Analogy in mathematics

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Sir Michael Atiyah

...mathematics can I think be viewed as the science of analogy.

Ulam reporting Banach

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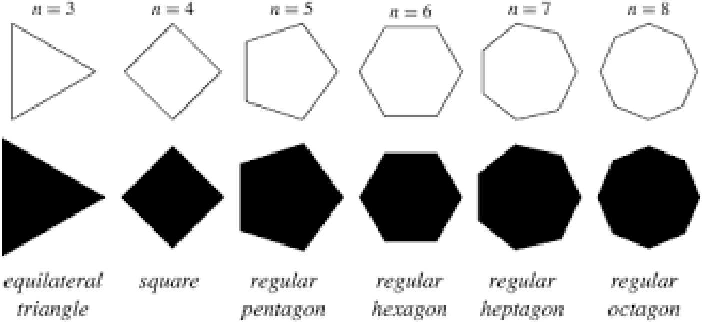
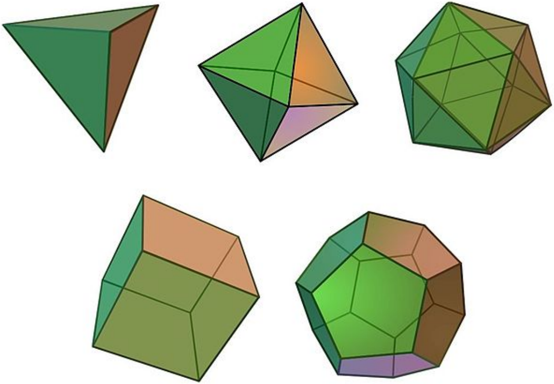
A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies.

Regular polygons



Regular polyhedra

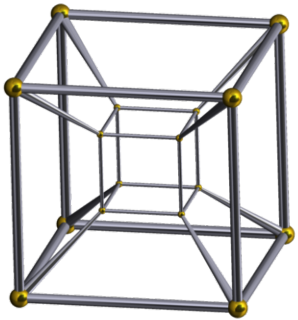


Regular polytopes

2	regular polygons	triangle	square	...	...
3	5 Platonic solids	tetrahed.	cube	octahed.	2 more

4 dimensional polytopes

The 8-cell



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n	simplex	hypercube	orthoplex
2	triangle	square	...
3	4-6-4	8-12-6	6-12-8
4	5-10-10-5	16-32-24-8	8-24-32-16
5	6-15-20-15-6	32-80-80-40-10	10-40-80-80-32

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3	4-6-4	8-12-6	6-12-8
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Vaguer analogy: infinite families with sporadics, such as simple Lie groups $(A_i, B_i, C_i, D_i, E_6, E_7, E_8, F_4, G_2)$.

Henri Poincaré

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But we can have too much of a good thing...

André Weil

Nothing is more fruitful, every mathematician knows it, than these obscure analogies, these disputes reflected from one theory to another, these secret caresses, these quarrels for no reason; indeed nothing gives the researcher more pleasure. The day comes when the illusion falls away; intuition turns into certainty, twin theories reveal their common source, before vanishing; as the Gita tells us, we achieve knowledge and indifference at the same time. Metaphysics has become mathematics, fit content for a treatise whose austere beauty we no longer find moving.

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Fortunately for researchers, as soon as the mists dissolve in one place, they descend elsewhere.

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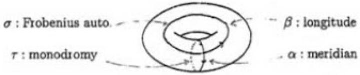
- Some are suggestive of a larger scheme, but resist complete systematisation (for now).

Suggestive analogies, such as arithmetic topology

number ring $\mathrm{Spec}(\mathcal{O}_k) \cup \{\infty\}$ $\mathrm{Spec}(\mathbf{Z}) \cup \{\infty\}$	$\longleftrightarrow$	3-manifold M S^3
prime : $\mathrm{Spec}(\mathbf{F}_\mathfrak{p}) \subset \mathrm{Spec}(\mathcal{O}_k)$ primes $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ infinite prime	$\longleftrightarrow$	knot $K : S^1 \subset M$ link $K_1 \cup \cdots \cup K_n$ end
$\mathfrak{p}$-adic integers $\mathrm{Spec}(\mathcal{O}_\mathfrak{p})$	$\longleftrightarrow$	tube n.b.d $V(K)$
$\mathfrak{p}$-adic field $\mathrm{Spec}(k_\mathfrak{p})$	$\longleftrightarrow$	torus $\partial V(K)$
$\pi_1(\mathrm{Spec}(\mathcal{O}_\mathfrak{p})) = \langle \sigma \rangle$	$\longleftrightarrow$	$\pi_1(V(K)) = \langle \beta \rangle$
$\pi_1^{tame}(\mathrm{Spec}(k_\mathfrak{p})) = \langle \tau, \sigma \mid \tau^{p-1}[\tau, \sigma] = 1 \rangle$	$\longleftrightarrow$	$\pi_1(\partial V(K)) = \langle \alpha, \beta \mid [\alpha, \beta] = 1 \rangle$



