Regular logic and regular fibrations*

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1 Regular fibrations

A regular fibration is a bifibration with fibred finite products, or equivalently a pseudofunctor $R\colon B^{\mathrm{op}} \to \mathbf{Cat}$, out of a category with finite products, that takes values in categories with finite products and where each $f^* = Rf$ has a left adjoint $\exists_f$ and (hence) preserves finite products. The latter condition is vacuous because the f^* are right adjoints, but we may also want to deal with those nearly-regular bifibrations where the base category has finite products but the fibres are merely monoidal, and in this case it is important to require that the f^* are strong monoidal (of course, they are automatically lax monoidal by virtue of being right adjoints).

A morphism of regular fibrations is the obvious thing: a product-preserving morphism of fibrations.

Our regular fibrations are those of [Pav96]. A very similar definition is given in [Jac99], the only difference being that the latter sort of regular fibration is required to have all fibres preordered.

The connection with regular categories is that a category $\mathbf{C}$ is regular if and only if the projection $\mathrm{cod}\colon \mathrm{Mon}\,\mathbf{C} \to \mathbf{C}$ that sends $S \hookrightarrow X$ to X is a regular fibration. For our purposes, a regular category is one that has finite limits and pullback-stable images.

If $\mathbf{C}$ is a regular category, then the adjunctions $\exists_f \dashv f^*$ come from pullbacks and images in $\mathbf{C}$ [Joh02, lemma 1.3.1] as does the Frobenius property [*op. cit.*, lemma 1.3.3]. The terminal object of $\mathrm{Sub}(X) = \mathrm{Mon}(\mathbf{C})_X$ is the identity 1_X on X, and binary products in the fibres $\mathrm{Sub}(X)$ are given by pullback. The products are preserved by reindexing functors f^* because (the f^* are right adjoints but also because) a cone over the diagram for $f^*(S \wedge S')$ can be rearranged into a cone over that for $f^*S \wedge f^*S'$, giving the two the same universal property. The projection cod clearly preserves these products. The Beck–Chevalley condition follows from pullback-stability of images in $\mathbf{C}$.

Conversely, suppose $\mathrm{Mon}\,\mathbf{C} \to \mathbf{C}$ is a regular fibration. We need to show that $\mathbf{C}$ has equalizers (to get finite limits) and pullback-stable images. But the equalizer of $f, g\colon A \rightrightarrows B$ is $(f,g)^*\Delta$. For images, let $\mathrm{im}\, f = \exists_f 1$ as in [Joh02, lemma 1.3.1]. Pullback-stability follows from the Beck–Chevalley condition.

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2 Regular logic

Regular logic is the fragment of first-order predicate logic that uses only the connectives $\top$ for truth, $\wedge$ for conjunction and $\exists$ for existential quantification. We will mostly follow [See83].

2.1 Language

A (regular) signature is a collection $X, Y, \ldots$ of sorts, together with a collection of typed predicate and function symbols. A type is a finite sequence $X_1, X_2, \ldots$ of sorts, and types will also be denoted $X, Y, \ldots$. If P is a predicate of type X we may write $P \colon X$, and similarly $f \colon X \to Y$ indicates the type of f. Every signature contains at least the equality predicate $=_X \colon X, X$.

We assume given an inexhaustible supply of free variables $x, x', y, y' \ldots$ and bound variables $\xi, \xi', \upsilon, \upsilon' \ldots$ of each sort, with the notation extended to types so that a variable of type X, Y is the same as a pair x, y of variables of sorts X and Y. A context is a finite list $x \colon X, y \colon Y, \ldots$ of sorted variables, or equivalently a single variable $z \colon X, Y, \ldots$.

A term is either a variable, a tuple of terms or a function symbol f applied to a term, all with the obvious well-typedness constraints. Every term lives in a context, which is assumed to contain every variable in the term, perhaps together with 'dummy' variables that don't. We write $t[x]$ to indicate that x is the context of t, and $t[s]$ to denote the obvious substitution.

A formula is either the constant $\top$, a predicate symbol $P(t)$ applied to a term, the conjunction $\phi \wedge \psi$ of two formulas, a quantified formula $\exists\xi.\phi$ or the substitution $\phi[t]$ of the term t into the formula ϕ, defined in the usual way. Every formula lives in a context, which we assume contains (perhaps strictly) all of its free variables, and we write $\phi[x]$ for this.

