ESI The Erwin Schrödinger International Institute for Mathematical Physics
Boltzmanngasse 9 A-1090 Wien, Austria

Lectures on Graded Differential Algebras and Noncommutative Geometry

M. Dubois-Violette

Vienna, Preprint ESI 842 (2000)

February 11, 2000

Supported by Federal Ministry of Science and Transport, Austria
Available via anonymous ftp or gopher from FTP.ESI.AC.AT
or via WWW, URL: http://www.esi.ac.at

LECTURES ON GRADED DIFFERENTIAL ALGEBRAS AND NONCOMMUTATIVE GEOMETRY

Michel DUBOIS-VIOLETTE

Laboratoire de Physique Théorique [1]
Université Paris XI, Bâtiment 210
F-91 405 Orsay Cedex, France
patricia@th.u-psud.fr

December 3, 1999

Abstract

These notes contain a survey of some aspects of the theory of graded differential algebras and of noncommutative differential calculi as well as of some applications connected with physics. They also give a description of several new developments.

LPT-ORSAY 99/100

Keywords : graded differential algebras; categories of algebras; bimodules; noncommutative differential calculus; noncommutative symplectic geometry.

MSC classification: 16D20, 18B99, 51P05, 53A99, 81Q99.

To be published in the Proceedings of the Workshop on Noncommutative Differential Geometry and its Application to Physics, Shonan-Kokusaimura, Japan, May 31 - June 4, 1999.

[1] Unité Mixte de Recherche du Centre National de la Recherche Scientifique - UMR 8627

Contents

1 Introduction

The correspondence between “spaces” and “commutative algebras” is by now familiar in mathematics and in theoretical physics. This correspondence allows an algebraic translation of various geometrical concepts on spaces in terms of the appropriate algebras of functions on these spaces. Replacing these commutative algebras by noncommutative algebras, i.e. forgetting commutativity, leads then to noncommutative generalizations of geometries where notions of “spaces of points” are not involved. Such a noncommutative generalization of geometry was a need in physics for the formulation of quantum theory and the understanding of its relations with classical physics. In fact, the relation between spectral theory and geometry has been implicitely understood very early in physics.

Gel'fand's transformation associates to each compact topological space X the algebra $C(X)$ of complex continuous functions on X. Equipped with the sup norm, $C(X)$ is a commutative unital C^*-algebra. One of the main points of Gel'fand theory is that *the correspondence* $X \mapsto C(X)$ *defines an equivalence between the category of compact topological spaces and the category of commutative unital* C^**-algebras.* The compact space X is then identified to the spectrum of $C(X)$, (i.e. to the set of homomorphisms of unital $*$-algebras of $C(X)$ into $\mathbb{C}$ equipped with the weak topology). Let X be a compact space and let $\mathcal{E}(X)$ denote the category of finite rank complex vector bundles over X. To any vector bundle E of $\mathcal{E}(X)$ one can associate the $C(X)$-module $\Gamma(E)$ of all continuous sections of E. The module $\Gamma(E)$ is a finite projective $C(X)$-module and the Serre-Swan theorem asserts that *the correspondence* $E \mapsto \Gamma(E)$ *defines an equivalence between the category* $\mathcal{E}(X)$ *and the category* $\mathcal{P}(C(X))$ *of finite projective* $C(X)$*-modules.* Thus the compact spaces and the complex vector bundles over them can be replaced by the commutative unital C^*-algebras and the finite projective modules over them. In this sense noncommutative unital C^*-algebras provide “noncommutative generalizations” of compact spaces whereas the notion of finite projective right module over them is a corresponding generalization of the notion of complex vector bundle. It is worth noticing here that for the latter generalization one can use as well left modules but these are not the only possibilities (see below) and that something else has to be used for the generalization of the notion of real vector bundle.

Remark 1. Let X be an arbitrary topological space, then the algebra $C^{\mathbf{b}}(X)$ of complex continuous bounded functions on X is a C^*-algebra if one equips it with the sup norm. In view of Gel'fand theory one has $C^{\mathbf{b}}(X) = C(\hat{X})$ (as C^*-algebras), where $\hat{X}$ denotes the spectrum of $C^{\mathbf{b}}(X)$. The spectrum $\hat{X}$ is a compact space and the evaluation defines a continuous mapping $e : X \mapsto \hat{X}$ with dense image ($\overline{e(X)} = \hat{X}$). The compact space $\hat{X}$ is called *the Stone-Čech compactification of* X and the pair $(e, \hat{X})$ is characterized (uniquely up to an isomorphism) by the following universal property: *For any continuous mapping $f : X \mapsto Y$ of X into a compact space Y there is a unique continuous mapping $\hat{f} : \hat{X} \mapsto Y$ such that $f = \hat{f} \circ e$.* Notice that $e : X \mapsto \hat{X}$ is generally not injective and that it is an isomorphism, i.e. $X = \hat{X}$, if and only if X is compact. The above universal property means that $\hat{X}$ is the biggest compactification of X . For instance if X is locally compact then e is injective, i.e. $X \subset \hat{X}$ canonically, but $\hat{X}$ is generally much bigger than the one point compactification $X \cup \{\infty\}$ of X, (e.g. for $X = \mathbb{R}$ the canonical projection $\hat{\mathbb{R}} \to \mathbb{R} \cup \{\infty\}$ has a huge inverse image of ∞).

If instead of (compact) topological spaces one is interested in the geometry of measure spaces, what replaces algebras of continuous functions are of course algebras of measurable functions. In this case the class of algebras is the class of commutative W^*-algebras (or von Neumann algebras). The noncommutative generalizations are therefore provided by general (noncommutative) W^*-algebras. It has been shown by A. Connes that the corresponding noncommutative measure theory (i.e. the theory of von Neumann algebras) has a very rich structure with no classical (i.e. commutative) counterpart (e.g. the occurrence of a canonical dynamical system) [12].

In the case of differential geometry, it is more or less obvious that the appropriate class of commutative algebras are algebras of smooth functions. Indeed if X is a smooth manifold and if $\mathcal{C}$ is the algebra of complex smooth function on X, ($\mathcal{C} = C^\infty(X)$), one can reconstruct X with its smooth structure and the objects attached to X, (differential forms, etc.), by starting from $\mathcal{C}$ considered as an abstract (commutative) unital $*$-algebra. As a set X can be identified with the set of characters of $\mathcal{C}$, i.e. with the set of homomorphisms of unital $*$-algebras of $\mathcal{C}$ into $\mathbb{C}$; its differential structure is connected with the abundance of derivations of $\mathcal{C}$ which identify with the smooth vector fields on X as well known. In fact, in [50], J.L. Koszul gave a powerful algebraic generalization of differential geometry in terms of a commutative (associative) algebra $\mathcal{C}$, of $\mathcal{C}$-modules and connections (called derivation laws there) on these modules. For the applications to differential geometry, $\mathcal{C}$ is of course the algebra of smooth functions on a smooth manifold and the $\mathcal{C}$-modules are modules of smooth sections of smooth vector bundles over the manifold.

In this approach what generalizes the vector fields are the derivations of $\mathcal{C}$ (into itself). The space $\mathrm{Der}(\mathcal{C})$ of all derivations of $\mathcal{C}$ is a Lie algebra and a $\mathcal{C}$-module, both structures being connected by $[X, fY] = f[X, Y] + X(f)Y$ for $X, Y \in \mathrm{Der}(\mathcal{C})$ and $f \in \mathcal{C}$. Using the latter property one can extract, (by $\mathcal{C}$-multilinearity), a graded differential algebra generalizing the algebra of differential forms, from the graded differential algebra $C_\wedge(\mathrm{Der}(\mathcal{C}), \mathcal{C})$ of $\mathcal{C}$-valued Chevalley-Eilenberg cochains of the Lie algebra $\mathrm{Der}(\mathcal{C})$ (with its canonical action on $\mathcal{C}$). This construction admits a generalization to the noncommutative case; it is the *derivation-based* differential calculus ([25], [26], [27],[34] [35]) which will be described below. As will be explained (see also [26] and [27]) this is the right differential calculus for quantum mechanics, in particular we shall show that the corresponding noncommutative symplectic geometry is exactly what is needed there.

For commutative algebras, there is another well-known generalization of the calculus of differential forms which is the Kähler differential calculus [6], [43], [52], [58]. This differential calculus is "universal" and consequently functorial for the category of (associate unital) commutative algebras. In these lectures we shall give a generalization of the Kähler differential calculus for the noncommutative algebras. By its very construction, this differential calculus will be functorial for the algebra-homomorphisms mapping the centers into the centers. More precisely this differential calculus will be shown to be the universal differential calculus for the category of algebra $\mathbf{Alg}_Z$ whose objects are the unital associative $\mathbb{C}$-algebras and whose morphisms are the homomorphisms of unital algebras mapping the centers into the centers. This differential calculus generalizes the Kähler differential calculus in the sense that it reduces to it for a commutative (unital associative $\mathbb{C}$) algebra. This latter property is in contrast with what happens for the so-called *universal differential calculus*, which is universal for the category $\mathbf{Alg}$ of unital associative $\mathbb{C}$-algebras and of *all* unital algebra-homomorphisms, the construction of which will be recalled in these lectures.

Concerning the generalizations of the notion of module over a commutative algebra $\mathcal{C}$ when one replaces it by a noncommutative algebra $\mathcal{A}$, there are the notion of right $\mathcal{A}$-module and the dual notion of left $\mathcal{A}$-module, but since a module over a commutative algebra is also canonically a bimodule (of a certain kind) and since a commutative algebra coincides with its center, there is a notion of bimodule over $\mathcal{A}$ and also the notion of module over the center $Z(\mathcal{A})$ of $\mathcal{A}$ which are natural. The "good choices" depend on the kind of problems involved. Again categorial notions can be of some help. As will be explained in these lectures, for each category of algebras there is a notion of bimodule over the objects of the category. Furthermore, for the category **Algcom** of unital commutative associative $\mathbb{C}$-algebras the notion of bimodule just reduces to the notion of module. Again, like for the universal differential calculus, for the notion of bimodule it is immaterial for a commutative algebra $\mathcal{C}$ whether one considers $\mathcal{C}$ as an object of **Algcom** or of $\mathbf{Alg}_Z$ whereas the notion of bimodule over $\mathcal{C}$ in **Alg** is much wider.

This problem of the choice of the generalization of the notion of module over a commutative algebra $\mathcal{C}$ when $\mathcal{C}$ is replaced by a noncommutative algebra $\mathcal{A}$ is closely connected with the problem of the noncommutative generalization of the classical notion of reality. If $\mathcal{C}$ is the algebra of complex continuous functions on a topological space or the algebra of complex smooth functions on a smooth manifold, then it is a $*$-algebra and the (real) algebra of real functions is the real subspace $\mathcal{C}^h$ of hermitian (i.e. $*$-invariant) elements of $\mathcal{C}$. More generally if $\mathcal{C}$ is a commutative associative complex $*$-algebra the set $\mathcal{C}^h$ of hermitian elements of $\mathcal{C}$ is a commutative associative real algebra. Conversely if $\mathcal{C}_\mathbb{R}$ is a commutative associative real algebra, then its complexification $\mathcal{C}$ is canonically a commutative associative complex $*$-algebra and one has $\mathcal{C}^h = \mathcal{C}_\mathbb{R}$. In fact the correspondence $\mathcal{C} \mapsto \mathcal{C}^h$ defines an equivalence between the category of commutative associative complex $*$-algebras and the category of commutative associative real algebras, (the morphisms of the first category being the $*$-homomorphisms). This is in contrast with what happens for noncommutative algebras. Recall that an associative complex $*$-algebra is an associative complex algebra $\mathcal{A}$ equipped with an antilinear involution $x \mapsto x^*$ such that $(xy)^* = y^*x^*$, $(\forall x, y \in \mathcal{A})$. From the fact that the involution reverses the order of the product it follows that the real subspace $\mathcal{A}^h$ of hermitian elements of a complex associative $*$-algebra is generally not stable by the product but only by the symmetrized Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. Thus $\mathcal{A}^h$ is not (generally) an associative algebra but is a real *Jordan algebra*. Therefore, one has two natural choices for the generalization of an algebra of real functions : either the real Jordan algebra $\mathcal{A}^h$ of hermitian elements of a complex associative $*$-algebra $\mathcal{A}$ which plays the role of the algebra of complex functions or a real associative algebra. In these lectures we take the first choice which is dictated by quantum theory (and spectral theory). This choice has important consequences on the possible generalizations of real vector bundles and, more generally, of modules over commutative real algebras.

Let $\mathcal{C}$ be a commutative associative $*$-algebra and let $\mathcal{M}^h$ be a $\mathcal{C}^h$-module. The complexified $\mathcal{M} = \mathcal{M}^h \oplus i\mathcal{M}^h = \mathcal{M}^h \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathcal{M}^h$ is canonically a $\mathcal{C}$-module. Furthermore there is a canonical antilinear involution $(\Phi + i\Psi) \mapsto (\Phi + i\Psi)^* = \Phi - i\Psi$ $(\Phi, \Psi \in \mathcal{M}^h)$ for which $\mathcal{M}^h$ is the set of $*$-invariant elements. This involution is compatible with the one of $\mathcal{C}$ in the sense that one has $(x\Phi)^* = x^*\Phi^*$ for $x \in \mathcal{C}$ and $\Phi \in \mathcal{M}$; $\mathcal{M}$ will be said to be a *$*$-module over the commutative $*$-algebra $\mathcal{C}$*. In view of the above discussion what generalizes $\mathcal{C}$ is a noncommutative $*$-algebra $\mathcal{A}$ and we have to generalize the $*$-module $\mathcal{M}$ and its "real part" $\mathcal{M}^h$. However it is clear that there is no noncommutative generalization of a $*$-module over $\mathcal{A}$ as right or left module. The reason is that, since the involution of $\mathcal{A}$ reverses the order in products, it intertwines between actions of $\mathcal{A}$ and actions of the opposite algebra $\mathcal{A}^0$, i.e. between a structure of right (resp. left) module and a structure of left (resp. right) module. Fortunately, as already mentioned, a $\mathcal{C}$-module is canonically a bimodule (of a certain kind) and the above compatibility condition can be equivalently written $(x\Phi)^* = \Phi^* x^*$. This latter condition immediately generalizes for $\mathcal{A}$, namely a *$*$-bimodule over the $*$-algebra $\mathcal{A}$* is a bimodule $\mathcal{M}$ over $\mathcal{A}$ equipped with an antilinear involution $\Phi \mapsto \Phi^*$ such that $(x\Phi y)^* = y^*\Phi^* x^*$, $(\forall x, y \in \mathcal{A}, \forall \Phi \in \mathcal{M})$. The real subspace $\mathcal{M}^h = \{\Phi \in \mathcal{M} | \Phi^* = \Phi\}$ of the $*$-invariant element of $\mathcal{M}$ can play the role of the sections of a real vector bundle (for some specific kind of $*$-bimodule $\mathcal{M}$). Since a commutative algebra is its center, one can also generalize $*$-modules over $\mathcal{C}$ by $*$-modules over the center $Z(\mathcal{A})$ of $\mathcal{A}$ and modules over $\mathcal{C}^h$ by modules over $Z(\mathcal{A})^h$. In a sense these two types of generalizations of the reality (for modules) are dual ([34], [27]) as we shall see later. The main message of this little discussion is that notions of reality force us to consider bimodules and not only right or left modules as generalization of vector bundles, [34], [27], [18], [61].

Remark 2. One can be more radical. Instead of generalizing an associative commutative $\mathbb{R}$-algebra $\mathcal{C}_{\mathbb{R}}$ by the Jordan algebra $\mathcal{A}^h$ of hermitian elements of an associative complex $*$-algebra $\mathcal{A}$, one can more generally choose to generalize $\mathcal{C}_{\mathbb{R}}$ by a real Jordan algebra $\mathcal{J}_{\mathbb{R}}$ (not a priori a special one). The corresponding generalization of a $\mathcal{C}_{\mathbb{R}}$-module could be then a *Jordan bimodule over* $\mathcal{J}_{\mathbb{R}}$ [44] instead of the real subspace of a $*$-bimodule over $\mathcal{A}$, (what is a Jordan bimodule will be explained later). We however refrain to do that because it is relatively complicated technically for a slight generalization practically.

In these lectures we shall be interested in noncommutative versions of differential geometry where the algebra of smooth complex functions on a smooth manifold is replaced by a noncommutative associative unital complex $*$-algebra $\mathcal{A}$. Since there are commutative $*$-algebras of this sort which are not (and cannot be) algebras of smooth functions on smooth manifolds, one cannot expect that an arbitrary $*$-algebra as above is a good noncommutative generalization of an algebra of smooth functions. What is involved here is the generalization of the notion of smootheness. It is possible to characterize among the unital commutative associative complex $*$-algebras the ones which are isomorphic to algebras of smooth functions, however there are several inequivalent noncommutative generalizations of this characterization and no one is universally accepted. Thus although it is an interesting subject on which work is currently in progress [30], we shall not discuss it here. This means that if the algebra $\mathcal{A}$ is not "good enough", some of our constructions can become a little trivial.

The plan of these notes is the following. After this introduction, in Section 2 we recall the definition of graded differential algebras and of various concepts related to them; we state in particular the result of D. Sullivan concerning the structure of connected finitely generated free graded comutative differential algebras and we review H. Cartan's notion of operation of a Lie algebra in a graded differential algebra. In Section 3, we explain the equivalence between the category of finite dimensional Lie algebras and the category of the free connected graded commutative differential algebras which are finitely generated in degree 1 (i.e. exterior algebras of finite dimensional spaces equipped with differentials); we describe several examples related to Lie algebras such as the Chevalley-Eilenberg complexes, the Weil algebra (and we state the result defining the Weil homomorphism) and we introduce the graded differential algebras of the derivation-based calculus. In Section 4, we start in an analogous way as in Section 3, that is we explain the equivalence between the category of finite dimensional associative algebras and the category of free connected graded differential algebras which are generated in degree 1 (i.e. tensor algebras of finite dimensional spaces equipped with differentials); we describe examples related to associative algebras such as Hochschild complexes. In Section 5, we introduce categories of algebras and we define the associated notions of bimodules which we follow on several relevant examples. In Section 6 we recall the notion of first order differential calculus over an algebra and we introduce our generalization of the module of Kähler differentials and discuss its functorial properties; we also recall in this section the definition and properties of the universal first order calculus. In Section 7 we introduce the higher order differential calculi and discuss in particular the universal one as well as our generalization of Kähler exterior forms; we give in particular their universal properties and study their functorial properties. In Section 8 we introduce another new differential calculus, the diagonal calculus, which, although not functorial, is characterized by a universal property and we compare it with the other differential calculi attached to an algebra. In Section 9 we define and study noncommutative Poisson and symplectic structures and show their relation with quantum theory. In Section 10 we describe the theory of connections on modules and on bimodules; in the latter case we recall in particular the generalization of the proposal of J. Mourad (concerning linear connections) and describe its basic properties and its relations with the theory of first-order operators in bimodules. In Section 11 we discuss in some examples the relations between connections in the noncommutative setting and classical Yang-Mills-Higgs models. Section 12 which serves as conclusion contains some further remarks concerning in particular the differential calculus on quantum groups.

Apart from in §5, an *algebra* without other specification shall always mean a unital associative complex algebra and by a $*$-*algebra* without other specification we shall mean a unital associative complex $*$-algebra. Given two algebras $\mathcal{A}$ and $\mathcal{B}$ in this sense, a $(\mathcal{A},\mathcal{B})$-*bimodule* is a vector space $\mathcal{M}$ equipped with linear maps $\mathcal{A}\otimes\mathcal{M}\rightarrow\mathcal{M}$ and $\mathcal{M}\otimes\mathcal{B}\rightarrow\mathcal{M}$ denoted by $a\otimes m\mapsto am$ and $m\otimes b\mapsto mb$ respectively such that $(aa')m=a(a'm)$, $m(bb')=(mb)b'$, $(am)b=a(mb)$, $\mathbb{1}m=m$ and $m\mathbb{1}=m$, $\forall a,a'\in\mathcal{A}$, $\forall b,b'\in\mathcal{B}$, $\forall m\in\mathcal{M}$ where $\mathbb{1}$ denotes the unit of $\mathcal{A}$ as well as the one of $\mathcal{B}$. In Section 5 we shall define for a more general algebra $\mathcal{A}$ a notion of $\mathcal{A}$-bimodule which is relative to a category of algebras; the notion of $(\mathcal{A},\mathcal{A})$-bimodule as above is the notion of $\mathcal{A}$-bimodule for the category **Alg** of unital associative complex algebras. A *complex* $\mathfrak{C}$ will be a $\mathbb{Z}$-graded vector space (over $\mathbb{C}$) equipped with a homogeneous endomorphism d of degree ± 1 and such that $d^2 = 0$. If d is of degree -1, $\mathfrak{C}$ is said to be a *chain complex*, its elements are called *chains* and d is called the *boundary*; if d is of degree $+1$, $\mathfrak{C}$ is said to be a *cochain complex*, its elements are called *cochains* and d is called the *coboundary*. The graded vector space $H(\mathfrak{C}) = \mathrm{Ker}(d)/\mathrm{Im}(d)$ is called the *homology* of $\mathfrak{C}$ if $\mathfrak{C}$ is a chain complex and the *cohomology* of $\mathfrak{C}$ if $\mathfrak{C}$ is a cochain complex.

2 Graded differential algebras

A *graded algebra* will be here a unital associative complex algebra $\mathfrak{A}$ which is a $\mathbb{Z}$-graded vector space $\mathfrak{A} = \oplus_{n\in\mathbb{Z}} \mathfrak{A}^n$ such that $\mathfrak{A}^m.\mathfrak{A}^n \subset \mathfrak{A}^{m+n}$. A *homomorphism of graded algebras* will be a homomorphism of the corresponding graded vector spaces (i.e. a homogeneous linear mapping of degree 0) which is also a homomorphism of unital algebras. A graded algebra $\mathfrak{A}$ is said to be *graded commutative* if one has $xy = (-1)^{mn} yx$, $\forall x \in \mathfrak{A}^m$ and $\forall y \in \mathfrak{A}^n$. Most graded algebras involved in these lectures will be $\mathbb{N}$-graded, i.e. $\mathfrak{A}^n = 0$ for $n \leq -1$. A graded algebra $\mathfrak{A}$ is said to be 0-*connected* or *connected* if it is $\mathbb{N}$-graded with $\mathfrak{A}^0 = \mathbb{C}\mathbb{1}$, where $\mathbb{1}$ denotes the unit of $\mathfrak{A}$. An example of connected graded algebra is the tensor algebra over $\mathbb{C}$ of a complex vector space E which will be denoted by $T(E)$. In this example, the graduation is the tensorial degree which means that the degree 1 is given to the elements of E. The exterior algebra $\bigwedge(E)$ of E is an example of connected graded commutative algebra, (the graduation being again induced by the tensorial degree).

More generally let $C = \oplus_n C^n$ be a $\mathbb{Z}$-graded complex vector space and let $T(C)$ be the tensor algebra of C. One has $C \subset T(C)$ and we equip the algebra $T(C)$ with the unique grading of algebra which induces on C the original grading. Since this is not the usual grading of the tensor algebra we shall denote the corresponding graded algebra by $\mathfrak{T}(C)$. The graded algebra $\mathfrak{T}(C)$ is characterized (uniquely up to an isomorphism) by the following universal property: *Any homomorphism of graded vector spaces* $\alpha : C \to \mathfrak{A}$ *of the graded vector space* C *into a graded algebra* $\mathfrak{A}$ *extends* *uniquely as a homomorphism of graded algebras* $\mathfrak{T}(\alpha) : \mathfrak{T}(C) \to \mathfrak{A}$. Let $\mathcal{I}$ be the graded two-sided ideal of $\mathfrak{T}(C)$ generated by the graded commutators $\psi_r \otimes \varphi_s - (-1)^{rs}\varphi_s \otimes \psi_r$ with $\psi_n, \varphi_n \in C^n$ and let $\mathfrak{F}(C)$ denote the quotient graded algebra $\mathfrak{T}(C)/\mathcal{I}$. Then $\mathfrak{F}(C)$ is a graded commutative algebra which contains again C as graded subspace. The graded commutative algebra $\mathfrak{F}(C)$ is characterized (uniquely up to an isomorphism) by the following universal property, (which is the graded commutative counterpart of the above one): *Any homomorphism of graded vector spaces* $\alpha : C \to \mathfrak{A}$ *of the graded vector space* C *into a graded commutative algebra* $\mathfrak{A}$ *extends uniquely as a homomorphism of graded commutative algebras* $\mathfrak{F}(\alpha) : \mathfrak{F}(C) \to \mathfrak{A}$. Notice that $\mathfrak{T}(C)$ (resp. $\mathfrak{F}(C)$) is connected if and only if $C^n = 0$ for $n \leq 0$ and that $\mathfrak{T}(C) = T(C)$ (resp. $\mathfrak{F}(C) = \bigwedge(C)$) as graded algebras if and only if $C^n = 0$ for $n \neq 1$. Notice also that, as algebra $\mathfrak{F}(C) = \bigwedge(\oplus_r C^{2r+1}) \otimes S(\oplus_s C^{2s})$ where $S(E)$ denotes the symmetric algebra of the vector space E. The graded algebra $\mathfrak{T}(C)$ will be refered to as *the free graded algebra generated by the graded vector space* C whereas the graded algebra $\mathfrak{F}(C)$ will be refered to as *the free graded commutative algebra generated by the graded vector space* C. Finally, a *finitely generated free graded algebra* will be a graded algebra of the form $\mathfrak{T}(C)$ for some finite dimensional graded vector space C whereas an algebra of the form $\mathfrak{F}(C)$ for some finite dimensional graded vector space C will be called a *finitely generated free graded commutative algebra*.

If $\mathfrak{A}$ and $\mathfrak{A}'$ are two graded algebras, their *tensor product* $\mathfrak{A} \otimes \mathfrak{A}'$ will be here their *skew tensor product* which means that the product in $\mathfrak{A} \otimes \mathfrak{A}'$ is defined by $(x \otimes x')(y \otimes y') = (-1)^{m'n} xy \otimes x'y'$ for $x' \in \mathfrak{A}'^{m'}$, $y \in \mathfrak{A}^n$, $x \in \mathfrak{A}$ and $y' \in \mathfrak{A}'$. With this convention, the tensor product of two (or more) graded commutative algebras is again a graded commutative algebra. If C and C' are $\mathbb{Z}$-graded complex vector spaces one has $\mathfrak{F}(C \oplus C') = \mathfrak{F}(C) \otimes \mathfrak{F}(C')$.

By a *graded ∗-algebra* we here mean a graded algebra $\mathfrak{A} = \oplus_n \mathfrak{A}^n$ equipped with an involution $x \mapsto x^*$ satisfying
(i) $x \in \mathfrak{A}^n \Rightarrow x^* \in \mathfrak{A}^n$ (homogeneity of degree $= 0$)
(ii) $(\lambda x + y)^* = \bar{\lambda} x^* + y^*$, $\forall x, y \in \mathfrak{A}$ and $\forall \lambda \in \mathbb{C}$ (antilinearity)
(iii) $(xy)^* = (-1)^{mn} y^* x^*$, $\forall x \in \mathfrak{A}^m$ and $\forall y \in \mathfrak{A}^n$.

Notice that Property (iii) implies that if $\mathfrak{A}$ is graded commutative then one has $(xy)^* = x^* y^*$, $(\forall x, y \in \mathfrak{A})$.

