Modal and Linear HoTT for Physics

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This conference

We have been called upon here to explore

The profound interplay between Quantum Gravity and Computation

My contribution today: some commentary on the Sati–Schreiber program

Acknowledgement

I've been following Urs Schreiber's work for 17 years, since we co-founded the blog, *The n-Category Café*, with John Baez, and then the wiki, *nLab* and *nForum*.

It's been a great pleasure and privilege to watch these ideas being formulated in such an open fashion.

Program of Sati, Schreiber + co-authors

- A proposal for M-theory, the long-sought non-perturbative quantum theory whose limiting cases are perturbative string theories.
- Charge quantization of the C-field occurs in a certain cohomology theory, namely, twisted equivariant differential non-abelian cohomotopy theory.
- This cohomology theory may be understood via modal homotopy type theory, an extension by 'modalities' of the new foundational language, HoTT.

Hypothesis H

Hypothesis H

Sati 13, Fiorenza-Sati-S. 19b,19c

$\mathbf{CovariantPhaseSpace}_{\text{M-Theory}} =$

spacetimes		
formalized as:	G_{ADE}–orbi $\mathbb{R}^{10,1\vert\mathbf{32}}$–folds ($\mathcal{X}$) super-orbifold	
equipped with:	**0) gravity**	**1) C-field**
formalized as:	Pin^{+}-structure (E, Ψ) super-vielbein	differential forms (G_4, G_7) flux densities
subject to:	**Einstein equations**	**Page equation** flux is in ~~rationalized~~
equivalently to:	super-torsion $= 0$	J-twisted Cohomotopy
	Candiello-Lechner 93, Howe 97	FSS 19b,19c, SS 19a,19b,19c

Hypothesis H: (FSS 19b,FSS 19c) *The C-field is charge quantized in J-twisted Cohomotopy cohomology theory.*

Consequences of Hypothesis H

Consequences of the hypothesis are elaborated in a series of articles establishing anomaly cancellation conditions, extending to the differential and the equivariant forms of cohomotopy cohomology.

A very rich array of mathematics is encountered along the way, requiring the development of new mathematical theory, see e.g., M/F-Theory as Mf-Theory, Proper Orbifold Cohomology and The Character Map in Nonabelian Cohomology – Twisted, Differential, and Generalized

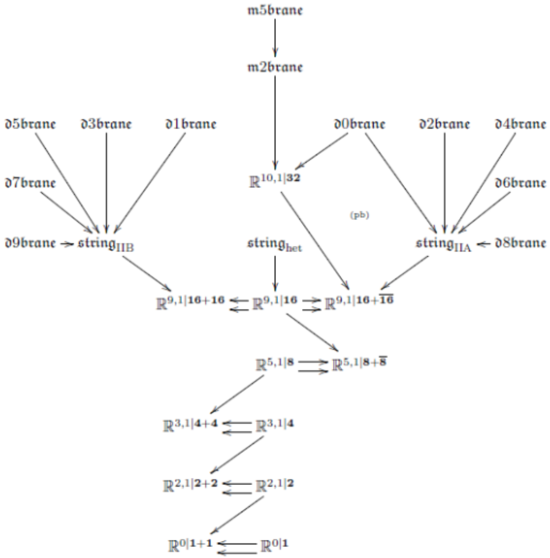
Much more to work out

> *Currently in all our discussions, the dynamical (gravitational) field is ignored, but we have sketched how to bring it in: one will have to phrase 11d-sugra as a super-Cartan geometry and demand that the differential Cohomotopy cocycle is super-tangent-space wise equal to the canonical S^4-cocycle on super-Minkowski spacetime. (Schreiber, personal communication)*

(Note the higher Cartan geometry of extended 11d-supergravity here.)

