$\mathrm{CON}(\mathrm{ETCS} - \mathrm{AC} + \neg \mathrm{WISC})$

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By considering a variant on forcing using a symmetric model for a proper class-sized group, we show that the very weak choice principle WISC—the statement that there is at most a *set* of incomparable surjections onto every set—is independent of the rest of the axioms of the categorical set theory ETCS-AC. Our result applies to any set theory which gives rise to a well-pointed boolean topos with nno. The proof does not rely on the axiom of choice, nor does it make any large cardinal assumptions.

We refer to [Shu10] for the background of the stack semantics, in particular section 7, the pertinent details of which are briefly outlined in the appendix. We pause only to record a lemma whose proof follows that of lemma 7.3 in *loc cit*.

Lemma 1. *Let* E *be a locally connected topos. Then then if for any* connected *object* V, *arrow* $p\colon V \to U$ *and* $A \in \mathrm{Obj}(S/V)$ *we have* $V \Vdash p^*\phi(A)$, *then* $U \Vdash (\forall X)\phi(X)$.

Here 'locally connected' is meant to refer to a base topos **set** that is well-pointed (hence boolean) topos with nno. We will refer to the objects of **set** as 'sets', but without an implication that these sets arise from a particular set of axioms, unless otherwise specified. Also we will assume all toposes will come with an nno.

We use the following formulation of WISC due to François Dorais.

The proof that this is equivalent to the usual definition of WISC works in any well-pointed topos [Dor12]

WISC (in set). *For every set* X *there is a set* Y *such that for every surjection* $q\colon Z \to X$ *there is a map* $s\colon Y \to Z$ *such that* $q \circ s\colon Y \to X$ *is a surjection.*

This version of WISC translates into the stack semantics as follows, where we have made the simplifying assumption that our topos is locally connected, and so we can apply lemma 1:

$$\begin{aligned}
&\forall\, X \to U,\ U \text{ connected},\\
&\exists\, V \overset{p}{\twoheadrightarrow} U,\ Y \to V,\\
&\forall\, W \overset{q}{\to} V,\ W \text{ connected},\ Z \overset{g}{\twoheadrightarrow} W \times_U X,\\
&\exists\, T \overset{r}{\twoheadrightarrow} W,\ T \times_V Y \overset{l}{\to} T \times_W Z,\\
&\text{the map } T \times_V Y \overset{l}{\to} T \times_W Z \to T \times_U X \text{ is an epi.}
\end{aligned}$$

Note also that «is an epi» is a proposition whose statement in the stack semantics is equivalent to the external statement.

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We will give a boolean **set**-topos E that is locally connected and in which the following statement holds:

$$
\begin{aligned}
&\exists\, X \to U,\ U \text{ connected},\\
&\forall\, V \overset{p}{\twoheadrightarrow} U,\ Y \to V,\\
&\exists\, W \overset{q}{\to} V,\ W \text{ connected},\ Z \overset{g}{\twoheadrightarrow} W \times_U X,\\
&\forall\, T \overset{r}{\twoheadrightarrow} W,\ T \times_V Y \overset{l}{\to} T \times_W Z,\\
&\text{the map } T \times_V Y \overset{l}{\to} T \times_W Z \to T \times_U X \text{ is not epi.}
\end{aligned}
$$

We will without any loss of generality take U terminal, and we can assume that $\pi_0(Y) \to \pi_0(V)$ is onto, otherwise it is trivial to find W such that the conclusion holds (take W to be any subobject of V disjoint to $\mathrm{im}(Y \to V)$). Since we are restricting to the case of W connected, we can assume that V is connected, because the other components do not contribute anything. Also, it is enough to consider connected T, since to show the map $T \times_{V_i} Y \to T \times X$ is not epi it is enough to show it is not epi over each connected component T_j of T.

As far as the existential statements go, we will let $X = \mathbb{N}_d$, the natural number object of E, and $q = \mathrm{id}_V$ ($\mathbb{N}_d$ is the image of the nno $\mathbb{N}$ of **set** under the inverse image part of the geometric morphism $E \to$ **set**). Recall also that there is a map $\mathcal{N} \to \mathbb{N}$ in **set** which is the *generic finite cardinal*. The fibre over $n \in \mathbb{N}$ has n elements, and will be denoted $[n]$.