2.2 Logic

We will use the usual natural-deduction rules. Conjunction is governed by

$$\frac{\phi \qquad \psi}{\phi \wedge \psi} \qquad\qquad \frac{\phi \wedge \psi}{\phi} \qquad \frac{\phi \wedge \psi}{\psi}$$

truth by

$$\frac{\phi}{\top}$$

existentials by

$$\frac{\phi[t]}{\exists\xi.\phi[\xi]} \qquad\qquad \frac{\exists\xi.\phi[\xi] \qquad \begin{array}{c}\overline{\phi[x]}\\ \vdots \\ \psi\end{array}}{\psi}$$

where on the right x is not free in ψ, and equality by

$$\frac{}{t = t} \qquad\qquad \frac{t = s \qquad \phi[t]}{\phi[s]}$$

The notion of context is easily extended to derivations. Observe that the rules for $\exists$ are the only rules that do not preserve the contexts of formulas.

Derivations using these rules may be composed:

$$\begin{array}{c}\phi\\ \vdots\\ \psi\end{array} \;,\; \begin{array}{c}\psi\\ \vdots\\ \chi\end{array} \;\mapsto\; \begin{array}{c}\phi\\ \vdots\\ \psi\\ \vdots\\ \chi\end{array}$$

as long as both derivations have the same context, and this composition is clearly associative, with units the identity derivations ϕ. We may write $p\colon \phi \overset{x}{\Longrightarrow} \psi$ to indicate that p is a derivation of ψ from the assumption ϕ with context x, and thus arrive at the rules

$$\frac{}{1_\phi\colon \phi \overset{x}{\Longrightarrow} \phi} \qquad\qquad \frac{p\colon \phi \overset{x}{\Longrightarrow} \psi \qquad q\colon \psi \overset{x}{\Longrightarrow} \chi}{q \circ p\colon \phi \overset{x}{\Longrightarrow} \chi}$$

The substitution $p[t]$ of a term $t\colon Y \to X$ into a derivation $p[x]$ with x free is defined in the obvious way, and an induction over the structure of derivations shows that the 'substitute t' mapping t^* is a functor from the category of derivations in the context x to derivations in the context y that commutes with the finite-product structure given by the following.

If $p_i\colon \phi \overset{x}{\Longrightarrow} \psi_i$ for $i = 1, 2$, then we may use the $\wedge$-introduction rule to form a derivation $\langle p_1, p_2\rangle\colon \phi \overset{x}{\Longrightarrow} \psi_1 \wedge \psi_2$, and conversely given a derivation p of the latter type the elimination rules give $\pi_i \circ p\colon \phi \overset{x}{\Longrightarrow} \psi_i$. Imposing the ($\beta$- and η-)equalities

$$\pi_i\langle p_1, p_2\rangle = p_i \qquad\qquad \langle \pi_1 p, \pi_2 p\rangle = p$$

then gives a 'bijective' rule

$$\frac{p_1\colon \phi \overset{x}{\Longrightarrow} \psi_1 \qquad p_2\colon \phi \overset{x}{\Longrightarrow} \psi_2}{\overline{\langle p_1, p_2\rangle\colon \phi \overset{x}{\Longrightarrow} \psi_1 \wedge \psi_2}}$$

where to move from bottom to top we compose with π_i, and this gives binary products in each category of derivations. As for $\top$, we will say that any derivation $p\colon \phi \overset{x}{\Longrightarrow} \top$ is equal to the canonical $!_\phi\colon \phi \overset{x}{\Longrightarrow} \top$, making $\top$ the terminal object in each category of derivations.

Similarly, there is a β rule for equality:

$$\frac{\overline{t = t} \qquad \begin{array}{c}\vdots\\ \phi[t]\end{array}}{\phi[t]} \quad = \quad \begin{array}{c}\vdots\\ \phi[t]\end{array}$$

and an η rule:

$$\begin{array}{c}p\,\vdots\\ t = t'\\ q[t,t']\,\vdots\\ \phi[t,t']\end{array} \quad = \quad \frac{\begin{array}{c}p\,\vdots\\ t = t'\end{array} \qquad \begin{array}{c}\overline{t = t}\\ q[t,t]\,\vdots\\ \phi[t,t]\end{array}}{\phi[t,t']}$$

and these set up a bijection

$$\frac{\phi, x = x' \overset{x,x'}{\Longrightarrow} \psi[x,x']}{\phi \overset{x}{\Longrightarrow} \psi[x,x]} \qquad (*)$$

between derivations of the indicated types [Jac99]. There is also a 'coherence' rule

$$\dfrac{\dfrac{\vdots}{t = t}}{\dfrac{\top}{t = t}} \qquad = \qquad \begin{array}{c}\vdots\\ t = t\end{array}$$

which makes sure that $\top_X \equiv x = x$, so that $x = x$ is the terminal object in the category of derivations over X.