For a graded algebra $\mathfrak{A}$, there is, beside the notion of derivation, the notion of antiderivation: A linear mapping $\theta : \mathfrak{A} \to \mathfrak{A}$ is called an *antiderivation of* $\mathfrak{A}$ if it satisfies $\theta(xy) = \theta(x)y + (-1)^m x\theta(y)$ for any $x \in \mathfrak{A}^m$ and $y \in \mathfrak{A}$. However the best generalizations of the notions of center and of derivations are the following graded generalizations. The *graded center* $Z_{\text{gr}}(\mathfrak{A})$ of $\mathfrak{A}$ is the graded subspace of $\mathfrak{A}$ generated by the homogeneous elements $x \in \mathfrak{A}^m$ ($m \in \mathbb{Z}$) satisfying $xy = (-1)^{mn}yx$, $\forall y \in \mathfrak{A}^n$ and $\forall n \in \mathbb{Z}$, (i.e. $Z_{\text{gr}}(\mathfrak{A})$ is the graded commutant of $\mathfrak{A}$ in $\mathfrak{A}$). The graded center is a graded subalgebra of $\mathfrak{A}$ which is graded commutative. A *graded derivation of degree k of* $\mathfrak{A}$, ($k \in \mathbb{Z}$), is a homogeneous linear mapping $X : \mathfrak{A} \to \mathfrak{A}$ which is of degree k and satisfies $X(xy) = X(x)y + (-1)^{km}xX(y)$ for $x \in \mathfrak{A}^m$ and $y \in \mathfrak{A}$. Thus a homogeneous graded derivation of even (resp. odd) degree is a derivation (resp. antiderivation). The vector space of all these graded derivations of degree k will be denoted by $\text{Der}^k_{\text{gr}}(\mathfrak{A})$ and the graded vector space $\text{Der}_{\text{gr}}(\mathfrak{A}) = \oplus_{k\in\mathbb{Z}} \text{Der}^k_{\text{gr}}(\mathfrak{A})$ of all graded derivations is a graded Lie algebra for the *graded commutator* $[X,Y]_{\text{gr}} = XY - (-1)^{k\ell}YX$, $X \in \text{Der}^k_{\text{gr}}(\mathfrak{A})$, $Y \in \text{Der}^\ell_{\text{gr}}(\mathfrak{A})$. If $x \in \mathfrak{A}^m$, one defines a graded derivation of degree m of $\mathfrak{A}$, denoted by $\text{ad}_{\text{gr}}(x)$, by setting $\text{ad}_{\text{gr}}(x)y = xy - (-1)^{mn}yx = [x,y]_{\text{gr}}$ for $y \in \mathfrak{A}^n$. The graded subspace of $\text{Der}_{\text{gr}}(\mathfrak{A})$ generated by these $\text{ad}(x)$, (when x runs over $\mathfrak{A}^m$ and m runs over $\mathbb{Z}$), is denoted by $\text{Int}_{\text{gr}}(\mathfrak{A})$ and its elements are called *inner graded derivations of* $\mathfrak{A}$. It is an ideal of the graded Lie algebra $\text{Der}_{\text{gr}}(\mathfrak{A})$ and the quotient graded Lie algebra will be denoted by $\text{Out}_{\text{gr}}(\mathfrak{A})$. Notice that the graded center $Z_{\text{gr}}(\mathfrak{A})$ is stable by the graded derivations of $\mathfrak{A}$ and that this leads to a canonical homomorphism $\text{Out}_{\text{gr}}(\mathfrak{A}) \to \text{Der}_{\text{gr}}(Z_{\text{gr}}(\mathfrak{A}))$ since the inner graded derivations vanish on $Z_{\text{gr}}(\mathfrak{A})$. If $\mathfrak{A}$ is a graded $*$-algebra, then $Z_{\text{gr}}(\mathfrak{A})$ is stable by the involution, (i.e. it is a graded $*$-subalgebra of $\mathfrak{A}$), one defines in the obvious manner an involution on $\text{Der}_{\text{gr}}(\mathfrak{A})$ and one has then $(\text{ad}_{\text{gr}}(x))^* = -\text{ad}_{\text{gr}}(x^*)$ for $x \in \mathfrak{A}$. One recovers the usual ungraded notions for an ordinary (ungraded) algebra $\mathcal{A}$ by considering $\mathcal{A}$ as a graded algebra which has non zero elements only in degree 0.

Finally a *graded differential algebra* is a graded algebra $\mathfrak{A} = \oplus_n \mathfrak{A}^n$ equipped with an antiderivation d of degree 1 satisfying $d^2 = 0$, (i.e. d is linear, $d(xy) = d(x)y + (-1)^m xd(y)$ $\forall x \in \mathfrak{A}^m$ and $\forall y \in \mathfrak{A}$, $d(\mathfrak{A}^n) \subset \mathfrak{A}^{n+1}$ and $d^2 = 0$); d is the *differential* of the graded differential algebra. Notice that then the graded center $Z_{\text{gr}}(\mathfrak{A})$ of $\mathfrak{A}$ is stable by the differential d and that it is therefore a graded differential subalgebra of $\mathfrak{A}$ which is graded commutative. A *graded differential $*$-algebra* will be a graded differential algebra

$\mathfrak{A}$ which is also a graded $*$-algebra such that $d(x^*) = (d(x))^*$, $\forall x \in \mathfrak{A}$.

Given a graded differential algebra $\mathfrak{A}$ its *cohomology* $H(\mathfrak{A})$ is a graded algebra. Indeed the antiderivation property of d implies that $\mathrm{Ker}(d)$ is a subalgebra of $\mathfrak{A}$ and that $\mathrm{Im}(d)$ is a two-sided ideal of $\mathrm{Ker}(d)$ and the homogeneity of d implies that they are graded. If $\mathfrak{A}$ is graded commutative then $H(\mathfrak{A})$ is also graded commutative and if $\mathfrak{A}$ is a graded differential $*$-algebra then $H(\mathfrak{A})$ is a graded $*$-algebra.

If $\mathfrak{A}'$ and $\mathfrak{A}''$ are two graded differential algebras their tensor product $\mathfrak{A}' \otimes \mathfrak{A}''$ will be the tensor product of the graded algebras equipped with the differential d defined by

$$d(x' \otimes x'') = d(x') \otimes x'' + (-1)^{n'} x' \otimes dx'', \ \forall x' \in \mathfrak{A}'^{n'} \text{ and } \forall x'' \in \mathfrak{A}''.$$

For the cohomology, one has the Künneth formula [60]

$$H(\mathfrak{A}' \otimes \mathfrak{A}'') = H(\mathfrak{A}') \otimes H(\mathfrak{A}'')$$

for the corresponding graded algebra.

<u>Remark 3</u>. More generally if $\mathfrak{A}'$ and $\mathfrak{A}''$ are (co)chain complexes of vector spaces with (co)boundaries denoted by d, then one defines a (co)boundary d on the graded vector space $\mathfrak{A}' \otimes \mathfrak{A}''$ by the same formula as above and one has the Künneth formula $H(\mathfrak{A}' \otimes \mathfrak{A}'') = H(\mathfrak{A}') \otimes H(\mathfrak{A}'')$ for the corresponding graded vector spaces of (co)homologies [60].

Let $\mathfrak{A}$ be a graded differential algebra which is connected, i.e. such that $\mathfrak{A} = \mathbb{C}\mathbb{1} \oplus \mathfrak{A}^+$ where $\mathfrak{A}^+$ is the ideal of elements of strictly positive degrees. Then $\mathfrak{A}$ will be said to be *minimal* or to be a *minimal graded differential algebra* if it satisfies the *condition of minimality* [59]:

$$d\mathfrak{A} \subset \mathfrak{A}^+ . \mathfrak{A}^+ \ \ \text{(minimal condition)}.$$

A *free graded differential algebra* is a graded differential algebra which is of the form $\mathfrak{T}(C)$ for some graded vector space C as a graded algebra whereas a *free graded commutative differential algebra* is a graded differential algebra which is of the form $\mathfrak{F}(C)$ as a graded algebra.

For instance if $\mathfrak{C}$ is a cochain complex, its coboundary extends uniquely as a differential of $\mathfrak{T}(\mathfrak{C})$ and also as a differential of $\mathfrak{F}(\mathfrak{C})$. The corresponding graded differential algebra which will be again denoted by $\mathfrak{T}(\mathfrak{C})$ and $\mathfrak{F}(\mathfrak{C})$ when no confusion arises will be refered to respectively as *the free graded differential algebra generated by the complex* $\mathfrak{C}$ and *the free graded commutative differential algebra generated by the complex* $\mathfrak{C}$. One can show (by using the Künneth formula) that one has in cohomology $H(\mathfrak{T}(\mathfrak{C})) = \mathfrak{T}(H(\mathfrak{C}))$ and $H(\mathfrak{F}(\mathfrak{C})) = \mathfrak{F}(H(\mathfrak{C}))$. We let the reader guess the universal properties which characterize $\mathfrak{T}(\mathfrak{C})$ and $\mathfrak{F}(\mathfrak{C})$ and to deduce from these the functorial character of the construction. A free graded (resp. graded commutative) differential algebra will be said to be *contractible* if it is of the form $\mathfrak{T}(\mathfrak{C})$ (resp. $\mathfrak{F}(\mathfrak{C})$) for a cochain complex (of vector spaces) $\mathfrak{C}$ such that $H(\mathfrak{C}) = 0$ (trivial cohomology). In Theorem 1 below we shall be interested in free graded commutative contractible differential algebras which are connected and finitely generated; such a differential algebra is a finite tensor product $\otimes_\alpha \mathfrak{F}(\mathbb{C}e_\alpha \oplus \mathbb{C}de_\alpha)$ with the e_α of degrees ≥ 1 (connected property).

Concerning the structure of connected finitely generated free graded commutative differential algebras, one has the following result [59].

THEOREM 1 *Every connected finitely generated free graded commutative differential algebra is the tensor product of a unique minimal one and a unique contractible one.*

This result has been for instance an important constructive ingredient in the computation of the local B.R.S. cohomology of gauge theory [37], [24].

There is probably a similar statement for the non graded commutative case (i.e. for connected finitely generated free graded differential algebras) in which the tensor product is replaced by the free product of unital algebras.

An *operation of a Lie algebra* $\mathfrak{g}$ *in a graded differential algebra* $\mathfrak{A}$ [9], [41] is a linear mapping $X \mapsto i_X$ of $\mathfrak{g}$ into the space of antiderivations of degree -1 of $\mathfrak{A}$ such that one has $(\forall X, Y \in \mathfrak{g})$
(i) $i_X i_Y + i_Y i_X = 0$ i.e. $[i_X, i_Y]_{\text{gr}} = 0$
$(ii) L_X i_Y - i_Y L_X = i_{[X,Y]}$ i.e. $[L_X, i_Y]_{\text{gr}} = i_{[X,Y]}$
where L_X denotes the derivation of degree 0 of $\mathfrak{A}$ defined by

$$L_X = i_X d + d i_X = [d, i_X]_{\text{gr}}$$

for $X \in \mathfrak{A}$. Property (ii) above implies
(iii) $L_X L_Y - L_Y L_X = L_{[X,Y]}$, $(\forall X, Y \in \mathfrak{g})$
which means that $X \mapsto L_X$ is a Lie algebra-homomorphism of $\mathfrak{g}$ into the Lie algebra of derivations of degree 0 of $\mathfrak{A}$. The definition implies that L_X commutes with the differential d for any $X \in \mathfrak{g}$.

Given an operation of $\mathfrak{g}$ in $\mathfrak{A}$ as above, an element x of $\mathfrak{A}$ is said to be *horizontal* if $i_X(x) = 0$ $(\forall X \in \mathfrak{g})$, *invariant* if $L_X(x) = 0$ $(\forall X \in \mathfrak{g})$ and *basic* if it is both horizontal and invariant i.e. if $i_X(x) = 0 = L_X(x)$ $(\forall X \in \mathfrak{g})$. The set $\mathfrak{A}_H$ of horizontal elements is a graded subalgebra of $\mathfrak{A}$ stable by the representation $X \mapsto L_X$ of $\mathfrak{g}$. The set $\mathfrak{A}_I$ of invariant elements is a graded differential subalgebra of $\mathfrak{A}$ and the set $\mathfrak{A}_B$ of basic elements is a graded differential subalgebra of $\mathfrak{A}_I$ (and therefore also of $\mathfrak{A}$). The cohomologies of $\mathfrak{A}_I$ and $\mathfrak{A}_B$ are called respectively *invariant cohomology* and *basic cohomology of* $\mathfrak{A}$ and are denoted by $H_I(\mathfrak{A})$ and $H_B(\mathfrak{A})$.

A prototype of graded differential algebra is the graded differential algebra $\Omega(M)$ of differential forms on a smooth manifold M. We shall discuss various generalizations of it in these lectures. Let P be a smooth principal bundle with structure group G and with basis M. One defines an operation $X \mapsto i_X$ of the Lie algebra $\mathfrak{g}$ of G in the graded differential algebra $\Omega(P)$ of differential forms on P by letting i_X be the contraction by the vertical vector field corresponding to $X \in \mathfrak{g}$. Then the elements of $\Omega(P)_H$ are the horizontal forms in the usual sense, $\Omega(P)_I$ is the differential algebra of the differential forms which are invariant by the action of G on P whereas the graded differential algebra $\Omega(P)_B$ is canonically isomorphic to the graded differential algebra $\Omega(M)$ of differential forms on the basis. The terminology adopted above for operations comes from this fundamental example. In [24], [25] very different kinds of operations of Lie algebras in graded differential algebras have been considered.

3 Examples related to Lie algebras

Let $\mathfrak{g}$ be a finite dimensional complex vector space with dual space $\mathfrak{g}^*$. Let $X, Y \mapsto [X, Y]$ be an antisymmetric bilinear product on $\mathfrak{g}$, i.e. a linear mapping $[\cdot,\cdot] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ of the second exterior power of $\mathfrak{g}$ into $\mathfrak{g}$. The dual of the bracket $[\cdot,\cdot]$ is a linear mapping of $\mathfrak{g}^*$ into $\bigwedge^2 \mathfrak{g}^* (= (\bigwedge^2 \mathfrak{g})^*)$ and such a linear mapping of $\mathfrak{g}^*$ into $\bigwedge^2 \mathfrak{g}^*$ has a unique extension as a graded derivation δ of degree 1 of the exterior algebra $\bigwedge \mathfrak{g}^*$. Conversely, given a graded derivation δ of degree 1 of $\bigwedge \mathfrak{g}^*$, the dual of $\delta : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$ is a bilinear antisymmetric product on $\mathfrak{g}(= (\mathfrak{g}^*)^*)$ and δ is the unique graded derivation of degree 1 of $\bigwedge \mathfrak{g}^*$ which extends the dual of this antisymmetric product. Thus to give an antisymmetric product $[\cdot,\cdot]$ on $\mathfrak{g}$ is the same thing as to give a graded derivation δ of degree 1 of the exterior algebra $\bigwedge \mathfrak{g}^*$. For notational reasons one usually introduces the antiderivation $d = -\delta$, i.e. the unique antiderivation of $\bigwedge \mathfrak{g}^*$ such that

$$d(\omega)(X, Y) = -\omega([X, Y])$$

for $\omega \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$. We shall call d the antiderivation of $\bigwedge \mathfrak{g}^*$ corresponding to the bilinear antisymmetric product on $\mathfrak{g}$.

LEMMA 1 *The bilinear antisymmetric product $[\cdot,\cdot]$ on $\mathfrak{g}$ satisfies the Jacobi identity if and only if the corresponding antiderivation d of $\bigwedge \mathfrak{g}^*$ satisfies $d^2 = 0$.*

i.e. $\mathfrak{g}$ is a Lie algebra if and only if $\bigwedge \mathfrak{g}^*$ is a graded differential algebra (for the d corresponding to the bracket of $\mathfrak{g}$).

Proof. One has $d^2 = \frac{1}{2}[d, d]_{gr}$ so d^2 is a derivation (a graded derivation of degree 2) of $\bigwedge \mathfrak{g}^*$. Since, as unital algebra $\bigwedge \mathfrak{g}^*$ is generated by $\mathfrak{g}^*$, $d^2 = 0$ is equivalent to $d^2(\mathfrak{g}^*) = 0$. On the other hand by definition one has $d(\omega)(X, Y) = -\omega([X, Y])$, for $\omega \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$, and, by the antiderivation property one has for $X, Y, Z \in \mathfrak{g}$

$$3! d^2(\omega)(X, Y, Z) = (d(\omega)(X, [Y, Z]) - d(\omega)([X, Y], Z)) + \text{ cycl } (X, Y, Z)$$

i.e. $d^2(\omega)(X, Y, Z) = \omega([[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y])$. Therefore $d^2(\omega) = 0\ \forall \omega \in \mathfrak{g}^*$ is equivalent to the Jacobi identity for $[\cdot,\cdot]$. $\square$

Thus to give a finite dimensional Lie algebra is the same thing as to give the exterior algebra of a finite dimensional vector space equipped with a differential, that is to give a finitely generated free graded commutative differential algebra which is generated in degree 1. Such a graded differential algebra is automatically connected and minimal. This is why, as pointed out in [59], the connected finitely generated free graded commutative differential algebras which are minimal constitute a natural categorical closure of finite dimensional Lie algebras. In fact such generalizations of Lie algebras occur in some physical models [5].

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, then *the cohomology* $H(\mathfrak{g})$ *of* $\mathfrak{g}$ is the cohomology of $\bigwedge \mathfrak{g}^*$. More generally, $\bigwedge \mathfrak{g}^*$ is the basic building block to construct the cochain complexes for the cohomology of $\mathfrak{g}$ with values in representations.

Assume that $\mathfrak{g}$ is the Lie algebra of a Lie group G. Then by identifying $\mathfrak{g}$ with the Lie algebra of left invariant vector fields on G one defines a canonical homomorphism of $\Lambda\mathfrak{g}^*$ into the graded differential algebra $\Omega(G)$ of differential forms on G, (in fact onto the algebra of left invariant forms). This induces a homomorphism of $H(\mathfrak{g})$ into the cohomology $H(G)$ of differential forms on G which is an isomorphism when G is compact.

In the following, we consider the symmetric algebra $S\mathfrak{g}^*$, (i.e. the algebra of polynomials on $\mathfrak{g}$), to be evenly graded by giving the degree two to its generators, i.e. by writing $(S\mathfrak{g}^*)^{2n} = S^n\mathfrak{g}^*$ and $(S\mathfrak{g}^*)^{2n+1} = 0$. With this convention $S\mathfrak{g}^*$ is graded commutative and one defines the graded commutative algebra $W(\mathfrak{g})$ by $W(\mathfrak{g}) = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$. Let (E_α) be a basis of $\mathfrak{g}$ with dual basis (E^α) and let us define correspondingly generators A^α and F^α of $W(\mathfrak{g})$ by $A^\alpha = E^\alpha \otimes \mathbb{1}$ and $F^\alpha = \mathbb{1} \otimes E^\alpha$ so that $W(\mathfrak{g})$ is just the free connected graded commutative algebra (freely) generated by the A^α's in degree 1 and the F^α's in degree 2. It is convenient to introduce the elements A and F of $\mathfrak{g} \otimes W(\mathfrak{g})$ defined by $A = E_\alpha \otimes A^\alpha$ and $F = E_\alpha \otimes F^\alpha$. One then defines the elements dA^α and dF^α of $W(\mathfrak{g})$ by setting

$$\begin{aligned} dA &= E_\alpha \otimes dA^\alpha &= -\tfrac{1}{2}[A,A] + F \\ dF &= E_\alpha \otimes dF^\alpha &= -[A,F] \end{aligned}$$

where the bracket is the graded Lie bracket obtained by combining the bracket of $\mathfrak{g}$ with the graded commutative product of $W(\mathfrak{g})$. One then extends d as an antiderivation of $W(\mathfrak{g})$ of degree 1. One has $d^2 = 0$, and since an alternative free system of homogeneous generators of $W(\mathfrak{g})$ is provided by the A^α's and the dA^α's, $W(\mathfrak{g})$ is a connected free graded commutative differential algebra which is contractible and which is refered to as the *Weil algebra* of the Lie algebra $\mathfrak{g}$ [9], [41]. It is straightforward to verify that one defines an operation of $\mathfrak{g}$ in $W(\mathfrak{g})$ by setting $i_X(A^\alpha) = X^\alpha$ and $i_X(F^\alpha) = 0$

for $X = X^\alpha E_\alpha \in \mathfrak{g}$ and by extending i_X as an antiderivation of $W(\mathfrak{g})$. Since $W(\mathfrak{g})$ is contractible, its cohomology is trivial; the same is true for the invariant cohomology $H_I(W(\mathfrak{g}))$ of $W(\mathfrak{g})$, i.e. one has $H^0_I(W(\mathfrak{g})) = \mathbb{C}$ and $H^n_I(W(\mathfrak{g})) = 0$ for $n \geq 1$ [9] (see also in [24]). The graded subalgebra of horizontal elements of $W(\mathfrak{g})$ is obviously $\mathbb{1} \otimes S\mathfrak{g}^*$ so it follows that the graded subalgebra of basis elements of $W(\mathfrak{g})$ is just $\mathbb{1} \otimes \mathcal{I}_S(\mathfrak{g})$ where $\mathcal{I}_S(\mathfrak{g})$ denotes the algebra of invariant polynomials on $\mathfrak{g}$ (with the degree 2n given in $W(\mathfrak{g})$ to the homogeneous polynomials of degree n). On the other hand one has $d(\mathbb{1} \otimes \mathcal{I}_S(\mathfrak{g})) = 0$ and it is easily seen that the corresponding homomorphism $\mathbb{1}\otimes\mathcal{I}_S(\mathfrak{g}) \to H_B(W(\mathfrak{g}))$ onto the basic cohomology of $W(\mathfrak{g})$ is an isomorphism. Therefore, one has $H^{2n}_B(W(\mathfrak{g})) = \mathcal{I}^n_S(\mathfrak{g})$ and $H^{2n+1}_B(W(\mathfrak{g})) = 0$, where $\mathcal{I}^n_S(\mathfrak{g})$ denotes the space of invariant homogeneous polynomials of degree n on $\mathfrak{g}$. Let now P be a smooth principal bundle with basis M and with structure group G such that its Lie algebra is $\mathfrak{g}$. One has the canonical operation $X \to i_X$ of $\mathfrak{g}$ in $\Omega(P)$ defined at the end of last section. Given a connection $\omega = E_\alpha \otimes \omega^\alpha \in \mathfrak{g} \otimes \Omega^1(P)$ on P, there is a unique homomorphism of graded differential algebras $\Psi : W(\mathfrak{g}) \to \Omega(P)$ such that $\Psi(A^\alpha) = \omega^\alpha$. This homomorphism satisfies $\Psi(i_X(w)) = i_X(\Psi(w))$ for any $X \in \mathfrak{g}$ and $w \in W(\mathfrak{g})$. It follows that it induces a homomorphism in basic cohomomogy $\varphi : H_B(W(\mathfrak{g})) \to H_B(P)$, i.e. a homomorphism of $\mathcal{I}_S(\mathfrak{g})$ into the cohomology $H(M)$ of the basis M of P, such that $\varphi(\mathcal{I}^n_S(\mathfrak{g})) \subset H^{2n}(M)$, (it is an homormorphism of commutative algebras). One has $\mathrm{Im}(\varphi) \subset H^{ev}(M) = \oplus_p H^{2p}(M)$.

THEOREM 2 *The above homomorphism $\varphi : \mathcal{I}_S(\mathfrak{g}) \to H^{ev}(M)$ does not depend on the choice of the connection ω on P.*

That is φ only depends on P; it is called the *Weil homomorphism* of the principal bundle P. Before leaving this subject, it is worth noticing here that there is a very interesting noncommutative (or quantized) version of the Weil algebra of $\mathfrak{g}$ in the case where $\mathfrak{g}$ admits a nondegenerate invariant symmetric bilinear form, i.e. for $\mathfrak{g}$ reductive, where $S\mathfrak{g}^*$ is replaced by the enveloping algebra $U(\mathfrak{g})$ and where $\Lambda\mathfrak{g}^*$ is replaced by the Clifford algebra $C\ell(\mathfrak{g})$ of the bilinear form, which has been introduced and studied in [1].

In these lectures the Lie algebras involved will be generally not finite dimensional and some care must be taken with respect to duality and tensor products. For instance, if $\mathfrak{g}$ is not finite dimensional then the dual of the Lie bracket $[\cdot,\cdot] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ is a linear mapping $\delta : \mathfrak{g}^* \to (\bigwedge^2 \mathfrak{g})^*$ and one only has an inclusion $\bigwedge^2 \mathfrak{g}^* \subset (\bigwedge^2 \mathfrak{g})^*$. In the following we give the formulation adapted to this more general situation.

Let $\mathfrak{g}$ be a Lie algebra, let E be a representation space of $\mathfrak{g}$ (i.e. a $\mathfrak{g}$-module or, as will be explained in Section 5, a $\mathfrak{g}$-bimodule for the category **Lie** of Lie algebras) and let $X \mapsto \pi(X) \in \mathrm{End}(E)$ denote the action of $\mathfrak{g}$ on E. An *E-valued (Lie algebra) n-cochain of* $\mathfrak{g}$ is a linear mapping $X_1 \wedge \cdots \wedge X_n \mapsto \omega(X_1, \dots, X_n)$ of $\bigwedge^n \mathfrak{g}$ into E. The vector space of these n-cochains will be denoted by $C^n_\wedge(\mathfrak{g}, E)$. One defines a homogeneous endomorphism d of degree 1 of the $\mathbb{N}$-graded vector space $C_\wedge(\mathfrak{g}, E) = \oplus_n C^n_\wedge(\mathfrak{g}, E)$ of all E-valued cochains of $\mathfrak{g}$ by setting

$$\begin{aligned} d(\omega)(X_0, \dots, X_n) &= \sum_{k=0}^n (-1)^k \pi(X_k)\omega(X_0, \overset{k}{\check{\dots}}, X_n) \\ &+ \sum_{0\leq r<s\leq n} (-1)^{r+s}\omega([X_r, X_s], X_0 \overset{r}{\check{\dots}} \overset{s}{\check{\dots}} X_n) \end{aligned}$$

for $\omega \in C^n_\wedge(\mathfrak{g}, E)$ and $X_i \in \mathfrak{g}$. It follows from the Jacobi identity and from $\pi(X)\pi(Y) - \pi(Y)\pi(X) = \pi([X, Y])$ that $d^2 = 0$. Thus equipped with d, $C_\wedge(\mathfrak{g}, E)$ is a cochain complex and its cohomology, denoted by $H(\mathfrak{g}, E)$, is called the *E-valued cohomology of* $\mathfrak{g}$. When $E = \mathbb{C}$ and π is the trivial representation $\pi = 0$, it is the *cohomology* $H(\mathfrak{g})$ of $\mathfrak{g}$. One verifies that if $\mathfrak{g}$ is finite dimensional, it is the same as the cohomology of $\bigwedge \mathfrak{g}^*$; in fact in this case one has $C_\wedge(\mathfrak{g}, E) = E \otimes \bigwedge \mathfrak{g}^*$.

Assume now that E is an algebra $\mathcal{A}$ (unital, associative, complex) and that $\mathfrak{g}$ acts on $\mathcal{A}$ by derivations, i.e. that one has $\pi(X)(xy) = \pi(X)(x)y + x\pi(X)(y)$ for $X \in \mathfrak{g}$ and $x, y \in \mathcal{A}$. Then $C_\wedge(\mathfrak{g}, \mathcal{A})$ is canonically a graded differential algebra. Indeed the product is obtained by taking the product in $\mathcal{A}$ after evaluation and then antisymmetrizing whereas, the derivation property of the action of $\mathfrak{g}$ implies that d is an antiderivation. The trivial representation $\pi = 0$ in $\mathbb{C}$ is of this kind, this is why $H(\mathfrak{g})$ is a graded algebra.

More generally, the vector space $\mathrm{Der}(\mathcal{A})$ of all derivations of $\mathcal{A}$ into itself is a Lie algebra and therefore $C_\wedge(\mathrm{Der}(\mathcal{A}), \mathcal{A})$ is a graded-differential algebra. Furthermore, $\mathrm{Der}(\mathcal{A})$ is also a module over the center $Z(\mathcal{A})$ of $\mathcal{A}$ and one has $[X, zY] = z[X, Y] + X(z)Y$ from which it follows that the graded subalgebra $\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$ of $C_\wedge(\mathrm{Der}(\mathcal{A}), \mathcal{A})$ which consists of $Z(\mathcal{A})$-multilinear cochains is stable by the differential and is therefore a graded differential subalgebra of $C_\wedge(\mathrm{Der}(\mathcal{A}),\mathcal{A})$. Since $\underline{\Omega}^0_{\mathrm{Der}}(\mathcal{A})=\mathcal{A}$, a smaller differential subalgebra is the smallest differential subalgebra $\Omega_{\mathrm{Der}}(\mathcal{A})$ of $C_\wedge(\mathrm{Der}(\mathcal{A}),\mathcal{A})$ containing $\mathcal{A}$. When M is a "good" smooth manifold (finite dimensional, paracompact, etc.) and $\mathcal{A}=C^\infty(M)$ then $\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$ and $\Omega_{\mathrm{Der}}(\mathcal{A})$ both coincide with the graded differential algebra $\Omega(M)$ of differential forms on M. In general, the inclusion $\Omega_{\mathrm{Der}}(\mathcal{A})\subset\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$ is a strict one even when $\mathcal{A}$ is commutative (e.g. for the smooth functions on a ∞-dimensional manifold). The differential calculus over $\mathcal{A}$ (see in Sections 7, 8) using $\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$ (or $\Omega_{\mathrm{Der}}(\mathcal{A})$) as generalization of differential forms will be refered to as the *derivation-based calculus*, [25], [26], [27], [28], [29], [33], [34], [35], [36]. If $\mathcal{A}$ is a $*$-algebra, one defines an involution $X\mapsto X^*$ on $\mathrm{Der}(\mathcal{A})$ by setting $X^*(a)=(X(a^*))^*$ and an involution $\omega\mapsto\omega^*$ on $C_\wedge(\mathrm{Der}(\mathcal{A}),\mathcal{A})$ by setting $\omega^*(X_1,\dots,X_n)=(\omega(X_1^*,\dots,X_n^*))^*$. So equipped $C_\wedge(\mathrm{Der}(\mathcal{A}),\mathcal{A})$ is a graded differential $*$-algebra and $\Omega_{\mathrm{Der}}(\mathcal{A})$ as well as $\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$ are stable by the involution and are therefore also graded differential $*$-algebras.

