Cubical sets as a classifying topos

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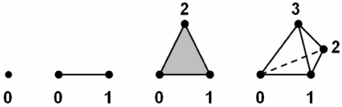
Simplicial sets

Univalence modeled in Kan fibrations of simplicial sets.
Simplicial sets are a standard example of a classifying topos.
Joyal/Johnstone: geometric realization as a geometric morphism.

Challenge: computational interpretation of univalence and higher inductive types.
Solution (Coquand et al): Cubical sets
Can we extend these methods?

Simplicial sets

Simplex category Δ:
finite ordinals and monotone maps
Simplicial sets $\hat{\Delta}$.
Geometric realization/Singular complex: $|-|:\widehat{\Delta}\to Top : S$

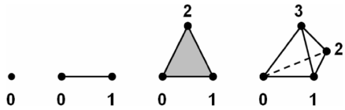


The pair $|-|\dashv S$ behaves as a geometric functor.
E.g. $|-|$ is left exact (pres fin lims).
However, Top is not a topos.
Johnstone: use topological topos instead.

Roughly: points, equalities, equalities between equalities, ...

Geometric realization of simplicial sets

Simplices are constructed from the linear order on $\mathbb{R}$ in Set.



Can be done in any topos with a linear order.
Geometric realization becomes a geometric morphism by moving from spaces to toposes.
Equivalence of cats:

$$Orders(\mathcal{E}) \to Hom(\mathcal{E}, \hat{\Delta})$$

assigns to an order I in $\mathcal{E}$, the geometric realization defined by I.
Simplicial sets classify the *geometric* theory of strict linear orders.

Cubical sets

$\mathbb{2}$: poset with two elements

$\Box$: full subcategory of Cat with obj powers of $\mathbb{2}$.

$$\begin{tikzcd} 00 \arrow[r, "\leqslant"] \arrow[d, "\leqslant"] & 01 \arrow[d, "\leqslant"] \\ 10 \arrow[r, "\leqslant"] & 11 \end{tikzcd}$$

Duality finite posets and distibutive lattices.
$\mathbb{2}$ is the ambimorphic object here:
poset maps into $\mathbb{2}$ pick out 'opens'
DL-maps select the 'points'.
Stone duality between powers of $\mathbb{2}$ and
free finitely generate distributive lattices (copowers of DL1)

Nerve

Nerve construction:
Embedding of $\square$ into Cat.
Hence, $Cat \to \widehat{\square}$, defined by $C \mapsto \hom(-, C)$.
This is fully faithful (Awodey).
$\square$ is a dense subcategory of Cat.

Alternative nerve construction, using Lawvere theories.

Lawvere theory

Classifying categories for Cartesian categories.
Alternative to monads in CS (Plotkin-Power)
For algebraic theory T, the *Lawvere theory* Θ_T^{op} is the opposite of the category of free finitely generated models.
models of T in any finite product category category E correspond to product-preserving functors $m : \Theta_T^{op} \to E$.
$m(n)$ consists of the n-tuples in the model m.
A map $1 \to T(2)$, gives a map $m^2 \to m^1$, as both are T-algebras.
E.g. $* \mapsto (x \wedge y)$, defines $(x, y) \mapsto (x \wedge y)$.

Nerve construction and Lawvere theories

Alternative nerve construction, using Lawvere theories.
Consider DL the free distributive lattice monad on Fin.
Then Θ_{DL}^{op} is the Lawvere theory for distributive lattices.
The inclusion of distributive lattices into $\widehat{\Theta_{DL}}$ is fully faithful.
The image consists of those presheaves satisfying the Segal condition.
$\widehat{\Theta_{DL}} = \check{\Box}$ are the cocubical sets.

Classifying topos

Λ_T: finitely *presented* T-models.
$\Lambda_T \to Set$ is the classifying topos.
This topos contains a generic T-algebra.
T-algebras in any topos $\mathcal{F}$ correspond to *left exact left adjoint* functors from the classifying topos to $\mathcal{F}$.

Classifying topos

Example:
T a propositional geometric theory (=formal topology).
$Sh(T)$ is the classifying topos.

Set^{Fin} classifies the Cartesian theory with one sort.
Used for variable binding (Fiore, Plotkin,Turi, Hofmann).
Replaces Pitts' use of nominal sets for the cubical model.
TYPES 2014. Now diagonals.
Nominal sets classify decidable unfinite sets.

Moerdijk: Connes' cyclic sets classify abstract circles.

Classifying topos of cubical sets

Let $\Theta = \square^{op}$ be the category of *free finitely generated* DL-algebras
Let Λ the category of *finitely presented* ones.
We have a fully faithful functor $f : \Theta \to \Lambda$.
This gives a geometric embedding $\phi : \check{\Theta} \to \check{\Lambda}$

Classifying topos of cubical sets

The subtopos $\check{\Theta}$ of the classifying topos for DL-algebras is given by a quotient theory, the theory of the model $\mathbb{I} := \phi^* M$,
the DL-algebra $\mathbb{I}(m) := m$ for each $m \in \Theta$.

Each free finitely generated DL-algebra has the disjunction property $(a \vee b = 1 \vdash a = 1, b = 1)$
This properties is geometric and hence also holds for $\mathbb{I}$.

Geometric realization for cubical sets

Theorem (Johnstone-Wraith)

Let T be an algebraic theory, then the topos $\widecheck{\Theta_{DL}}$ classifies the geometric theory of flat T-models.

In particular, $\widehat{\square}$ classifies flat distributive lattices.

Need to show that [0,1] is a flat DL-algebra.

Geometric realization as a geometric morphism

Lem: Every linear order D defines a flat distributive lattice.
Hence, we have a geometric morphism $\widehat{\Delta} \to \widehat{\square}$.
Let $\mathcal{E}$ be Johnstone's topological topos.

Theorem (Cubical geometric realization)

There is a geometric morphism $r : \mathcal{E} \to \hat{\square}$ defined using the flat distributive lattice $[0,1]$.

We obtain the familiar formulas for both simplicial and topological realization.

Related work

Independently, Awodey showed that Cartesian cubical sets (without connections or reversions) classify strictly bipointed objects.
Much of Awodey's constructions of the cubical methods can be extended based on $\mathbb{I}$ and should give Coquand's model?

Categorical models of Id-types

Application: We have an ETT with an internal ‘interval’ $\mathbb{I}$.
van den Berg, Garner. Path object categories.
Usual path composition is only h-associative.
Moore paths can have arbitrary length.
category freely generated from paths of length one.
Moore paths: strict associativity, but non-strict involution.
Docherty: Id-types in cubical sets with $\vee$, but no diagonals.

Apply vdB/G-D construction. However, work internally in the topos of cubical sets using the generic DL-algebra $\mathbb{I}$.
Simplifies computation substantially.
North: Π-h-tribe.

Conclusion

- Towards a more categorical description of the cubical model Id-types.
- Cubical sets as a classifying topos.
- Cubical geometric realization