A (regular) theory over a signature is given by a collection of axioms (derivation constants) together with a collection of equations between derivations built from those axioms and the above rules. The terms of a signature, together with the equational axioms $t = t'$ of a theory over that signature, give rise to a category B_T with finite products — the 'multisorted Lawvere theory' associated to the theory. In this category an object is a type $X_1, X_2, \ldots, X_n$, and a morphism from $X_1, X_2, \ldots, X_n$ to $Y_1, Y_2, \ldots, Y_m$ is given by an m-tuple $\langle t_1, t_2, \ldots, t_m\rangle$ of terms, where each $t_i\colon X_1, X_2, \ldots, X_n \to Y_i$. Thus a theory T gives rise to a pseudofunctor $T\colon {B_T}^{\mathrm{op}} \to \mathbf{Cat}$, which takes a type to the finite-product category of formulas and terms whose context is of that type, and takes a term $t\colon X \to Y$ to the substitution functor $t^*\colon T_Y \to T_X$.

We want to show that a regular theory T gives rise to a bifibration $E_T \to B_T$, that is, that for each term $t\colon X \to Y$, the functor t^* has a left adjoint $\exists_t$. Define the latter on formulas as

$$\exists_t \phi = \exists\xi.(t[\xi] = y \wedge \phi[\xi])$$

Let $t\colon X \to Y$ be any term; it suffices to show that for any $\phi[x]$ of type X there is a universal $\eta^t_\phi\colon \phi \overset{x}{\Longrightarrow} t^*\exists_t\phi$; that is, for any equivalence class of proofs $p\colon \phi \overset{x}{\Longrightarrow} t^*\psi$, there is a unique $\hat{p}\colon \exists_t\phi \overset{y}{\Longrightarrow} \psi$ such that $t^*\hat{p} \circ \eta^t_\phi$ is equal to p. The derivation η^t_ϕ is obtained by forming the derivation

$$\dfrac{\dfrac{\dfrac{x = x' \qquad \dfrac{}{t[x] = t[x]}}{t[x'] = t[x]} \qquad \dfrac{x = x' \qquad \phi[x]}{\phi[x']}}{t[x'] = t[x] \wedge \phi[x']}}{\exists\xi.(t[\xi] = t[x] \wedge \phi[\xi])}$$

of type $\phi[x], x = x' \overset{x,x'}{\Longrightarrow} t^*\exists_t\phi$ and using the bijection $(*)$ above to get rid of the hypothesis $x = x'$. Given $p\colon \phi \overset{x}{\Longrightarrow} t^*\psi$, let $\hat{p}$ be

$$
\dfrac{\exists\xi.(t[\xi]=y\wedge\phi[\xi]) \qquad \dfrac{\dfrac{\overline{t[x]=y\wedge\phi[x]}}{t[x]=y} \qquad \begin{array}{c}\dfrac{\overline{t[x]=y\wedge\phi[x]}}{\phi[x]}\\ \vdots \\ \psi[t[x]]\end{array}}{\psi[y]}}{\psi[y]}
$$

The β and η equalities given above show that the composite $t^*\hat{p}\circ\eta^t_\phi$ is equal to p, and uniqueness of $\hat{p}$ follows from the normal form theorem for natural deduction [Pra06]. So we have another bijection

$$
\dfrac{\exists_t\phi \overset{y}{\Longrightarrow} \psi}{\phi \overset{x}{\Longrightarrow} t^*\psi}
$$

In particular, we have the usual rewriting rules, as given in [See83]:

$$
\dfrac{\dfrac{\begin{array}{c} p\,\vdots \\ \phi[t]\end{array}}{\exists\xi.\phi[\xi]} \qquad \begin{array}{c}\overline{\phi[x]}\\ \vdots\, q[x] \\ \psi\end{array}}{\psi}
\quad = \quad
\begin{array}{c} p\,\vdots \\ \phi[t] \\ q[t]\,\vdots \\ \psi \end{array}
$$

and

$$
\begin{array}{c} p\,\vdots \\ \exists\xi.\phi[\xi] \\ q\,\vdots \\ \psi \end{array}
\quad = \quad
\dfrac{\begin{array}{c} p\,\vdots \\ \exists\xi.\phi[\xi]\end{array} \qquad \begin{array}{c}\dfrac{\overline{\phi[x]}}{\exists\xi.\phi[\xi]} \\ q\,\vdots \\ \psi\end{array}}{\psi}
$$