One defines a linear mapping $X\mapsto i_X$ of $\mathfrak{g}$ into the homogeneous endomorphisms of degree -1 of $C_\wedge(\mathfrak{g},E)$ by setting $i_X(\omega)(X_1,\dots,X_{n-1})=\omega(X,X_1,\dots,X_{n-1})$ for $\omega\in C^n_\wedge(\mathfrak{g},E)$ and $X_i\in\mathfrak{g}$. Then $X\mapsto L_X=i_Xd+di_X$ is a representation of $\mathfrak{g}$ in $C_\wedge(\mathfrak{g},E)$ by homogeneous endomorphisms of degree 0 which extends the original representation π in $E=C^0_\wedge(\mathfrak{g},E)$, i.e. $L_X\restriction E=\pi(X)$ for $X\in\mathfrak{g}$. In the case where E is an algebra $\mathcal{A}$ and where $\mathfrak{g}$ acts by derivations on $\mathcal{A}$, we have seen that $C_\wedge(\mathfrak{g},\mathcal{A})$ is a graded differential algebra and it is easy to show that $X\mapsto i_X$ is an operation of the Lie algebra $\mathfrak{g}$ in the graded differential algebra $C_\wedge(\mathfrak{g},\mathcal{A})$; in fact properties (i) and (ii) of operations (see last section) hold already in $C_\wedge(\mathfrak{g},E)$ for any $\mathfrak{g}$-module E.

In particular one has the operation $X\mapsto i_X$ of the Lie algebra $\mathrm{Der}(\mathcal{A})$ in the graded differential algebra $C_\wedge(\mathrm{Der}(\mathcal{A}),\mathcal{A})$ defined as above. It is not hard to verify that the graded differential subalgebras $\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$ and $\Omega_{\mathrm{Der}}(\mathcal{A})$ are stable by the i_X ($X\in\mathrm{Der}(\mathcal{A})$). The corresponding operations will be refered to as *the canonical operations of* $\mathrm{Der}(\mathcal{A})$ *in* $\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$ *and in* $\Omega_{\mathrm{Der}}(\mathcal{A})$.

4 Examples related to associative algebras

Let $\mathcal{A}$ be a finite dimensional complex vector space with dual space $\mathcal{A}^*$ and let $x,y\mapsto xy$ be an arbitrary bilinear product on $\mathcal{A}$, i.e. a linear mapping

$\otimes^2\mathcal{A} \to \mathcal{A}$ where $\otimes^2\mathcal{A}$ denotes the second tensor power of $\mathcal{A}$. The dual of the product is a linear mapping of $\mathcal{A}^*$ into $\otimes^2\mathcal{A}^*$ and again such a linear mapping uniquely extends as a graded derivation δ of degree 1 of the tensor algebra $T(\mathcal{A}^*) = \bigoplus_{n\geq 0} \otimes^n \mathcal{A}^*$. Conversely, given such a graded derivation δ of degree 1 (i.e. an antiderivation of degree 1) of $T(\mathcal{A}^*)$, the dual mapping of the restriction $\delta : \mathcal{A}^* \to \otimes^2\mathcal{A}^*$ of δ to $\mathcal{A}^*$ is a bilinear product on $\mathcal{A}$ which is such that δ is obtained from it by the above construction. Thus, to give a bilinear product on $\mathcal{A}$ is the same thing as to give an antiderivation of degree 1 of $T(\mathcal{A}^*)$. Again, for notational reasons, it is usual to consider the antiderivation $d = -\delta$, i.e. the unique antiderivation of $T(\mathcal{A}^*)$ such that

$$d(\omega)(x,y) = -\omega(xy)$$

for $\omega \in \mathcal{A}^*$ and $x, y \in \mathcal{A}$. We shall call this d the antiderivation of $T(\mathcal{A}^*)$ corresponding to the bilinear product of $\mathcal{A}$.

LEMMA 2 *The bilinear product on $\mathcal{A}$ is associative if and only if the corresponding antiderivation of $T(\mathcal{A}^*)$ satisfies $d^2 = 0$.*

i.e. $\mathcal{A}$ is an associative algebra if and only if $T(\mathcal{A}^*)$ is a graded differential algebra (for the d corresponding to the product of $\mathcal{A}$).

Proof. By definition, one has for $\omega \in \mathcal{A}^*$ and $x, y, z \in \mathcal{A}$

$$d(d(\omega))(x, y.z) = d(\omega)(x, yz) - d(\omega)(xy, z) = \omega((xy)z - x(yz)).$$

Therefore the product of $\mathcal{A}$ is associative if and only if d^2 vanishes on $\mathcal{A}^*$ but this is equivalent to $d^2 = 0$ since d^2 is a derivation and since the (unital) graded algebra $T(\mathcal{A}^*)$ is generated by $\mathcal{A}^*$. $\square$

Therefore to give a finite dimensional associative algebra is the same thing as to give a finitely generated free graded differential algebra which is generated in degree 1. Again such a graded differential algebra is automatically connected and minimal. The situation is very similar to the one of last section except that here one has not graded commutativity. So one can consider in particular that the connected finitely generated free graded differential algebras which are minimal constitute a natural categorical closure of finite dimensional associative algebras, i.e. a natural generalization of the notion of associative algebra.

Let $\mathcal{A}$ be a finite dimensional associative algebra; we shall see that if $\mathcal{A}$ has a unit then the cohomology of the graded differential algebra $T(\mathcal{A}^*)$ is trivial. Nevertheless $T(\mathcal{A}^*)$ is the basic building block to construct the Hochschild cochain complexes. Namely if $\mathcal{M}$ is a $(\mathcal{A},\mathcal{A})$-bimodule then the graded vector space of $\mathcal{M}$-valued Hochschild cochains of $\mathcal{A}$ is the graded space $\mathcal{M}\otimes T(\mathcal{A}^*)$ and the Hochschild coboundary d_H is given by

$$\begin{aligned} d_H(\omega)(x_0,\dots,x_n) &= x_0\omega(x_1,\dots,x_n)+(I_\mathcal{M}\otimes d)(\omega)(x_0,\dots,x_n)\\ &+(-1)^{n+1}\omega(x_0,\dots,x_{n-1})x_n \end{aligned}$$

for $\omega\in\mathcal{M}\otimes(\otimes^n\mathcal{A}^*)$ and $x_i\in\mathcal{A}$.

In these lectures we shall have to deal with infinite dimensional algebras like algebras of smooth functions and their generalizations so again (as in last section) one has to take some care of duality and tensor products.

Let $\mathcal{A}$ be now an arbitrary associative algebra and let $C(\mathcal{A})$ denote the graded vector space of multilinear forms on $\mathcal{A}$, i.e. $C(\mathcal{A})=\oplus_n C^n(\mathcal{A})$ where $C^n(\mathcal{A})=(\otimes^n\mathcal{A})^*$ is the dual of the n-th tensor power of $\mathcal{A}$. One has $T(\mathcal{A}^*)\subset C(\mathcal{A})$ and the equality $T(\mathcal{A}^*)=C(\mathcal{A})$ holds if and only if $\mathcal{A}$ is finite dimensional. The product of $T(\mathcal{A}^*)$ (i.e. the tensor product) canonically extends to $C(\mathcal{A})$ which so equipped is a graded algebra. Furthermore minus the dual of the product of $\mathcal{A}$ is a linear mapping of $C^1(\mathcal{A})=\mathcal{A}^*$ into $C^2(\mathcal{A})=(\mathcal{A}\otimes\mathcal{A})^*$ which also canonically extends as an antiderivation d of $C(\mathcal{A})$ which is a differential as consequence of the associativity of the product of $\mathcal{A}$. It is given by:

$$d\omega(x_0,\dots,x_n)=\sum_{k=1}^n(-1)^k\omega(x_0,\dots,x_{i-1}x_i,\dots,x_n)$$

for $\omega\in C^n(\mathcal{A})$ and $x_i\in\mathcal{A}$. The graded differential algebra $C(\mathcal{A})$ is the generalization of the above $T(\mathcal{A}^*)$ for an infinite dimensional algebra $\mathcal{A}$. As announced before the cohomology of $C(\mathcal{A})$ is trivial whenever $\mathcal{A}$ has a unit.

LEMMA 3 *Let $\mathcal{A}$ be a unital associative algebra (over $\mathbb{C}$). Then the cohomology $H(C(\mathcal{A}))$ of $C(\mathcal{A})$ is trivial in the sense that one has:*

$$H^0(C(\mathcal{A}))=\mathbb{C} \text{ and } H^n(C(\mathcal{A}))=0 \text{ for } n\geq 1.$$

Proof. By definition $C(\mathcal{A})$ is connected so $H^0(C(\mathcal{A})) = \mathbb{C}$ is obvious. For $\omega \in C^n(\mathcal{A})$ with $n \geq 1$ let us define $h(\omega) \in C^{n-1}(\mathcal{A})$ by $h(\omega)(x_1, \dots, x_{n-1}) = \omega(\mathbb{1}, x_1, \dots, x_{n-1})$, $\forall x_i \in \mathcal{A}$. One has

$$d(h(\omega)) + h(d(\omega)) = \omega \text{ for any } \omega \in C^n(\mathcal{A}) \text{ with } n \geq 1$$

which implies $H^n(C(\mathcal{A})) = 0$ for $n \geq 1$. $\square$

If $\mathcal{M}$ is a $(\mathcal{A}, \mathcal{A})$-bimodule, then the graded vector space of *$\mathcal{M}$-valued Hochschild cochains of $\mathcal{A}$* is the graded vector space $C(\mathcal{A}, \mathcal{M})$ of multilinear mappings of $\mathcal{A}$ into $\mathcal{M}$, i.e. $C^n(\mathcal{A}, \mathcal{M})$ is the space of linear mappings of $\otimes^n \mathcal{A}$ into $\mathcal{M}$, equipped with the Hochschild coboundary d_H defined by

$$\begin{aligned} d_H(\omega)(x_0, \dots, x_n) = x_0 \omega(x_1, \dots, x_n) \quad & + d(\omega)(x_0, \dots, x_n) \\ & + (-1)^{n+1} \omega(x_0, \dots, x_{n-1}) x_n \end{aligned}$$

for $\omega \in C^n(\mathcal{A}, \mathcal{M})$, $x_i \in \mathcal{A}$ and where d is "the obvious extension" to $C(\mathcal{A}, \mathcal{M})$ of the differential d of $C(\mathcal{A})$. When $\mathcal{A}$ is finite dimensional all this reduces to the previous definitions, in particular in this case one has $C(\mathcal{A}, \mathcal{M}) = \mathcal{M} \otimes T(\mathcal{A}^*)$. The cohomology $H(\mathcal{A}, \mathcal{M})$ of $C(\mathcal{A}, \mathcal{M})$ is the *$\mathcal{M}$-valued Hochschild cohomology of $\mathcal{A}$* or the *Hochschild cohomology of $\mathcal{A}$ with coefficients in $\mathcal{M}$*. The $\mathcal{M}$-valued Hochschild cochains of $\mathcal{A}$ which vanishes whenever one of their arguments is the unit $\mathbb{1}$ of $\mathcal{A}$ are said to be *normalized Hochschild cochains*. The graded vector space $C_0(\mathcal{A}, \mathcal{M})$ of $\mathcal{M}$-valued normalized Hochschild cochains is stable by the Hochschild coboundary d_H and it is well known and easy to show that the injection of $C_0(\mathcal{A}, \mathcal{M})$ into $C(\mathcal{A}, \mathcal{M})$ induces an isomorphism in cohomology, i.e. the cohomology of $C_0(\mathcal{A}, \mathcal{M})$ is again $H(\mathcal{A}, \mathcal{M})$. Notice that a $\mathcal{M}$-valued Hochschild 1-cocycle (i.e. an element of $C^1(\mathcal{A}, \mathcal{M})$ in $\mathrm{Ker}(d_H)$) is a derivation δ of $\mathcal{A}$ in $\mathcal{M}$, and that it is automatically normalized. If $\mathcal{N}$ is another $(\mathcal{A}, \mathcal{A})$-bimodule then the tensor product over $\mathcal{A}$ of $\mathcal{M}$ and $\mathcal{N}$, $(\mathcal{M}, \mathcal{N}) \mapsto \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$, induces a product $(\alpha, \beta) \mapsto \alpha \cup \beta$, *the cup product* $\cup : C(\mathcal{A}, \mathcal{M}) \otimes C(\mathcal{A}, \mathcal{N}) \to C(\mathcal{A}, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$ such that $C^m(\mathcal{A}, \mathcal{M}) \cup C^n(\mathcal{A}, \mathcal{N}) \subset C^{m+n}(\mathcal{A}, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$ defined by

$$(\alpha \cup \beta)(x_1, \dots, x_{m+n}) = \alpha(x_1, \dots, x_m) \underset{\mathcal{A}}{\otimes} \beta(x_{m+1}, \dots, x_{m+n})$$

for $\alpha \in C^m(\mathcal{A}, \mathcal{M}), \beta \in C^n(\mathcal{A}, \mathcal{N})$ and $x_i \in \mathcal{A}$. If $\mathcal{P}$ is another $(\mathcal{A}, \mathcal{A})$-bimodule and if $\gamma \in C^p(\mathcal{A}, \mathcal{P})$, one has:$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$. Furthermore one has $d_H(\alpha \cup \beta) = d_H(\alpha) \cup \beta + (-1)^m \alpha \cup d(\beta)$ for $\alpha \in C^m(\mathcal{A}, \mathcal{M})$,

$\beta \in C(\mathcal{A}, \mathcal{N})$. This implies in particular that $C(\mathcal{A}, \mathcal{A})$ is a graded differential algebra (when equipped with the cup product and with d_H). In fact, $C(\mathcal{A}, \mathcal{A})$ has a very rich structure which was first described in [40]. As pointed out in [40], its cohomology $H(\mathcal{A}, \mathcal{A})$ which inherits from this structure is graded commutative (as graded algebra for the cup product). The cohomology $H(\mathcal{A}, \mathcal{A})$ is a sort of graded commutative Poisson algebra.

A unital associative algebra $\mathcal{A}$ is said to be of *Hochschild dimension* n if n is the smaller integer such that $H^k(\mathcal{A}, \mathcal{M}) = 0$ for any $k \geq n+1$ and any $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{M}$. The Hochschild dimension of the algebra $\mathbb{C}[X_1, \dots, X_n]$ of complex polynomials with n indeterminates is n. If one considers $\mathcal{A}$ as the generalization of the algebra of smooth functions on a noncommutative space then its Hochschild dimension n is the analog of the dimension of the noncommutative space.

In spite of the triviality of the cohomology of $C(\mathcal{A})$, several complexes with nontrivial cohomologies can be extracted from it. Let $\mathcal{S} : C(\mathcal{A}) \to C(\mathcal{A})$ and $\mathcal{C} : C(\mathcal{A}) \to C(\mathcal{A})$ be linear mappings defined by

$$\mathcal{S}(\omega)(x_1, \dots, x_n) = \sum_{\pi \in \mathcal{S}_n} \varepsilon(\pi) \omega(x_{\pi(1)}, \dots, x_{\pi(n)})$$

and

$$\mathcal{C}(\omega)(x_1, \dots, x_n) = \sum_{\gamma \in \mathcal{C}_n} \varepsilon(\gamma) \omega(x_{\gamma(1)}, \dots, x_{\gamma(n)})$$

for $\omega \in C^n(\mathcal{A})$, $x_i \in \mathcal{A}$ and where $\mathcal{S}_n$ is the group of permutations of $\{1, \dots, n\}$ and $\mathcal{C}_n$ is the subgroup of cyclic permutations, ($\varepsilon(\pi)$ denoting the signature of the permutation π). The mapping $C(\mathcal{A}) \stackrel{\mathcal{S}}{\to} \mathcal{S}(C(\mathcal{A}))$ is a homomorphism of graded differential algebras of $C(\mathcal{A})$ onto the graded differential algebra $C_\wedge(\mathcal{A}_{\text{Lie}})$ of Lie algebra cochains of the underlying Lie algebra $\mathcal{A}_{\text{Lie}}$ with values in the trivial representation of $\mathcal{A}_{\text{Lie}}$ in $\mathbb{C}$; (Notice that the product of $C_\wedge(\mathcal{A}_{\text{Lie}})$ is not induced by the inclusion $C_\wedge(\mathcal{A}_{\text{Lie}}) \subset C(\mathcal{A})$). The cohomology of $\text{Im}(\mathcal{S}) = C_\wedge(\mathcal{A}_{\text{Lie}})$ is therefore the Lie algebra cohomology of $\mathcal{A}_{\text{Lie}}$. On the other hand, (see Lemma 3 in [13] part II), one has $\mathcal{C} \circ d = d_H \circ \mathcal{C}$ where d_H is the Hochschild coboundary of $C(\mathcal{A}, \mathcal{A}^*)$ and therefore $(\text{Im}(\mathcal{C}), d_H)$ is a complex the cohomology of which coincides with *the cyclic cohomology* $H_\lambda(\mathcal{A})$ *of* $\mathcal{A}$ up to a shift -1 in degree [13].

Let us define for $a \in \mathcal{A}$ the homogeneous linear mapping i_a of degree -1 of $C(\mathcal{A})$ into itself by setting

$$i_a(\omega)(x_1, \dots, x_{n-1}) = \sum_{k=0}^{n-1} (-1)^k \omega(x_1, \dots, x_k, a, x_{k+1}, \dots, x_{n-1})$$

2103 for $\omega \in C^n(\mathcal{A})$ with $n \geq 1$ and $x_i \in \mathcal{A}$, and by setting $i_a(C^0(\mathcal{A})) = 0$. For each $a \in \mathcal{A}$, i_a is an antiderivation of $C(\mathcal{A})$ and it is easy to verify that $a \mapsto i_a$ is an operation of the Lie algebra $\mathcal{A}_{\text{Lie}}$ in the graded differential algebra $C(\mathcal{A})$. The homotopy h used in the proof of Lemma 3 commutes with the L_a's which implies that the invariant cohomology $H_I(C(\mathcal{A}))$ of $C(\mathcal{A})$ is also trivial. The basic cohomology of $C(\mathcal{A})$ for this operation has been called *basic cohomology of* $\mathcal{A}$ and denoted by $H_B(\mathcal{A})$ in [31]. It is given by the following theorem [31]

THEOREM 3 *The basic cohomology $H_B(\mathcal{A})$ of $\mathcal{A}$ identifies with the algebra $\mathcal{I}_S(\mathcal{A}_{Lie})$ of invariant polynomials on the Lie algebra $\mathcal{A}_{Lie}$ where the degree $2n$ is given to the homogeneous polynomials of degree n, that is $H^{2n}_B(\mathcal{A}) = \mathcal{I}^n_S(\mathcal{A}_{Lie})$ and $H^{2n+1}_B(\mathcal{A}) = 0$.*

The proof of this theorem which is not straightforward uses a familiar trick in equivariant cohomology to convert the operation i of $\mathcal{A}_{\text{Lie}}$ into a differential.

Two algebras $\mathcal{A}$ and $\mathcal{B}$ (associative unital, etc.) are said to be *Morita equivalent* if there is a $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{U}$ and a $(\mathcal{B}, \mathcal{A})$-bimodule $\mathcal{V}$ such that one has an isomorphism of $(\mathcal{A}, \mathcal{A})$-bimodules $\mathcal{U} \otimes_{\mathcal{B}} \mathcal{V} \simeq \mathcal{A}$ and an isomorphism of $(\mathcal{B}, \mathcal{B})$-bimodules $\mathcal{V} \otimes_{\mathcal{A}} \mathcal{U} \simeq \mathcal{B}$. This is an equivalence relation and this induces an equivalence between the category of right $\mathcal{A}$-modules (resp. left $\mathcal{A}$-modules, $(\mathcal{A}, \mathcal{A})$-bimodules) and the category of right $\mathcal{B}$-modules (resp. left $\mathcal{B}$-modules, $(\mathcal{B}, \mathcal{B})$-bimodules). The algebras $M_m(\mathcal{A})$ and $M_n(\mathcal{A})$ of $m \times m$ matrices and of $n \times n$ matrices with entries in $\mathcal{A}$ are Morita equivalent for any $m, n \in \mathbb{N}$; in fact the $(M_m(\mathcal{A}), M_n(\mathcal{A}))$-bimodule $M_{mn}(\mathcal{A})$ of rectangular $m \times n$ matrices and the $(M_n(\mathcal{A}), M_m(\mathcal{A}))$-bimodule $M_{nm}(\mathcal{A})$ of rectangular $n \times m$ matrices with entries in $\mathcal{A}$ are such that $M_n(\mathcal{A}) = M_{nm}(\mathcal{A}) \otimes_{M_m(\mathcal{A})} M_{mn}(\mathcal{A})$ and $M_m(\mathcal{A}) = M_{mn}(\mathcal{A}) \otimes_{M_n(\mathcal{A})} M_{nm}(\mathcal{A})$, (the tensor products over $M_m(\mathcal{A})$ and $M_n(\mathcal{A})$ being canonically the usual matricial products).

An important property of Hochschild cohomology and cyclic cohomology (and of the corresponding homologies) is their Morita invariance [45], [52], [60]. More precisely if $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent with $\mathcal{U}$ and $\mathcal{V}$ as above and if $\mathcal{M}$ is a $(\mathcal{A},\mathcal{A})$-bimodule (resp. $\mathcal{N}$ is a $(\mathcal{B},\mathcal{B})$-bimodule) one has a canonical isomorphism $H(\mathcal{A},\mathcal{M}) \simeq H(\mathcal{B},\mathcal{V}\otimes_{\mathcal{A}}\mathcal{M}\otimes_{\mathcal{A}}\mathcal{U})$, (resp. $H(\mathcal{B},\mathcal{N}) \simeq H(\mathcal{A},\mathcal{U}\otimes_{\mathcal{B}}\mathcal{N}\otimes_{\mathcal{B}}\mathcal{V})$) in Hochschild cohomology and also $H_\lambda(\mathcal{A}) \simeq H_\lambda(\mathcal{B})$ in cyclic cohomology. In contrast, the Lie algebra cohomology $H(\mathcal{A}_{\text{Lie}})$ and the basic cohomology $H_B(\mathcal{A})$ are not Morita invariant since for instance for $\mathcal{A} = M_n(\mathbb{C})$ they depend on the number $n \in \mathbb{N}$ whereas $M_n(\mathbb{C})$ is Morita equivalent to $\mathbb{C}$.

5 Categories of algebras

In this section we consider general algebras over $\mathbb{C}$. That is by an *algebra* we here mean a complex vector space $\mathcal{A}$ equipped with a bilinear product $m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$. Given two such algebras $\mathcal{A}$ and $\mathcal{B}$, an *algebra homomorphism* of $\mathcal{A}$ into $\mathcal{B}$ is a linear mapping $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(m(x\otimes y)) = m(\varphi(x)\otimes\varphi(y))$, $(\forall x, y \in \mathcal{A})$, i.e. $\varphi\circ m = m\circ(\varphi\otimes\varphi)$.

Let us define the category $\mathbf{A}$ to be the category such that the class $\text{Ob}(\mathbf{A})$ of its objects is the class of all algebras (in the above sense) and such that for any $\mathcal{A},\mathcal{B} \in \text{Ob}(\mathbf{A})$ the set $\text{Hom}_{\mathbf{A}}(\mathcal{A},\mathcal{B})$ of morphisms from $\mathcal{A}$ to $\mathcal{B}$ is the set of all algebra homomorphisms of $\mathcal{A}$ into $\mathcal{B}$.

A subcategory of $\mathbf{A}$ will be called a *category of algebras*. Thus a category $\mathbf{C}$ is a category of algebras if $\text{Ob}(\mathbf{C})$ is a subclass of $\text{Ob}(\mathbf{A})$ and if, for any $\mathcal{A},\mathcal{B} \in \text{Ob}(\mathbf{C})$, one has $\text{Hom}_{\mathbf{C}}(\mathcal{A},\mathcal{B}) \subset \text{Hom}_{\mathbf{A}}(\mathcal{A},\mathcal{B})$. We now list some categories of algebras which will be used later.

1. The category **Alg** of unital associative algebras: $\text{Ob}(\mathbf{Alg})$ is the class of all complex unital associative algebras and for any $\mathcal{A},\mathcal{B} \in \text{Ob}(\mathbf{Alg})$, $\text{Hom}_{\mathbf{Alg}}(\mathcal{A},\mathcal{B})$ is the set of all algebra homomorphisms mapping the unit of $\mathcal{A}$ onto the unit of $\mathcal{B}$.

2. The category $\mathbf{Alg}_Z$ is the subcategory of **Alg** defined by $\text{Ob}(\mathbf{Alg}_Z) = \text{Ob}(\mathbf{Alg})$ and for any $\mathcal{A},\mathcal{B} \in \text{Ob}(\mathbf{Alg}_Z)$, $\text{Hom}_{\mathbf{Alg}_Z}(\mathcal{A},\mathcal{B})$ is the set of all $\varphi \in \text{Hom}_{\mathbf{Alg}}(\mathcal{A},\mathcal{B})$ mapping the center $Z(\mathcal{A})$ of $\mathcal{A}$ into the center $Z(\mathcal{B})$ of $\mathcal{B}$,

i.e. such that $\varphi(Z(\mathcal{A})) \subset Z(\mathcal{B})$.

3. The category **Jord** of complex unital Jordan algebras: Ob(**Jord**) is the class of all complex unital Jordan algebras and for any $\mathcal{A}, \mathcal{B} \in$ Ob(**Jord**), $\mathrm{Hom}_{\mathbf{Jord}}(\mathcal{A}, \mathcal{B})$ is the set of all algebra homomorphisms mapping the unit of $\mathcal{A}$ onto the unit of $\mathcal{B}$.

4. The category **Algcom** of unital associative and commutative algebras: Ob(**Algcom**) is the class of all complex unital associative commutative algebras and for any $\mathcal{A}, \mathcal{B} \in$ Ob(**Algcom**), $\mathrm{Hom}_{\mathbf{Algcom}}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{\mathbf{Alg}}(\mathcal{A}, \mathcal{B})$.

5. The category **Lie** of Lie algebras: Ob(**Lie**) is the class of all complex Lie algebras and for any $\mathcal{A}, \mathcal{B} \in$ Ob(**Lie**), $\mathrm{Hom}_{\mathbf{Lie}}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{\mathbf{A}}(\mathcal{A}, \mathcal{B})$.

Remark 4. If $\mathcal{A} \in$ Ob(**Alg**) and $\mathcal{B} \in$ Ob(**Algcom**), one has $\mathrm{Hom}_{\mathbf{Alg}}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{\mathbf{Alg}_Z}(\mathcal{A}, \mathcal{B})$.

On the other hand if $\mathcal{A}$ and $\mathcal{B}$ are objects of **Algcom** then

$$\mathrm{Hom}_{\mathbf{Algcom}}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{\mathbf{Jord}}(\mathcal{A}, \mathcal{B}).$$

Thus **Algcom** is a *full subcategory* of **Alg**, of $\mathbf{Alg}_Z$ and of **Jord**, i.e. for any $\mathcal{A}, \mathcal{B} \in$ Ob(**Algcom**) one has :
$\mathrm{Hom}_{\mathbf{Algcom}}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{\mathbf{Alg}}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{\mathbf{Alg}_Z}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{\mathbf{Jord}}(\mathcal{A}, \mathcal{B})$

In order to discuss reality conditions we shall also need categories of $*$-algebras. By a $*$-algebra we here mean a general complex algebra $\mathcal{A}$ as above equipped with an antilinear involution $x \mapsto x^*$ such that $m(x \otimes y)^* = m(y^* \otimes x^*)$, (i.e. such that it reverses the order in the product). If $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras, a $*$-algebra homomorphism of $\mathcal{A}$ into $\mathcal{B}$ is an algebra homomorphism φ of $\mathcal{A}$ into $\mathcal{B}$ which preserves the involutions, i.e. $\varphi(x^*) = \varphi(x)^*$ for $x \in \mathcal{A}$. One defines the category of algebras $*$-**A** to be the category where Ob($*$-**A**) is the class of $*$-algebras and such that for any $\mathcal{A}, \mathcal{B} \in$ Ob($*$-**A**), $\mathrm{Hom}_{*\text{-}\mathbf{A}}(\mathcal{A}, \mathcal{B})$ is the set of $*$-algebra homomorphisms of $\mathcal{A}$ into $\mathcal{B}$. A subcategory of $*$-**A** will be called a *category of $*$-algebras* and one defines in the obvious manner the categories of $*$-algebras $*$-**Alg**, $*$-$\mathbf{Alg}_Z$, $*$-**Jord**, $*$-**Algcom**, $*$-**Lie** corresponding to the above examples 1, 2, 3, 4, 5.