All of this doesn't just arrive out of the blue – there's an origin story...
(see nLab: Hegel's *The Science of Logic*)

How M-branes emerge from the superpoint



How the superpoint emerges from nothing

				id	$\dashv$	id			
				$\vee$		$\vee$			
				$\rightrightarrows$	$\dashv$	$\rightsquigarrow$	/	e	bosonic / fermionic
solidity				$\bot$		$\bot$			
				$\rightsquigarrow$	$\dashv$	R	$= \mathrm{loc}_{\mathbb{R}^{0\vert 1}}$		rheonomic
				$\vee$	super	$\vee$			
	infinitesimal / reduced	$\bar{\mathfrak{R}}$	/	$\mathfrak{R}$	$\dashv$	$\Im$			
elasticity				$\bot$		$\bot$			
	infinitesimal shape			$\Im$	infinitesimal quality $\dashv$	&			étalé
				$\vee$		$\vee$			
	shape	$\mathrm{loc}_{\mathbb{R}} =$		$\int$	quality $\dashv$	$\flat$	/	$\bar{\flat}$	content(flat / rational)
cohesion				$\bot$	gaugemeasure	$\bot$			
	discrete	$\bar{\flat}$	/	$\flat$	quantity $\dashv$	$\sharp$	$= \mathrm{loc}_{\neg\neg}$		continuous(intensive / extensive)
				$\vee$	ground	$\vee$			
				$\emptyset$	$\dashv$	$*$			

The computation?

From this program there has recently emerged an approach to quantum computation:

- Building on results on defect branes in string/M-theory and on their holographically dual anyonic defects in condensed matter theory, they provide a paradigm for simulating and verifying topological quantum computing architectures with high-level certification languages aware of the actual physical principles of realistic topological quantum hardware.
- Then a theory of linear homotopy types extends this scheme to a full-blown quantum programming/certification language in which topological quantum gates may be compiled into verified quantum circuits with quantum measurement gates and classical control.

Exposition of the latter

Abstract

We lay out a language paradigm, `QS`, for quantum programming and quantum information theory – rooted in the algebraic topology of stable homotopy types – which has the following properties, deemed necessary and probably sufficient for the eventual goal of heavy-duty quantum computation:

- **Application**: in its 0-sector, `QS` is cross-translatable with the established quantum programming scheme `Quipper`, including support for classical control (dynamic lifting via dependent linear types) such as by quantum measurement outcomes which are handled monadically as in the widely used `zxCalculus`.
- **Compilation**: but `QS` is embedded in (is just syntactic sugar for) a universal quantum certification language `LHoTT`, being a novel linear enhancement of the established formal (programming/certification) language scheme of Homotopy Type Theory (`HoTT`).
- **Certification**: as such, `QS` introduces a previously missing method of formal verification of general classically controlled quantum programs, e.g. it verifies quantum axioms such as the deferred measurement principle.
- **Stabilization**: in its higher sector, `QS` natively models hardware-level topologically stabilized quantum computation such as by realistic anyonic braid gates, verifying their conformal field theoretic properties.
- **Realization**: in fact, `QS` naturally interfaces with the holographic quantum theory of topologically ordered quantum materials that are thought to eventually provide topologically stabilized quantum hardware.

In developing these results we find a pleasant unification of *quantum logic* (linear types), *epistemic modal logic* (possible worlds), *quantum interpretations* (many worlds), and *twisted cohomology* (parameterized spectra) & *motives* (six-operations) – which may be of interest in itself. ("`QS`" stands both for "Quantum Systems language" and for the sphere spectrum "QS^0".)

Myers-Sati-Schreiber, QS: Quantum Programming via Linear Homotopy Types

Not just about quantum computing

Generally, our thesis is that the conceptual foundation not just of quantum computing but in fact of fundamental quantum physics generally is in linear homotopy theory. (p. 5)

It may seem overambitious that in a treatise on quantum programming, we should have anything to say about problems in quantum field theory, but we offer the inclined reader an argument (exposition in ...) that the solutions to these fundamental problems share a common root in linear homotopy theory and as such lend themselves to formulation in LHoTT [linear homotopy type theory]. (p. 8)

Some relevant articles

- Introduction to Hypothesis H: Non-perturbative quantum theory as candidate for M-theory, applicable via holography to topologically ordered solid states for topological quantum computation; confined quarks in hadrons.
- Topological Quantum Gates in Homotopy Type Theory: The specification of realistic topological quantum gates, operating by anyon defect braiding in topologically ordered quantum materials. (Sec. 5.1 contains a good intro to HoTT.)
- Entanglement of Sections: The pushout of entangled and parameterized quantum information: Categorical semantics for the linear-multiplicative fragment of Linear Homotopy Type Theory.
- QS: Quantum Programming via Linear Homotopy Types: A new language paradigm, QS, for quantum programming and quantum information theory. QS as embedded in Linear Homotopy Type Theory. .