Thus we consider the category-theoretical statement

$$
\begin{aligned}
&\forall\, Y \twoheadrightarrow V,\ V \text{ connected},\\
&\exists\, Z \twoheadrightarrow V \times \mathbb{N}_d,\\
&\forall\, T \twoheadrightarrow V,\ T \text{ connected, and } T \times_V Y \to T \times_V Z,\\
&\text{the map } T \times_V Y \to T \times_V Z \to T \times \mathbb{N}_d \text{ is not epi.}
\end{aligned}
\tag{1}
$$

Recall that given a *large group* $\mathbb{G}$, i.e. a one-object groupoid which is not locally small (relative to **set**), there is a topos $\mathbb{G}$**set** of sets with an action by $\mathbb{G}$. It comes with a triple of adjoint functors, namely

$$\mathbb{G}\mathbf{set} \xrightarrow{\pi_0} \mathbf{set} \xrightarrow{(-)_d} \mathbb{G}\mathbf{set} \xrightarrow{u} \mathbf{set},$$

with $\pi_0 \dashv (-)_d \dashv u$. The topos we will construct will be a subcategory of $\mathbb{G}$**set** which, speaking informally, consists of continuous actions for a certain topology on (a specific) $\mathbb{G}$, given by a normal filter. Since defining a filter on a proper class can be a dubious exercise without the right foundations, we will take a somewhat different tack.

Given our base topos **set**, we can consider the category of objects in **set** equipped with a linear order with no infinite descending chains, which we shall call ordinals, in analogy with material set theory. The usual Burali-Forti argument—which requires no Choice—tells us there is a large category O with objects ordinals and arrows the order-preserving injections onto initial segments. This large category is a preorder, a linear order and even has no infinite (strictly!) descending chains. That there are multiple representatives for a particular order type, that is, non-identical isomorphic ordinals, does not cause any problems.

Next, consider the functor $O \to$ **Grp** which sends an ordinal α to the set $\mathbb{Z}^\alpha$ of functions to $\mathbb{Z}$ with pointwise addition, and the arrow $\alpha \hookrightarrow \beta$ to the 'extend by zero' homomorphism, $\mathbb{Z}^\alpha \hookrightarrow \mathbb{Z}^\beta$. The choice of $\mathbb{Z}$ is not so important here; any group G in **set** with transitive actions on arbitrarily large finite sets would be sufficient. Note that we also have a homomorphism $\mathbb{Z}^\beta \to \mathbb{Z}^\alpha$ given by restricting along $\alpha \hookrightarrow \beta$, which is a retract onto the subgroup $\mathbb{Z}^\alpha$.

We would like to take the colimit of this functor to get $\mathbb{G} = \mathrm{colim}_O\, \mathbb{Z}^\alpha$, but this is clearly impossible without some finessing of what is meant by the colimit here. Luckily, we are interested in objects of **set** with an action by $\mathbb{G}$, and moreover, not just any such actions, but those for which each stabiliser is in some specified subgroups which can be explicitly defined. A useful analogy to keep in mind is that of a direct sum of infinitely many factors of $\mathbb{Z}$, except we replace the finitely many non-trivial entries with set-many and the infinitely many factors with 'proper class-many'.

First, we restrict to $\mathbb{G}$-sets with finite orbits, or more formally:

Definition 2. A $\mathbb{G}$-set A is called *small* if there is a map $\pi_0(A) \to \mathbb{N}$ and a pullback square

$$\begin{tikzcd} u(A) \arrow[r] \arrow[d] & \mathcal{N} \arrow[d] \\ \pi_0(A) \arrow[r] & \mathbb{N} \end{tikzcd}$$

Thus we make the restriction that stabilisers are finite-index subgroups. Such subgroups can be described by 'functions' $d\colon \mathcal{O} \to [N]$, such that isomorphic ordinals are mapped to the same element. The set $[N]$ is to be thought of as the set of subgroups $\{\mathbb{Z}, 2\mathbb{Z}, \ldots, N\mathbb{Z}\}$ of $\mathbb{Z}$. The subgroup corresponding to $d\colon \mathcal{O} \to [N]$ is then the (formal) colimit of the diagram

In the event that $\mathbb{Z}$ is replaced by some other group G, we consider functions to a finite set of finite-index subgroups of G

$$\mathcal{O} \to \mathbf{Ab}, \qquad \alpha \mapsto \prod_{\beta \hookrightarrow \alpha} d(\beta)\mathbb{Z}$$

and the order of this subgroup is the lowest common multiple of $d(\alpha)$ as α ranges over $\mathcal{O}$. We call the function d a *local depth function*.