Let **C** be a category of algebras and let $\mathcal{A}$ be an object of **C** with product denoted by $a \otimes a' \mapsto aa'$ $(a, a' \in a)$. A complex vector space $\mathcal{E}$ will be said to be a $\mathcal{A}$-*bimodule for* $\mathbf{C}$ if there are linear mappings $\mathcal{A} \otimes \mathcal{E} \to \mathcal{E}$ and $\mathcal{E} \otimes \mathcal{A} \to \mathcal{E}$, denoted by $a \otimes e \mapsto ae$ and $e \otimes a \mapsto ea$ ($a \in \mathcal{A}, e \in \mathcal{E}$) respectively, such that the direct sum $\mathcal{A} \oplus \mathcal{E}$ equipped with the product

$$(a \oplus e) \otimes (a' \oplus e') \mapsto aa' \oplus (ae' + ea')$$

is an object of $\mathbf{C}$ and such that the canonical linear mappings

$$i : \mathcal{A} \to \mathcal{A} \oplus \mathcal{E} \text{ and } p : \mathcal{A} \oplus \mathcal{E} \to \mathcal{A}$$

defined by $i(a) = a \oplus 0$ and $p(a \oplus e) = a$ ($\forall a \in \mathcal{A}$ and $\forall e \in \mathcal{E}$) are morphisms of $\mathbf{C}$. In other words $\mathcal{E}$ is a $\mathcal{A}$-bimodule for $\mathbf{C}$ if $\mathcal{A} \oplus \mathcal{E}$ is equipped with a bilinear product vanishing on $\mathcal{E} \otimes \mathcal{E}$ and such that $\mathcal{A} \oplus \mathcal{E} \in \mathrm{Ob}(\mathbf{C})$, $i \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{A}, \mathcal{A} \oplus \mathcal{E})$ and $p \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{A} \oplus \mathcal{E}, \mathcal{A})$.

For the category $\mathbf{A}$ this notion of bimodule is not very restrictive. In fact, if $\mathcal{A}$ is an algebra (i.e. $\mathcal{A} \in \mathrm{Ob}(\mathbf{A})$) then a $\mathcal{A}$-bimodule for $\mathbf{A}$ is simply a complex vector space $\mathcal{E}$ with two bilinear mappings corresponding to linear mappings $\mathcal{A} \otimes \mathcal{E} \to \mathcal{E}$ and $\mathcal{E} \otimes \mathcal{A} \to \mathcal{E}$ as above. These two linear mappings will be always denoted by $a \otimes e \mapsto ae$ and $e \otimes a \mapsto ea$ and called *left* and *right action of* $\mathcal{A}$ *on* $\mathcal{E}$. Let us describe what restrictions occur for the categories of algebras of examples 1, 2, 3, 4, 5.

1. Let $\mathcal{A}$ be a unital associative complex algebra with product denoted by $a \otimes a' \mapsto aa'$ and unit denoted by $\mathbb{1}$. Then, $\mathcal{E}$ is a $\mathcal{A}$-bimodule for $\mathbf{Alg}$ if and only if one has

$$\begin{array}{lll} (i) & (aa')e = a(a'e) \quad \text{and} \quad \mathbb{1}e = e \\ (ii) & e(aa') = (ea)a' \quad \text{and} \quad e\mathbb{1} = e \\ (iii) & (ae)a' = a(ea') \end{array}$$

for any $a, a' \in \mathcal{A}$ and $e \in \mathcal{E}$. Conditions (i) express the fact that $\mathcal{E}$ is a left $\mathcal{A}$-module in the usual sense, conditions (ii) express the fact that $\mathcal{E}$ is a right $\mathcal{A}$-module in the usual sense whereas, completed with the compatibility condition (iii), all these conditions express the fact that $\mathcal{E}$ is a $(\mathcal{A}, \mathcal{A})$-bimodule in the usual sense for unital associative algebras.

2. Let $\mathcal{A}$ be as in 1 above. Then $\mathcal{E}$ is a $\mathcal{A}$-bimodule for $\mathbf{Alg}_Z$ if and only if it is a $\mathcal{A}$-bimodule for $\mathbf{Alg}$ such that one has

$$ze = ez$$

for any element z of the center $Z(\mathcal{A})$ of $\mathcal{A}$ and $e \in \mathcal{E}$. This condition expresses that as $(Z(\mathcal{A}), Z(\mathcal{A}))$-bimodule, $\mathcal{E}$ is the underlying bimodule of a $Z(\mathcal{A})$-module. Such $(\mathcal{A}, \mathcal{A})$-bimodules were called *central bimodules* over $\mathcal{A}$ in [34], [35] (see also in [27]). We shall keep this terminology here and call central bimodule a bimodule for $\mathbf{Alg}_Z$.

Let $\mathcal{E}$ be a $\mathcal{A}$-bimodule for $\mathbf{Alg}$ (i.e. a $(\mathcal{A}, \mathcal{A})$-bimodule). One can associate to $\mathcal{E}$ two $\mathcal{A}$-bimodules for $\mathbf{Alg}_Z$ (i.e. two central bimodules) $\mathcal{E}^Z$ and $\mathcal{E}_Z$. The bimodule $\mathcal{E}^Z$ is the biggest $(\mathcal{A}, \mathcal{A})$-subbimodule of $\mathcal{E}$ which is central and we denote by i^Z the canonical inclusion of $\mathcal{E}^Z$ into $\mathcal{E}$ whereas $\mathcal{E}_Z$ is the quotient of $\mathcal{E}$ by the $(\mathcal{A}, \mathcal{A})$-subbimodule $[Z(\mathcal{A}), \mathcal{E}]$ generated by the $ze - ez$ where z is in the center $Z(\mathcal{A})$ of $\mathcal{A}$, $e \in \mathcal{E}$ and we denote by p_Z the canonical projection of $\mathcal{E}$ onto $\mathcal{E}_Z$. The pair $(\mathcal{E}^Z, i^Z)$ is characterized by the following universal property: *For any* $(\mathcal{A}, \mathcal{A})$*-bimodule homomorphism* $\Phi : \mathcal{N} \to \mathcal{E}$ *of a central bimodule* $\mathcal{N}$ *into* $\mathcal{E}$*, there is a unique* $(\mathcal{A}, \mathcal{A})$*-bimodule homomorphism* $\Phi^Z : \mathcal{N} \to \mathcal{E}^Z$ *such that* $\Phi = i^Z \circ \Phi^Z$. The pair $(\mathcal{E}_Z, p_Z)$ is characterized by the following universal property: *For any* $(\mathcal{A}, \mathcal{A})$*-bimodule homomorphism* $\varphi : \mathcal{E} \to \mathcal{M}$ *of* $\mathcal{E}$ *into a central bimodule* $\mathcal{M}$ *there is a unique* $(\mathcal{A}, \mathcal{A})$*-bimodule homomorphism* $\varphi_Z : \mathcal{E}_Z \to \mathcal{M}$ *such that* $\varphi = \varphi_Z \circ p_Z$. In functorial language, this means that $\mathcal{E} \mapsto \mathcal{E}^Z$ is a right adjoint and that $\mathcal{E} \mapsto \mathcal{E}_Z$ is a left adjoint of the canonical functor I_Z from the category of $\mathcal{A}$-bimodules for $\mathbf{Alg}_Z$ in the category of $\mathcal{A}$-bimodules for $\mathbf{Alg}$. Notice also that $\mathcal{E}$ is central if and only if $\mathcal{E} = \mathcal{E}_Z$ which is equivalent to $\mathcal{E} = \mathcal{E}^Z$ and that if $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{A}$-bimodules for $\mathbf{Alg}_Z$ (i.e. two central bimodules) then one has $(\mathcal{M} \otimes \mathcal{N})_Z = \mathcal{M} \otimes_{Z(\mathcal{A})} \mathcal{N}$. One has the further following stability properties for the $\mathcal{A}$-bimodules for $\mathbf{Alg}_Z$: Every subbimodule of a central bimodule is central, every quotient of a central bimodule is central and any product of central bimodules is central. For all this, we refer to [35].

3. Let $\mathcal{J}$ be a complex unital Jordan algebra with product denoted by $x \otimes y \mapsto x \bullet y$ $(x, y \in \mathcal{J})$ and unit $\mathbb{1}$. Then $\mathcal{E}$ is a $\mathcal{J}$-bimodule for **Jord** if and only if one has

$$
\begin{array}{ll}
(i) & xe = ex \quad \text{and} \quad \mathbb{1}e = e \\
(ii) & x((x \bullet x)e) = (x \bullet x)(xe) \\
(iii) & ((x \bullet x) \bullet y)e - (x \bullet x)(ye) = 2((x \bullet y)(xe) - x(y(xe)))
\end{array}
$$

for any $x, y \in \mathcal{J}$ and $e \in \mathcal{E}$. Such a bimodule for **Jord** is called a *Jordan module* over $\mathcal{J}$ [44] which is natural since, in view of (i), there is only one bilinear mapping of $\mathcal{J} \times \mathcal{E}$ into $\mathcal{E}$.

4. Let $\mathcal{C}$ be a unital associative commutative complex algebra. Then $\mathcal{E}$ is a $\mathcal{C}$-bimodule for **Algcom** if and only if it is a $\mathcal{C}$-bimodule for **Alg** such that one has

$$ce = ec$$

for any $c \in \mathcal{C}$ and $e \in \mathcal{E}$. This means that a $\mathcal{C}$-bimodule for **Algcom** is the same thing as (the underlying bimodule of) a $\mathcal{C}$-module in the usual sense. Since the center of $\mathcal{C}$ coincides with $\mathcal{C}$, $Z(\mathcal{C}) = \mathcal{C}$, this implies that it is also the same thing as a $\mathcal{C}$-bimodule for $\mathbf{Alg}_Z$, as announced in the introduction. Notice that in the case of a $\mathcal{C}$-bimodule for **Alg** one generally has $ce \neq ec$.

5. Let $\mathfrak{g}$ be a complex Lie algebra with product (Lie bracket) denoted by $X \otimes Y \mapsto [X, Y]$ for $X, Y \in \mathfrak{g}$. Then, $\mathcal{E}$ is a $\mathfrak{g}$-bimodule for **Lie** if and only if one has

$$\begin{array}{ll} (i) & Xe = -eX \\ (ii) & [X,Y]e = X(Ye) - Y(Xe) \end{array}$$

for any $X, Y \in \mathfrak{g}$ and $e \in \mathcal{E}$. Condition (i) shows that again there is only one bilinear mapping of $\mathfrak{g} \times \mathcal{E}$ into $\mathcal{E}$ and (ii) means that $\mathcal{E}$ is the space of a linear representation of $\mathfrak{g}$; Thus a $\mathfrak{g}$-bimodule for **Lie** is what is usually called a $\mathfrak{g}$-module (or a linear representation of $\mathfrak{g}$).

One defines in a similar way the notion of $\ast$-bimodule for a category $\ast$-$\mathbf{C}$ of $\ast$-algebras. Namely, if $\mathcal{A} \in \mathrm{Ob}(\ast\text{-}\mathbf{C})$, a complex vector space $\mathcal{E}$ will be said to be a *$\mathcal{A}$-$\ast$-bimodule for $\ast$-$\mathbf{C}$* if $\mathcal{A} \oplus \mathcal{E}$ is equipped with a structure of $\ast$-algebra with product vanishing on $\mathcal{E} \otimes \mathcal{E}$ such that $\mathcal{A} \oplus \mathcal{E} \in \mathrm{Ob}(\ast\text{-}\mathbf{C})$, $i \in \mathrm{Hom}_{\ast\text{-}\mathbf{C}}(\mathcal{A}, \mathcal{A} \oplus \mathcal{E})$ and $p \in \mathrm{Hom}_{\ast\text{-}\mathbf{C}}(\mathcal{A} \oplus \mathcal{E}, \mathcal{A})$.

One can easily describe what is a $\ast$-bimodule for the various categories of $\ast$-algebras. If $\mathcal{A}$ is a $\ast$-algebra, we also denote by $\mathcal{A}$ the algebra obtained by "forgetting the involution". If $\mathcal{A}$ is an object of $\ast$-**Alg** then a $\mathcal{A}$-$\ast$-bimodule for $\ast$-**Alg** is a $\mathcal{A}$-bimodule $\mathcal{E}$ for **Alg** which is equipped with an antilinear involution $e \mapsto e^\ast$ such that $(xey)^\ast = y^\ast e^\ast x^\ast$ for $x, y \in \mathcal{A}$ and $e \in \mathcal{E}$, i.e. it is what has been called in the introduction a $\ast$-bimodule over the (unital associative complex) $\ast$-algebra $\mathcal{A}$. A $\mathcal{A}$-$\ast$-bimodule for $\ast$-$\mathbf{Alg}_Z$ is then just such a $*$-bimodule over $\mathcal{A}$ which is central. If $\mathcal{C}$ is a unital associative complex commutative $*$-algebra, then a $\mathcal{C}$-$*$-bimodule for $*$-**Algcom** is just what has been called a $*$-module over the (unital associative complex) commutative $*$-algebra $\mathcal{C}$.

One can proceed similarily with real algebras. However to be in conformity with the point of view of the introduction concerning reality, we shall work with $*$-algebras and, eventually, extract their hermitian parts as well as the hermitian parts of the $*$-bimodules over them.

6 First order differential calculi

Throughout the following $\mathcal{A}$ denotes a unital associative complex algebra. A pair (Ω^1, d) where Ω^1 is a $(\mathcal{A},\mathcal{A})$-bimodule (i.e. a $\mathcal{A}$-bimodule for **Alg**) and where $d : \mathcal{A} \to \Omega^1$ is a derivation of $\mathcal{A}$ into Ω^1, that is a linear mapping which satisfies (the Leibniz rule)

$$d(xy) = d(x)y + xd(y)$$

for any $x, y \in \mathcal{A}$, will be called a *first order differential calculus over $\mathcal{A}$ for* **Alg** or simply a *first order differential calculus over $\mathcal{A}$* [61]. If furthermore Ω^1 is a central bimodule (i.e. a $\mathcal{A}$-bimodule for $\mathbf{Alg}_Z$), we shall say that (Ω^1, d) is a *first order differential calculus over $\mathcal{A}$ for* $\mathbf{Alg}_Z$. One can more generally define the notion of first order differential calculus over $\mathcal{A}$ for any category $\mathbf{C}$ of algebras such that $\mathcal{A} \in \mathrm{Ob}(\mathbf{C})$.

<u>Remark 5</u>. If Ω^1 is a $\mathcal{A}$-bimodule for $\mathbf{C}$ a derivation $d : \mathcal{A} \to \Omega^1$ can be defined to be a linear mapping such that $a \mapsto a \oplus d(a)$ is in $\mathrm{Hom}_{\mathbf{C}}(\mathcal{A}, \mathcal{A} \oplus \Omega^1)$. However, for the category $\mathbf{Alg}_Z$ this does not impose restrictions on first order differential calculus. Indeed if Ω^1 is a central bimodule and if $d : \mathcal{A} \to \Omega^1$ is a derivation one has $d(z)a + zd(a) = d(za) = d(az) = ad(z) + d(a)z$ for any $a \in \mathcal{A}$ and z in the center $Z(\mathcal{A})$ of $\mathcal{A}$, i.e. $d(z)a = ad(z)$ since, by "centrality", $zd(a) = d(a)z$; again, by centrality $z\omega = \omega z$, $\forall z \in Z(\mathcal{A})$ and $\forall \omega \in \Omega^1$, which finally implies $(z \oplus d(z))(a \oplus \omega) = (a \oplus \omega)(z \oplus d(z))$ and therefore $z \oplus d(z) \in Z(\mathcal{A} \oplus \Omega^1)$ for any $z \in Z(\mathcal{A})$ which means that the linear mapping $a \mapsto a \oplus d(a)$ is in $\mathrm{Hom}_{\mathbf{Alg}_Z}(\mathcal{A}, \mathcal{A} \oplus \Omega^1)$.

We shall refer to d as the *first order differential*; by definition it is a Ω^1-valued Hochschild cocycle of degree 1 of $\mathcal{A}$, i.e. $d \in Z^1_H(\mathcal{A}, \Omega^1)$. Examples of first order differentials are thus provided by Hochschild coboundaries i.e. given by $d(x) = \tau x - x\tau$ $(\forall x \in \mathcal{A})$ for some $\tau \in \Omega^1$. We shall now explain that there are "universal first order differential calculi" for **Alg** and for $\mathbf{Alg}_Z$ which define respectively functors from **Alg** and from $\mathbf{Alg}_Z$ in the corresponding categories of first order differential calculi. For the case of a commutative algebra, there is also a well-known universal first order differential calculus for **Algcom** which is the universal derivation into the module of Kähler differentials ([6], [52], [58]). We shall see however that it reduces to the universal calculus for $\mathbf{Alg}_Z$ (Corollary 1).

Let m be the product of $\mathcal{A}, (x, y) \mapsto m(x \otimes y) = xy$ and let $\Omega^1_u(\mathcal{A})$ be the kernel of m, i.e. one has the short exact sequence

$$0 \to \Omega^1_u(\mathcal{A}) \stackrel{\subseteq}{\to} \mathcal{A} \otimes \mathcal{A} \stackrel{m}{\to} \mathcal{A} \to 0$$

of $(\mathcal{A}, \mathcal{A})$-bimodules ($\mathcal{A}$-bimodules for **Alg**). Define $d_u : \mathcal{A} \to \Omega^1_u(\mathcal{A})$ by $d_u(x) = \mathbb{1} \otimes x - x \otimes \mathbb{1}$, $\forall x \in \mathcal{A}$. One verifies easily that d_u is a derivation. The first order differential calculus $(\Omega^1_u(\mathcal{A}), d_u)$ over $\mathcal{A}$ is characterized uniquely (up to an isomorphism) by the following universal property [10], [6].

PROPOSITION 1 *For any first order differential calculus (Ω^1, d) over $\mathcal{A}$, there is a unique bimodule homomorphism i_d of $\Omega^1_u(\mathcal{A})$ into Ω^1 such that $d = i_d \circ d_u$.*

Proof. $\Omega^1_u(\mathcal{A})$ is generated by $d_u(\mathcal{A})$ as left module since $x^\alpha \otimes y_\alpha$ with $x^\alpha y_\alpha = 0$ is the same thing as $x^\alpha d(y_\alpha)$. On the other hand $d_u(\mathbb{1}) = 0 (= d_u(\mathbb{1}^2) = 2d_u(\mathbb{1}))$. Therefore one has a surjective left $\mathcal{A}$-module homomorphism of $\mathcal{A} \otimes (\mathcal{A}/\mathbb{C}\mathbb{1})$ onto $\Omega^1_u(\mathcal{A})$, $x \otimes \dot{y} \mapsto x d_u(y)$, which is easily shown to be an isomorphism. Then $x d_u(y) \mapsto x d(y)$ defines a left $\mathcal{A}$-module homomorphism i_d of $\Omega^1_u(\mathcal{A})$ into Ω^1 which is easily shown to be a bimodule homomorphism by using the Leibniz rule for d_u and for d. One clearly has $d = i_d \circ d_u$. Uniqueness is straightforward. $\square$

Concerning the image of i_d, let us notice the following easy lemma.

LEMMA 4 *Let (Ω^1, d) be a first order differential calculus over $\mathcal{A}$. The* *following conditions are equivalent.*

(i) Ω^1 *is generated by* $d\mathcal{A}$ *as left* $\mathcal{A}$*-module.*

(ii) Ω^1 *is generated by* $d\mathcal{A}$ *as right* $\mathcal{A}$*-module.*

(iii) Ω^1 *is generated by* $d\mathcal{A}$ *as* $(\mathcal{A},\mathcal{A})$*-bimodule.*

$(iiii)$ *The homomorphism* i_d *is surjective, i.e.* $\Omega^1 = i_d(\Omega^1_u(\mathcal{A}))$.

Proof. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ follows from (Leibniz rule)

$$ud(v)w = ud(vw) - uvd(w) = d(uv)w - d(u)vw$$

for $u, v, w \in \mathcal{A}$ whereas the equivalence $(iii) \Leftrightarrow (iiii)$ is straightforward from the definitions. $\square$

Remark 6. Proposition 1 claims that there is a unique bimodule homomorphism i_d of $\Omega^1_u(\mathcal{A})$ into Ω^1 mapping the $\Omega^1_u(\mathcal{A})$-valued Hochschild 1-cocycle d_u on the Ω^1-valued Hochschild 1-cocycle d. One can complete the statement by the following: *The* Ω^1*-valued Hochschild 1-cocycle* d *is a Hochschild coboundary, (i.e. there is a* $\tau \in \Omega^1$ *such that* $d(a) = \tau a - a\tau$ *for any* $a \in \mathcal{A}$*), if and only if* i_d *has an extension* $\tilde{i}_d$ *as a bimodule homomorphism of* $\mathcal{A} \otimes \mathcal{A}$ *into* Ω^1, [8]. In fact τ is then $\tilde{i}_d(\mathbb{1} \otimes \mathbb{1})$, which gives essentially the proof.

The first order differential calculus $(\Omega^1_u(\mathcal{A}), d_u)$ *is universal for* **Alg**, it is usually simply called *the universal first order differential calculus over* $\mathcal{A}$. From Proposition 1 follows the functorial property.

PROPOSITION 2 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be algebras and let* $\varphi : \mathcal{A} \to \mathcal{B}$ *be a homomorphism, (i.e. let* $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbf{Alg})$ *and let* $\varphi \in \mathrm{Hom}_{\mathbf{Alg}}(\mathcal{A},\mathcal{B})$*), then there is a unique linear mapping* $\Omega^1_u(\varphi)$ *of* $\Omega^1_u(\mathcal{A})$ *into* $\Omega^1_u(\mathcal{B})$ *satisfying* $\Omega^1_u(\varphi)(x\omega y) = \varphi(x)\Omega^1_u(\varphi)(\omega)\varphi(y)$ *for any* $x, y \in \mathcal{A}$ *and* $\omega \in \Omega^1_u(\mathcal{A})$ *and such that* $d_u \circ \varphi = \Omega^1_u(\varphi) \circ d_u$.

Proof. One equips $\Omega^1_u(\mathcal{B})$ of a structure of $(\mathcal{A},\mathcal{A})$-bimodule by setting $x\lambda y = \varphi(x)\lambda\varphi(y)$ for $x, y \in \mathcal{A}$ and $\lambda \in \Omega^1_u(\mathcal{B})$. Then $d = d_u \circ \varphi$ is a derivation of $\mathcal{A}$ into the $(\mathcal{A},\mathcal{A})$-bimodule $\Omega^1_u(\mathcal{B})$, i.e. $(\Omega^1_u(\mathcal{B}), d)$ is a first order differential calculus over $\mathcal{A}$, and the result follows from Proposition 1 with $\Omega^1_u(\varphi) = i_d$. $\square$

One can summarize the content of Proposition 2 by the following: *For any homomorphism* $\varphi : \mathcal{A} \to \mathcal{B}$ *(of unital associative* $\mathbb{C}$*-algebras) there is a unique* $(\mathcal{A},\mathcal{A})$*-bimodule homomorphism* $\Omega^1_u(\varphi) : \Omega^1_u(\mathcal{A}) \to \Omega^1_u(\mathcal{B})$ *for which the diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\ \downarrow{\scriptstyle d_u} & & \downarrow{\scriptstyle d_u} \\ \Omega^1_u(\mathcal{A}) & \xrightarrow{\Omega^1_u(\varphi)} & \Omega^1_u(\mathcal{B}) \end{array}$$

is commutative. All this was for the category **Alg**, we now pass to $\mathbf{Alg}_Z$.

Let $[Z(\mathcal{A}), \Omega^1_u(\mathcal{A})]$ be the subbimodule of $\Omega^1_u(\mathcal{A})$ defined by

$$[Z(\mathcal{A}), \Omega^1_u(\mathcal{A})] = \{z\omega - \omega z | z \in Z(\mathcal{A}), \omega \in \Omega^1_u(\mathcal{A})\}.$$

By definition the quotient $\Omega^1_Z(\mathcal{A}) = \Omega^1_u(\mathcal{A})/[Z(\mathcal{A}), \Omega^1_u(\mathcal{A})]$ is a central bimodule i.e. a $\mathcal{A}$-bimodule for $\mathbf{Alg}_Z$. Let $p_Z : \Omega^1_u(\mathcal{A}) \to \Omega^1_Z(\mathcal{A})$ be the canonical projection and let $d_Z : \mathcal{A} \to \Omega^1_Z(\mathcal{A})$ be defined by $d_Z = p_Z \circ d_u$. Then d_Z is again a derivation so $(\Omega^1_Z(\mathcal{A}), d_Z)$ is a first order differential calculus over $\mathcal{A}$ for $\mathbf{Alg}_Z$. It is characterized uniquely (up to an isomorphism) among the first order differential calculi over $\mathcal{A}$ for $\mathbf{Alg}_Z$ by the following universal property [35].

PROPOSITION 3 *For any first order differential calculus (Ω^1, d) over $\mathcal{A}$ for $\mathbf{Alg}_Z$, there is a unique bimodule homomorphism i_d of $\Omega^1_Z(\mathcal{A})$ into Ω^1 such that $d = i_d \circ d_Z$; i.e. there is a unique morphism of first order differential calculi over $\mathcal{A}$ for $\mathbf{Alg}_Z$ from $(\Omega^1_Z(\mathcal{A}), d_Z)$ to (Ω^1, d).*

Proof. The unique bimodule homomorphism $i_d : \Omega^1_u(\mathcal{A}) \to \Omega^1$ of Proposition 1 vanishes on $[Z(\mathcal{A}), \Omega^1_u(\mathcal{A})]$ since Ω^1 is central. Therefore it factorizes as $\Omega^1_u(\mathcal{A}) \stackrel{p_Z}{\to} \Omega^1_Z(\mathcal{A}) \to \Omega^1$

through a unique bimodule homomorphism, again denoted i_d, of $\Omega^1_Z(\mathcal{A})$ into Ω^1 for which one has $d = i_d \circ d_Z$. Again, uniqueness is obvious. $\square$

Remark 7. Proposition 3 can be slightly improved. One can replace the assumption "(Ω^1, d) over $\mathcal{A}$ for $\mathbf{Alg}_Z$" by "(Ω^1, d) over $\mathcal{A}$ such that $zd(a) = d(a)z$ for any $a \in \mathcal{A}$ and $z \in Z(\mathcal{A})$" in the statement. That is, what is important is that the subbimodule of Ω^1 generated by $d\mathcal{A}$ is central.

The first order differential calculus $(\Omega^1_Z(\mathcal{A}), d_Z)$ will be called *the universal first order differential calculus over* $\mathcal{A}$ *for* $\mathbf{Alg}_Z$. Concerning the functorial property of this first order differential calculus, Proposition 2 has the following counterpart for $\mathbf{Alg}_Z$.

PROPOSITION 4 *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras as above and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism such that $\varphi(Z(\mathcal{A})) \subset Z(\mathcal{B})$, (i.e. let $\mathcal{A}, \mathcal{B} \in$* $\mathrm{Ob}(\mathbf{Alg}_Z)$ *and let $\varphi \in \mathrm{Hom}_{\mathbf{Alg}_Z}(\mathcal{A}, \mathcal{B})$), then there is a unique linear mapping $\Omega^1_Z(\varphi)$ of $\Omega^1_Z(\mathcal{A})$ into $\Omega^1_Z(\mathcal{B})$ satisfying $\Omega^1_Z(\varphi)(x\omega y) = \varphi(x)\Omega^1_Z(\varphi)(\omega)\varphi(y)$ for any $x, y \in \mathcal{A}$ and $\omega \in \Omega^1_Z(\mathcal{A})$ and such that $d_Z \circ \varphi = \Omega^1_Z(\varphi) \circ d_Z$.*

Proof. Again, as in the proof of Proposition 2, $\Omega^1_Z(\mathcal{B})$ is a $(\mathcal{A}, \mathcal{A})$-bimodule by setting $x\lambda y = \varphi(x)\lambda\varphi(y)$ for $x, y \in \mathcal{A}$ and $\lambda \in \Omega^1_Z(\mathcal{B})$. Thus Proposition 4 follows from Proposition 3 if one can show that this bimodule is central i.e. if $\varphi(z)\lambda = \lambda\varphi(z)$ for any $z \in Z(\mathcal{A})$ and $\lambda \in \Omega^1_Z(\mathcal{B})$. This however follows from the fact that $\Omega^1_Z(\mathcal{B})$ is central over $\mathcal{B}$ and that φ maps the center $Z(\mathcal{A})$ of $\mathcal{A}$ into the center $Z(\mathcal{B})$ of $\mathcal{B}$. $\square$

Again this can be summarized (by identifying $\Omega^1_Z(\mathcal{B})$ with a central bimodule over $\mathcal{A}$ via φ) as : *For any $\varphi \in Hom_{\mathbf{Alg}_Z}(\mathcal{A}, \mathcal{B})$, there is a unique homomorphism of $\mathcal{A}$-bimodules for $\mathbf{Alg}_Z$, $\Omega^1_Z(\varphi) : \Omega^1_Z(\mathcal{A}) \to \Omega^1_Z(\mathcal{B})$, for which the diagram*

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\
\downarrow{\scriptstyle d_Z} & & \downarrow{\scriptstyle d_Z} \\
\Omega^1_Z(\mathcal{A}) & \xrightarrow{\Omega^1_Z(\varphi)} & \Omega^1_Z(\mathcal{B})
\end{array}$$

is commutative.