Alongside the odd contribution to the research program...

- David Corfield, Hisham Sati, Urs Schreiber

  Fundamental weight systems are quantum states

  Letters in Mathematical Physics (2023, in print)

  download:

  - pdf
  - arXiv:2105.02871

Abstract. Weight systems on chord diagrams play a central role in knot theory and Chern-Simons theory; and more recently in stringy quantum gravity. We highlight that the noncommutative algebra of horizontal chord diagrams is canonically a star-algebra, and ask which weight systems are positive with respect to this structure; hence we ask: Which weight systems are quantum states, if horizontal chord diagrams are quantum observables? We observe that the fundamental $\mathfrak{gl}(n)$-weight systems on horizontal chord diagrams with N strands may be identified with the Cayley distance kernel at inverse temperature $\beta = \ln(n)$ on the symmetric group on N elements. In contrast to related kernels like the Mallows kernel, the positivity of the Cayley distance kernel had remained open. We characterize its phases of indefinite, semi-definite and definite positivity, in dependence of the inverse temperature β; and we prove that the Cayley distance kernel is positive (semi-)definite at $\beta = \ln(n)$ for all $n = 1, 2, 3, \cdots$. In particular, this proves that all fundamental $\mathfrak{gl}(n)$-weight systems are quantum states, and hence so are all their convex linear combinations. We close with briefly recalling how, under our "Hypothesis H", this result impacts on the identification of bound states of multiple M5-branes.

there's the task of providing some philosophical context

If the program goes to plan, we are seeing emerge:

A new logic for a new mathematics for a new physics.

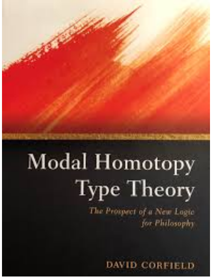
See my Thomas Kuhn, Modern Mathematics and the Dynamics of Reason.

While each stage meets a certain resistance, it's remarkable to see how the logic speaks to the physics.

Computational trinitarianism identifies the ends (see also Baez and Stay's *Rosetta stone*).

What is linear HoTT?

- What is HoTT?
- What is modal HoTT?
- What is the linear modality?



Homotopy type theory – a new foundational language

(Homotopy type) theory = Homotopy (type theory)

- Theory of *homotopy types*
- A *type theory* that is homotopical

Homotopy (type theory)

HoTT is a constructive dependent type theory

- Elements of types correspond to proofs of propositions correspond to programs carrying out specified tasks.
- Types may depend on other types, tasks may depend on the way other tasks can be fulfilled: $x : A \vdash B(x) : Type$
- Note a type of types (indeed an infinite series) $Type_i$.
- Type formation: $\mathbf{0}$, $\mathbf{1}$, sum type $A + B$, product type $A \times B$, function type $[A, B]$, ...
- Two important constructions are dependent sum (pair/co-product), $\sum_{x:A} B(x)$ and dependent product (function), $\prod_{x:A} B(x)$.
- Identity types: $A : Type, a, b : A \vdash Id_A(a, b) : Type$

A dictionary

Type formation is governed by *formation-introduction-elimination-computation* rules, such as for product.

	type theory	category theory
	syntax	semantics
	natural deduction	universal construction
	product type	***product***
type formation	$\frac{\vdash A : \mathrm{Type} \quad \vdash B : \mathrm{Type}}{\vdash A \times B : \mathrm{Type}}$	$A, B \in \mathcal{C} \Rightarrow A \times B \in \mathcal{C}$
term introduction	$\frac{\vdash a : A \quad \vdash b : B}{\vdash (a,b) : A \times B}$	$\begin{array}{ccccc} & & Q & & \\ & {}^{a}\swarrow & \downarrow_{(a,b)} & \searrow^{b} & \\ A & & A \times B & & B \end{array}$
term elimination	$\frac{\vdash t : A \times B}{\vdash p_1(t) : A} \quad \frac{\vdash t : A \times B}{\vdash p_2(t) : B}$	$\begin{array}{ccc} & Q & \\ & \downarrow^{t} & \\ A \overset{p_1}{\leftarrow} & A \times B & \overset{p_2}{\rightarrow} B \end{array}$
computation rule	$p_1(a,b) = a \quad p_2(a,b) = b$	$\begin{array}{ccccc} & & Q & & \\ & {}^{a}\swarrow & \downarrow_{(a,b)} & \searrow^{b} & \\ A & \overset{p_1}{\longleftarrow} & A \times B & \overset{p_2}{\rightarrow} & B \end{array}$

(For some Brandomian reflections on this, see my Type-theoretic Expressivism slides.)

