The ubiquity of modal types

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SYCO1

20 September, 2018

A common phenomenon

- Philosophers will think about a family of concepts and try to theorize and then perhaps formalize.
- Other disciplines develop these theories and formalisms.
- Philosophers continue along their own path without paying attention to descendent theories.

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- Goal is to explore alethic, epistemic, doxastic, deontological, temporal... modalities.
- They might consider the differences, if any, between *physical*, *metaphysical* and *logical* necessity and possibility.
- Technically, still largely in the era of modal logics (K, S4, S5, etc.) and Kripke models for semantics.

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- Philosophers' modalities for different uses: Model-checking (temporal). Multi-agent systems (epistemic).
- Technically, use of sub-structural logics, coalgebra, labelled transition systems, bisimulations, adjunctions,...

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- Kripke models, 1959 (presheaves over states).
- Metaphysical phase - possible worlds, e.g., Kripke, *Naming and Necessity* (1970/80), David Lewis, *On the Plurality of Worlds* (1986).

Naturally there were efforts to develop a first-order modal logic, leading to questions about, say, the relationship between $\exists\Diamond$ and $\Diamond\exists$.

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- It is possible that something is P.

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A different solution has the relationship made trivial by allowing quantification over all possible things.

Sheaf semantics to the rescue

Modal logicians have devoted the overwhelming majority of their inquiries to propositional modal logic and achieved a great advancement. In contrast, the subfield of quantified modal logic has been arguably much less successful. Philosophical logicians–most notably Carnap, Kripke, and David Lewis–have proposed semantics for quantified modal logic; but frameworks seem to keep ramifying rather than to converge. This is probably because building a system and semantics of quantified modal logic involves too many choices of technical and conceptual parameters, and perhaps because the field is lacking in a good methodology for tackling these choices in a unifying manner. The remainder of this chapter illustrates how the essential use of category theory helps this situation, both mathematically and philosophically. (Kishida 2017, p. 192)

Or jump to modal HoTT?

Propositions as types $\rightarrow$ Propositions as some types

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Propositions as types $\rightarrow$ Propositions as some types

...	...
2	2-groupoid
1	groupoid
0	set
-1	mere proposition
-2	

Common constructions applied to the hierarchy provide propositional logic, first-order logic and a structural set theory at the lower levels.

Modal HoTT

$$\begin{array}{ccc} \text{Logic} & \rightarrow & \text{Modal Logic} \\ \downarrow & & \downarrow \\ \text{HoTT} & \rightarrow & \end{array}$$

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$$

Near thing?

« Higher Structures in Göttingen IV | Main | 3000 and One Things to Think About »

February 25, 2010

(Infinity, 1)-logic

Posted by David Corfield

We're having a chat over **here** about what an $(\infty, 1)$-logic might look like. The issue is that if we can extract a (1)-logic from ordinary toposes, shouldn't there be an $(\infty, 1)$-**logic** to be extracted from $(\infty, 1)$-toposes. This post originated as an ordinary comment, but as things have gone a little quiet at the Café (10 days without a post!), I thought I'd promote it.

Won't there be a sense in which this internal logic to an $(\infty, 1)$-topos will have to be interpretable as a 'logic of space'? If Set is an especially nice topos which being **well-pointed** allows us to understand 1-logic internally and externally, we might hope that the well-pointed $(\infty, 1)$-topos of ∞-groupoids, ∞Gpd, does the same for $(\infty, 1)$-logic, given ∞-groupoids are models for spaces.

Had I not known anything about logic beyond that it is a language used to formulate statements in a domain and to represent valid reasoning, then had I been asked to extract such a thing given the definition of a topos, that would seem to be a tricky task. But give me Set and I can make a start.

Lawvere on quantifiers

For **H** a topos (or ∞-topos) and $f : X \to Y$ an arrow in **H** induces a 'base change', f^*, between slices (categories of dependent types):

$$(\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}/X \underset{f_*}{\overset{\overset{f_!}{\rightarrow}}{\overset{\overset{f^*}{\leftarrow}}{\rightarrow}}} \mathbf{H}/Y$$

This base change has dependent sum and product as left and right adjoint.

Modal logic

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- The unit of the monad is the injection of a world where P holds into all such worlds.
- The counit of the comonad applies a function proving P at each world to this world.

Accessible worlds

More generally, we might consider an equivalence relation: $W \to V$, then

- Necessarily P holds at a world if P holds at all related worlds.
- Possibly P holds at a world if it holds at some related world.

General modal types

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Let's consider things through another map:

$$spec : Animal \to Species$$

Then for an *Animal*-dependent type, $Leg(x)$:

- $\bigcirc_{spec} Leg(Fido)$ is the set of legs of dogs
- $\Box_{spec} Leg(Fido)$ is the set of choices of a leg for each dog.

Examples of the latter include 'the last leg to have left the ground(x)', and 'front right leg(x)'.

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The latter is definable in terms of the species *Dog*, part of the blueprint for being a member of the species,

- $s : Species \vdash BodyPart(s) : Type$
- front right leg: $BodyPart(Dog)$
- $spec^* BodyPart(x)$ is a type dependent on $x : Animal$.