However, we are not interested in all subgroups corresponding to local depth functions of this form, but those whose support is a set, where the support of d is all the ordinals α at which $d(\alpha) \neq 1$. Thus they are described by actual functions (i.e. arrows of **set**) $\alpha \to [N]$ for some $\alpha \in \mathcal{O}$. Given $\beta \hookrightarrow \alpha$ and a pair of local depth functions $d_1\colon \alpha \to [N]$, $d_2\colon \beta \to [M]$, they describe the same subgroup if and only if $d_2^{1\cdots 1}(\gamma) = d_1(\gamma)$ for all $\gamma \hookrightarrow \alpha$ where $d_2^{1\cdots 1}$ is the function d_2 extended by the constant value 1 on $\alpha - \beta$. We thus have canonical representatives $d\colon \alpha \to [N]$ by taking the least such α and $[N]$, and we shall say the local depth function d is in *minimal form*. Given a local depth function d we let K_d denote the subgroup of $\mathbb{Z}^\alpha$ corresponding to d.

The map $\mathbb{G} \to \mathbb{Z}^\alpha$ is the universal map arising from the description of $\mathbb{G}$ as a formal colimit and the retracts $\mathbb{Z}^\beta \to \mathbb{Z}^\alpha$ for all $\beta \hookleftarrow \alpha$

Reasoning informally, we have the result that giving a set X with an action of $\mathbb{G} = \operatorname{colim}_{\mathcal{O}} \mathbb{Z}^\alpha$, such that all stabilisers are contained in a subgroup K_d given by the function $d\colon \alpha \to [N]$, is the same as specifying X together with the restricted action $\mathbb{G} \to \mathbb{Z}^\alpha \to \mathrm{Aut}(X)$.

If we take the point of view that we are defining continuous group actions via a normal filter on $\mathbb{G}$, then the filter consists of those subgroups given by functions $d\colon \alpha \to [N]$ which are not eventually constant at 1 (it *can* be eventually constant at any other value, or if $\alpha = 0$, take only the value 1). But from the point of view of defining a subcategory of $\mathbb{G}$**set**, we can just specify that we are interested in sets X with an action of some $\mathbb{Z}^\alpha$ for which the kernel of $\mathbb{Z}^\alpha \to \mathrm{Aut}(X)$ is given by a local depth function in minimal form. Note, however, that given any pair of actions $\mathbb{Z}^\alpha \to \mathrm{Aut}(X)$, $\mathbb{Z}^\beta \to \mathrm{Aut}(Y)$ (or more generally any set of actions), we can consider X and Y to have canonical $\mathbb{Z}^{\alpha \vee \beta}$-actions, where $\alpha \vee \beta$ is the meet of α and β in $\mathcal{O}$.

So now we arrive at our formal definition.

Note that α is necessarily a successor ordinal, else it would not have a top element. We could refer to such ordinals as well-ordered internals.

Definition 3. Given the well-pointed topos **set**, let $\mathbb{Z}^{<\mathrm{Ord}}$**set** be the category with objects triples $(X, \rho\colon \mathbb{Z}^\alpha \to \mathrm{Aut}(X), d\colon \alpha \to [N])$ (or more briefly, (X, ρ, d)) consisting of a set X, an action ρ and a local depth function d in minimal form such that $\ker \rho = K_d$. Arrows of $\mathbb{Z}^{<\mathrm{Ord}}$**set**,

$$(X, \rho\colon \mathbb{Z}^\alpha \to \mathrm{Aut}(X), d\colon \alpha \to [N]) \to (Y, \lambda\colon \mathbb{Z}^\beta \to \mathrm{Aut}(X), d'\colon \beta \to [M]),$$

are given by functions $X \to Y$ which are $\mathbb{Z}^{\alpha \vee \beta}$-equivariant .