(Homotopy type) theory

Synthetic treatment of abstract spatial structure – homotopy types.

- A structurally invariant theory of ∞-groupoids, structure emerging from iterated identity types.
- Dependent types correspond to spaces sitting over another space.
- Dependent sum corresponds to the *total* space.
- Dependent product corresponds to the type of *sections*
- Physics: principal bundles, gauge-of-gauge transformations.

(Cf. Mike Shulman's Homotopy type theory: the logic of space)

HoTT subsumes logic

- Propositional logic: (-1)-types $=$ propositions
- Predicate logic: (-1)-types depending on 0-types $=$ predicates on sets.
- Structural set theory: 0-types.
- HoTT: dependent ∞-types.

HoTT is the internal language of an ∞-topos

HoTT offers the opportunity for a *synthetic* mathematical treatment of spaces as homotopy types.

Modality

Philosophers and computer scientists have sought *modal* variants of propositional and predicate logic.

It was natural then to expect a *modal* HoTT.

Examples of these modalities may be use to capture important mathematical structure synthetically, such as cohesion and smoothness.

A modal HoTT is the internal language of a system of ∞-toposes

Modalities are kinds of monad and comonad, operators arising from adjunctions.

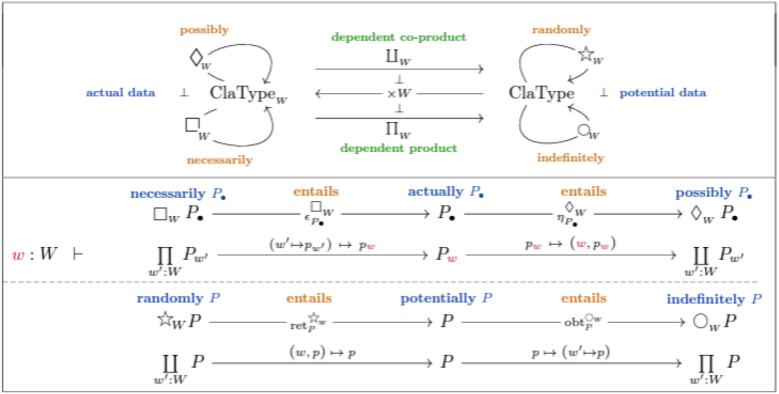
Native modality

One natural pair of examples to consider are the modalities arising from dependent sum and dependent product.

- $w : World \vdash A(w) : Prop$
- $\prod_{w:World} A(w), \sum_{w:World} A(w)$
- 'For all worlds, A'.
- 'The worlds where A', truncated to 'In some world, A'.
- From these we derive the operators on world-dependent types which act as necessity and possibility.
- E.g., $w : World \vdash A(w) : Type$, then $w : World \vdash \Box A(w) : Type$.

Cf. Ch. 4 of my Modal HoTT book for discussion.

As it appears in QS



(Eq. 157)

But one may also specify modal operators for various purposes, e.g., cohesive and linear structure.

Modal HoTT

1. HoTT: synthetic language to describe structure
2. *Cohesive* HoTT: synthetic language for differential and equivariant structure, differential cohomology of (higher) gauge theory.
3. *Linear* HoTT: synthetic language for 'linear' structure (infinitesimal, tangent, abelian, stable, etc.), quantum information

((1) Shulman; (2) Sati-Schreiber; (3) Myers-Sati-Schreiber)

Category-theoretic semantics

- HoTT: ∞-topos
- Modal HoTT: systems of ∞-toposes and geometric morphisms
- Cohesive HoTT: adjoint quadruple between two ∞-toposes
- Linear HoTT: bireflective inclusion of one ∞-topos inside another

