is interesting to note that prominent physicists warn us from adopting the mathematical statements as being the physics, rather than forming a tool to depict facets of, or providing a computational tool to, physics (see David Mermin, "What's bad about this habit" in Physics Today, May 2009). Thus, physicists may consider the mathematics as a tool only, (pure) mathematicians may consider mathematics as the real thing and applied Platonists see the connection between the two realities.

I have refrained from referring to the issue of mathematics versus applied mathematics, yet a comparison is called for. In my opinion there are no applied mathematicians. There are mathematicians who care about the mathematical aspects of the science they are doing (hence including most of the so-called applied mathematicians) and there are those that apply mathematics but care only of the application. The deviation of physical reality (nature) from the mathematical reality (Plato or not) is primarily a mathematical issue. Those I know that actively follow applied Platonism are mathematicians.

Does the notion of applied Platonism shed light on mathematical Platonism? It may. Consider, for instance, the question of whether there exist chapters in mathematics that cannot take part in the exploration of nature via the applied Platonism route. Would measurable cardinals form such a chapter? No, or so I believe; any mathematical pattern that emerges in the human brain is, potentially, a component in the applied Platonism paradigm. But this is probably the source of a new debate.

Zvi Artstein [zvi.artstein@weizmann.ac.il] received his PhD from the Hebrew University of Jerusalem under the supervision of Robert J. Aumann. Following a post-doctoral position at Brown University he joined the Weizmann Institute of Science where, on top of his research, he held several administrative positions. He is currently a member of the Execute Committee of the EMS.

Nominalism versus Realism

David Corfield

This brief essay will certainly not be an interpretative exercise to discern Plato's philosophical views on the nature of mathematics. That would require the kind of subtle exegesis that can be found in Colin McLarty's "'Mathematical Platonism' Versus Gathering the Dead" (McLarty, 2005), which makes an excellent case for saying that the position closest to contemporary Platonism is voiced not by Socrates but by Glaucon, his interlocutor for much of *The Republic*. Instead, the essay will discuss what should be made of two different uses in contemporary philosophy of mathematics of the distinction: nominalism *versus* realism. First, a description is needed of what will be called here the *external* and *internal* forms of this distinction.

As the term suggests, participants in the *external* nominalism/realism debate look on at mathematics from the outside. They see a great uniformity amongst different pieces of mathematics, whether it be adding 2 and 2, calculating the Fourier transform of a function or proving Fermat's Last Theorem. The philosophically salient activity of mathematicians appears to the externalist to be that of establishing the truth of certain propositions. The question then is what makes these propositions true. Such statements appear to refer to entities and to state properties that hold for them. But what then are these entities? Those who take them to be existing abstract objects are termed 'realist' or 'Platonist'; those who would think we've been misled by the outward grammar of the propositions and that we need make no reference to any form of entity are termed 'nominalist'.

Now when it comes to the *internal* nominalism/realism distinction, it is not so much that we need to take sides in a debate. What interests those who make the distinction above all is rather the thought that some pieces of mathematical theory are worthier than others. Let us explore this thought in the hands of Imre Lakatos. Forgetting all you may have heard about Lakatos as one of the first social constructivists about mathematics, consider the following quotation:

> *"As far as naive classification is concerned, nominalists are close to the truth when claiming that the only thing that polyhedra have in common is their name. But after a few centuries of proofs and refutations, as the theory of polyhedra develops and theoretical classification replaces naive classification, the balance changes in favour of the realist." (Lakatos, 1976: 92n)*

This Footnote appears in Lakatos' famous dialogue 'Proofs and Refutations' and asserts his claim that if we properly subject our mathematical reasoning to a thorough toughening-up process ('dialectic' to give it its fancy name) then we can arrive at more adequate conceptions. Poincaré's late nineteenth century definition is better justified than those of his predecessors earlier in the century.

