On the Cubical Model of Homotopy Type Theory
— *work in progress* —

Steve Awodey
Carnegie Mellon University

Deutsche Mathematiker-Vereinigung
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Why Cubical HoTT?

- Basic MLTT has a **constructive character** that makes it well-suited for use in computational proof assistants: strong normalization of terms, decidability of type-checking, decidability of judgemental equality, canonicity, etc.

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- But when we add new **axioms** like Univalence and HITs, this constructive character is spoiled. Instances of UA cannot always be eliminated, and new primitive terms of higher Id-type need not reduce to normal forms.
- A "**normalization up to homotopy**" algorithm could partially restore the constructive character of the system.
- But, as recently shown by **Coquand** et al., a system with additional cubical structure seems to allow for such extensions while still retaining a constructive character.
- This could lead to a proof of normalization up to homotopy for the **original** system via an interpretation. Moreover, it could also serve on its own as the basis of a new generation of proof assistants based on **cubical** HoTT.

Cubical HoTT: Recent work

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- Cubical sets are a **combinatorial** model of homotopy theory, introduced by Kan and still used in algebraic topology. Like the more familiar simplicial sets, they provide a more **algebraic** setting to study the homotopy theory of spaces.
- Voevodsky's original model of UA used classical **simplicial** sets and is **not constructive**. Known models of HITs are also based on **classical methods** from the theory of ∞-toposes.

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Cubes rule!

- The cubical model suggests enriching the type theory itself with some additional **cubical operations** and **equations** which are present in the model, and which allow calculations that are otherwise available only "up-to-homotopy". This makes the system **more computational**.

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- Brunerie and Licata (LICS 2015) have a variant system of **cubical HoTT** in which e.g. the proof that $\mathbf{T}^2 \simeq \mathbf{S}^1 \times \mathbf{S}^1$ is short and sweet (in contrast to the original "heroic" proof in plain HoTT first given by Sojakova in 2013).

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- The cubical setting seems to be **better suited to HoTT** than the simplicial one (or the globular one). It may also be of some use in **homotopy theory** (cf. recent work by Jardine, Grandis, Williamson, and others).

Variations on cubical sets

The category of **cubical sets** is the functor category

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The more structure one puts into the index category of cubes, the more "algebraic" the resulting model of type theory will be.

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Write the bipointed sets:

$$[n] = \{0, x_1, ..., x_n, 1\}$$

So $\mathbb{C}$ has the objects: $[0], [1], ..., [n], ...$, which we regard dually as the basic **n-cubes**.

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They can also be represented syntactically as the terms of a very simple **algebraic theory**.

Cartesian cubical sets

Definition

The category **cSet** of (cartesian) **cubical sets** is the **presheaves** on $\mathbb{C}$. It is thus equal to the **covariant** functors on $\mathbb{B}$,

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The **interval** object $\mathtt{I} = \mathrm{hom}_{\mathbb{C}}(-, [1])$ generates all the other cubes, which are closed under finite products and satisfy:

$$\mathtt{I}^n \times \mathtt{I}^m \cong \mathtt{I}^{n+m}\,.$$

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The interval $1 + 1 \to \mathtt{I}$ in **cSet** is **universal**, in the following sense.

Theorem (A. 2015)

The category **cSet** *of cubical sets is the* **classifying topos** *for strictly bipointed objects* $(X, a, b, a \neq b)$.

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- Moreover, the geometric realization **cSet** $\longrightarrow$ **Top** preserves finite products.

Path spaces in cubical sets

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The object $A^{\mathtt{I}}$ has the special property,

$$\begin{aligned} A^{\mathtt{I}}_n \;&\cong\; \mathrm{hom}(\mathtt{I}^n, A^{\mathtt{I}}) \;\cong\; \mathrm{hom}(\mathtt{I}^n \times \mathtt{I}, A) \\ &\cong\; \mathrm{hom}(\mathtt{I}^{n+1}, A) \;\cong\; A_{n+1}. \end{aligned}$$

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This **combinatorial specification** makes this path object very well-behaved. For example, it has not only a **left** adjoint ("cylinder") but also a **right** adjoint,

$$X \times \mathtt{I} \;\dashv\; Y^{\mathtt{I}} \;\dashv\; Z_{\mathtt{I}}\,.$$

Path spaces in cubical sets

Lemma

The interval $\mathtt{I}$ *in* **cSet** *satisfies the "domain equation"*

$$\mathtt{I}^{\mathtt{I}} \cong \mathtt{I} + 1 \, .$$

Something similar happens in the object classifier and in the Schanuel topos. We can use this to calculate the right adjoint $Z_{\mathtt{I}}$.

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Corollary

For the "amazing right adjoint" $Z_{\mathtt{I}}$*, we have:*

$$\begin{aligned} Z_{\mathtt{I}}(n) &\cong \mathrm{Hom}(\,\mathtt{I}^n,\, Z_{\mathtt{I}}) \;\cong\; \mathrm{Hom}(\,(\mathtt{I}^n)^{\mathtt{I}},\, Z\,) \\ &\cong \mathrm{Hom}(\,(\mathtt{I}^{\mathtt{I}})^n,\, Z) \;\cong\; \mathrm{Hom}(\,(\mathtt{I}+1)^n,\, Z) \\ &\cong \mathrm{Hom}(\,\mathtt{I}^n + C^n_{n-1}\mathtt{I}^{n-1} + \cdots + C^n_1\mathtt{I} + 1,\, Z) \\ &\cong Z_n \times Z_{n-1}^{C^n_{n-1}} \times \cdots \times Z_1^{C^n_1} \times Z_0, \end{aligned}$$

where $C^n_k = \binom{n}{k}$ *is the usual binomial coefficient.*

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This implies some new type-theoretic **equations** and **conditions**, such as:

$$\mathtt{Id}_{\mathtt{Id}_A} = (A^{\mathtt{I}})^{\mathtt{I}} \cong A^{\mathtt{I}\times\mathtt{I}}\,,$$
$$\mathtt{Id}_{A+B} = (A+B)^{\mathtt{I}} \;\cong\; A^{\mathtt{I}} + B^{\mathtt{I}} = \mathtt{Id}_A + \mathtt{Id}_B\,,$$

and generally, the $\mathtt{Id}$-type of a colimit is a colimit of $\mathtt{Id}$-types.