Proposition 3 has the following corollary

COROLLARY 1 *If $\mathcal{A}$ is commutative, $\Omega^1_Z(\mathcal{A})$ identifies canonically with the module of Kähler differentials $\Omega^1_{\mathcal{A}|\mathbb{C}}$ and d_Z identifies with the corresponding universal derivation.*

Proof. The proof is straightforward since, for a commutative algebra $\mathcal{A}$, a central bimodule is just (the underlying bimodule of) a $\mathcal{A}$-module and then, Proposition 3 just reduces to the universal property which characterizes the first order Kähler differential calculus (see e.g. in [6], [52], [58]). □

Remark 8. If $\mathcal{A}$ is commutative, the module of Kähler differentials $\Omega^1_{\mathcal{A}|\mathbb{C}}$ is known to be a version of differential 1-forms. There is however a little subtility. In fact $\Omega^1_{\mathcal{A}|\mathbb{C}}$ is the quotient of $\Omega^1_u(\mathcal{A})$ which is a commutative algebra (a subalgebra of $\mathcal{A}\otimes\mathcal{A}$) by the ideal $(\Omega^1_u(\mathcal{A}))^2$. If $\mathcal{A}$ is the algebra of smooth functions $C^\infty(M)$ on a manifold M, this means that $\Omega^1_{\mathcal{A}|\mathbb{C}}$ is the algebra of functions in $\mathcal{A}\otimes\mathcal{A} = C^\infty(M)\otimes C^\infty(M)$ vanishing on the diagonal of $M\times M$ modulo functions vanishing to order one on the diagonal of $M\times M$. On the other hand it is clear (by using the Taylor expansion around the diagonal) that the ordinary differential 1-forms are smooth functions of $C^\infty(M\times M)$ vanishing on the diagonal of $M\times M$ modulo the functions vanishing to order one on the diagonal of $M\times M$. The subtility here lies in the fact that without completion of the tensor product, the inclusion $C^\infty(M)\otimes C^\infty(M)\subset C^\infty(M\times M)$ is a strict one so there is generally a slight difference between $\Omega^1_{C^\infty(M)|\mathbb{C}}$ and the module $\Omega^1(M)$ of smooth 1-forms on M. Apart from this, one can consider that $(\Omega^1_Z(\mathcal{A}), d_Z)$ generalizes the ordinary first order differential calculus. This is in contrast to what happens for $(\Omega^1_u(\mathcal{A}), d_u)$. Indeed if $\mathcal{A}$ is an algebra of functions on a set S containing more than one element, $(\mathrm{card}(S)>1)$, then $\Omega^1_u(\mathcal{A})$ consists of functions on $S\times S$ which vanish on the diagonal and is therefore *not* the underlying bimodule of a module (nonlocality) whereas $(d_u f)(x,y) = f(y)-f(x)$ $(x,y\in S)$ is the finite difference.

7 Higher order differential calculi

Let $\mathcal{A}$ be as before a unital associative complex algebra. A $\mathbb{N}$-graded differential algebra $\Omega = \oplus_{n\geq 0}\Omega^n$ such that the subalgebra Ω^0 of its elements of degree 0 coincides with $\mathcal{A}$, $\Omega^0 = \mathcal{A}$, will be called a *differential calculus over $\mathcal{A}$ for* **Alg** or simply a *differential calculus over $\mathcal{A}$*. If furthermore the Ω^n $(n\in\mathbb{N})$ are central bimodules over $\mathcal{A}$, (i.e. $\mathcal{A}$-bimodules for $\mathbf{Alg}_Z$), Ω will be said to be a *differential calculus over $\mathcal{A}$ for* $\mathbf{Alg}_Z$.

Let us define the $(\mathcal{A},\mathcal{A})$-bimodules $\Omega^n_u(\mathcal{A})$ for $n\geq 0$ by $\Omega^0_u(\mathcal{A}) = \mathcal{A}$

and by $\Omega^n_u(\mathcal{A}) = \Omega^1_u(\mathcal{A}) \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \Omega^1_u(\mathcal{A})$ (n factors) for $n \geq 1$. The direct sum $\Omega_u(\mathcal{A}) = \oplus_{n\geq 0} \Omega^n_u(\mathcal{A})$ is canonically a graded algebra, it is *the tensor algebra over* $\mathcal{A}$, $T_{\mathcal{A}}(\Omega^1_u(\mathcal{A}))$, *of the* $(\mathcal{A},\mathcal{A})$*-bimodule* $\Omega^1_u(\mathcal{A})$. The derivation $d_u : \mathcal{A} \to \Omega^1_u(\mathcal{A})$ has a unique extension, again denoted by d_u, as a differential of $\Omega_u(\mathcal{A})$: In fact, it is known on $\mathcal{A} = \Omega^0_u(\mathcal{A})$ and $d^2_u = 0$ fixes it on $d_u(\mathcal{A})$ to be 0 so it is known on the generators of $\Omega_u(\mathcal{A})$ and the extension by the antiderivation property to the whole $\Omega_u(\mathcal{A})$ is well defined and unique; moreover, d^2_u is a derivation vanishing on the generators and therefore $d^2_u = 0$. So equipped, $\Omega_u(\mathcal{A})$ is a graded differential algebra [46] which is characterized uniquely (up to an isomorphism) by the following universal property.

PROPOSITION 5 *Any homomorphism φ of unital algebras of $\mathcal{A}$ into the subalgebra Ω^0 of elements of degree 0 of a graded differential algebra Ω has a unique extension $\tilde{\varphi} : \Omega_u(\mathcal{A}) \to \Omega$ as a homomorphism of graded differential algebras.*

Proof. The (Ω^0, Ω^0)-bimodule Ω^1 can be considered as a $(\mathcal{A},\mathcal{A})$-bimodule by setting $x\lambda y = \varphi(x)\lambda\varphi(y)$ for $x, y \in \mathcal{A}$ and $\lambda \in \Omega^1$ and then $d \circ \varphi$ defines a derivation of $\mathcal{A}$ into Ω^1. Therefore, by Proposition 1, there is a unique bimodule homomorphism $\varphi^1 : \Omega^1_u(\mathcal{A}) \to \Omega^1$ such that $d \circ \varphi = \varphi^1 \circ d_u : \mathcal{A} \to \Omega^1$ (namely $\varphi^1 = i_{d\circ\varphi}$). The property of $\Omega_u(\mathcal{A})$ to be the tensor algebra $T_{\mathcal{A}}(\Omega^1_u(\mathcal{A}))$ implies that φ and φ^1 uniquely extend as a homomorphism $\tilde{\varphi} : \Omega_u(\mathcal{A}) \to \Omega$ of graded algebras. By construction one has $\tilde{\varphi} \circ d_u = d \circ \tilde{\varphi}$ on $\mathcal{A}$ and on $d_u\mathcal{A}$ where it vanishes which implies $\tilde{\varphi} \circ d_u = d \circ \tilde{\varphi}$ everywhere by the antiderivation property of d_u and d. □

The graded differential algebra $\Omega_u(\mathcal{A})$ is called (in view of the above universal property) *the universal differential calculus over* $\mathcal{A}$ (it is universal for **Alg**). The functorial property follows immediately: *For any homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ (of unital $\mathbb{C}$-algebras), there is a unique homomorphism $\Omega_u(\varphi) : \Omega_u(\mathcal{A}) \to \Omega_u(\mathcal{B})$ of graded differential algebra which extends φ (i.e. $\varphi = \Omega_u(\varphi) \upharpoonright \mathcal{A}$).* This defines the covariant functor Ω_u from the category **Alg** to the category **Dif** of graded differential algebras (the morphisms being the homomorphisms of graded differential algebras preserving the units).

Proposition 5 is clearly a generalization of Proposition 1. There is another useful generalization of the universality of the Hochschild 1-cocycle $a \mapsto d_u(a)$ (which is the content of Proposition 1) and of Remark 6 which is described in

[8] (see also in [22]) and which we now review (Proposition 6 below). First, notice that $(a_1, \dots, a_n) \mapsto d_u(a_1) \dots d_u(a_n)$ is a $\Omega^n_u(\mathcal{A})$-valued Hochschild n-cocycle which is normalized (i.e. which vanishes whenever one of the a_i is the unit $\mathbb{1}$ of $\mathcal{A}$). Second, notice that the short exact sequence of Section 6 (before Proposition 1) has the following generalization for $n \geq 1$

$$0 \to \Omega^n_u(\mathcal{A}) \stackrel{\subsetneq}{\to} \mathcal{A} \otimes \Omega^{n-1}_u(\mathcal{A}) \stackrel{m}{\to} \Omega^{n-1}_u(\mathcal{A}) \to 0$$

2184 as short exact sequence of $(\mathcal{A}, \mathcal{A})$-bimodules, where m is the left multiplication by elements of $\mathcal{A}$ of elements of $\Omega^{n-1}_u(\mathcal{A})$, (the inclusion is canonical). One has the following [8].

PROPOSITION 6 *Let $\mathcal{M}$ be a $(\mathcal{A}, \mathcal{A})$-bimodule and let $(a_1, \dots, a_n) \mapsto c(a_1, \dots, a_n)$ be a normalized $\mathcal{M}$-valued Hochschild n-cocyle. Then, there is a unique bimodule homomorphism $i_c : \Omega^n_u(\mathcal{A}) \to \mathcal{M}$ such that*

$$c(a_1, \dots, a_n) = i_c(d_u(a_1) \dots d_u(a_n)), \ \forall a_i \in \mathcal{A}.$$

Furthermore, c is a Hochschild coboundary if and only if i_c has an extension $\tilde{i}_c$ as a bimodule homomorphism of $\mathcal{A} \otimes \Omega^{n-1}_u(\mathcal{A})$ into $\mathcal{M}$.

Proof. We only give here some indications and refer to [8] for the detailed proof. The proof of the first part proceeds exactly as the proof of Proposition 1: One first shows that the mapping $a_0 \otimes a_1 \otimes \dots \otimes a_n \mapsto a_0 d_u(a_1) \dots d_u(a_n)$ induces a left module isomorphism of $\mathcal{A} \otimes (\otimes^n(\mathcal{A}/\mathbb{C}\mathbb{1}))$ onto $\Omega^n_u(\mathcal{A})$ which implies that $a_0 d_u(a_1) \dots d_u(a_n) \mapsto a_0 c(a_1, \dots, a_n)$ defines a left module homomorphism i_c of $\Omega^n_u(\mathcal{A})$ into $\mathcal{M}$; the cocycle property of c then implies that i_c is a bimodule homomorphism. Again uniqueness is straightforward. Concerning the last part, if there is an extension $\tilde{i}_c$ to $\mathcal{A} \otimes \Omega^{n-1}_u(\mathcal{A})$, then c is the Hochschild coboundary of $(a_1, \dots, a_{n-1}) \mapsto \tilde{i}_c(\mathbb{1} \otimes d_u(a_1) \dots d_u(a_{n-1}))$ and conversely, if c is the coboundary of a normalized $(n-1)$-cochain c' then one defines an extension $\tilde{i}_c$ by setting $\tilde{i}_c(\mathbb{1} \otimes d_u(a_1) \dots d_u(a_{n-1})) = c'(a_1, \dots, a_{n-1})$. $\square$

Thus, for each integer $n \geq 1$, the normalized n-cocycle $d_u^{\cup n}$, defined by $d_u^{\cup n}(a_1, \dots, a_n) = d_u(a_1) \dots d_u(a_n)$, is universal among the normalized Hochschild n-cocyles.

By its very construction, $\Omega_u(\mathcal{A})$ is a graded subalgebra of the tensor algebra over $\mathcal{A}$, $T_{\mathcal{A}}(\mathcal{A} \otimes \mathcal{A})$, of the $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{A} \otimes \mathcal{A}$. Indeed $T^n_{\mathcal{A}}(\mathcal{A} \otimes \mathcal{A})$ is the $(n+1)$-th tensor power (over $\mathbb{C}$) $\otimes^{n+1}\mathcal{A}$ of $\mathcal{A}$ whereas $\Omega^n_u(\mathcal{A}) = T^n_{\mathcal{A}}(\Omega^1_u(\mathcal{A}))$ is the intersection of the kernels of the $(\mathcal{A},\mathcal{A})$-bimodule homomorphisms $m_k : \otimes^{n+1}\mathcal{A} \to \otimes^n\mathcal{A}$ defined by

$$\begin{array}{ll} m_1(x_0 \otimes \cdots \otimes x_n) & = x_0x_1 \otimes x_2 \otimes \cdots \otimes x_n \\ \dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots \\ m_n(x_0 \otimes \cdots \otimes x_n) & = x_1 \otimes \cdots \otimes x_{n-2} \otimes x_{n-1}x_n \end{array}$$

(i.e. m_k is the product in $\mathcal{A}$ of the consecutive factors x_{k-1} and x_k). It turns out that the differential of $\Omega_u(\mathcal{A})$ has an extension, again denoted by d_u, as a differential of $T_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})$ which is defined by

$$d_u(x_0 \otimes \cdots \otimes x_n) = \sum_{k=0}^{n+1} (-1)^k \quad x_0 \otimes \cdots \otimes x_{k-1} \otimes \mathbb{1} \otimes x_k \otimes \cdots \otimes x_n$$

for $x_i \in \mathcal{A}$ where the first term of the summation is $\mathbb{1} \otimes x_0 \otimes \cdots \otimes x_n$ and the last term is $(-1)^{n+1}x_0 \otimes \cdots \otimes x_n \otimes \mathbb{1}$ (by convention). So equipped $T_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})$ is a graded differential algebra, in fact a differential calculus over $\mathcal{A}$, and $\Omega_u(\mathcal{A})$ is a graded-differential subalgebra.

LEMMA 5 *The cohomologies of $T_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})$ and of $\Omega_u(\mathcal{A})$ are trivial in the sense that one has : $H^0(T_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})) = H^0(\Omega_u(\mathcal{A})) = \mathbb{C}$ and $H^n(T_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})) = H^n(\Omega_u(\mathcal{A})) = 0$ for $n \geq 1$.*

Proof. Define $\delta : \mathbb{C} \to \mathcal{A}$ by $\delta(\lambda) = \lambda\mathbb{1}$, one has $d_u \circ \delta = 0$ so

$$0 \to \mathbb{C} \xrightarrow{\delta} \mathcal{A} \xrightarrow{d_u} \mathcal{A}\otimes\mathcal{A} \xrightarrow{d_u} \cdots \xrightarrow{d_u} \otimes^{n+1}\mathcal{A} \xrightarrow{d_u} \otimes^{n+2}\mathcal{A} \xrightarrow{d_u} \dots$$

is a cochain complex with coboundary d being d_u or δ. Let ω be a linear form on $\mathcal{A}$ such that $\omega(\mathbb{1}) = 1$ and define k by $k(\mathbb{C}) = 0$ and by $k(x_0 \otimes \cdots \otimes x_n) = \omega(x_0)x_1 \otimes \cdots \otimes x_n$ for $n \geq 0$. One has $kd + dk = I$ which implies $H^n(T_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})) = 0$ for $n \geq 1$ and $H^0(T_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})) = H^0(\Omega_u(\mathcal{A})) = \mathbb{C}$. Then $H^n(\Omega_u(\mathcal{A})) = 0$ for $n \geq 1$ follows from the fact that one has $k(\Omega^n_u(\mathcal{A})) \subset \Omega^{n-1}_u(\mathcal{A})$ for $n \geq 1$. □

Remark 9. The graded differential algebra $C(\mathcal{A},\mathcal{A})$ of $\mathcal{A}$-valued Hochschild cochains of $\mathcal{A}$ (see in Section 4) is a differential calculus over $\mathcal{A}$. Therefore, by Proposition 5 there is a unique homomorphism Φ of $\Omega_u(\mathcal{A})$ into $C(\mathcal{A},\mathcal{A})$ of graded differential algebras which induces the identity mapping of $\mathcal{A}$ onto itself. This homomorphism extends to $T_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})$ i.e. as an homomorphism $\Phi : T_{\mathcal{A}}(\mathcal{A} \otimes \mathcal{A}) \to C(\mathcal{A}, \mathcal{A})$ of graded differential algebras which is given by $\Phi(x_0 \otimes \cdots \otimes x_n)(y_1, \dots, y_n) = x_0 y_1 x_1 \dots y_n x_n$, [55]. Notice that $\Phi(\Omega_u(\mathcal{A}))$ is contained in the graded differential subalgebra $C_0(\mathcal{A}, \mathcal{A})$ of the normalized cochains of $C(\mathcal{A}, \mathcal{A})$.

In Section 6 we have defined the central bimodule $\Omega^1_Z(\mathcal{A})$ to be the quotient of $\Omega^1_u(\mathcal{A})$ by the bimodule $[Z(\mathcal{A}), \Omega^1_u(\mathcal{A})]$ and the derivation d_Z of $\mathcal{A}$ into $\Omega^1_Z(\mathcal{A})$ to be the image of $d_u : \mathcal{A} \to \Omega^1_u(\mathcal{A})$. Let I_Z be the closed two-sided ideal of $\Omega_u(\mathcal{A})$ generated by $[Z(\mathcal{A}), \Omega^1_u(\mathcal{A})]$ i.e. the two-sided ideal generated by $[Z(\mathcal{A}), \Omega^1_u(\mathcal{A})]$ and $d_u([Z(\mathcal{A}), \Omega^1_u(\mathcal{A})])$. The space I_Z is a graded ideal which is closed and such that $I_Z \cap \Omega^1_u(\mathcal{A}) = [Z(\mathcal{A}), \Omega^1_u(\mathcal{A})]$ which implies that the quotient $\Omega_Z(\mathcal{A})$ is a graded differential algebra which coincides in degree 1 with the above $\Omega^1_Z(\mathcal{A})$ and that its differential (the image of d_u) extends $d_Z : \mathcal{A} \to \Omega^1_Z(\mathcal{A})$; this differential will be also denoted by d_Z. By construction, $\Omega_Z(\mathcal{A})$ is, as graded algebra, a quotient of the tensor algebra over $\mathcal{A}$ of the central bimodule $\Omega^1_Z(\mathcal{A})$; on the other hand it is easily seen that tensor products over $\mathcal{A}$ of central bimodules and quotients of central bimodules are again central bimodules [35] so the $(\mathcal{A}, \mathcal{A})$-bimodules $\Omega^n_Z(\mathcal{A})$ are central bimodules ($\Omega_Z(\mathcal{A}) = \oplus_n \Omega^n_Z(\mathcal{A})$) and therefore the graded differential algebra $\Omega_Z(\mathcal{A})$ is a differential calculus over $\mathcal{A}$ for $\mathbf{Alg}_Z$. Proposition 5 has the following counterpart for $\Omega_Z(\mathcal{A})$.

PROPOSITION 7 *Any homomorphism φ of unital algebras of $\mathcal{A}$ into the subalgebra Ω^0 of elements of degree 0 of a graded differential algebra Ω which is such that $\varphi(z)d(\varphi(x)) = d(\varphi(x))\varphi(z)$ for any $z \in Z(\mathcal{A})$ and $x \in \mathcal{A}$, (d being the differential of Ω), has a unique extension $\tilde{\varphi}_Z : \Omega_Z(\mathcal{A}) \to \Omega$ as a homomorphism of graded differential algebras.*

Proof. By Proposition 5, φ has a unique extension $\tilde{\varphi} : \Omega_u(\mathcal{A}) \to \Omega$ as homomorphism of graded differential algebras. On the other hand $\varphi(z)d(\varphi(x)) = d(\varphi(x))\varphi(z)$ for $z \in Z(\mathcal{A})$ and $x \in \mathcal{A}$ implies that $\tilde{\varphi}$ vanishes on $[Z(\mathcal{A}), \Omega^1_u(\mathcal{A})]$ and therefore also on I_Z since it is a homomorphism of graded differential algebras. Thus $\tilde{\varphi}$ factorizes through a homomorphism $\tilde{\varphi}_Z : \Omega_Z(\mathcal{A}) \to \Omega$ of graded differential algebras which extends φ. Uniqueness is also straightforward here. □

Proposition 7 has the following corollaries.

COROLLARY 2 *For any differential calculus Ω over $\mathcal{A}$ for* $\mathbf{Alg}_Z$*, there* *is a unique homomorphism $j_\Omega : \Omega_Z(\mathcal{A}) \to \Omega$ of differential algebras which induces the identity mapping of $\mathcal{A}$ onto itself.*

In other words $\Omega_Z(\mathcal{A})$ is universal among the differential calculi over $\mathcal{A}$ for $\mathbf{Alg}_Z$ and this universal property characterizes it (up to an isomorphism). This is why we shall refer to $\Omega_Z(\mathcal{A})$ as *the universal differential calculus over $\mathcal{A}$ for $\mathbf{Alg}_Z$*.

COROLLARY 3 *Any homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of unital algebras mapping the center $Z(\mathcal{A})$ of $\mathcal{A}$ into the center $Z(\mathcal{B})$ of $\mathcal{B}$ has a unique extension $\Omega_Z(\varphi) : \Omega_Z(\mathcal{A}) \to \Omega_Z(\mathcal{B})$ as a homomorphism of graded differential algebras.*

In fact Ω_Z is a covariant functor from the category $\mathbf{Alg}_Z$ to the category **Dif** of graded differential algebras.

In Section 2 it was pointed out that the graded center of a graded algebra is stable by the graded derivations. This implies in particular that the graded center $Z_{\mathrm{gr}}(\Omega)$ of a graded differential algebra Ω is a graded differential subalgebra of Ω which is graded commutative. We have defined a differential calculus over $\mathcal{A}$ for $\mathbf{Alg}_Z$ to be a graded differential algebra Ω such that $\Omega^0 = \mathcal{A}$ and such that the center $Z(\mathcal{A})$ of $\mathcal{A}(= \Omega^0)$ is contained in the center of Ω i.e. in its graded center $Z_{\mathrm{gr}}(\Omega)$ since its elements are of degree zero in Ω. It follows that if Ω is a differential calculus over $\mathcal{A}$ for $\mathbf{Alg}_Z$ then the center $Z(\mathcal{A})$ of $\mathcal{A}$ generates a graded differential subalgebra of Ω which is graded commutative and is in fact a graded differential subalgebra of the graded center $Z_{\mathrm{gr}}(\Omega)$ of Ω. This applies in particular to Ω_Z. If $\mathcal{A}$ is commutative then $\Omega_Z(\mathcal{A})$ is graded commutative since it is generated by $\mathcal{A}$ which coincides then with its center. In this case Proposition 7 has the following corollary.

COROLLARY 4 *If $\mathcal{A}$ is commutative $\Omega_Z(\mathcal{A})$ identifies canonically with the graded differential algebra $\Omega_{\mathcal{A}|\mathbb{C}}$ of Cartan-de Rham-Kähler exterior differential forms.*

Proof. Let us recall that $\Omega_{\mathcal{A}|\mathbb{C}}$ is the exterior algebra over $\mathcal{A}$ of the module $\Omega^1_{\mathcal{A}|\mathbb{C}}$ of Kähler differential, $\Lambda_{\mathcal{A}}\Omega^1_{\mathcal{A}|\mathbb{C}}$, equipped with the unique differential extending the universal derivation of $\mathcal{A}$ into $\Omega^1_{\mathcal{A}|\mathbb{C}}$. From this definition and the universality of the derivation of $\mathcal{A}$ into $\Omega^1_{\mathcal{A}|\mathbb{C}}$ (which identifies, in view of Corollary 1, with $d_Z : \mathcal{A} \to \Omega^1_Z(\mathcal{A})$) it follows that $\Omega_{\mathcal{A}|\mathbb{C}}$ is characterized

by the following universal property: *Any homomorphism ψ of $\mathcal{A}$ into the subalgebra Ω^0 of the elements of degree 0 of a graded commutative differential algebra Ω has a unique extension $\tilde{\psi} : \Omega_{\mathcal{A}|\mathbb{C}} \to \Omega$ as a homomorphism of graded (commutative) differential algebras.*
Let us come back to the proof of Corollary 4. Since $\Omega_Z(\mathcal{A})$ is graded commutative with $\Omega^0_Z(\mathcal{A}) = \mathcal{A}$, the above universal property implies that there is a unique homomorphism of graded differential algebras of $\Omega_{\mathcal{A}|\mathbb{C}}$ into $\Omega_Z(\mathcal{A})$ which induces the identity mapping of $\mathcal{A}$ onto itself. On the other hand Proposition 7 (or Corollary 2) implies that there is a unique homomorphism of graded differential algebras of $\Omega_Z(\mathcal{A})$ into $\Omega_{\mathcal{A}|\mathbb{C}}$ which induces the identity of $\mathcal{A}$ onto itself. Using again these two universal properties, it follows that the above homomorphisms are inverse isomorphisms. □

If $\mathcal{A}$ is commutative the cohomology of $\Omega_Z(\mathcal{A}) = \Omega_{\mathcal{A}|\mathbb{C}}$ if often called the de Rham cohomology ([52], [43]) in spite of the fact that as explained in Remark 8, for $\mathcal{A} = C^\infty(M)$, $\Omega_{\mathcal{A}|\mathbb{C}}$ can be slightly different from the algebra of smooth differential forms and that therefore there is an ambiguity. Nevertheless $\Omega_Z(\mathcal{A})$ can be considered as a generalization of the graded differential algebra of differential forms which has the great advantage that the correspondence $\mathcal{A} \mapsto \Omega_Z(\mathcal{A})$ is functorial (Corollary 3). In contrast to the cohomology of $\Omega_u(\mathcal{A})$, (see Lemma 5), the cohomology $H_Z(\mathcal{A})$ of $\Omega_Z(\mathcal{A})$ is generally non trivial. Since $H_Z(\mathcal{A})$ is a noncommutative generalization of the de Rham cohomology and since, by construction, $\mathcal{A} \mapsto H_Z(\mathcal{A})$ is a covariant functor from the category $\mathbf{Alg}_Z$ to the category of graded algebras, it is natural to study the properties of this cohomology.

Let $\mathrm{Der}(\mathcal{A})$ denote the vector space of all derivations of $\mathcal{A}$ into itself. This vector space is a Lie algebra for the bracket $[\cdot,\cdot]$ defined by $[X,Y](a) = X(Y(a)) - Y(X(a))$ for $X, Y \in \mathrm{Der}(\mathcal{A})$ and $a \in \mathcal{A}$. In view of Proposition 1, (universal property of $(\Omega^1_u(\mathcal{A}), d_u)$), for each $X \in \mathrm{Der}(\mathcal{A})$ there is a unique bimodule homomorphism $i_X : \Omega^1_u(\mathcal{A}) \to \mathcal{A}$ for which $X = i_X \circ d_u$. This homomorphism of $\Omega^1_u(\mathcal{A})$ into $\mathcal{A} = \Omega^0_u(\mathcal{A})$ has a unique extension as an antiderivation of $\Omega_u(\mathcal{A}) = T_{\mathcal{A}}(\Omega^1_u(\mathcal{A}))$. This antiderivation which will be again denoted by i_X is of degree -1, (i.e. it is a graded derivation of degree -1). It is not hard to verify that $X \mapsto i_X$ is an operation of the Lie algebra $\mathrm{Der}(\mathcal{A})$ in the graded differential algebra $\Omega_u(\mathcal{A})$, (see Section 2 for the definition). The corresponding Lie derivative $L_X = i_X d_u + d_u i_X$ is for $X \in \mathrm{Der}(\mathcal{A})$ a derivation of degree 0 of $\Omega_u(\mathcal{A})$ which extends X. This operation will be refered to as *the canonical operation of* $\mathrm{Der}(\mathcal{A})$ *in* $\Omega_u(\mathcal{A})$.

Let $X \in \mathrm{Der}(\mathcal{A})$ be a derivation of $\mathcal{A}$ and let $z \in Z(\mathcal{A})$ and $\omega \in \Omega^1_u(\mathcal{A})$ one has

$$i_X([z,\omega]) = [z, i_X(\omega)] = 0$$

and

$$i_X(d([z,\omega])) = L_X([z,\omega]) = [X(z),\omega] + [z, L_X(\omega)] = [z, L_X(\omega)]$$

since $Z(\mathcal{A})$ is stable by the derivations of $\mathcal{A}$. This implies that $i_X(I_Z) \subset I_Z$ and therefore that the antiderivation i_X passes to the quotient and defines an antiderivation of degree -1 of $\Omega_Z(\mathcal{A})$ which will be again denoted by i_X. Notice that this (abuse of) notation is coherent with the one used in Proposition 3, ($\mathcal{A}$ is obviously a central bimodule). The corresponding mapping $X \mapsto i_X$ of $\mathrm{Der}(\mathcal{A})$ into the antiderivations of degree -1 of $\Omega_Z(\mathcal{A})$ is again an operation (the quotient of the one in $\Omega_u(\mathcal{A})$) which will be refered to as *the canonical operation of* $\mathrm{Der}(\mathcal{A})$ *in* $\Omega_Z(\mathcal{A})$.

Finally if $\mathcal{A}$ is a $*$-algebra, $T_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})$ is a graded differential $*$-algebra if one equips it with the involution defined by $(x_0 \otimes \cdots \otimes x_n)^* = (-1)^{\frac{n(n+1)}{2}} x_n^* \otimes \cdots \otimes x_0^*$. Since $\Omega_u(\mathcal{A})$ is stable by this involution, it is also a graded differential $*$-algebra, [61]. Furthermore $[Z(\mathcal{A}), \Omega^1_u(\mathcal{A})]$ is $*$-invariant which implies that the involution of $\Omega_u(\mathcal{A})$ passes to the quotient and induces an involution on $\Omega_Z(\mathcal{A})$ for which $\Omega_Z(\mathcal{A})$ also becomes a graded differential $*$-algebra. More generally in this case, a differential calculus Ω over $\mathcal{A}$ will always be assumed to be equipped with an involution extending the involution of $\mathcal{A}$ and such that it is a graded differential $*$-algebra, (notice that if Ω is generated by $\mathcal{A}$ such an involution is unique).