For $E_T \to B_T$ to be a regular fibration, it must satisfy the Frobenius and Beck–Chevalley conditions. The former means that for any term t the canonical map $\exists_t(\phi\wedge t^*\psi) \overset{y}{\Longrightarrow} (\exists_t\phi)\wedge\psi$ is an isomorphism. This canonical map is given [Joh02, definition D1.3.1(i)] by

$$
\dfrac{\dfrac{\phi\wedge t^*\psi \overset{x}{\Longrightarrow} t^*\psi}{\exists_t(\phi\wedge t^*\psi) \overset{y}{\Longrightarrow} \psi} \qquad \dfrac{\dfrac{\phi\wedge t^*\psi \overset{x}{\Longrightarrow} \phi \qquad \dfrac{\exists_t\phi \overset{y}{\Longrightarrow} \exists_t\phi}{\phi \overset{x}{\Longrightarrow} t^*\exists_t\phi}}{\phi\wedge t^*\psi \overset{x}{\Longrightarrow} t^*\exists_t\phi}}{\exists_t(\phi\wedge t^*\psi) \overset{x}{\Longrightarrow} \exists_t\phi}}{\exists_t(\phi\wedge t^*\psi) \overset{x}{\Longrightarrow} (\exists_t\phi)\wedge\psi}
$$

So we must insist that in B_T the above proof, call it f, have a formal inverse $f^{-1}\colon(\exists_t\phi)\wedge\psi \overset{x}{\Longrightarrow} \exists_t(\phi\wedge t^*\psi)$, adding to the equations above $f^{-1}f=1$ and $ff^{-1}=1$.

The Beck–Chevalley condition asks that for any pullback $tu=sv$ in B_T, the mate of the isomorphism $u^*t^*\cong v^*s^*$ in **Cat** is again invertible. Now B_T need not have all pullbacks, but there are some that it must have by virtue of having finite products:

$$\begin{array}{ccc} X & \xrightarrow{\langle X,t\rangle} & X\times Y \\ {\scriptstyle t}\big\downarrow & \text{(A)} & \big\downarrow{\scriptstyle t\times Y} \\ Y & \xrightarrow[\Delta]{} & Y\times Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{X} & X \\ {\scriptstyle X}\big\downarrow & \text{(B)} & \big\downarrow{\scriptstyle \Delta} \\ X & \xrightarrow[\Delta]{} & X\times X \end{array}$$

and

$$\begin{array}{ccc} X'\times X & \xrightarrow{X'\times t} & X'\times Y \\ {\scriptstyle t'\times X}\big\downarrow & \text{(C)} & \big\downarrow{\scriptstyle t'\times Y} \\ Y'\times X & \xrightarrow[Y'\times t]{} & Y'\times Y \end{array}$$

Also, if $tu = sv$ is a pullback, then so is its product with any object:

$$\begin{array}{ccc} P\times Z & \xrightarrow{u\times Z} & X\times Z \\ {\scriptstyle v\times Z}\big\downarrow & \text{(D)} & \big\downarrow{\scriptstyle t\times Z} \\ X'\times Z & \xrightarrow[s\times Z]{} & Y\times Z \end{array}$$

By [See83, Theorem, §8], if a hyperdoctrine satisfies Beck–Chevalley for these types of pullback, then it satisfies the condition for any pullback $tu = sv$ if and only if it proves

$$t[m] = s[m'] \Longrightarrow \exists\xi.(u[\xi] = m \wedge v[\xi] = m')$$

and

$$u[p] = u[p'], v[p] = v[p'] \Longrightarrow p = p'$$

that is, if the hyperdoctrine 'knows' that the diagram is a pullback. Seely's proof goes through unchanged for a bifibration with fibred finite products, like our T.

The Beck–Chevalley condition for (B) asks that η^{Δ} be invertible. An inverse is given by

$$\dfrac{\exists\xi.\Delta[\xi] = \Delta[x] \wedge \phi[\xi] \qquad \dfrac{\dfrac{(x',x') = (x,x) \wedge \phi[x']}{(x',x') = (x,x)} \qquad \dfrac{(x',x') = (x,x) \wedge \phi[x']}{\phi[x']}}{\phi[x]}}{\phi[x]}$$

That this derivation is a left inverse for η^{Δ}_{ϕ} is easy to show, using the β-reductions given above, and conversely that it is a right inverse follows from the η-reductions for $\wedge$, $\exists$ and $=$.

As for the other types of pullback, the Beck–Chevalley condition for these is shown as in [See83, §4]. So in order to prove that the syntactic model $T\colon {B_T}^{\mathrm{op}} \to$ **Cat** satisfies the full condition, it suffices to show that T recognizes pullbacks in the sense above. But this is practically trivial: for a pullback $tu = sv$ in B_T, the mediating morphism automatically exists for any commuting square over t and s, while the second sequent follows from its uniqueness.

We can now perform the usual rites of categorical logic: a model of a regular theory T in a regular fibration $E \to B$ is a morphism of regular fibrations from $E_T \to B_T$ to $E \to B$, and it is easy to see that this is equivalent to the traditional notion. Soundness is automatic, as is completeness, because if a sequent is true in every model then it is true in the syntactic model and thence provable.

References

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