various related settings**: Coquand and Co., Harper and Co., Gambino-Sattler, Orton-Pitts, Frumin-van den Berg, Brunerie-Licata, Awodey-Frey-Hofstra, ...
- A **theoretical foundation** for HoTT should use the tools and concepts of Quillen model categories (not just the jargon) to compare and relate these different approaches.
- Note that **Voevodsky's model** makes essential use of the Quillen model structure on simplicial sets.
- I will now show how to put a Quillen model structure on the **Cartesian** cubical sets.

Dedekind cubical sets

- The **Dedekind cube category** $\mathbb{D}$ may be defined as the Lawvere theory of (bounded) distributive lattices.
- Concretely, it is the full subcategory $\mathbb{D} \hookrightarrow \mathsf{Cat}$ on the finite powers $\mathbb{2}^n$ of the **walking arrow** category $\mathbb{2} = (0 \to 1)$.
- Recent work of Gambino-Sattler and Sattler can be used to put a Quillen model structure on the **Dedekind cubical sets**,

$$\widehat{\mathbb{D}} = \mathsf{Set}^{\mathbb{D}^{op}}.$$

- The **cofibrations** may be taken to be all monos. The **fibrations** are the maps with the RLP against all pushout-products $c \otimes \delta : D \to \mathbb{I}^{n+1}$ of a cofibration $c : C \longrightarrow \mathbb{I}^n$ and an endpoint inclusion $\delta : 1 \longrightarrow \mathbb{I}$.
- This QMS can be made **algebraic** using a **polynomial monad** related to the one in Clive's work on algebraic type theory.

Lifting the QMS from Dedekind to Cartesian

- Since the Dedekind 1-cube has two points $\mathbb{1} \rightrightarrows \mathbb{2}$, there is a **comparison functor** $c : \mathbb{C} \longrightarrow \mathbb{D}$ from Cartesian to Dedekind cubes. It takes the Cartesian 1-cube to the Dedekind 1-cube and preserves all finite products.
- This induces a **triple of adjoints** $c_! \dashv c^* \dashv c_* : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ on presheaves, of which the left-most functor $c_!$ is left-exact.
- We then apply a recent result of Hess, Kedziorek, Riehl, and Shipley (reproved in Garner, Kedziorek, and Riehl, 2018) to "lift" the QMS along $c_! : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ to get a **left-lifted model structure** on $\widehat{\mathbb{C}}$. The required left acyclicity condition is easily verified in this special case. Thus we have:

Theorem

There is a **Quillen model structure** *on the Cartesian cubical sets* $\widehat{\mathbb{C}}$ *with* **cofibrations** *the monomorphisms and* **weak equivalences** *those maps* $f : A \longrightarrow B$ *with* $c_! f$ *a weak equivalence in* $\widehat{\mathbb{D}}$*.*

Cartesian versus Dedekind cubes

Why not just use the Dedekind cubes?

How many maps $\mathbb{I}^n \longrightarrow \mathbb{I}^m$ are there?

Table : $\mathrm{Hom}(\mathbb{I}^n, \mathbb{I})$

n	Cartesian	Dedekind
0	2	2
1	3	3
2	4	6
3	5	20

So far pretty close ...

Cartesian versus Dedekind cubes

Table : $\mathrm{Hom}(\mathbb{I}^n, \mathbb{I})$

n	Cartesian	Dedekind
0	2	2
1	3	3
2	4	6
3	5	20
4	6	168
5	7	7,581
6	8	7,828,354
7	9	2,414,682,040,998
8	10	56,130,437,228,687,557,907,788

OMG!

Cartesian versus Dedekind cubes

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0	2	2
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8	10	56,130,437,228,687,557,907,788
n	n+2	?

These are called the **Dedekind numbers**, and their calculation is called **Dedekind's problem**. These are the only values known!

Cartesian versus Dedekind cubes

For comparison:

Table : $\mathrm{Hom}(\mathbb{I}^n, \mathbb{I})$

n	Cartesian	Dedekind	Simplicial
0	2	2	2
1	3	3	3
2	4	6	4
3	5	20	5
4	6	168	6
5	7	7,581	7
6	8	7,828,354	8
7	9	2,414,682,040,998	9
8	10	56,130,437,228,687,557,907,788	10
n	n+2	?	n+2

Should we really build a proof assistant on the Dedekind cubes?

Future Work: Cartesian ∞-Topos

We now have a "Cartesian (realizability) ∞-topos" on the basis of this QMS on the Cartesian cubical sets (or assemblies). Future work includes:

- Analyze the lifted model structure on $\widehat{\mathbb{C}}$ to describe the weak equivalences directly. **Conjecture:** they are the weak homotopy equivalences under geometric realization.
- Classically, we do have that $X \longrightarrow X^{\mathbb{I}}$ is a trivial cofibration whenever X is fibrant, so that path spaces $X^{\mathbb{I}}$ agree with identity types. Investigate this in the cubical assemblies.
- Investigate the LOPS universe construction in this setting.

Thanks!