Even though we cannot define a quotient $\mathbb{G}/H$ for H a stabiliser of a $\mathbb{G}$-set, given K_d for some $d\colon \alpha \to [N]$ we let $\mathbb{Z}^{<\mathrm{Ord}}/K_d$ denote the $\mathbb{Z}^{<\mathrm{Ord}}$-set $\mathbb{Z}^\alpha/K_d$. Moreover, every transitive $\mathbb{Z}^{<\mathrm{Ord}}$-set is of this form. Given a map of $\mathbb{Z}^{<\mathrm{Ord}}$-sets, we find that there is a constraint on their local depth functions.

Lemma 4. *Let $\mathbb{Z}^{<\mathrm{Ord}}/K_d \to \mathbb{Z}^{<\mathrm{Ord}}/K_\delta$ be an equivariant map of small transitive $\mathbb{Z}^{<\mathrm{Ord}}$-sets. Then we necessarily have $K_d \leqslant K_\delta$ as subgroups of some $\mathbb{Z}^\alpha$, and for every $\beta \leqslant \alpha$ we have $d(\beta) = m_\beta \delta(\beta)$ with $m_\beta \geqslant 1$.*

We also need to consider what taking pullbacks (of transitive small $\mathbb{Z}^{<\mathrm{Ord}}$-sets) looks like from the point of view of local depth functions. It suffices to consider transitive $\mathbb{Z}^{<\mathrm{Ord}}$-sets since, as we shall see, $\mathbb{Z}^{<\mathrm{Ord}}$**set** is a cocomplete topos hence infinitary extensive.

Lemma 5. *The orbit $\mathbb{Z}^{<\mathrm{Ord}}/(K_{d_1} \cap K_{d_2})$ of the pullback $\mathbb{Z}^{<\mathrm{Ord}}/K_{d_1} \times_{\mathbb{Z}^{<\mathrm{Ord}}/K_{d_3}} \mathbb{Z}^{<\mathrm{Ord}}/K_{d_2}$ containing the canonical basepoint has local depth function d satisfying*

$$d(\beta) = \mathrm{lcm}\{d_1(\beta), d_2(\beta)\}, \quad \forall \beta \leqslant \alpha.$$

We can combine these two lemmas, to see that if we have a morphism

$$\mathbb{Z}^{<\mathrm{Ord}}/K_{d_1} \times_{\mathbb{Z}^{<\mathrm{Ord}}/K_{d_3}} \mathbb{Z}^{<\mathrm{Ord}}/K_{d_2} \to \mathbb{Z}^{<\mathrm{Ord}}/K_\delta$$

we must have $\mathrm{lcm}\{d_1(\beta), d_2(\beta)\} = m_\beta \delta(\beta)$.

We have to show that $\mathbb{Z}^{<\mathrm{Ord}}$**set** is a topos with the right sort of properties in order to interpret the set theory underlying **set**

Proposition 6. *The category $\mathbb{Z}^{<\mathrm{Ord}}$**set** is a locally small, connected, atomic cocomplete **set**-topos.*

Proof. First, there is a forgetful functor to **set** sending (X, ρ, d) to X, which has a left adjoint $(-)_d$ sending X to $(X, \mathbb{Z}^0 \to \mathrm{Aut}(X), d\colon 0 \to [1])$. Furthermore, this left adjoint is fully faithful, and so if we know $\mathbb{Z}^{<\mathrm{Ord}}$**set** is a topos, it is connected. Note that there is an additional left adjoint π_0 to $(-)_d$ sending (X, ρ, d) to $X/\mathbb{Z}^\alpha$.

To show we have a topos, notice that given a cospan, we can form its pullback in **set**, then equip it with the action of the largest of the groups $\mathbb{Z}^\alpha$ involved. We clearly have a terminal object, hence finite limits. The set [2] equipped with the trivial action is a subobject classifier in the usual way, and we define the internal hom to be the set of all functions equipped with the conjugation action. We thus have a topos, and the functor $(-)_d$ is easily seen to be logical, hence an atomic topos. Standard theory tells us that it is automatic that $\mathbb{Z}^{<\mathrm{Ord}}$**set** is locally small and cocomplete as a **set**-topos. □

Thus the stack semantics in $\mathbb{Z}^{<\mathrm{Ord}}$**set** give us an interpretation of the set theory underlying **set**, by the results of [Shu10], minus any form of Choice that may hold in **set**. However, we don't yet quite have enough to assert the negation of any choice principle.