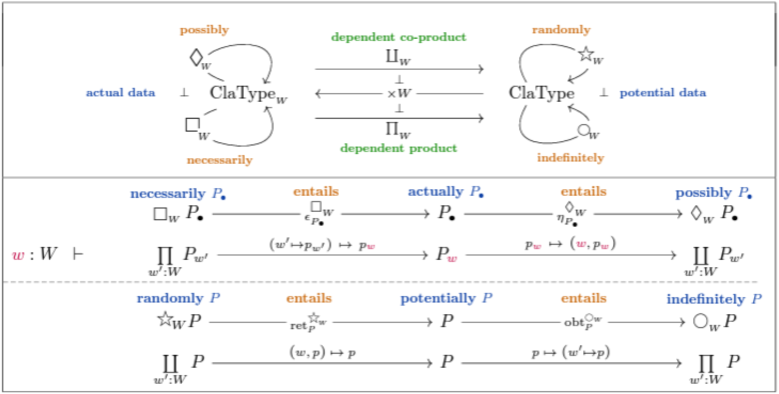
Linear HoTT

- Surprise to find that the 'tangent' ∞-category of an ∞ -topos is an ∞-topos.
- Parameterized spectra as tangent to ∞-groupoids.
- Inclusion of 0-spectra is left and right adjoint to projection.
- This situation has the nature of an infinitesimal thickening.
- Over finite sets, direct sum and direct product coincide.

Two operators

$$
\begin{array}{ccccc}
W & \mapsfrom & \begin{bmatrix} \mathcal{H}_\bullet \\ \downarrow \\ W \end{bmatrix} & & \\
\mathrm{ClaType} & \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} & \mathrm{Type} & \circlearrowright \natural & \text{classically} \\
W & \mapsto & \begin{bmatrix} 0_\bullet \\ \downarrow \\ W \end{bmatrix} & &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\bigoplus\limits_w \mathcal{H}_w & \mapsfrom & \begin{bmatrix} \mathcal{H}_\bullet \\ \downarrow \\ W \end{bmatrix} & & \\
\mathrm{QuType} & \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} & \mathrm{Type} & \circlearrowright \triangleright & \text{quantumly} \\
\mathcal{H} & \mapsto & \begin{bmatrix} \mathcal{H} \\ \downarrow \\ * \end{bmatrix} & &
\end{array}
$$

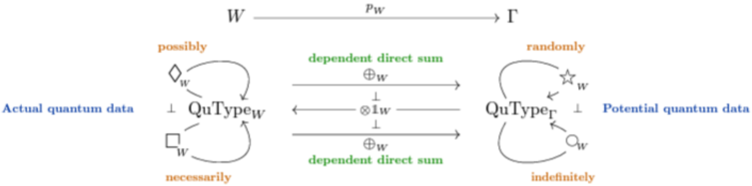
Classically



(Eq. 157)

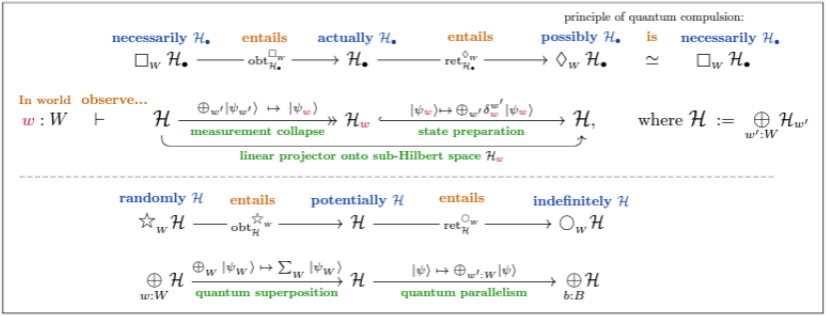
Now for the quantum version: from *possible* worlds to *many* worlds.

Quantumly



(Eq. 158)

Quantumly



(Eq. 161)

No cloning/no deleting

Linear HoTT maintains features of the multiplicative conjunction of linear logic:

Quantum Phenomena	Linear Type Inference	Linear maps in Linear algebra...
No-cloning theorem	Absence of contraction rule	...use their argument at most once.
No-deleting theorem	Absence of weakening rule	...use their argument at least once.

(18)

Entanglement

A quantum programming language captures the ideas of quantum computation in a linear type theory.