8 Diagonal and derivation-based calculi

Let $\mathcal{A}$ be a unital associative complex algebra and let $\mathcal{M}$ be an arbitrary $(\mathcal{A},\mathcal{A})$-bimodule. Then the set $\mathrm{Hom}^{\mathcal{A}}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ of all bimodule homomorphisms of $\mathcal{M}$ into $\mathcal{A}$ is a module over the center $Z(\mathcal{A})$ of $\mathcal{A}$ which will be refered to as the $\mathcal{A}$-*dual of* $\mathcal{M}$ and denoted by $\mathcal{M}^{*_{\mathcal{A}}}$, [34], [27]. Conversely, if $\mathcal{N}$ is a $Z(\mathcal{A})$-module the set $\mathrm{Hom}_{Z(\mathcal{A})}(\mathcal{N},\mathcal{A})$ of all $Z(\mathcal{A})$-module homorphisms of $\mathcal{N}$ into $\mathcal{A}$ is canonically a $(\mathcal{A},\mathcal{A})$-bimodule which will be also refered to as the *$\mathcal{A}$-dual of $\mathcal{N}$* and denoted by $\mathcal{N}^{*_\mathcal{A}}$. The $\mathcal{A}$-dual of a $Z(\mathcal{A})$-module is clearly a central bimodule over $\mathcal{A}$ so the above duality between $(\mathcal{A},\mathcal{A})$-bimodules and $Z(\mathcal{A})$-modules can be restricted to a duality between the central bimodules over $\mathcal{A}$ and the $Z(\mathcal{A})$-modules. This latter duality generalizes the duality between modules over a commutative algebra, [34], [27]. Indeed, if $\mathcal{A}$ is commutative both central bimodules over $\mathcal{A}$ and $Z(\mathcal{A})$-modules coincide with $\mathcal{A}$-modules and the above duality is then the usual duality between $\mathcal{A}$-modules. Let us come back to the general situation and let $\mathcal{M}$ be a $(\mathcal{A},\mathcal{A})$-bimodule; then one obtains by evaluation a *canonical homomorphism* of $(\mathcal{A},\mathcal{A})$-bimodule $c:\mathcal{M}\rightarrow\mathcal{M}^{*_\mathcal{A}*_\mathcal{A}}$ of $\mathcal{M}$ into *its $\mathcal{A}$-bidual* $\mathcal{M}^{*_\mathcal{A}*_\mathcal{A}}=(\mathcal{M}^{*_\mathcal{A}})^{*_\mathcal{A}}$.

LEMMA 6 *The following properties (a) and (b) are equivalent for a $(\mathcal{A},\mathcal{A})$-bimodule $\mathcal{M}$.*
(a) The canonical homomorphism $c:\mathcal{M}\rightarrow\mathcal{M}^{*_\mathcal{A}*_\mathcal{A}}$ is injective.
(b) $\mathcal{M}$ is isomorphic to a subbimodule of $\mathcal{A}^I$ for some set I.

Proof. $(a)\Rightarrow(b)$. By definition $\mathcal{M}^{*_\mathcal{A}*_\mathcal{A}}$ is a subbimodule of $\mathcal{A}^I$ with $I=\mathrm{Hom}_{Z(\mathcal{A})}(\mathcal{M}^{*_\mathcal{A}},\mathcal{A})$ so if c is injective $\mathcal{M}$ is isomorphic to a subbimodule of $\mathcal{M}^{*_\mathcal{A}*_\mathcal{A}}$ and therefore also to a subbimodule of $\mathcal{A}^I$.

$(b)\Rightarrow(a)$. Let φ be a bimodule homomorphism of $\mathcal{A}$ into itself. One has $\varphi(a)=a\varphi(\mathbb{1})=\varphi(\mathbb{1})a$ which implies $\varphi(\mathbb{1})\in Z(\mathcal{A})$. Conversely any $z\in Z(\mathcal{A})$ defines a bimodule homomorphism φ of $\mathcal{A}$ into itself by setting $\varphi(a)=az$ (i.e. $\varphi(\mathbb{1})=z$). It follows that $\mathcal{A}^{*_\mathcal{A}}=Z(\mathcal{A})$. Let Φ be a $Z(\mathcal{A})$-module homomorphism of $Z(\mathcal{A})$ into $\mathcal{A}$. Then $\Phi(z)=z\Phi(\mathbb{1})$ with $\Phi(\mathbb{1})\in\mathcal{A}$. Conversely any $a\in\mathcal{A}$ defines such a $Z(\mathcal{A})$-module homomorphism Φ by setting $\Phi(z)=za$ (i.e. $\Phi(\mathbb{1})=a$). It follows that $Z(\mathcal{A})^{*_\mathcal{A}}=\mathcal{A}$ and therefore $\mathcal{A}^{*_\mathcal{A}*_\mathcal{A}}=\mathcal{A}$. This immediately implies that if $\mathcal{M}\subset\mathcal{A}^I$ as subbimodule then $c:\mathcal{M}\rightarrow\mathcal{M}^{*_\mathcal{A}*_\mathcal{A}}$ is injective. $\square$

An $(\mathcal{A},\mathcal{A})$-bimodule $\mathcal{M}$ satisfying the equivalent conditions of Lemma 6 will be said to be a *diagonal bimodule* over $\mathcal{A}$, [34], [35] (see also in [27]). A diagonal bimodule is central but the converse is not generally true. The $\mathcal{A}$-dual of an arbitrary $Z(\mathcal{A})$-module is a diagonal bimodule. Every subbimodule of a diagonal bimodule is diagonal, every product of diagonal bimodules is diagonal and the tensor product over $\mathcal{A}$ of two diagonal bimodules is diagonal.

If $\mathcal{A}$ is commutative, a diagonal bimodule over $\mathcal{A}$ is simply a $\mathcal{A}$-module such that the canonical homomorphism in its bidual is injective. In particular in this case a projective module is diagonal (as a bimodule for the underlying structure).

Remark 10. It is a fortunate circumstance which is easy to verify that, for a $Z(\mathcal{A})$-module $\mathcal{N}$, the biduality does not depend on $\mathcal{A}$ but only on $Z(\mathcal{A})$. That is one has $\mathcal{N}^{*_\mathcal{A}*_\mathcal{A}} = \mathcal{N}^{**}$ and the canonical homomorphism $c : \mathcal{N} \to \mathcal{N}^{**}$ obtained by evaluation for the $\mathcal{A}$-duality reduces to the usual one for a module over the commutative algebra $Z(\mathcal{A})$.

Let $\mathcal{M}$ be a $(\mathcal{A},\mathcal{A})$-bimodule then the canonical image $c(\mathcal{M})$ of $\mathcal{M}$ in its $\mathcal{A}$-bidual is a diagonal bimodule. The diagonal bimodule $c(\mathcal{M})$ is the universal "diagonalization" of $\mathcal{M}$ in the sense that it is characterized (among the diagonal bimodules over $\mathcal{A}$) by the following universal property, [34], [35].

PROPOSITION 8 *For any homomorphism of $(\mathcal{A},\mathcal{A})$-bimodules $\varphi : \mathcal{M} \to \mathcal{N}$ of $\mathcal{M}$ into a diagonal bimodule $\mathcal{N}$ over $\mathcal{A}$, there is a unique homomorphism of $(\mathcal{A},\mathcal{A})$-bimodules $\varphi_c : c(\mathcal{M}) \to \mathcal{N}$ such that $\varphi = \varphi_c \circ c$.*

Proof. In view of the definition and Lemma 6 (b), it is sufficient to prove the statement for $\mathcal{N} = \mathcal{A}^I$ for some set I, which is then equivalent to the statement for $\mathcal{N} = \mathcal{A}$. On the other hand, for $\mathcal{N} = \mathcal{A}$, $\varphi \in \mathrm{Hom}^{\mathcal{A}}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) = \mathcal{M}^{*_\mathcal{A}}$ and one has $\varphi(m) =< c(m), \varphi >= \varphi_c(c(m))$ for $m \in \mathcal{M}$ (by the definitions of $\mathcal{M}^{*_\mathcal{A}*_\mathcal{A}}$ and of the evaluation c) which defines φ_c uniquely. □

One has $c(\Omega^1_u(\mathcal{A})) = c(\Omega^1_Z(\mathcal{A}))$ and we shall denote by $\Omega^1_{\mathrm{Diag}}(\mathcal{A})$ this diagonal bimodule and by d_{Diag} the derivation $c \circ d_u$ (or equivalently $c \circ d_Z$) of $\mathcal{A}$ into $\Omega^1_{\mathrm{Diag}}(\mathcal{A})$.

PROPOSITION 9 *For any first order differential calculus (Ω^1, d) over $\mathcal{A}$ such that Ω^1 is diagonal, there is a unique bimodule homomorphism i_d of $\Omega^1_{\mathrm{Diag}}(\mathcal{A})$ into Ω^1 such that $d = i_d \circ d_{\mathrm{Diag}}$.*

Proof. In view of the above universal property of $c(\Omega^1_u(\mathcal{A}))$, the corresponding canonical homomorphism of $\Omega^1_u(\mathcal{A})$ into Ω^1 (as in Proposition 1) factorizes through a unique homomorphism $i_d : \Omega^1_{\mathrm{Diag}}(\mathcal{A}) \to \Omega^1$. □

In other words, the derivation $d_{\mathrm{Diag}} : \mathcal{A} \to \Omega^1_{\mathrm{Diag}}(\mathcal{A})$ of $\mathcal{A}$ into the diagonal bimodule $\Omega^1_{\mathrm{Diag}}(\mathcal{A})$ is universal for the derivations of $\mathcal{A}$ into diagonal bimodules over $\mathcal{A}$.

Let us recall (see Section 3) that the vector space $\mathrm{Der}(\mathcal{A})$ of all derivations of $\mathcal{A}$ into itself is a Lie algebra and also a $Z(\mathcal{A})$-module and that $\Omega_{\mathrm{Der}}(\mathcal{A})$ was defined to be the graded differential subalgebra of $C_\wedge(\mathrm{Der}(\mathcal{A}), \mathcal{A})$ generated by $\mathcal{A}$ whereas $\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$ was defined to be the graded differential subalgebra of $C_\wedge(\mathrm{Der}(\mathcal{A}), \mathcal{A})$ which consists of cochains of $\mathrm{Der}(\mathcal{A})$ which are $Z(\mathcal{A})$-multilinear. Clearly $C^n_\wedge(\mathrm{Der}(\mathcal{A}), \mathcal{A})$ is diagonal so the first order differential calculus $(C^1_\wedge(\mathrm{Der}(\mathcal{A}), \mathcal{A}), d)$ satisfies the conditions of Proposition 9 which implies that there is a unique bimodule homomorphism i_d of $\Omega^1_{\mathrm{Diag}}(\mathcal{A})$ into $C^1_\wedge(\mathrm{Der}(\mathcal{A}), \mathcal{A})$ for which $d = i_d \circ d_{\mathrm{Diag}}$.

PROPOSITION 10 *The homomorphism $i_d : \Omega^1_{\mathrm{Diag}}(\mathcal{A}) \to C^1_\wedge(Der(\mathcal{A}), \mathcal{A})$ is injective, so by identifying $\Omega^1_{\mathrm{Diag}}(\mathcal{A})$ with its image (by i_d), one has canonically:*

$$\Omega^1_{\mathrm{Diag}}(\mathcal{A}) = \Omega^1_{\mathrm{Der}}(\mathcal{A}), (\Omega^1_{\mathrm{Diag}}(\mathcal{A}))^{*_\mathcal{A}} = \mathrm{Der}(\mathcal{A}) \textit{ and } (\Omega^1_{\mathrm{Diag}}(\mathcal{A}))^{*_\mathcal{A}*_\mathcal{A}} = \underline{\Omega}^1_{\mathrm{Der}}(\mathcal{A}).$$

Proof. Applying Proposition 9 for $\Omega^1 = \mathcal{A}$ leads to the identification $\mathrm{Hom}^{\mathcal{A}}_{\mathcal{A}}(\Omega^1_{\mathrm{Diag}}(\mathcal{A}), \mathcal{A}) = \mathrm{Der}(\mathcal{A})$ that is $(\Omega^1_{\mathrm{Diag}}(\mathcal{A}))^{*_\mathcal{A}} = \mathrm{Der}(\mathcal{A})$, (notice that one has also $(\Omega^1_u(\mathcal{A}))^{*_\mathcal{A}} = \mathrm{Der}(\mathcal{A})$). By definition one has $\underline{\Omega}^1_{\mathrm{Der}}(\mathcal{A}) = \mathrm{Hom}_{Z(\mathcal{A})}(\mathrm{Der}(\mathcal{A}), \mathcal{A})$ that is $\underline{\Omega}^1_{\mathrm{Der}}(\mathcal{A}) = (\mathrm{Der}(\mathcal{A}))^{*_\mathcal{A}}$ and therefore $(\Omega^1_{\mathrm{Diag}}(\mathcal{A}))^{*_\mathcal{A}*_\mathcal{A}} = \underline{\Omega}^1_{\mathrm{Der}}(\mathcal{A})$. On the other hand one has $i_d(\Omega^1_{\mathrm{Diag}}(\mathcal{A})) = \Omega^1_{\mathrm{Der}}(\mathcal{A})$ since $\Omega^1_{\mathrm{Der}}(\mathcal{A})$ is generated by $\mathcal{A}$ (as bimodule). The injectivity of i_d follows from the fact that $\Omega^1_{\mathrm{Diag}}(\mathcal{A})$ is diagonal i.e. that the canonical homomorphism in its $\mathcal{A}$-bidual is injective. □

Notice that by definition one also has $(\bigwedge^n_{Z(\mathcal{A})} \mathrm{Der}(\mathcal{A}))^{*_\mathcal{A}} = \underline{\Omega}^n_{\mathrm{Der}}(\mathcal{A})$.

Let I_{Diag} be the closed two-sided ideal of $\Omega_u(\mathcal{A})$ generated by the kernel of the canonical homomorphism c of $\Omega^1_u(\mathcal{A})$ into its $\mathcal{A}$-bidual. The ideal I_{Diag} is graded such that $I_{\mathrm{Diag}} \cap \Omega^0_u(\mathcal{A}) = 0$ and $I_{\mathrm{Diag}} \cap \Omega^1_u(\mathcal{A}) = \mathrm{Ker}(c)$ which implies that the quotient $\Omega^1_u(\mathcal{A})/I_{\mathrm{Diag}}$ is a graded differential algebra which is a differential calculus over $\mathcal{A}$ and coincides in degree 1 with $c(\Omega^1_u(\mathcal{A})) = \Omega^1_{\mathrm{Diag}}(\mathcal{A})$. This differential calculus will be refered to as the *diagonal calculus* and denoted by $\Omega_{\mathrm{Diag}}(\mathcal{A})$. The differential of $\Omega_{\mathrm{Diag}}(\mathcal{A})$ is the image of d_u

and extends the derivation $d_{\rm Diag} : \mathcal{A} \to \Omega^1_{\rm Diag}(\mathcal{A})$; this differential will be also denoted by $d_{\rm Diag}$. Proposition 5 and Proposition 7 have the following counterpart for $\Omega_{\rm Diag}(\mathcal{A})$.

PROPOSITION 11 *Any homomorphism φ of unital algebras of $\mathcal{A}$ into the subalgebra Ω^0 of elements of degree 0 of a graded differential algebra Ω which is such that $d(\mathcal{A})$ spans a diagonal bimodule over $\mathcal{A}$ (for the $(\mathcal{A},\mathcal{A})$-bimodule structure on Ω^1 induced by φ) has a unique extension $\tilde{\varphi}_{\rm Diag} : \Omega_{\rm Diag}(\mathcal{A}) \to \Omega$ as a homomorphism of graded differential algebras.*

Proof. By Proposition 5, φ has a unique extension $\tilde{\varphi} : \Omega_u(\mathcal{A}) \to \Omega$ as homomorphism of graded differential algebras. On the other hand the assumption means that $d : \mathcal{A} \to \tilde{\varphi}(\Omega^1_u(\mathcal{A}))$ is a derivation and that $\tilde{\varphi}(\Omega^1_u(\mathcal{A}))$ is a diagonal bimodule over $\mathcal{A}$ so, in view of Proposition 9, the homomorphism $\tilde{\varphi} : \Omega^1_u(\mathcal{A}) \to \Omega^1$ factorizes through a homomorphism $\tilde{\varphi}^1_{\rm Diag} : \Omega^1_{\rm Diag}(\mathcal{A}) \to \Omega^1$. Thus $\tilde{\varphi}$ vanishes on $\mathrm{Ker}(c)$ and therefore on I_Z since it is a homomorphism of graded differential algebras so it factorizes through a homomorphism $\tilde{\varphi}_{\rm Diag} : \Omega_{\rm Diag}(\mathcal{A}) \to \Omega$ of graded differential algebras. Uniqueness is again straightforward. □

Thus $\Omega_{\rm Diag}(\mathcal{A})$ is also characterized by a universal property like $\Omega_u(\mathcal{A})$ and $\Omega_Z(\mathcal{A})$ but in contrast to the cases of $\Omega_u(\mathcal{A})$ and $\Omega_Z(\mathcal{A})$, the correspondence $\mathcal{A} \mapsto \Omega_{\rm Diag}(\mathcal{A})$ has no obvious functorial property. The reason for this is the fact that the diagonal bimodules are not the bimodules for a category of algebras in the sense explained in Section 5.

Proposition 11 implies in particular that one has a unique homomorphism of graded differential algebra of $\Omega_{\rm Diag}(\mathcal{A})$ into $\Omega_{\rm Der}(\mathcal{A})$ which extends the identity mapping of $\mathcal{A}$ onto itself. This homomorphism $\Omega_{\rm Diag}(\mathcal{A}) \to \Omega_{\rm Der}(\mathcal{A})$ is surjective since $\Omega_{\rm Der}(\mathcal{A})$ is generated by $\mathcal{A}$ as differential algebra. Furthermore in degree 1 it is, in view of Proposition 10, a bimodule isomorphism of $\Omega^1_{\rm Diag}(\mathcal{A})$ onto $\Omega^1_{\rm Der}(\mathcal{A})$. However, for $m \geq 2$, the corresponding bimodule homomorphism of $\Omega^m_{\rm Diag}(\mathcal{A})$ onto $\Omega^m_{\rm Der}(\mathcal{A})$ is not generally injective (i.e. it has a non trivial kernel).

For instance when $\mathcal{A}$ coincides with the algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices one has

$$\Omega_u(M_n(\mathbb{C})) = \Omega_Z(M_n(\mathbb{C})) = \Omega_{\mathrm{Diag}}(M_n(\mathbb{C})) \simeq$$

$$C_0(M_n(\mathbb{C}), M_n(\mathbb{C})) = M_n(\mathbb{C}) \otimes T\mathfrak{sl}(n,\mathbb{C})^*$$

whereas

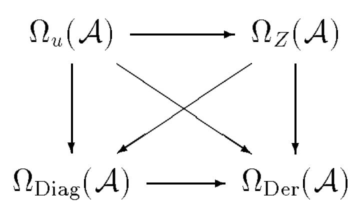
$$\Omega_{\mathrm{Der}}(M_n(\mathbb{C})) = C_\wedge(\mathfrak{sl}(n,\mathbb{C}), M_n(\mathbb{C})) = M_n(\mathbb{C}) \otimes \bigwedge \mathfrak{sl}(n,\mathbb{C})^*.$$

In fact, in this case, the homomorphism Φ of Remark 9 is an isomorphism which induces the isomorphism of $\Omega_u(M_n(\mathbb{C}))$ onto the differential algebra $C_0(M_n(\mathbb{C}), M_n(\mathbb{C}))$ of normalized Hochschild cochains; the latter being identical as graded algebra to the tensor product $M_n(\mathbb{C}) \otimes T\mathfrak{sl}(n,\mathbb{C})^*$ of $M_n(\mathbb{C})$ with the tensor algebra over $\mathbb{C}$ of the dual of $\mathfrak{sl}(n,\mathbb{C})$, (concerning $\Omega^1_{\mathrm{Der}}(M_n(\mathbb{C})) = \Omega^1_u(M_n(\mathbb{C}))$, and $\Omega_{\mathrm{Der}}(M_n(\mathbb{C})) = C_\wedge(\mathfrak{sl}(n,\mathbb{C}), M_n(\mathbb{C}))$, see in [25]).

In the case where $\mathcal{A}$ is the algebra $C^\infty(M)$ of smooth functions on a good smooth manifold (finite dimensional paracompact, etc.) then one has $\Omega_{\mathrm{Diag}}(C^\infty(M)) = \Omega_{\mathrm{Der}}(C^\infty(M))$ $(= \underline{\Omega}_{\mathrm{Der}}(C^\infty(M)))$.

It is not hard to show that the operations of the Lie algebra $\mathrm{Der}(\mathcal{A})$ in $\Omega_u(\mathcal{A})$ and in $\Omega_Z(\mathcal{A})$ pass to the quotient to define an operation of $\mathrm{Der}(\mathcal{A})$ in the graded differential algebra $\Omega_{\mathrm{Diag}}(\mathcal{A})$ which will be again refered to as *the canonical operation of* $\mathrm{Der}(\mathcal{A})$ *in* $\Omega_{\mathrm{Diag}}(\mathcal{A})$. Furthermore, all these operations of $\mathrm{Der}(\mathcal{A})$ pass to the quotient to define an operation of $\mathrm{Der}(\mathcal{A})$ in $\Omega_{\mathrm{Der}}(\mathcal{A})$ which coincides with the canonical operation of $\mathrm{Der}(\mathcal{A})$ in $\Omega_{\mathrm{Der}}(\mathcal{A})$ defined in Section 3.

One has the following commutative diagram of surjective homomorphisms of graded differential algebras which is also a diagram of homomorphisms of the operations of $\mathrm{Der}(\mathcal{A})$.



Furthermore, if $\mathcal{A}$ is a $*$-algebra there is a canonical involution on $\Omega_{\text{Diag}}(\mathcal{A})$ such that this diagram is also a diagram of graded differential $*$-algebras, (the involutions of $\Omega_u(\mathcal{A}), \Omega_Z(\mathcal{A})$ and $\Omega_{\text{Der}}(\mathcal{A})$ have been defined previously in Section 7 and Section 3).

9 Noncommutative symplectic geometry and quantum mechanics

Let $\mathcal{A}$ be as before a unital associative complex algebra. A *Poisson bracket* on $\mathcal{A}$ is a Lie bracket which is a biderivation on $\mathcal{A}$ (for its associative product). That is $(a,b) \mapsto \{a,b\}$ is a Poisson bracket if it is a bilinear antisymmetric mapping of $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ (i.e. a linear mapping of $\bigwedge^2 \mathcal{A}$ into $\mathcal{A}$) which satisfies

$$\begin{array}{ll} \{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0 & \text{(Jacobi identity)} \\ \{a,bc\} = \{a,b\}c + b\{a,c\} & \text{(derivation property)} \end{array}$$

for any elements a,b,c of $\mathcal{A}$. Equipped with such a Poisson bracket, $\mathcal{A}$ is refered to as a *Poisson algebra*, [38].

There is a lot of classical commutative Poisson algebras, for instance the symmetric algebra $S(\mathfrak{g})$ (over $\mathbb{C}$) of a (complex) Lie algebra $\mathfrak{g}$, the algebra $C^\infty(M)$ of smooth functions on a symplectic manifold, etc.. For a noncommutative algebra $\mathcal{A}$, a generic type of Poisson bracket $\{\cdot,\cdot\}$ is obtained by setting for $a,b \in \mathcal{A}$

$$\{a,b\} = \frac{i}{\hbar}[a,b]$$

where $[a,b]$ denotes the commutator in $\mathcal{A}$, i.e. $[a,b] = ab - ba$, and where $\hbar \in \mathbb{C}$ is any non zero complex number. We have put a $i \in \mathbb{C}$ in front of the right-hand side of the above formula in order that in the case where $\mathcal{A}$ is a $*$-algebra *the Poisson bracket is real*, i.e. satisfies $\{a,b\}^* = \{a^*, b^*\}$, whenever $\hbar$ is real. The reason why the Poisson brackets proportional to the commutator are quite generic (in the noncommutative case) is connected to the following lemma [38].

LEMMA 7 *Let $\mathcal{A}$ be a Poisson algebra, then one has $[a,b]\{c,d\} = \{a,b\}[c,d]$ and more generally $[a,b]x\{c,d\} = \{a,b\}x[c,d]$ for any elements a,b,c,d and x of $\mathcal{A}$.*

Proof. The first identity is obtained by developing $\{ac, bd\}$ in two different orders by using the biderivation property. The second (more general since $\mathbb{1} \in \mathcal{A}$) identity is obtained by replacing c by xc in the first identity, by developing and by using again the first identity. □

For more details concerning the "generic side" of Poisson brackets proportional to the commutator we refer to [38]. We simply observe here that this is the type of Poisson brackets which occurs in quantum mechanics.

Our aim is now to develop a (noncommutative) generalization of symplectic structures related to the above Poisson brackets. One should start from a notion of differential form i.e. from a differential calculus Ω over $\mathcal{A}$. Since for a Poisson bracket $x \mapsto \{a, x\}$ is an element of $\mathrm{Der}(\mathcal{A})$ for any $a \in \mathcal{A}$, it is natural to assume that one has an operation $X \mapsto i_X$ of the Lie algebra $\mathrm{Der}(\mathcal{A})$ in the graded differential algebra Ω. Furthermore we wish to take into account the structure of $Z(\mathcal{A})$-module of $\mathrm{Der}(\mathcal{A})$ so we require that Ω is a central bimodule over $\mathcal{A}$ and that $X \mapsto i_X$ is a $Z(\mathcal{A})$-linear mapping of $\mathrm{Der}(\mathcal{A})$ into $\mathrm{Der}^{-1}_{\mathrm{gr}}(\Omega)$. Notice that this $Z(\mathcal{A})$-linearity is well defined since Ω central is equivalent to $Z(\mathcal{A}) \subset Z^0_{\mathrm{gr}}(\Omega)$, (see in Section 2 for the notations). Having such a differential calculus, one defines a homomorphism λ of Ω into $\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$ by setting $\lambda(\omega)(X_1, \dots, X_n) = i_{X_n} \dots i_{X_1}\omega$ for $\omega \in \Omega^n$. The fact that this defines a homomorphism of graded differential algebra of Ω into $C_\wedge(\mathrm{Der}(\mathcal{A}), \mathcal{A})$ follows from the general properties of operations whereas the fact that the image of λ is contained in $\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$ follows from the $Z(\mathcal{A})$-linearity. It turns out that even if one uses a general differential calculus Ω for the symplectic structures, the only relevant parts for the corresponding Poisson structures are the images by λ in $\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$, (see e.g. in [38]). One is then led to the definitions of [26], or more precisely to the following slight generalizations [27].

An element ω of $\underline{\Omega}^2_{\mathrm{Der}}(\mathcal{A})$ will be said to be *nondegenerate* if, for any $x \in \mathcal{A}$, there is a derivation $\mathrm{Ham}(x) \in \mathrm{Der}(\mathcal{A})$ such that one has $\omega(X, \mathrm{Ham}(x)) = X(x)$ for any $X \in \mathrm{Der}(\mathcal{A})$. Notice that if ω is nondegenerate then $X \mapsto i_X\omega$ is an injective linear mapping of $\mathrm{Der}(\mathcal{A})$ into $\underline{\Omega}^1_{\mathrm{Der}}(\mathcal{A})$ but that the converse is not true; the condition for ω to be nondegenerate is stronger than the injectivity of $X \mapsto i_X\omega$. If M is a manifold, an element $\omega \in \underline{\Omega}^2_{\mathrm{Der}}(C^\infty(M))$ is an ordinary 2-form on M and it is nondegenerate in the above sense if and only if the 2-form ω is nondegenerate in the classical sense (i.e. everywhere nondegenerate).

Let $\omega \in \underline{\Omega}^2_{\mathrm{Der}}(\mathcal{A})$ be nondegenerate, then for a given $x \in \mathcal{A}$ the derivation $\mathrm{Ham}(x)$ is unique and $x \mapsto \mathrm{Ham}(x)$ is a linear mapping of $\mathcal{A}$ into $\mathrm{Der}(\mathcal{A})$.

A closed nondegenerate element ω of $\underline{\Omega}^2_{\mathrm{Der}}(\mathcal{A})$ will be called a *symplectic structure for* $\mathcal{A}$.