It is here that we use the structure of the specific large group we are considering. Because we have restricted the possible stabilisers of $\mathbb{Z}^{<\mathrm{Ord}}$-sets in two different directions, we can always find subgroups which are in some sense distant from any given set of subgroups, where we make the blanket assumption that we only consider subgroups given by local depth functions. This notion of 'distant' is given by the following non-symmetric relation.

Definition 7. For any group G, with subgroups $H, L \leqslant G$, we say L is *separated from* H if and only if for all subgroups K, $H \cap K \leqslant L \cap K$ implies $K \leqslant L$.

Example 8. Let $G = \mathbb{Z}^\alpha$, and let $\beta > \alpha$. let $\{H_i\}_{i\in I}$ be any set of subgroups of $\mathbb{Z}^\alpha$, considered as subgroups of $\mathbb{Z}^\beta$ by the standard inclusion. Then the subgroups given by the local depth function $\delta^n_\beta \colon \beta \to [n]$ defined as

$$\delta^n_\beta(\gamma) = \begin{cases} n & \text{if } \gamma = \beta; \\ 1 & \text{if } \gamma < \beta \end{cases}$$

are separated from every H_i.

Since we always have the ability to pass to a larger ordinal, given any set of subgroups given by local depth functions, we can always find subgroups separated from that set. This is the key property in our proof. Informally, we can state it as follows:

Definition 9 (informal)**.** Let $\mathbb{G}$ be a large group with a normal filter $\mathcal{F}$ consisting of finite-index subgroups. We say $(\mathbb{G}, \mathcal{F})$ has *many deep subgroups* if given any set of subgroups $\{H_i\}_{i\in I} \subset \mathcal{F}$ there is a set of subgroups $\{K_n\} \subset \mathcal{F}$ separated from every H_i, and such that the set $\{|\mathbb{G}/K_n|\} \subset \mathbb{N}$ is unbounded.

More formally we can state it as a result about our topos $\mathbb{Z}^{<\mathrm{Ord}}$**set**.

Lemma 10. *Let $\{H_i\}_{i\in I}$ be any set of subgroups of $H \leqslant \mathbb{Z}^\alpha$, $\beta > \alpha$ and let $K < \mathbb{Z}^\beta$ be a subgroup separated from each H_i. Then if $L \leqslant H$ and*

$$\coprod_{i\in I} \mathbb{Z}^{<\mathrm{Ord}}/L \times_{\mathbb{Z}^{<\mathrm{Ord}}/H} \mathbb{Z}^{<\mathrm{Ord}}/H_i \to \mathbb{Z}^{<\mathrm{Ord}}/L \times \mathbb{Z}^{<\mathrm{Ord}}/K$$

is a map of $\mathbb{Z}^{<\mathrm{Ord}}$-sets, we have $L \leqslant K$.

The proof is just a straightforward application of lemmas 4 and 5. We thus arrive at our main result.

Theorem 11. *The statement of WISC in the stack semantics in $\mathbb{Z}^{<\mathrm{Ord}}$**set** fails.*

Proof. We will show that (1) holds. Given $Y = (Y, \rho^\alpha, d) \to \mathbb{Z}^{<\mathrm{Ord}}/H$, let β be an ordinal such that $\beta > \alpha$. Let $K(n) = K_{\delta^n_\beta}$,

$$Z = \coprod_{n\in\mathbb{N}} \mathbb{Z}^{<\mathrm{Ord}}/K(n)$$

and define the map

$$g \colon \mathbb{Z}^{<\mathrm{Ord}}/H \times Z \to \mathbb{Z}^{<\mathrm{Ord}}/H \times \mathbb{N}_d.$$