Bunched classical/quantum type theory and EPR phenomena. And yet, a comprehensive programming language implementing such *linear type theories* of *combined* classical and quantum data had remained elusive all along: The type-theoretic subtlety here is that with the classical conjunction ($\times$) being accompanied by a linear multiplicative conjunction ($\otimes$), then contexts on which terms and their types should depend are no longer just linear lists of (dependent) classical products

$$\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$$

a classical type-context (tuples of classical data)

but may be nested ("bunched") such products, alternating with linear multiplicative conjunctions to form tree-structured expressions like this example:

$$\Gamma_1 \times (\Gamma_2 \otimes (\Gamma_3 \times \Gamma_4)) \times (\Gamma_5 \otimes \Gamma_6) \times (\Gamma_7 \otimes \Gamma_8 \otimes \Gamma_9)$$

a mixed classical/quantum type-context (tuples of classical data mixed with *entangled* quantum data).

While the idea of formulating such "bunched" type theories is not new [OP99][Py02][O'H03], its implementation has turned out to be tricky and the results unsatisfactory; see [Py08, §13.6][Ri22, p. 19]. The claim of the type theory introduced in [Ri22] is to have finally resolved this long-standing issue of formulating "bunched linear dependent type theory". Here we understand this as saying that a verifiable universal quantum programming language now exists (LHoTT, §2).

To put this into perspective it may be noteworthy that the root of this subtlety resolved by **LHoTT** corresponds to the hallmark phenomenon of quantum physics which famously puzzled the subject's founding fathers (Lit. 1.2), namely the *conditioning of physics on entangled quantum states* (known as the *EPR phenomenon*, e.g. [Sel88]):

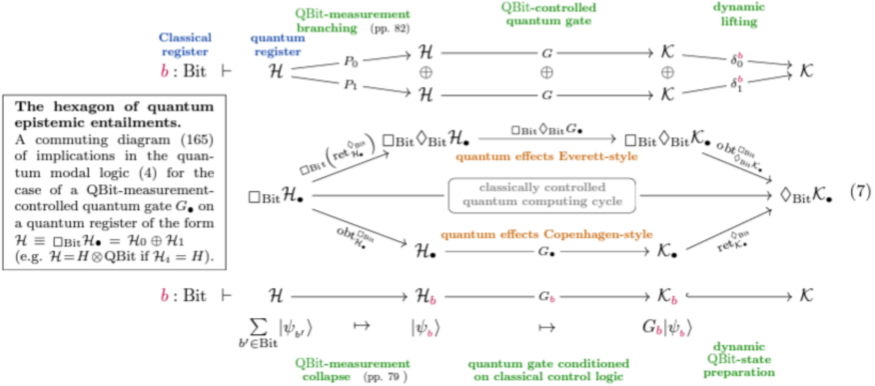
Under the correspondence between dependent linear type theory and quantum information theory, the existence of bunched typing contexts involving linearly multiplicative conjunctions $\otimes$ corresponds to the conditioning of protocols on entangled quantum states and hence to what in quantum physics are known as EPR phenomena.

Bunched logic	**EPR phenomena**
Typing contexts built via multiplicative conjunction ($\otimes$)	Physics conditioned on entangled quantum states

(Literature 1.5)

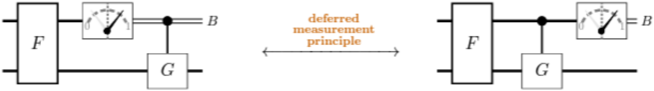
Equivalence of Everett and Copenhagen styles

Formal logic of quantum measurement effects. Remarkably, unwinding the logical rules of this epistemic quantum logic (6) reveals that it knows all about the state collapse after quantum measurement including formal proof of its equivalence to *branching* into "many worlds" (Lit. 1.2):



Deferred measurement

Deferred measurement principle. Since quantum measurement turns quantum data into classical data, it intertwines quantum control with classical control. Concretely, a statement known as the *deferred measurement principle* asserts that any quantum circuit containing intermediate (mid-circuit) quantum measurement gates followed by gates conditioned on the measurement outcome is equivalent to a circuit where all measurements are "deferred" to the last step of the computation



(9)

there as an "axiom" of quantum computation. We prove below (Prop. 3.17) that the deferred measurement principle (9) is verified in the data-typing of quantum processes provided in `LHoTT`.