LEMMA 8 *Let ω be a symplectic structure for $\mathcal{A}$ and let us define an antisymmetric bilinear bracket on $\mathcal{A}$ by $\{x, y\} = \omega(\mathrm{Ham}(x), \mathrm{Ham}(y))$. Then $(x, y) \mapsto \{x, y\}$ is a Poisson bracket on $\mathcal{A}$.*

Proof. One has $\{x, yz\} = \{x, y\}z + y\{x, y\}$ for $x, y, z \in \mathcal{A}$. Furthermore one has the identity

$$d\omega(\mathrm{Ham}(x), \mathrm{Ham}(y), \mathrm{Ham}(z)) = -\{x, \{y, z\}\} - \{y, \{z, x\}\} - \{z, \{x, y\}\}$$

which implies the Jacobi identity since $d\omega = 0$. □

Let ω be a symplectic structure for $\mathcal{A}$, then one has

$$[\mathrm{Ham}(x), \mathrm{Ham}(y)] = \mathrm{Ham}(\{x, y\}),$$

i.e. Ham is a Lie-algebra homomorphism of $(\mathcal{A}, \{,\})$ into $\mathrm{Der}(\mathcal{A})$. We shall refer to the above bracket as *the Poisson bracket associated to the symplectic structure* ω. If $\mathcal{A}$ is a $*$-algebra and if furthermore ω is real, i.e. $\omega = \omega^*$, then this Poisson bracket is real and $\mathrm{Ham}(x^*) = (\mathrm{Ham}(x))^*$ for any $x \in \mathcal{A}$.

An algebra $\mathcal{A}$ equipped with a symplectic structure will be refered to as a *symplectic algebra*. Thus, symplectic algebras are particular Poisson algebras.

Remark 11. Let $\mathcal{A}$ be an arbitrary Poisson algebra with Poisson bracket $(x, y) \mapsto \{x, y\}$; one defines a linear mapping $\mathrm{Ham} : \mathcal{A} \to \mathrm{Der}(\mathcal{A})$ by $\mathrm{Ham}(x)(y) = \{x, y\}$, (i.e. $\mathrm{Ham}(x) = \{x, \cdot\}$), for $x, y \in \mathcal{A}$. In this general setting one also has the identity $[\mathrm{Ham}(x), \mathrm{Ham}(y)] = \mathrm{Ham}(\{x, y\})$ since it is equivalent to the Jacobi identity for the Poisson bracket.

If M is a manifold, a symplectic structure for $C^\infty(M)$ is just a symplectic form on M. Since there are manifolds which do not admit symplectic form, one cannot expect that an arbitrary $\mathcal{A}$ admits a symplectic structure.

Assume that $\mathcal{A}$ has a trivial center $Z(\mathcal{A}) = \mathbb{C}\mathbb{1}$ and that all its derivations are inner (i.e. of the form $ad(x), x \in \mathcal{A}$). Then one defines an element ω of $\underline{\Omega}^2_{\text{Der}}(\mathcal{A})$ by setting $\omega(ad(ix), ad(iy)) = i[x,y]$. It is easily seen that ω is a symplectic structure for which one has $\text{Ham}(x) = ad(ix)$ and $\{x,y\} = i[x,y]$. If furthermore $\mathcal{A}$ is a $*$-algebra, then this symplectic structure is real ($\omega = \omega^*$). Although a little tautological, this construction is relevant for quantum mechanics.

Let $\mathcal{A}$ be, as above, a complex unital $*$-algebra with a trivial center and only inner derivations and assume that there exists a linear form τ on $\mathcal{A}$ which is central, i.e. $\tau(xy) = \tau(yx)$, and normalized by $\tau(\mathbb{1}) = 1$. Then one defines an element $\theta \in \underline{\Omega}^1_{\text{Der}}(\mathcal{A})$ by $\theta(ad(ix)) = x - \tau(x)\mathbb{1}$. One has $(d\theta)(ad(ix), ad(iy)) = i[x,y]$, i.e. $\omega = d\theta$, so in this case the symplectic form ω is exact. As examples of such algebras one can take $\mathcal{A} = M_n(\mathbb{C})$, (a factor of type I_n), with $\tau = \frac{1}{n}$ trace, or $\mathcal{A} = \mathcal{R}$, a von Neumann algebra which is a factor of type II_1 with τ equal to the normalized trace. The algebra $M_n(\mathbb{C})$ is the algebra of observables of a quantum spin $s = \frac{n-1}{2}$ while $\mathcal{R}$ is the algebra used to describe the observables of an infinite assembly of quantum spins; two typical types of quantum systems with no classical counterpart.

Let us now consider the C.C.R. algebra (canonical commutative relations) $\mathcal{A}_{CCR}$ [26]. This is the complex unital $*$-algebra generated by two hermitian elements q and p satisfying the relation $[q,p] = i\hbar\mathbb{1}$. This algebra is the algebra of observables of the quantum counterpart of a classical system with one degree of freedom. We keep here the positive constant $\hbar$ (the Planck constant) in the formula for comparison with classical mechanics, although the algebra for $\hbar \neq 0$ is isomorphic to the one with $\hbar = 1$. We restrict attention to one degree of freedom to simplify the notations but the discussion extends easily to a finite number of degrees of freedom. This algebra has again only inner derivations and a trivial center so $\omega(ad(\frac{i}{\hbar}x), ad(\frac{i}{\hbar}y)) = \frac{i}{\hbar}[x,y]$

defines a symplectic structure for which $\text{Ham}(x) = ad(\frac{i}{\hbar}x)$ and $\{x,y\} = \frac{i}{\hbar}[x,y]$ which is the standard quantum Poisson bracket. In this case one can express ω in terms of the generators q and p and their differentials [26], [27]:

$$\omega = \sum_{n\geq 0} \left(\frac{1}{i\hbar}\right)^n \frac{1}{(n+1)!}[\dots[dp, \underbrace{p], \dots, p]}_{n}[\dots[dq, \underbrace{q], \dots, q]}_{n}$$

Notice that this formula is meaningful because if one inserts two derivations $ad(ix), ad(iy)$ in it, only a finite number of terms contribute to the sum. In contrast to the preceding case, here the symplectic form is not exact, i.e. it corresponds to a non vanishing element of $H^2(\underline{\Omega}_{\mathrm{Der}}(\mathcal{A}_{CCR}))$ which is therefore non trivial. This was guessed in [26] on the basis of the nonexistence of a finite trace (i.e. central linear form) on $\mathcal{A}_{CCR}$ and finally proved in [38]. For $\hbar = 0$, q and p commute and the algebra reduces to the algebra of complex polynomial functions on the phase space $\mathbb{R}^2$. Furthermore the limit of $\{x, y\} = \frac{i}{\hbar}[x, y]$ at $\hbar = 0$ reduces to the usual classical Poisson bracket as well known and, by using the above formula, one sees that the formal limit of ω at $\hbar = 0$ is $dpdq$. This limit is however very singular since the limit algebra is the algebra of complex polynomials in two indeterminates, the limit symplectic form is exact and not every derivation is hamiltonian in contrast to what happens for $\mathcal{A}_{CCR}$ (i.e. for $\hbar \neq 0$).

10 Theory of connections

Throughout this section, $\mathcal{A}$ is a unital associative complex algebra and Ω is a differential calculus over $\mathcal{A}$, that is a graded differential algebra such that $\Omega^0 = \mathcal{A}$ with differential denoted by d.

Let $\mathcal{M}$ be a left $\mathcal{A}$-module; a Ω-*connection* on $\mathcal{M}$ (or simply a *connection* on $\mathcal{M}$ if no confusion arises) is a linear mapping $\nabla : \mathcal{M} \to \Omega^1 \otimes_{\mathcal{A}} \mathcal{M}$ such that one has

$$\nabla(am) = a\nabla(m) + d(a) \otimes_{\mathcal{A}} m$$

for any $a \in \mathcal{A}$ and $m \in \mathcal{M}$, ($\Omega^1 \otimes_{\mathcal{A}} \mathcal{M}$ being equipped with its canonical structure of left $\mathcal{A}$-module). One extends ∇ to $\Omega \otimes_{\mathcal{A}} \mathcal{M}$ by setting $\nabla(\omega \otimes_{\mathcal{A}} m) = (-1)^n \omega\nabla(m) + d(\omega) \otimes_{\mathcal{A}} m$ for $\omega \in \Omega^n$ and $m \in \mathcal{M}$ ($\Omega \otimes_{\mathcal{A}} \mathcal{M}$ is canonically a left Ω-module). It then follows from the definitions that ∇^2 is a left Ω-module endomorphism of $\Omega \otimes_{\mathcal{A}} \mathcal{M}$ which implies that its restriction $\nabla^2 : \mathcal{M} \to \Omega^2 \otimes_{\mathcal{A}} \mathcal{M}$ to $\mathcal{M}$ is a homomorphism of left $\mathcal{A}$-modules; this homomorphism is called the *curvature* of the connection ∇.

Not every left $\mathcal{A}$-module admits a connection. If $\mathcal{M}$ is the free $\mathcal{A}$-module $\mathcal{A}\otimes E$, where E is some complex vector space, then $\nabla = d\otimes I_E$ is a connection on $\mathcal{A}\otimes E$ which has a vanishing curvature (such a connection with zero curvature is said to be flat). If $\mathcal{M}\subset \mathcal{A}\otimes E$ is a direct summand of a free $\mathcal{A}$-module $\mathcal{A}\otimes E$ and if $P:\mathcal{A}\otimes E\to\mathcal{M}$ is the corresponding projection, then $\nabla = P\circ(d\otimes I_E)$ is a connection on $\mathcal{M}$. Thus a projective module admits (at least one) a connection. In the case where Ω is the universal differential calculus $\Omega_u(\mathcal{A})$ the converse is also true: It was shown in [22] that a (left) $\mathcal{A}$-module admits a $\Omega_u(\mathcal{A})$-connection if and only if it is projective.

One defines in a similar manner Ω-connections on right modules. Namely if $\mathcal{N}$ is a right $\mathcal{A}$-module, a Ω-connection on $\mathcal{N}$ is a linear mapping ∇ of $\mathcal{N}$ into $\mathcal{N}\otimes_\mathcal{A}\Omega^1$ such that $\nabla(na)=\nabla(n)a+n\otimes_\mathcal{A} d(a)$ for any $n\in\mathcal{N}$ and $a\in\mathcal{A}$.

Let $\mathcal{M}$ be a left $\mathcal{A}$-module, then its dual $\mathcal{M}^*$ (i.e. the set of left $\mathcal{A}$-module homomorphisms of $\mathcal{M}$ into $\mathcal{A}$) is a right $\mathcal{A}$-module. We denote by $< m,n >\in\mathcal{A}$ the evaluation of $n\in\mathcal{M}^*$ on $m\in\mathcal{M}$. Let ∇ be a Ω-connection on $\mathcal{M}$, then one defines a unique linear mapping ∇^* of $\mathcal{M}^*$ into $\mathcal{M}^*\otimes_\mathcal{A}\Omega^1$ by setting (with obvious notations)

$$< m,\nabla^*(n) >= d(< m,n >)- < \nabla(m),n >$$

for any $m\in\mathcal{M}$ and $n\in\mathcal{M}^*$. It is easy to verify that ∇^* is a Ω-connection on the right module $\mathcal{M}^*$ which will be refered to as *the dual connection* of ∇. One defines in the same way the dual connection of a connection on a right module.

Our aim is now to recall the definitions of hermitian modules over a $*$-algebra $\mathcal{A}$ and of hermitian connections. We assume that $\mathcal{A}$ is a $*$-algebra such that the convex cone $\mathcal{A}^+$ generated by the a^*a ($a\in\mathcal{A}$) is a strict cone i.e. such that $\mathcal{A}^+\cap(-\mathcal{A}^+)=0$. This property is satisfied for instance by $*$-algebras of operators in Hilbert spaces. A *hermitian structure* on a right $\mathcal{A}$-module $\mathcal{M}$ [14] is a sesquilinear mapping $h:\mathcal{M}\times\mathcal{M}\to\mathcal{A}$ such that one has:

$$\begin{array}{ll}(i) & h(ma,nb)=a^*h(m,n)b,\quad \forall m,n\in\mathcal{M} \text{ and } \forall a,b\in\mathcal{A}\\ (ii) & h(m,m)\in\mathcal{A}^+,\quad \forall m\in\mathcal{M} \text{ and } h(m,m)=0\Rightarrow m=0.\end{array}$$

A right $\mathcal{A}$-module $\mathcal{M}$ equipped with a hermitian structure will be refered to as a *hermitian module* over $\mathcal{A}$. If $\mathcal{M}$ is a hermitian module over $\mathcal{A}$, a *hermitian connection* on $\mathcal{M}$ is a connection ∇ on the right $\mathcal{A}$-module $\mathcal{M}$ such that one has

$$d(h(m,n)) = h(\nabla m, n) + h(m, \nabla n)$$

for any $m, n \in \mathcal{M}$ with obvious notations. We have chosen to define hermitian structures on right modules for notational reasons, (we prefer the convention of physicists for sesquilinearity, i.e. linearity in the second argument); one can define similarily hermitian structures and connections for left modules.

Let $\mathcal{M}$ be a right $\mathcal{A}$-module. The group $\mathrm{Aut}(\mathcal{M})$ of all module automorphisms of $\mathcal{M}$ acts on the affine space of all connections on $\mathcal{M}$ via $\nabla \mapsto \nabla^U = U \circ \nabla \circ U^{-1}$, $U \in \mathrm{Aut}(\mathcal{M})$, (one canonically has $\mathrm{Aut}(\mathcal{M}) \subset \mathrm{Aut}(\mathcal{M} \otimes_{\mathcal{A}} \Omega^1)$). If furthermore $\mathcal{A}$ is a $*$-algebra as above and if h is a hermitian structure on $\mathcal{M}$, then the subgroup of $\mathrm{Aut}(\mathcal{M})$ of all automorphisms U which preserve h, i.e. such that $h(Um, Un) = h(m,n)$ for $m, n \in \mathcal{M}$, will be denoted by $\mathrm{Aut}(\mathcal{M}, h)$ and called the *gauge group* whereas its elements will be called *gauge transformations*; it acts on the real affine space of hermitian connections on $\mathcal{M}$.

As pointed out before, one-sided modules are not sufficient and one needs bimodules for a lot of reasons. Firstly, in the case where $\mathcal{A}$ is a $*$-algebra, one needs $*$-bimodules to formulate and discuss reality conditions [34], [18], [27] (see also in the introduction). Secondly, a natural noncommutative generalization of linear connections should be connections on Ω^1, since Ω is taken as an analog of differential forms, but this is a $(\mathcal{A}, \mathcal{A})$-bimodule in an essential way. Thirdly, in order to have an analog of local couplings, one needs to have a tensor product over $\mathcal{A}$ since the latter is the noncommutative version of the local tensor product of tensor fields. In short one needs a theory of connections for bimodules and any of the above quoted problems shows that one-sided connections on bimodules (i.e. on bimodules considered as left or right modules) are of no help. The difficulty to define a Ω-connection on a $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{M}$ lies in the fact that a left $\mathcal{A}$-module connection on $\mathcal{M}$ sends $\mathcal{M}$ into $\Omega^1 \otimes_{\mathcal{A}} \mathcal{M}$ whereas a right $\mathcal{A}$-module connection on $\mathcal{M}$ sends $\mathcal{M}$ into $\mathcal{M} \otimes_{\mathcal{A}} \Omega^1$. A solution of this problem adapted to the case where $\mathcal{M} = \Omega^1$

has been given in [56] and generalized in [32] for arbitrary $(\mathcal{A},\mathcal{A})$-bimodules on the basis of an analysis of first order differential operators in bimodules. This led to the following definition [32].

Let $\mathcal{M}$ be a $(\mathcal{A},\mathcal{A})$-bimodule; a *left bimodule* Ω*-connection* on $\mathcal{M}$ is a left $\mathcal{A}$-module Ω-connection ∇ on $\mathcal{M}$ for which there is a bimodule homomorphism $\sigma : \mathcal{M}\otimes_{\mathcal{A}}\Omega^1 \to \Omega^1\otimes_{\mathcal{A}}\mathcal{M}$
such that

$$\nabla(ma) = \nabla(m)a + \sigma(m\otimes_{\mathcal{A}} d(a))$$

for any $a\in\mathcal{A}$ and $m\in\mathcal{M}$. Clearly σ is then unique under these conditions. One defines similarily a *right bimodule* Ω*-connection* on $\mathcal{M}$ to be a right $\mathcal{A}$-module Ω-connection ∇ on $\mathcal{M}$ for which there is a bimodule homomorphism $\sigma : \Omega^1\otimes_{\mathcal{A}}\mathcal{M}\to\mathcal{M}\otimes_{\mathcal{A}}\Omega^1$ such that

$$\nabla(am) = a\nabla(m) + \sigma(d(a)\otimes_{\mathcal{A}} m)$$

for any $a\in\mathcal{A}$ and $m\in\mathcal{M}$. When no confusion arises on Ω and on "left-right" we simply refer to this notion as *bimodule connection*.

In the case where $\mathcal{M}$ is the bimodule Ω^1 itself, a left bimodule Ω-connection is just the first part of the proposal of [56] for the definition of linear connections in noncommutative geometry; the second part of the proposal of [56] relates σ and the product $\Omega^1\otimes_{\mathcal{A}}\Omega^1\to\Omega^2$ so it makes sense only for $\mathcal{M}=\Omega^1$ and is there necessary to define the generalization of torsion.

It has been shown in [7] (Appendix A of [7]) that on general grounds, the above definition is just what is needed to define tensor products over $\mathcal{A}$ of bimodule connections and of left (right) bimodule connections with left (right) module connections. In fact, let ∇' be a left bimodule connection on the bimodule $\mathcal{M}'$ and let ∇'' be a connection on a left module $\mathcal{M}''$. Then one defines a connection ∇ on the left module $\mathcal{M}'\otimes_{\mathcal{A}}\mathcal{M}''$ by setting

$$\nabla = \nabla'\otimes_{\mathcal{A}} I_{\mathcal{M}''} + (\sigma'\otimes_{\mathcal{A}} I_{\mathcal{M}''})\circ(I_{\mathcal{M}'}\otimes_{\mathcal{A}}\nabla'')$$

where $\sigma' : \mathcal{M}'\otimes_{\mathcal{A}}\Omega^1\to\Omega^1\otimes_{\mathcal{A}}\mathcal{M}'$ is the bimodule homomorphism corresponding to ∇'. If furthermore $\mathcal{M}''$ is a $(\mathcal{A},\mathcal{A})$-bimodule and if ∇'' is a left bimodule connection with corresponding bimodule homomorphism

$\sigma'' : \mathcal{M}'' \otimes_{\mathcal{A}} \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \mathcal{M}''$, then ∇ is also a left bimodule connection with corresponding bimodule homomorphism σ given by

$$\sigma = (\sigma' \otimes_{\mathcal{A}} I_{\mathcal{M}''}) \circ (I_{\mathcal{M}'} \otimes_{\mathcal{A}} \sigma'')$$

of $\mathcal{M}' \otimes_{\mathcal{A}} \mathcal{M}'' \otimes_{\mathcal{A}} \Omega^1$ into $\Omega^1 \otimes_{\mathcal{A}} \mathcal{M}' \otimes_{\mathcal{A}} \mathcal{M}''$.

Let $\mathcal{M}$ be a $(\mathcal{A},\mathcal{A})$-bimodule and let $\mathcal{M}^*$ denote the dual of $\mathcal{M}$ considered as a left $\mathcal{A}$-module. Then $\mathcal{M}^*$ is a right $\mathcal{A}$-module as dual of a left $\mathcal{A}$-module, but it is in fact a bimodule if one defines the left action $m' \mapsto am'$ of $\mathcal{A}$ on $\mathcal{M}^*$ by $< m, am' >=< ma, m' >$ for any $m \in \mathcal{M}, a \in \mathcal{A}, m' \in \mathcal{M}^*$. If ∇ is a left bimodule Ω-connection on $\mathcal{M}$ then one verifies that ∇^* is a right bimodule Ω-connection on $\mathcal{M}^*$ [7] (Appendix B of [7]). Notice that this kind of duality between bimodules is different of the $\mathcal{A}$-duality between bimodules over $\mathcal{A}$ and modules over $Z(\mathcal{A})$ discussed in Section 8.

When $\mathcal{A}$ is a $*$-algebra, there is also a generalization of hermitian forms on $(\mathcal{A},\mathcal{A})$-bimodules which has been introduced on [57] and called *right hermitian forms* in [34] which is adapted for tensor products over $\mathcal{A}$. If $\mathcal{M}$ is a $(\mathcal{A},\mathcal{A})$-bimodule, then a right hermitian form on $\mathcal{M}$ is a hermitian form h on $\mathcal{M}$ considered as a right $\mathcal{A}$-module which is such that for the left multiplication by $a \in \mathcal{A}$ one has $h(m, an) = h(a^*m, n)$. One can then define the notion of right hermitian bimodule connection, (which is in particular a right bimodule connection).

We now explain the relation between the above notion of bimodule connection and the theory of first order operators in bimodules. Let $\mathcal{A}$ and $\mathcal{B}$ be unital associative complex algebras and let $\mathcal{M}$ and $\mathcal{N}$ be two $(\mathcal{A},\mathcal{B})$-bimodules. We denote by l_a the left multiplication by $a \in \mathcal{A}$ in $\mathcal{M}$ and in $\mathcal{N}$ and we denote by r_b the right multiplication by $b \in \mathcal{B}$ in $\mathcal{M}$ and in $\mathcal{N}$. A linear mapping D of $\mathcal{M}$ into $\mathcal{N}$ which is such that one has $[[D, l_a], r_b] = 0$ for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$ is called *a first-order operator* or *an operator of order 1* of $\mathcal{M}$ into $\mathcal{N}$ [16]. Notice that homomorphisms of left $\mathcal{A}$-modules of $\mathcal{M}$ into $\mathcal{N}$ as well as homomorphisms of right $\mathcal{B}$-modules of $\mathcal{M}$ into $\mathcal{N}$ are first-order operators of $\mathcal{M}$ into $\mathcal{N}$. The structure of first-order operators is given by the following theorem [32].

THEOREM 4 *Let $\mathcal{M}$ and $\mathcal{N}$ be two $(\mathcal{A},\mathcal{B})$-bimodules and let D be a first order operator of $\mathcal{M}$ into $\mathcal{N}$. Then, there is a unique $(\mathcal{A},\mathcal{B})$-bimodule ho-*

momorphism $\sigma_L(D)$ *of* $\Omega^1_u(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{M}$ *into* $\mathcal{N}$ *and there is a unique* $(\mathcal{A},\mathcal{B})$*-bimodule homomorphism* $\sigma_R(D)$ *of* $\mathcal{M}\otimes_{\mathcal{B}}\Omega^1_u(\mathcal{B})$ *into* $\mathcal{N}$ *such that one has:*

$$D(amb) = aD(m)b + \sigma_L(D)(d_u a\otimes m)b + a\sigma_R(D)(m\otimes d_u b)$$

for any $m\in\mathcal{M}$, $a\in\mathcal{A}$ *and* $b\in\mathcal{B}$.

For the proof and further informations, see in [32]. It is clear that $\sigma_L(D)$ and $\sigma_R(D)$ are the appropriate generalization of the notion of symbol in this setting. We shall refer to them as the *left* and the *right universal symbol of* D respectively.

Remark 12. The converse of Theorem 4 is also true. More precisely, let (Ω^1_L, d_L) be a first order differential calculus over $\mathcal{A}$, let (Ω^1_R, d_R) be a first order differential calculus over $\mathcal{B}$ and let $D:\mathcal{M}\to\mathcal{N}$ be a linear mapping then any of the following condition (1) or (2) implies that D is a first-order operator.

(1) There is a $(\mathcal{A},\mathcal{B})$-bimodule homomorphism $\sigma_L:\Omega^1_L\otimes_{\mathcal{A}}\mathcal{M}\to\mathcal{N}$ such that

$$D(am) = aD(m) + \sigma_L(d_L(a)\otimes m),\quad \forall m\in\mathcal{M} \text{ and } \forall a\in\mathcal{A}$$

(2) There is a $(\mathcal{A},\mathcal{B})$-bimodule homomorphism $\sigma_R:\mathcal{M}\otimes_{\mathcal{B}}\Omega^1_R\to\mathcal{N}$ such that

$$D(mb) = D(m)b + \sigma_R(m\otimes d_R(b)),\quad \forall m\in\mathcal{M} \text{ and } \forall b\in\mathcal{B}.$$

Let $\mathcal{M}$ be a $(\mathcal{A},\mathcal{A})$-bimodule and let ∇ be a left $\mathcal{A}$-module Ω-connection on $\mathcal{M}$. It is obvious that ∇ is a first-order operator of the $(\mathcal{A},\mathcal{A})$-bimodule $\mathcal{M}$ into the $(\mathcal{A},\mathcal{A})$-bimodule $\Omega^1\otimes_{\mathcal{A}}\mathcal{M}$. It follows therefore from the above theorem that there is a unique $(\mathcal{A},\mathcal{A})$-bimodule homomorphism $\sigma_R(\nabla)$ of $\mathcal{M}\otimes_{\mathcal{A}}\Omega^1_u(\mathcal{A})$ into $\Omega^1\otimes_{\mathcal{A}}\mathcal{M}$ such that one has

$$\nabla(ma) = \nabla(m)a + \sigma_R(\nabla)(m\otimes_{\mathcal{A}} d_u(a))$$

for any $m\in\mathcal{M}$ and $a\in\mathcal{A}$. Therefore, ∇ is a left bimodule Ω-connection on $\mathcal{M}$ if and only if $\sigma_R(\nabla)$ factorizes through a $(\mathcal{A},\mathcal{A})$-bimodule homomorphism $\sigma:\mathcal{M}\otimes_{\mathcal{A}}\Omega^1\to\Omega^1\otimes_{\mathcal{A}}\mathcal{M}$ as $\sigma_R(\nabla)=\sigma\circ(I_{\mathcal{M}}\otimes i_d)$ where $I_{\mathcal{M}}$ is the identity mapping of $\mathcal{M}$ onto itself and i_d is the unique $(\mathcal{A},\mathcal{A})$-bimodule homomorphism of $\Omega^1_u(\mathcal{A})$ into Ω^1 such that $d = i_d \circ d_u$ (see Proposition 1). This implies in particular that any left $\mathcal{A}$-module $\Omega_u(\mathcal{A})$-connection is a left bimodule $\Omega_u(\mathcal{A})$-connection.

In the case of the derivation-based differential calculus, there is an easy natural way to define connections on left and right modules and on central bimodules over $\mathcal{A}$, [34]. We describe it in the case of central bimodules (for left and for right modules, just forget multiplications on the right and on the left respectively). Let $\mathcal{M}$ be a central bimodule over $\mathcal{A}$, i.e. a $\mathcal{A}$-bimodule for $\mathbf{Alg}_Z$, a *(derivation-based) connection on* $\mathcal{M}$ is a linear maping $\nabla, X \mapsto \nabla_X$, of $\mathrm{Der}(\mathcal{A})$ into the linear endomorphisms of $\mathcal{M}$ such that

$$\nabla_{zX}(m) = z\nabla_X(m), \nabla_X(amb) = a\nabla_X(m)b + X(a)mb + amX(b)$$

for any $m \in \mathcal{M}$, $X \in \mathrm{Der}(\mathcal{A})$, $z \in Z(\mathcal{A})$ and $a, b \in \mathcal{A}$. One verifies that such a connection on the central bimodule $\mathcal{M}$ is a bimodule $\underline{\Omega}_{\mathrm{Der}}(\mathcal{A})$-connection on $\mathcal{M}$ in the previous sense with a well defined σ, (modulo some technical problems of completion of the tensor products $\underline{\Omega}^1_{\mathrm{Der}}(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{M}$ and $\mathcal{M}\otimes_{\mathcal{A}}\underline{\Omega}^1_{\mathrm{Der}}(\mathcal{A})$). The interest of this formulation is that curvature is straightforwardly defined and is a bimodule homomorphism [34]. We refer to [34] (and also to [27]) for more details and in particular for the relation with $\mathcal{A}$-duality. Furthermore, in this frame the notion of reality on connections is obvious. Assume that $\mathcal{A}$ is a $*$-algebra and that $\mathcal{M}$ is a central bimodule which is a $*$-bimodule over $\mathcal{A}$ then a (derivation-based) connection ∇ on $\mathcal{M}$ will be said to be *real* if one has $\nabla_X(m^*) = (\nabla_X(m))^*$ for any $m \in \mathcal{M}$ and any $X \in \mathrm{Der}_{\mathbb{R}}(\mathcal{A})$, i.e. $X \in \mathrm{Der}(\mathcal{A})$ with $X = X^*$.

The notion of reality in the general frame of bimodule Ω-connections is slightly more involved and will not be discussed here.

11 Classical Yang-Mills-Higgs models

An aspect with no counterpart in ordinary differential geometry of the theory of Ω-connections on $\mathcal{A}$-modules for a differential calculus Ω which is not graded commutative is the generic occurrence of inequivalent Ω-connections with vanishing curvature (on a fixed $\mathcal{A}$-module). By taking as algebra $\mathcal{A}$ the algebra of functions on space-time with values in some algebra $\mathcal{A}_0$, i.e. $\mathcal{A} = C^\infty(\mathbb{R}^{s+1}) \otimes \mathcal{A}_0$, this led to classical Yang-Mills-Higgs models based on noncommutative geometry in which the Higgs field is the part of the connection which is in the "noncommutative directions".

In the following, we display the case of Ω-connections on right modules over the algebra $\mathcal{A} = C^\infty(\mathbb{R}^{s+1}) \otimes M_n(\mathbb{C})$ of smooth $M_n(\mathbb{C})$-valued functions on $\mathbb{R}^{s+1}$ for $\Omega = \Omega_{\mathrm{Der}}(\mathcal{A})$.