So it is not that an individual is an internal realist or nominalist, just that there is a distinction to be made within the practice of mathematics between different assertions, definitions and ideas. Indeed, the distinction can be applied to one's own work. In the letter he wrote to his sister, the philosopher Simone Weil, André Weil (1940) describes how, when devising the axioms of a uniform space, it resembled to him the activity of a sculptor working with snow – the material did not resist. What unites uniform spaces is the mere fact that they all satisfy some axioms, rather than that they share a common essence. By contrast, in a further fragment of letter, attached in his *Collected Works* to the previous letter, Weil likens his work on the analogy between function fields and number fields to a sculptor working on hard stone, releasing the form from its prison. Now we are reaching for essential properties behind varied surface appearances.

So how are we to characterise the grounds for such an internal distinction? In the case of the sciences, we might imagine that we are right to make internal-to-practice distinctions between the reality of oxygen and non-reality of phlogiston, or perhaps between the reality of the inferiority complex construct and non-reality of the Oedipal complex, and we are right precisely because of external reality. We believe our concepts to have grasped something in the world. We like to ground our sense of the internal reality of aspects of a practice on external reality. But what can mathematicians count on to play this role? Physical interpretation may be thought to warrant the reality of some mathematics – e.g. natural numbers and beads or group representations and particles – but not all. We might then extend this warrant to include realisation in a game governed by symbolic rules. But in doing this we open the floodgates to all formal manipulation. Even in the case of recognised games, we might say that were I to prove something about chess, it is made true by the set of possible legal games. However, according to the internal distinction I'm alluding to, chess is not a part of mathematics, or at any rate far from what is most real. If the axioms of uniform spaces were devised so easily, then the arbitrariness of the rules of chess must strike us as all the more contingent. Where we expect a real concept to prove its mettle by leading us on to surprising discoveries elsewhere, Vaughan Jones' towers of subfactors and knot invariants being a good case, we don't expect a result concerning chess to be relevant to anything else.

Staying on the theme of games, the mathematician Alexandre Borovik once told me he thinks of mathematics as a *Massively-Multiplayer Online Role-Playing Game*. If so, it would show up very clearly the difference between internal and external viewpoints. Inside the game people are asking each other whether they were right about something they encountered in it – "When you entered the dungeon did you see that dragon in the fireplace or did I imagine it?" But someone observing them from the outside wants to shout: "You're not dealing with anything real. You've just got a silly virtual reality helmet on." External nominalists say the same thing, if more politely, to mathematical practitioners. But in an important way the analogy breaks down. Even if the players interact with the game to change its functioning in unforeseen ways, there were the original programmers who set the bounds for what is possible by the choices they made. When they release the next version of the game they will have made changes to allow new things to happen. In the case of mathematics, it's the players themselves who make these choices. There's no further layer outside.

What can we do then instead to pin down internal reality? Let us take as a starting point something that has often been noted about mathematics – its conservatism. Mathematics has been going on for an awfully long time. We recognise mathematical thinking in a culture that, over four millennia ago, could ask for the length of a field given that 11 times its area added to 7 times its length is 6 and 15/60, following a recipe which gives the answer 30/60 or 1/2. But note that this is not just a case of a result being recognisable by us. We also take it to be the kind of thing a young person should learn today. That mathematical topic is still important. Anyone with any hope of becoming a mathematician had better understand the formula for the quadratic equation. It may be one speck in a research mathematician's mind but it is still a recognisably good piece of mathematics, one capable of multiple elaborations, which may lead into deep waters, e.g. the formulae for the roots of cubic and quartic equations but not for the quintic, Galois theory and so on. On the other hand, there are things that are perfectly true about entities that are properly mathematical but which don't have this status. The fact that the number represented by 37 in our normal denary system is prime and remains prime when the digits are reversed is true but I would not say it is a part of mathematics. Just as I should say that a contingent physical fact that a neutrino from the sun passed through my body within a fraction of a second of a photon from Sirius being absorbed by my eye according to my frame of reference is not the concern of physics.