Notice that the content of this *equivalence between intermediate and deferred measurement collapse* (9) is not trivial without a good formalization; in fact it has historically been perceived as a *paradox*, namely this is essentially the paradox of "*Schrödinger's cat*" (where the cat plays the role of the intermediate controlled quantum gate). Moreover, the same paradox, in different words, was influentially offered in [Ev57a, pp. 4] as the main argument against the "Copenhagen interpretation" and for the "many-worlds interpretation" of quantum physics (cf. Lit. 1.2). Note that our same formalism which proves (9) also proves the equivalence (7) of these two "interpretations".

The deferred measurement principle is verified in the data-typing of quantum processes provided in LHoTT.

Subsuming previous work

- Proto-Quipper programs may be translated to LHoTT, so as to formally certify them.
- LHoTT/QS can be used for certifying (type-checking) ZX Calculus-protocols

Further topics

Having treated no-cloning/deleting, superposition/parallelism, quantization, entanglement, quantum gates with quantum measurement, Section 4 of the QS paper treats: the Born rule, mixed states, and quantum channels.

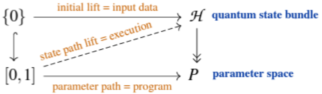
Using some more of the expressive possibilities of LHoTT, ($\mathbb{Z}_2$-equivariant types), it is possible to avoid specifying dagger structure to capture the Born rule.

Back to the conference question on computation

We're offered a scheme for computation:

The driving theme of our discussion is the observation/claim that:

> *Fundamental (quantum) computing processes are lifts of classical parameter paths, i.e. the programs, to (linear) maps of state spaces, i.e. the (quantum) gates.*



(19)

This may sound simple, but we claim it is profound (similar statements are in [ZR99][NDGD06][DN08][LW17]): Namely, it means that natural certification languages for low hardware-level (quantum) computation ought to natively know about *path lifting* (Lit. 2.30). This is unheard-of in traditional programming languages — but it is the hallmark (74) of homotopically-typed languages! (Lit. 2.27) Moreover, under such identification of low-level computation with path-lifting, homotopically-typed languages natively reflect the crucial reversibility (76) of fundamental quantum computational processes (Lit. 2.2).

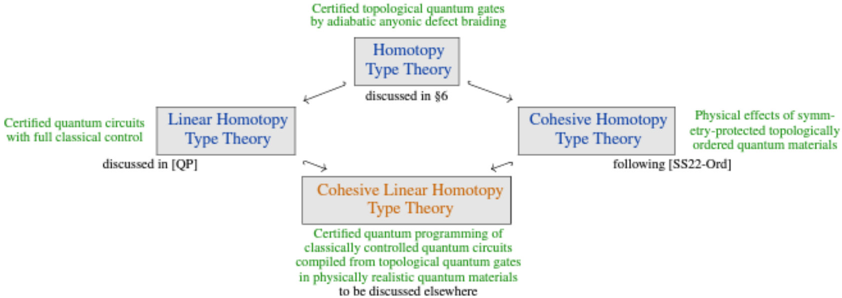
Myers-Sati-Schreiber, Topological Quantum Gates in Homotopy Type Theory

Computation and quantum gravity

Does their M-theoretic account participate in the computational aspects of this language of quantum computation?

Remember that linear HoTT has far richer semantics than given by finite-dimensional Hilbert spaces parameterized by sets.

Cohesive Linear HoTT



[1] Intriguingly, cohesive linear homotopy theory is also the semantic context in which to naturally make formal sense of the key ingredients of high energy physics, specifically of string/M-theory (cf. [Sc14b][SS20-Orb, p. 6]). This is in line with a deep relationship between strongly coupled quantum systems (such as anyonic topological order) and string/M-theory, cf. [SS22-Ord, Rem. 2.8][Sa23][Sc23].

Myers-Sati-Schreiber, Topological Quantum Gates in Homotopy Type Theory

Follow-up

The relation to M-theory is obtained by taking "external parameters" to be "positions (moduli) of defect branes" (specifically of M3-branes inside M5-branes) and "computational states" to be "quantum states of these defect branes", which under holographic CMT ought to translate to "positions of anyonic defects" and "topologically ordered quantum ground states of 2d materials with such defects".

In both cases the physical mechanism by which we actually go about moving these positions, as a physical process, is currently being disregarded. In condensed matter theory (CMT) and understanding defects as "band nodes in momentum space" it ought to happen by changing external parameters such as the material's strain. (Schreiber, personal communication)