Then for any $\mathbb{Z}^{<\mathrm{Ord}}/L \to \mathbb{Z}^{<\mathrm{Ord}}/H$, and function $\mathbb{Z}^{<\mathrm{Ord}}/K \times_{\mathbb{Z}^{<\mathrm{Ord}}/H} Y \to \mathbb{Z}^{<\mathrm{Ord}}/K \times Z$, the map

$$\mathbb{Z}^{<\mathrm{Ord}}/K \times_{\mathbb{Z}^{<\mathrm{Ord}}/H} Y \xrightarrow{(pr,q)} \mathbb{Z}^{<\mathrm{Ord}}/K \times \mathbb{N}_d,$$

is not surjective. It is enough to consider the image of q, as $\mathrm{im}(pr, q) = \mathbb{Z}^{<\mathrm{Ord}}/K \times \mathrm{im}(q)$. By lemma 10, we must have $K \leqslant K(n)$ for all $n \in \mathrm{im}(q)$. But K is given by a local depth function $d \colon \alpha \to [N]$ for some n, and so we have $n|N$ for all $n \in \mathrm{im}(q)$, and thus q is not surjective. □

Recall that ETCS is a set theory defined by specifying the properties of the category of sets, namely that it is a boolean well-pointed topos satisfying the axiom of choice. We can likewise specify a choiceless version, which is the theory of a boolean well-pointed topos, denoted ETCS–AC.

Corollary 12. *WISC is independent of ETCS–AC.*

The theory ETCS is equiconsistent with Bounded Zermelo set theory with Choice, BZC, and likewise ETCS–AC is equiconsistent with BZ.

Corollary 13. *WISC is independent of BZ.*

However, we cannot quite arrive at the following conjecture.

Conjecture 14. *The axiom WISC is independent of the ZF axioms.*

The reason for this is that when one forms a model of ZF from $\mathbb{Z}^{<\mathrm{Ord}}\mathbf{set}$, the well-founded relations which model the $\in$-relation on sets do not contain the non-trivial $\mathbb{Z}^{<\mathrm{Ord}}$-orbits. This is a problem analogous to that pointed out by Freyd in [Fre80], in that the there is a simple topos (in fact the inspiration for $\mathbb{Z}^{<\mathrm{Ord}}\mathbf{set}$) in which the internal axiom of choice fails, but this failure does not translate to the Fourman-Hayashi interpretation of ZF in that topos.

However, we do have a weaker result which is worth pointing out

Corollary 15. $\mathrm{Con}(\mathrm{ZF}) \Rightarrow \mathrm{Con}(\mathrm{ZFA} + \exists$ *proper class of atoms* $+ \neg\mathrm{WISC})$.

Proof. The proof relies on constructing relations which model the $\in$-relation, and which are extensional *except* for possibly set-many elements which are taken from the non-trivial $\mathbb{Z}^{<\mathrm{Ord}}$-orbits.

We start with a model of ZF as our base topos, and take the class A of atoms to be

$$A = \{(\alpha, d\colon \alpha \to [n], x) | \alpha \in \mathrm{Ord}, d \text{ is a local depth function}, x \in \mathbb{Z}^\alpha/d\mathbb{Z}\}$$

This is an Ord-indexed class where each ordinal α is assigned to only $\aleph_0 \cdot |\alpha|$ atoms. $\square$

Now we will build a topos, which will be inspired by $\mathbb{Z}^{<\mathrm{Ord}}\mathbf{set}$, in which we will be able to interpret ZF.

Let P be a poset $p \in P$, and $\downarrow p$ denote the poset of all elements $\leqslant p$. Recall the following definition.

Definition 16. Given a poset P the *dense topology* has as sieves on an object $p \in P$ those given by downward closed cofinal subsets $D \subset \downarrow p$.

The dense topology coincides with the *double negation topology* arising from the modal operator $\neg\neg$ on the subobject classifier in the topos of presheaves on P. We will be interested in posets which are products of posets with top element $\top$, where the products are indexed by well-ordered intervals. We will in fact be defining an analogue of $\mathbb{Z}^{<\mathrm{Ord}}\mathbf{set}$, where objects are pairs consisting of an ordinal and a sheaf on a product of posets indexed by that ordinal. This is the first stage in the construction.

Assume we have a definition of a poset P_α with top element, depending on the ordinal α. In practice this will be one of the standard posets which arise in forcing, but we can avoid that for now. For $p \in \prod_{\alpha\leqslant\lambda} P_\alpha$, which we define the *support* of p to be the set of those α such that the component of p at α is not equal to $\top$. Analogously, define $\mathrm{supp}_\lambda(p) \subset \mathrm{supp}(p)$ to be the subset consisting of $\alpha \leqslant \lambda$.