Let us first describe the situation for $\mathcal{A} = M_n(\mathbb{C})$. The derivations of $M_n(\mathbb{C})$ are all inner so the complex Lie algebra $\mathrm{Der}(M_n(\mathbb{C}))$ reduces to $\mathfrak{sl}(n)$ and the real Lie algebra $\mathrm{Der}_{\mathbb{R}}(M_n(\mathbb{C}))$ reduces to $\mathfrak{su}(n)$. As already mentioned in Section 8, one has

$$\Omega_{\mathrm{Der}}(M_n(\mathbb{C})) = C_\wedge(\mathrm{Der} M_n(\mathbb{C}), M_n(\mathbb{C})) = C_\wedge(\mathfrak{sl}(n), M_n(\mathbb{C}))$$

as can be shown directly [25] and as also follows from the formulas below. Let $E_k, k \in \{1, 2, \dots, n^2-1\}$ be a base of self–adjoint traceless $n \times n$–matrices. The $\partial_k = \mathrm{ad}(iE_k)$ form a basis of real derivations i.e. of $\mathrm{Der}_{\mathbb{R}}(M_n(\mathbb{C})) = \mathfrak{su}(n)$. One has $[\partial_k, \partial_\ell] = C^m_{k\ell}\partial_m$, the $C^m_{k\ell}$ are the corresponding structure constants of $\mathfrak{su}(n)$, (or $\mathfrak{sl}(n)$). Define $\theta^k \in \Omega^1_{\mathrm{Der}}(M_n(\mathbb{C}))$ by $\theta^k(\partial_\ell) = \delta^k_\ell \mathbb{1}$.The following formulas give a presentation of the graded differential algebra $\Omega_{\mathrm{Der}}(M_n(\mathbb{C}))$ [28], [26]:

$$\begin{aligned}
E_k E_\ell &= g_{k\ell}\mathbb{1} + (S^m_{k\ell} - \tfrac{i}{2} C^m_{k\ell}) E_m \\
E_k \theta^\ell &= \theta^\ell E_k \\
\theta^k \theta^\ell &= -\theta^\ell \theta^k \\
dE_k &= -C^m_{k\ell} E_m \theta^\ell \\
d\theta^k &= -\tfrac{1}{2} C^k_{\ell m} \theta^\ell \theta^m
\end{aligned}$$

where $g_{k\ell} = g_{\ell k}$, $S^m_{k\ell} = S^m_{\ell k}$ are real, $g_{k\ell}$ are the components of the Killing form of $\mathfrak{su}(n)$ and $C^m_{k\ell} = -C^m_{\ell k}$ are as above the (real) structure constants of $\mathfrak{su}(n)$.Formula giving the dE_k can be inverted and one has

$$\theta^k = -\frac{i}{n^2} g^{\ell m} g^{kr} E_\ell E_r dE_m$$

where $g^{k\ell}$ are the components of the inverse matrix of $(g_{k\ell})$. The element $\theta = E_k \theta^k$ of $\Omega^1_{\mathrm{Der}}(M_n(\mathbb{C}))$ is real, $\theta = \theta^*$, and independent of the choice of the E_k, in fact we already met θ in Section 9: $\theta(\mathrm{ad}(iA)) = A - \frac{1}{n}\mathrm{tr}(A)\mathbb{1}$ and $\omega = d\theta$ is the natural symplectic structure for $M_n(\mathbb{C})$. Furthermore θ is invariant, $L_X\theta = 0$, and any invariant element of $\Omega^1_{\mathrm{Der}}(M_n(\mathbb{C}))$ is a scalar multiple of θ. We call θ the *canonical invariant element* of $\Omega^1_{\mathrm{Der}}(M_n(\mathbb{C}))$. One has

$$dM = i[\theta, M],\ \forall M \in M_n(\mathbb{C})$$
$$d(-i\theta) + (-i\theta)^2 = 0.$$

The $*$–algebra $M_n(\mathbb{C})$ is simple with only one irreductible representation in $\mathbb{C}^n$. A general finite right–module (which is projective) is the space $M_{Kn}(\mathbb{C})$ of $K\times n$–matrices with right action of $M_n(\mathbb{C})$. Then $\mathrm{Aut}(M_{Kn}(\mathbb{C}))$ is the group $GL(K)$ acting by left matrix multiplication. The module $M_{Kn}(\mathbb{C})$ is naturally hermitian with $h(\Phi,\Psi) = \Phi^*\Psi$ where Φ^* is the $n\times K$ matrix hermitian conjugate to Φ. The gauge group is then the unitary group $U(K)(\subset GL(K))$. Here, there is a natural origin $\overset{0}{\nabla}$ in the space of connections given by $\overset{0}{\nabla}\Phi = -i\Phi\theta$ where $\Phi \in M_{Kn}(\mathbb{C})$ and where θ is the canonical invariant element of $\Omega^1_{\mathrm{Der}}(M_n(\mathbb{C}))$. The fact that this defines a connection follows from

$$\overset{0}{\nabla}(\Phi M) = (\overset{0}{\nabla}\Phi)M + \Phi i[\theta, M]$$

and from the above expression of dM for $M \in M_n(\mathbb{C})$. This connection is hermitian and its follows from the above expression for $d\theta$ that its curvature vanishes, i.e. $(\overset{0}{\nabla})^2 = 0$. Any connection ∇ is of the form $\nabla\Phi = \overset{0}{\nabla}\Phi + A\Phi$ where $A = A_k\theta^k$ with $A_k \in M_K(\mathbb{C})$ and $A\Phi$ means $A_k\Phi\otimes\theta^k$. The connection ∇ is hermitian if and only if the A_k are antihermitian i.e. $A_k^* = -A_k$. The curvature of ∇ is given by $\nabla^2\Phi = F\Phi$ $(= F_{k\ell}\Phi\otimes\theta^k\theta^\ell)$ with

$$F = \frac{1}{2}([A_k, A_\ell] - C^m_{k\ell}A_m)\theta^k\theta^\ell.$$

Thus $\nabla^2 = 0$ if and only if the A_k form a representation of the Lie algebra $\mathfrak{sl}(n)$ in $\mathbb{C}^K$ and two such connections are in the same $\mathrm{Aut}(M_{Kn}(\mathbb{C}))$–orbit if and only if the corresponding representations of $\mathfrak{sl}(n)$ are equivalent. This implies that the *gauge orbits of flat* ($\nabla^2 = 0$) *hermitian connections are in one–to–one correspondence with unitary classes of representations of* $\mathfrak{su}(n)$ *in* $\mathbb{C}^K$, [28]. For instance if $n = 2$, these orbits are labelled by the number of partitions of the integer K.

We now come to the case $\mathcal{A} = C^\infty(\mathbb{R}^{s+1}) \otimes M_n(\mathbb{C})$. Let x^μ, $\mu \in \{0, 1, \dots, s\}$, be the canonical coordinates of $\mathbb{R}^{s+1}$. One has $\Omega_{\mathrm{Der}}(C^\infty(\mathbb{R}^{s+1}) \otimes M_n(\mathbb{C})) = \Omega_{\mathrm{Der}}(C^\infty(\mathbb{R}^{s+1})) \otimes \Omega_{\mathrm{Der}}(M_n(\mathbb{C}))$ so one can split the differential as $d = d' + d''$ where d' is the differential along $\mathbb{R}^{s+1}$ and d'' is the differential of $\Omega_{\mathrm{Der}}(M_n(\mathbb{C}))$. A typical finite projective right module is $C^\infty(\mathbb{R}^{s+1}) \otimes M_{Kn}(\mathbb{C})$. This is an hermitian module with hermitian structure given by $h(\Phi, \Psi)(x) = \Phi(x)^*\Psi(x)$, $(x \in \mathbb{R}^{s+1})$. As a $C^\infty(\mathbb{R}^{s+1})$–module, this module is free (of rank $K.n$), so $d'\Phi$ is well defined for $\Phi \in C^\infty(\mathbb{R}^{s+1}) \otimes M_{Kn}(\mathbb{C})$. In fact, $d'\Phi(x) = \frac{\partial \Phi}{\partial x^\mu}(x) dx^\mu$. A connection on the $C^\infty(\mathbb{R}^{s+1}) \otimes M_n(\mathbb{C})$–module $C^\infty(\mathbb{R}^{s+1}) \otimes M_{Kn}(\mathbb{C})$ is of the form $\nabla\Phi = d'\Phi - i\Phi\theta + A\Phi$ with $A = A_\mu dx^\mu + A_k\theta^k$, where the A_μ and the A_k are $K \times K$ matrix valued functions on $\mathbb{R}^{s+1}$ (i.e. elements of $C^\infty(\mathbb{R}^{s+1}) \otimes M_K(\mathbb{C})$) and where $A\Phi(x) = A_\mu(x)\Phi(x)dx^\mu + A_k(x)\Phi(x)\theta^k$. Such a connection is hermitian if and only if the $A_\mu(x)$ and the $A_k(x)$ are antihermitian, $\forall x \in \mathbb{R}^{s+1}$. The curvature of ∇ is given by $\nabla^2\Phi = F\Phi$ where

$$\begin{aligned} F = \tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu \quad & +[A_\mu, A_\nu])dx^\mu dx^\nu \\ & +(\partial_\mu A_k + [A_\mu, A_k])dx^\mu\theta^k \\ & +\tfrac{1}{2}([A_k, A_\ell] - C^m_{k\ell}A_m)\theta^k\theta^\ell \end{aligned}$$

The connection ∇ is flat (i.e. $\nabla^2 = 0$) if and only if each term of the above formula vanishes which implies that ∇ is gauge equivalent to a connection for which one has $A_\mu = 0$, $\partial_\mu A_k = 0$ and $[A_k, A_\ell] = C^m_{k\ell}A$. Furthermore two such connections are equivalent if and only if the corresponding representations of $\mathfrak{su}(n)$ in $\mathbb{C}^K$ (given by the constant $K \times K$–matrices A_ℓ) are equivalent. So again, *the gauge orbits of flat hermitian connections are in one-to-one correspondence with the unitary classes of (antihermitian) representations of* $\mathfrak{su}(n)$ *in* $\mathbb{C}^K$. Again, in the case n=2, the number of such orbits is the number of partitions of the integer$K$ i.e.

$$\mathrm{card}\{(n_r) | \sum_r \quad n_r.r = K\}.$$

If we consider $\mathbb{R}^{s+1}$ as the $(s+1)$–dimensional space–time and if we replace the algebra of smooth functions on $\mathbb{R}^{s+1}$ by $C^\infty(\mathbb{R}^{s+1}) \otimes M_n(\mathbb{C})$ which we interpret as the algebra of "smooth functions on a noncommutative generalized space-time". It is clear, from the above expression for the curvature that the generalization of the (euclidean) Yang–Mills action for a hermitian connection ∇ on $C^\infty(\mathbb{R}^{s+1}) \otimes M_{Kn}(\mathbb{C})$ is

$$\begin{aligned}\|F\|^2 &= \int d^{s+1}x \,\mathrm{tr}\Big\{ \tfrac{1}{4}\sum(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])^2 \\ &+\tfrac{1}{2}\sum(\partial_\mu A_k + [A_\mu, A_k])^2 + \tfrac{1}{4}\sum([A_k, A_\ell] - C^m_{k\ell}A_m)^2\Big\}\end{aligned}$$

where the metrics of space-time is $g_{\mu\nu} = \delta_{\mu\nu}$ and where the basis E_k of hermitian traceless $n \times n$–matrices is chosen in such a way that $g_{k\ell} = \delta_{k\ell}$, i.e. $\mathrm{tr}(E_k E_\ell) = n\delta_{k\ell}$. This can be more deeply justified by introducing the analog of the Hodge involution on $\Omega_{\mathrm{Der}}(M_n(\mathbb{C}))$, the analog of the integration of elements of $\Omega^{n^2-1}_{\mathrm{Der}}(M_n(\mathbb{C}))$ (essentially the trace) and by combining these operations with the corresponding one on $\mathbb{R}^{s+1}$ to obtain a scalar product on $\Omega_{\mathrm{Der}}(C^\infty(\mathbb{R}^{s+1}) \otimes M_n(\mathbb{C}))$ etc. See in [28], [29] for more details.

The above action is the Yang–Mills action on the noncommutative space corresponding to $C^\infty(\mathbb{R}^{s+1}) \otimes M_n(\mathbb{C})$. However it can be interpreted as the action of a field theory on the $(s+1)$–dimensional space–time $\mathbb{R}^{s+1}$. At first sight, this field theory consists of a $U(K)$-Yang-Mills potential $A_\mu(x)$ minimally coupled with scalar fields $A_k(x)$ with values in the adjoint representation which interact among themselves through a quartic potential. The action is positive and vanishes for $A_\mu = 0$ and $A_k = 0$, but is also vanishes on other gauge orbits. Indeed $\|F\|^2 = 0$ is equivalent to $F = 0$, so the gauge orbits on which the action vanishes are labelled by unitary classes of representations of $\mathfrak{su}(n)$ in $\mathbb{C}^K$. By the standard semi–heuristic argument, these gauge orbits are interpreted as different vacua for the corresponding quantum theory. To specify a quantum theory, one has to choose one and to translate the fields in order that the zero of these translated fields corresponds to the chosen vacuum (i.e. is the corresponding zero of the action). The variables A_μ, A_k are thus adapted to the specific vacuum φ_0 corresponding to the trivial representation $A_k = 0$ of $\mathfrak{su}(n)$. If one chooses the vacuum φ_α corresponding to a representation $\overset{\alpha}{R}_k$ of $\mathfrak{su}(n)$, (i.e. one has $[\overset{\alpha}{R}_k, \overset{\alpha}{R}_\ell] = C^m_{k\ell}\overset{\alpha}{R}_n$), one must instead use the variables A_μ and $\overset{\alpha}{B}_k = A_k - \overset{\alpha}{R}_k$. Making this change of variable one observes that components of A_μ become massive and that the $\overset{\alpha}{B}_k$ have different masses; the whole mass spectrum depends on α. This is very analogous to the Higgs mechanism. Here however the gauge invariance is not broken, the non–invariance of the mass–terms of the A_μ is compensated by the fact that the gauge transformation of the $\overset{\alpha}{B}_k$ becomes inhomogeneous (they are components of a connection). Nevertheless, from the point of view of the space-time intepretation this is the Higgs mechanism and the A_k are Higgs fields.

The above models were the first ones of classical Yang-Mills-Higgs models based on noncommutative geometry. They certainly admit a natural supersymmetric extension since there is a natural extension of the derivation-based calculus to graded matrix algebras [42]. There is also another extension of the above calculus where $C^\infty(\mathbb{R}^{s+1})\otimes M_n(\mathbb{C})$ is replaced by the algebra $\Gamma\mathrm{End}(E)$ of smooth sections of the endomorphisms bundle of a (nontrivial) smooth vector bundle E (of rank n) admitting a volume over a smooth $((s+1)$-dimensional) manifold [33].

The use of the derivation-based calculus makes the above models quite rigid. By relaxing this i.e. by using other differential calculi Ω, other models based on noncommutative geometry which are closer to the classical version of the standard model have been constructed [15], [19], [21]. Furthermore there is an elegant way to combine the introduction of the (spinors) matter fields with the differential calculus and the metric [16] as well as with the reality conditions [18] in noncommutative geometry, (and also with the action principles [11]). Within this general set-up, one can probably absorb any classical model of gauge theory.

A problem arises for the quantization of these classical models based on noncommutative geometry. Namely is it possible to keep something of the noncommutative geometrical interpretation of these classical models at the quantum level? The best would be to find some B.R.S. symmetry [3] ensuring that (perturbative) quantization does not spoil the correspondence with noncommutative geometry. Unfortunately no such symmetry was discovered up to now. As long as no progress is obtained on this problem, the noncommutative geometrical interpretation of the gauge theory with Higgs field must be taken with some circumspection in spite of its appealing features.

12 Conclusion : Further remarks

Concerning the noncommutative generalization of differential geometry the point of view more or less explicit here is that the data are encoded in an algebra $\mathcal{A}$ which plays the role of the algebra of smooth functions. This is why although we have described various notions in terms of an arbitrary differential calculus Ω, we have studied in some details specific differential calculi "naturally" associated with $\mathcal{A}$ (i.e. which do not depend on other data than $\mathcal{A}$ itself) such as the universal differential calculus $\Omega_u(\mathcal{A})$, the generalization $\Omega_Z(\mathcal{A})$ of the Kähler exterior forms, the diagonal calculus $\Omega_{\text{Diag}}(\mathcal{A})$ and the derivation-based calculus. There are other possibilities, for instance some authors consider that the data are encoded in a graded differential algebra which plays the role of the algebra of smooth differential forms, e.g. [54]. This latter point of view can be taken into account here by using an arbitrary differential calculus Ω.

In all the above points of view, the generalization of differential forms is provided by a graded differential algebra. This is not always so natural. For instance it was shown in [46] (see also [47]) that the subspace $[\Omega_u(\mathcal{A}), \Omega_u(\mathcal{A})]_{\text{gr}}$ of graded commutators in $\Omega_u(\mathcal{A})$ is stable by d_u and that the cohomology of the cochain complex $\Omega_u(\mathcal{A})/[\Omega_u(\mathcal{A}), \Omega_u(\mathcal{A})]$ is closely related to the cyclic homology (it is contained in the reduced cyclic homology), and is also in several respects a noncommutative version of de Rham cohomology. This complex $\Omega_u(\mathcal{A})/[\Omega_u(\mathcal{A}), \Omega_u(\mathcal{A})]$ (which is generally not a graded algebra) is sometimes called *the noncommutative de Rham complex* [58]. It is worth noticing that, for $\mathcal{A}$ noncommutative, there is no tensor product over $\mathcal{A}$ between $\mathcal{A}$-modules (i.e. no analog of the tensor product of vector bundles) and that therefore the Grothendieck group $K_0(\mathcal{A})$ (of classes of projective $\mathcal{A}$-modules) has no product. Thus for $\mathcal{A}$ noncommutative $K_0(\mathcal{A}) \otimes \mathbb{C}$ is not an algebra and therefore there is no reason for a cochain complex such that its cohomology is a receptacle for the image of the Chern character of $K_0(\mathcal{A}) \otimes \mathbb{C}$ to be a graded algebra.

Also we did not describe here the approach to the differential calculus and to the metric aspects in noncommutative geometry based on generalized Dirac operators (spectral triples) [16], [17], [18] as well as the related supersymmetric approach of [39], see in O. Grandjean's lectures. In these notes we did not introduce specifically generalizations of linear connections and a fortiori not generalizations of riemannian structures.

Finally we did not discuss differential calculus for quantum groups, i.e. bicovariant differential calculus [61]. In the spirit of Section 2, let us define a *graded differential Hopf algebra* to be a graded differential algebra $\mathfrak{A}$ which is also a graded Hopf algebra with coproduct Δ such that $\Delta : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ is a homomorphism of graded differential algebras (i.e. in particular the differential d of $\mathfrak{A}$ satisfies the graded co-Leibniz rule), with counit ε such that $\varepsilon \circ d = 0$ and with antipode S homogeneous of degree 0 such that $S \circ d = d \circ S$. If $\mathfrak{A}$ is a graded differential Hopf algebra, then the subalgebra $\mathfrak{A}^0$ of elements of degree 0 of $\mathfrak{A}$ is an ordinary Hopf algebra, i.e. a quantum group, and $\mathfrak{A}$ is a bicovariant differential calculus over $\mathfrak{A}^0$. Notice that if G is a Lie group then the graded differential algebra $\Omega(G)$ of differential forms on G is in fact a graded differential Hopf algebra which is graded commutative, (in order to be correct, one has to complete the tensor product in the definition of the coproduct or to use, instead of $\Omega(G)$, the graded differential subalgebra of forms generated by the representative functions on G).

References

[1] A. Alekseev, E. Meinrenken, *The non-commutative Weil algebra*, math-DG/9903052 to appear in Inv. Math.

[2] Y. André, *Différentielles non-commutatives et théorie de Galois différentielle ou aux différences*, prépublication 221, Institut de Mathématiques de Jussieu (1999).

[3] C. Becchi, A. Rouet, R. Stora, *Renormalization models with broken symmetries*, in G. Velo and A.S. Wightman (eds), Renormalization Theory (Erice 1975), D. Reidel, Dordrecht, 1976.

[4] R. Bott, L.W. Tu, *Differential forms in algebraic topology*, Springer-Verlag 1982.

[5] S. Boukraa, *The BRS algebra of a free minimal differential algebra*, Nucl. Phys. **B303** (1988), 237-259.

[6] N. Bourbaki, *Algèbre I*, Chapitre III. Paris, Hermann 1970.

[7] K. Bresser, F. Müller-Hoissen, A. Dimakis, A. Sitarz, *Noncommutative geometry of finite groups*, J. of Physics A (Math. and General) **29** (1996), 2705-2735.

[8] A. Cap, A. Kriegl, P.W. Michor, J. Vanzura, *The Frölicher-Nijenhuis bracket in non commutative differential geometry*, Acta Math. Univ. Comenianae **LXII** (1993), 17-49.

[9] H. Cartan, *Notion d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie and La trangression dans un groupe de Lie et dans un espace fibré principal*, Colloque de topologie (Bruxelles 1950), Paris, Masson 1951.

[10] H. Cartan, S. Eilenberg, *Homological algebra*, Princeton University Press 1973.

[11] A. Chamseddine, A. Connes, *The spectral action principle*, Commun. Math. Phys. **186** (1997) 731-750.

[12] A. Connes, *Une classification des facteurs de type III*, Ann. Scient. E.N.S. 4ème Série **t.6** (1973), 133-252.

[13] A. Connes, *Non-commutative differential geometry*, Publ. IHES **62** (1986), 257-360.

[14] A. Connes, *C^* algèbres et géométrie différentielle*, C.R. Acad. Sci. Paris, **290**, Série A (1980), 599-604.

[15] A. Connes, *Essay on physics and noncommutative geometry*, in The interface of mathematics and particles physics, pp. 9-48, Oxford Univ. Press 1990.

[16] A. Connes, *Non-commutative geometry*, Academic Press, 1994.

[17] A. Connes, *Geometry from the spectral point of view*, Lett. Math. Phys. **34** (1995), 203-238.

[18] A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. **36** (1995), 6194-6231.

[19] A. Connes, J. Lott, *Particle models and noncommutative geometry*, Nucl. Phys. **B18** Suppl. (1990), 29-47.

[20] R. Coquereaux, *Noncommutative geometry and theoretical physics*, J. Geom. Phys. **6** (1989), 425-490.

[21] R. Coquereaux, *Higgs fields and superconnections*, in Differential Geometric Methods in Theoretical Physics, Rapallo 1990 (C. Bartocci, U. Bruzzo, R. Cianci, eds), Lecture Notes in Physics **375**, Springer Verlag 1991.

R. Coquereaux, R. Häußling, F. Scheck, *Algebraic connections on parallel universes*, Int. J. Mod. Phys. **A10** (1995), 89-98.

[22] J. Cuntz, D. Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (1995), 251-289.

[23] P.A.M. Dirac, *On quantum algebras*, Proc. Camb. Phil. Soc. **23** (1926), 412.

[24] M. Dubois-Violette, *The Weil-BRS algebra of a Lie algebra and the anomalous terms in gauge theory*, J. Geom. Phys., **3** (1987), 525-565.

[25] M. Dubois-Violette, *Dérivations et calcul différentiel non commutatif*, C.R. Acad. Sci. Paris, **307**, Série I (1988), 403-408.

[26] M. Dubois-Violette, *Non-commutative differential geometry, quantum mechanics and gauge Theory*, in Differential Geometric Methods in Theoretical Physics, Rapallo 1990 (C. Bartocci, U. Bruzzo, R. Cianci, eds), Lecture Notes in Physics **375**, Springer-Verlag 1991.

[27] M. Dubois-Violette, *Some aspects of noncommutative differential geometry*, in Contemporary Mathematics **203** (1997), 145-157.

[28] M. Dubois-Violette, R. Kerner, J. Madore, *Non-commutative differential geometry of matrix algebras*, J. Math. Phys. **31** (1990), 316-322.

[29] M. Dubois-Violette, R. Kerner, J. Madore, *Gauge bosons in a noncommutative geometry*, Phys. Lett. **B217** (1989), 485-488.

M. Dubois-Violette, R. Kerner, J. Madore, *Classical bosons in a noncommutative geometry*, Class. Quantum Grav. **6** (1989), 1709-1724.

M. Dubois-Violette, R. Kerner, J. Madore, *Non-commutative differential geometry and new models of gauge theory*, J. Math. Phys. **31** (1990), 323-330.

[30] M. Dubois-Violette, A. Kriegl, Y. Maeda, P.W. Michor, In preparation.

[31] M. Dubois-Violette, T. Masson, *Basic cohomology of associative algebras*, Journal of Pure and Applied Algebra, **114** (1996), 39-50.

[32] M. Dubois-Violette, T. Masson, *On the first order operators in bimodules*, Lett. Math. Phys. **37** (1996), 467-474.

[33] M. Dubois-Violette, T. Masson, *$SU(n)$-gauge theories in noncommutative differential geometry.* J. Geom. Phys., **25** (1998), 104-118.

[34] M. Dubois-Violette, P.W. Michor, *Connections on central bimodules in noncommutative geometry*, J. Geom. Phys. **20** (1996), 218-232.

[35] M. Dubois-Violette, P.W. Michor, *Dérivations et calcul différentiel noncommutatif.* II, C.R. Acad. Sci. Paris, **319**, Série I (1994), 927-931.

[36] M. Dubois-Violette, P.W. Michor, *More on the Frölicher-Nijenhuis bracket in noncommutative differential geometry*, Journal of Pure and Applied Algebra, **121** (1997), 107-135.

[37] M. Dubois-Violette, M. Talon, C.M. Viallet, *B.R.S. algebras. Analysis of consistency equations in gauge theory*, Commun. Math. Phys. **102** (1985), 105-122.

M. Dubois-Violette, M. Henneaux, M. Talon, C.M. Viallet, *General solution of the consistency equation*, Phys. Lett. **B289** (1992), 361-367.

[38] D.R. Farkas, G.. Letzter, *Ring theory from symplectic geometry*, Journal of Pure and Applied Algebra, **125** (1998), 155-190.

[39] J. Fröhlich, O. Grandjean, A. Recknagel, *Supersymmetric quantum theory, non-commutative geometry, and gravitation*, in Quantum Symmetries, Les Houches 1995 (A. Connes, K. Gawedzki, J. Zinn-Justin, eds), Elsevier 1998.

[40] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. Math. **78** (1963), 267-287.

[41] W. Greub, S. Halperin, R. Vanstone, *Connections, curvature, and cohomology*, Vol. III, Academic Press 1976.

[42] H. Grosse, G. Reiter, *Graded differential geometry of graded matrix algebras*, math-ph/9905018 to appear in J. Math. Phys..

[43] D. Husemoller, *Lectures on cyclic homology*, Springer-Verlag 1991.

[44] N. Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society 1968.

[45] N. Jacobson, *Basic algebra II*, second edition, Freeman and Co., New York 1989.

[46] M. Karoubi, *Homologie cyclique des groupes et algèbres*, C.R. Acad. Sci. Paris **297**, Série I, (1983) 381-384.

[47] M. Karoubi, *Homologie cyclique et K-théorie*, Astérisque **149** (SMF), 1987.

[48] D. Kastler, *Lectures on Alain Connes' non commutative geometry and applications to fundamental interactions*, in Infinite dimensional geometry, noncommutative geometry, operator algebras, fundamental interactions; Saint-François, Guadeloupe 1993 (R. Coquereaux, M. Dubois-Violette, P. Flad, eds), World Scientific 1995.

[49] J.L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. Fr. **78** (1950), 65-127.

[50] J.L. Koszul, *Fibre bundles and differential geometry*, Tata Institute of Fundamental Research, Bombay, 1960.

[51] G. Landi, *An introduction to noncommutative spaces and their geometries*, Springer-Verlag, 1997.

[52] J.-L. Loday, *Cyclic homology*, Springer-Verlag, New York 1992.

[53] J. Madore, *Noncommutative differential geometry and its physical applications*, Cambridge University Press 1995.

[54] G. Maltsiniotis, *Le langage des espaces et des groupes quantiques*, Commun. Math. Phys. **151** (1993), 275-302.

[55] T. Masson, *Géométrie non commutative et applications à la théorie des champs*, Thesis, Orsay 1995.

[56] J. Mourad, *Linear connections in noncommutative geometry*, Class. Quantum Grav. **12** (1995), 965-974.

[57] M. Rieffel. *Projective modules over higher dimensional noncommutative tori*, Can. J. Math **XL2** (1988), 257-338.

[58] P. Seibt, *Cyclic homology of algebras*, World Scientific Publishing Co., 1987.

[59] D. Sullivan, *Infinitesimal computations in topology*, Publ. IHES **47** (1977), 269-331.

[60] C.A. Weibel, *An introduction to homological algebra*, Cambridge University Press 1994.

[61] S.L.Woronowicz, *Differential calculus on compact matrix pseudogroups (quantum groups)*, Commun. Math. Phys. **122** (1989), 125-170.