Mathematics is the historical course of mathematical activity. Any good, sufficiently complete history of mathematics will tell the story of the solution of the quadratic. No good history would mention the fact about 37 and 73. Similarly any good, even rather brief, history of mathematics will tell the story of the complex numbers. This shows that another philosopher, José Benardete, although alive to the internal sense of realism, is hopelessly wrong when he says:

> *"Stated in realist terms, the extended number system [of the complex numbers – DC] is presumed in effect to stake out a 'natural kind' of reality. Far from 'carving reality at the joints', however, the system can be shown to feature a flagrantly gerrymandered fragment of heterogeneous reality that is hardly suited to enshrinement at the centre of a serious science like physics, not to mention a rigorous one like pure mathematics. Couched in these ultra-realist terms, the puzzle might be thought to be one that someone with more pragmatic leanings – the system works, doesn't it? – need not fret over; and in fact such a one might*

even look forward to exploiting it to the discomfort of the realist. Fair enough. I should be happy to have my discussion of this Rube Goldberg contraption (as the extended number system pretty much turns out to be) serve as a contribution to the quarrel between anti-realist and realist that is being waged on a broad front today." (Benardete 1989: 106)

If you pass the complex numbers off as pairs of real numbers, each of which is a Dedekind cut of rationals, each of which is an equivalence class of pairs of integers, each of which is an equivalence class of pairs of natural numbers, then gerrymandering is easy to argue for. But to do so you must ignore the whole story of mathematics.

What is it to assert that a piece of contemporary mathematics, say the attempt by Jacob Lurie to devise a homotopic geometry, is good mathematics? It is to say that "Time will tell" and if it does choose to tell, it will do so as a chapter in the story of mathematics. We can construe what Weil was saying above as the claim that the axioms for uniform spaces may not have that honour; they will appear at best as a minor character, or may be superseded by a better notion devised at a later date. His work on function and number fields, on the other hand, he predicts to have an historical permanence.

But whether actual history retains something isn't quite enough because we also work with a notion that a practice may make mistakes. It may dismiss things later seen as important, it may linger on things later seen as trivial and so on. But if this took place against a backdrop of rapidly and radically changing views as to the best organisation of mathematical thinking, there would be little sense that one's decisions could be right or wrong. What we have then is real history located somewhere between two extremes.

1. The history of a practice that demonstrates the ability to understand the path that led to the current situation. Profound conceptual transformations take place but only when justified by an explanation of what was partial about earlier views. They lead to unexpected discoveries in what appear to be unrelated fields. The practice uses historical research not to justify its present position but to challenge its current conceptions. Practitioners are ready to understand partiality in their own viewpoints by exposing their ideas to other practitioners. They make an effort to understand other viewpoints. There is a dynamic exchange with practices that use its results.
2. The field is divided into isolated communities looking to protect their own theories from outside scrutiny. Any conservatism is to be attributed to sociological and intellectual inertia. When changes take place it is due to arbitrary fashion. Conceptual changes never lead to light being thrown in unexpected places.

There is much more to be said here but the point is that, despite some appearances, the actual history of mathematics more closely resembles (1) than (2). To take one part of it, in the past I may have been more open to the idea that large parts of earlier mathematical thinking have been lost to us but I am less inclined to believe that now. Useful for me on this score was putting the following claim to the test:

"The foregoing analysis of the early geometric works of Klein and Lie is far from exhaustive. Nevertheless, it should suffice to make clear that during this relatively brief period they developed a wealth of interesting ideas, techniques, and results that are all but forgotten today." (Rowe, 1989, 264)

In a weblog discussion (Corfield 2009), it became abundantly clear that Rowe had overstated the case and that, as far as algebraic geometry is concerned at least, we should agree with Matthew Emerton's assertion:

"Overall, my sense is that algebraic geometers are aware of the richness of the past of their subject (not all individually of course, but collectively, as a group of researchers), and have made continual efforts over the decades, as their subject developed, to go back to past literature and comb it for ideas that have been temporarily forgotten or misunderstood, if only for the hope of finding a technique that will help them solve open problems of current interest." (Corfield, 2009)

To conclude, mathematical reality in the internal sense may be thought to be that which allows mathematics to proceed as a tradition of intellectual enquiry, that is, allows it to approximate the first of the two descriptions of practice given above.

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David Corfield [D.Corfield@kent.ac.uk] is a senior lecturer in philosophy at the University of Kent, author of Towards a Philosophy of Real Mathematics (Cambridge University Press, 2003) and host of the blog The n-Category Café (http://golem.ph.utexas.edu/category/).