Definition 17. We call an ordinal α *regular* if given any isomorphism $\alpha \simeq \coprod_{i\in I} X_i$ in **set** where X_i is not isomorphic to α for any $i \in I$, then $I \simeq \alpha$ (in **set**).

Definition 18. The *Easton product* $\prod^E_{\alpha\leqslant\lambda} P_\alpha$ is the subset of $\prod_{\alpha\leqslant\lambda} P_\alpha$ consisting of those elements p such that for every regular ordinal κ, $\mathrm{supp}_\kappa(p) < \kappa$.

Note that in the definition we could have stated 'for every regular ordinal $\kappa \leqslant \lambda$'.

THANKS...

... to Mike Shulman for suggesting the use of the stack semantics and patiently explaining their use to me.

Everything in this appendix is taken from [Shu10]. In what follows, S is only required to be a positive Heyting category, but we will assume we are working with a topos.

If U is an object of S we say that a formula of category theory ϕ with parameters in S/U is a *formula over* U. We have the base change functor $p^*: S/U \to S/V$ for any map $p: V \to U$ (technically, only after choosing a splitting of the fibred category $S^2 \to S$) and call the formula over V given by replacing each parameter of ϕ by its image under p^* the pullback of $p^*\phi$.

Definition 19. Given the ambient category S, and a sentence ϕ over U, we define the relation $U \Vdash \phi$ recursively as follows

- $U \Vdash (f = g) \leftrightarrow f = g$
- $U \Vdash \top$ always
- $U \Vdash \bot \leftrightarrow U \simeq 0$
- $U \Vdash (\phi \wedge \psi) \leftrightarrow U \Vdash \phi$ and $U \Vdash \psi$
- $U \Vdash (\phi \vee \psi) \leftrightarrow U = V \cup W$, where $i: V \hookrightarrow U$ and $j: W \hookrightarrow U$ are subobjects such that $V \Vdash i^*\phi$ and $W \Vdash j^*\psi$
- $U \Vdash (\phi \Rightarrow \psi) \leftrightarrow$ for any $p: V \to U$ such that $V \Vdash p^*\phi$, also $V \Vdash p^*\psi$
- $U \Vdash \neg\phi \leftrightarrow U \Vdash (\phi \Rightarrow \bot)$
- $U \Vdash (\exists X)\phi(X) \leftrightarrow \exists p: V \twoheadrightarrow U$ and $A \in \mathrm{Obj}(S/V)$ such that $V \Vdash p^*\phi(A)$
- $U \Vdash (\exists f: A \to B)\phi(f) \leftrightarrow \exists p: V \twoheadrightarrow U$ and $g: p^*A \to p^*B \in \mathrm{Mor}(S/V)$ such that $V \Vdash p^*\phi(g)$
- $U \Vdash (\forall X)\phi(X) \leftrightarrow$ for any $p: V \to U$ and $A \in \mathrm{Obj}(S/V)$, $V \Vdash p^*\phi(A)$
- $U \Vdash (\forall f: A \to B)\phi(f) \leftrightarrow$ for any $p: V \to U$ and $j: p^*A \to p^*B \in \mathrm{Mor}(S/V)$, $V \Vdash p^*\phi(j)$

If ϕ is a formula over 1 we say ϕ is *valid* if $1 \Vdash \phi$.

One of the main results from [Shu10] is that certain topos-like categories give rise to models of material set theory (i.e. variants of ZF(C)) if they are well-pointed, and for non-well-pointed topos-like categories, we can use the stack semantics to interpret the theory of a well-pointed category, thence a model of material set theory.

We are interested only in models of set theory in classical logic, thus we can make the assumption that the toposes we consider are boolean. In addition, we are considering locally connected toposes that are cocomplete, hence *infinitary extensive*: coproducts are disjoint and stable under pullback.

Under these assumptions [Shu10] shows that the interpretation via stack semantics is identical to that given by Fourman [Fou80] and Hayashi [Hay81] when these latter interpretations make sense. The benefit of working with the stack sematics is that one can work with the objects and arrows of the topos directly, rather than with members of an imitation of the cumulative hierarchy.

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