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The interpretation is thus **not** expected to be **conservative** — indeed, one hopes to determine some new **cubical laws** that may be soundly added to the original theory

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When does $A^{\mathtt{I}} \to A \times A$ *satisfy the rules for* `Id`*-types?*

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Theorem (A. 2015)

The path space $A^{\mathtt{I}} \to A \times A$ satisfies the rules for `Id`-types if

1. *The obect A is a Kan complex.*
2. *The dependent types $B \to A$ are Kan fibrations.*

The notions of **Kan complex** and **Kan fibration** are determined by the usual **box-filling conditions**.

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Proof.

1. Reduce `Id`-elim to **transport** and **contraction**.
2. Transport follows from path-lifting, i.e. 1-box filling.
3. Contraction follows from 1-box filling for $A^{\mathrm{I}} \to A \times A$.
4. 1-box filling in $A^{\mathrm{I}} \to A \times A$ is 2-box filling in A. □

Path spaces and identity types

The last step of the foregoing is a special case of the following:

Lemma

The following are equivalent for a cubical set A.

1. $(n+1)$*-box filling in* A,
2. n*-box filling in* $A^{\mathtt{I}} \to A \times A$,
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This can be used to prove a **converse** of the foregoing theorem: the box-filling conditions for cubical sets follow from the `Id`-rules together with Σ-types.

Cubical Lumsdaine

We can use the foregoing lemma to derive a **cubical version** of "Lumsdaine's Theorem" (aka "Lumsdaine-van den Berg-Garner"):

Theorem (A. 2015)

Every type A in MLTT gives rise to a **cubical** *∞-groupoid (a cubical set satisfying the box-filling conditions).*

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We first need to determine the **cubical nerve of a type** A, i.e. a cubical set $N(A)$:

$$N(A)_0 \leftleftarrows\!\!\rightarrow N(A)_1 \quad N(A)_2 \quad \cdots$$

with:

$$N(A)_n \;\cong\; \text{"}n\text{-cubes in } A\text{"}$$

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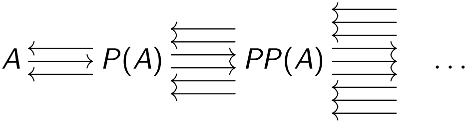
Since P also acts on maps by the "map on paths" operation, there are also the successive **images** of these maps under P:

$$\begin{array}{ccccccc}
A & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & P(A) & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & PP(A) & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & \ldots \\
 & & & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & \\
 & & & & & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} &
\end{array}$$

Cubical nerve of a type

Rearranging, we find the usual cubical **structure**:



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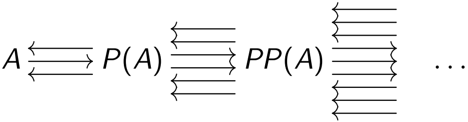

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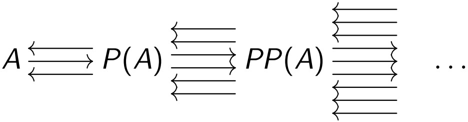

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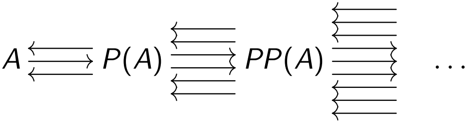

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In **cubical** type theory we expect to have a **cartesian** cubical nerve.

Cubical nerve of a category

A similar example is the **cubical nerve** $N(\mathbb{A})$ of a category $\mathbb{A}$.
As a "pathobject" we can take the arrow category:

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We also have the usual "realization $\dashv$ nerve" adjunction,

$$\mathbf{cSet} \overset{\longleftarrow}{\longrightarrow} \mathbf{Cat},$$

given by Kan extension along $\mathbb{C} \longrightarrow \mathbf{Cat}$, the cartesian classifying map of the interval $\mathbb{1} \to \mathbb{2} \leftarrow \mathbb{1}$ in $\mathbf{Cat}$.

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- As in **sSets**, the categories $\mathbb{A}$ with a Kan nerve $N(\mathbb{A})$ are exactly the **groupoids**.
- **Cubical analogues** of the **orientals**, the **homotopy coherent nerve**, and the notions of **quasicategory** and ∞-**topos** have not yet been studied.
- We expect the (cubical nerve of) the category of types in cubical homotopy type theory to be a cubical ∞-topos.

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- Since there is a (distinguished) path $p : \mathtt{Id}_{\mathtt{I}}(0, 1)$ in $\mathtt{I}$, and $C \to \mathtt{I}$ is a fibration, we have the **transport** map

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- This is the map $\mathtt{Id}_{\mathcal{U}}(A, B) \to A \simeq B$ which by UA is supposed to have an **inverse**.
- Given an equivalence $e : A \simeq B$, we can build a suitable fibration $A +_e B \to \mathtt{I}$ using the **mapping cylinder** construction from homotopy theory.