$k^\times \to \bigoplus_{\mathfrak{p}:\text{primes}} \mathbf{Z}$ $a \mapsto a\mathcal{O}_k$	$\longleftrightarrow$	$C_2(M, \mathbf{Z}) \xrightarrow{\partial} C_1(M, \mathbf{Z})$ $\Sigma \mapsto \partial\Sigma$
class group H_k	$\longleftrightarrow$	$H_1(M, \mathbf{Z})$
$\mathcal{O}_k^\times$	$\longleftrightarrow$	$H_2(M, \mathbf{Z})$
$\pi_1(\mathrm{Spec}(\mathcal{O}_k) \setminus \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\})$ max. Galois group with given ramification	$\longleftrightarrow$	$\pi_1(M \setminus K_1 \cup \cdots \cup K_n)$ link group
power residue symbol	$\longleftrightarrow$	linking number

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There are suggestive ‘dictionaries’ for this analogy. But let’s turn to the more precisely formulated end of analogy formation.

Category theory as the great systematizer

Emily Riehl:

> *Atiyah described mathematics as the "science of analogy." In this vein, the purview of category theory is mathematical analogy. Category theory provides a cross-disciplinary language for mathematics designed to delineate general phenomena, which enables the transfer of ideas from one area of study to another. The category-theoretic perspective can function as a simplifying abstraction, isolating propositions that hold for formal reasons from those whose proofs require techniques particular to a given mathematical discipline.*

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- This data can be represented in the plane.
- This is so for all such collections.

Products

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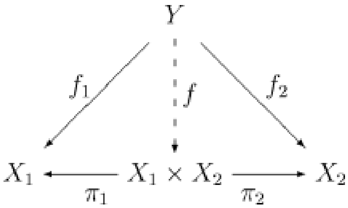
All instances of:



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- the set of points underlying the product of two spaces $=$ the product of the sets of points underlying each space.

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H must be 'lawlike'.

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Eventually a general theorem (a common ground) concerning such factorisation appears in the 19th century.

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> *There is impressive numerical evidence in its favour but certainly the best reason to believe that it is true comes from the analogy of number fields with function fields of curves over finite fields where the analogue of RC has first been proved by A. Weil. (Deninger 1994)*

[Developed further by Deligne.]

Caution needed, Munson

We intend to explain some of the intuition behind one incarnation of calculus of functors, namely the so-called "manifold calculus" due to Weiss and Goodwillie. Specifically, we will **highlight some analogies** *between the ordinary calculus of functions and the manifold calculus of functors. The* **trouble with analogies is that they are not equivalences**, *and some may lead the reader to want to push them further. Some may indeed be pushed further than we are currently aware, and some may lead to direct contradictions and/or bad intuition. Another risk is that it is considered bad manners to tell people how to categorize various ideas: part of our mathematical culture seems to be that we leave intuition for talks and personal communications and rigor and precision for our papers, and with good reason: we cannot anticipate the ways in which our work may be useful in the future, and so it may be best to convey it in as concise and precise a way as possible. We feel the relatively small risk of misleading the reader and the faux pas of making permanent intuitive notions by publishing them is a small price to pay for the possibility that this may entice some reader to learn more about these ideas and try to use them.*

The structure of the category...is much richer than the usual topology on the real line, **so analogies between functions and functors may seem a little weak**. *Still, there are a few things to say that may be helpful. We like to think of ... as the* **analog** *of the dense subset Q.*

To conclude

- Analogical reasoning is everywhere in maths.
- Category theory is good at capturing very general, precise analogies across fields.
- Some analogies run for a very long time without being turned into general theory.
- Analogies play a huge role in plausibility assessment.
- Care must be taken not to follow them blindly.