‘Front right leg’ is acting as a *rigid designator* over the animals which are dogs.

Recall that generally we have a map $\Box A \to A$, but not one from $A \to \Box A$.

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Given an element in $spec^* BodyPart(Fido)$, such as Fido's front right leg, we can name a similar body part for Fido's conspecifics, i.e., an element of $\Box_{spec} spec^* BodyPart(Fido)$.

[Note we're in a world where no animal has lost a leg. Or we might speak of Patch having lost his front right leg.]

These ‘rigid designators’ are elements of the sort of types, $A(w)$, for which there is a natural map, $A(w) \to \Box A(w)$, which is not the case for general world-dependent types.

Consider $W \to 1$, then for a non-dependent type, B, there’s a map $W^*B(w) \to \Box_W W^*B(w)$ sending $b : B$ to the constant section, $w \mapsto b$.

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It's all about knowing how to continue to counterparts in neighbouring worlds/fibres/dogs. If I point to the front right leg of a dog and show you another dog, you probably choose the same leg.

There's a short route from this construction to (formally integrable) partial differential equations, being told how behaviour carries over to infinitesimally neighbouring points.

Here we are in a differentiable context with a map $X \to \Im(X)$, identification of infinitesimal neighbourhoods.

The corresponding 'necessity' operator corresponds to forming the 'jet comonad', and coalgebras are PDEs.

Chestnut

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- The number of planets is 8.
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Applying the discipline of types avoids mistakes:

$\mathbb{N}$, $W^*\mathbb{N}$, $\Box_W(W^*\mathbb{N})$, $a^*(\Box_W(W^*\mathbb{N})) = \prod_W W^*\mathbb{N}$

Actualism and Higher-Order Worlds

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Hayaki arranges things through nested trees. The first sentence presents a level 1 world, the second a level 2 world.

In modal type theory we could imagine an approach via changes to the context.

$$\Gamma \;=\; x_0 : A_0,\; x_1 : A_1(x_0),\; x_2 : A_2(x_0, x_1),\; \ldots x_n : A_n(x_0, \ldots, x_{n-1}),$$

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We could base change, etc., relative to an initial segment of the context.

Counterfactuals could work by stripping back a context until the counterfactual antecedent can hold.

Temporal types

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With *Time* as an internal category, poset, linear order, we can generate some form of temporal type theory.

We'll have at least $b, e : Time_1 \to Time_0$ generating two adjoint triples to express the temporal operators - F, G, H, P.

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Composition between matching intervals allows for the expressivity of *until* and *since* by quantifying over ways to chop up intervals.

Adjoint triples

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Returning to the possibility/necessity situation ($W \to *$), cpmpositions may be made in a different order, generating

reader monad $\dashv$ *writer comonad*

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Not idempotent, but modalities (idempotent (co)monads) in opposition will arise in one of two ways from an adjoint triple:

- Two projections, one injection – bireflective subcategory $\bigcirc \dashv \Box$:
- One projection, two injections – essential subtopos $\Box \dashv \bigcirc$.

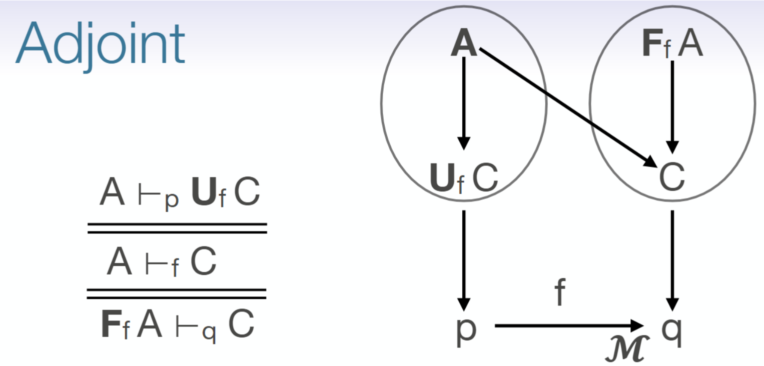
Physics with Urs Schreiber

				id	$\dashv$	id			
				$\vee$		$\vee$			
				$\rightrightarrows$	$\dashv$	$\rightsquigarrow$	$/$	e	bosonic/fermionic
solidity				$\perp$		$\perp$			
				$\rightsquigarrow$	$\dashv$	R	$= \mathrm{loc}_{\mathbb{R}^{0\|1}}$		rheonomic
				$\vee$	super	$\vee$			
	infinitesimal/reduced	$\overline{\Re}$	$/$	$\Re$	$\dashv$	$\Im$			
elasticity				$\perp$		$\perp$			
	infinitesimal shape			$\Im$	infinitesimal quality $\dashv$	$\&$			étalé
				$\vee$		$\vee$			
	shape	$\mathrm{loc}_{\mathbb{R}} =$		$\int$	quality $\dashv$	$\flat$	$/$	$\bar{\flat}$	content (flat/rational)
cohesion				$\perp$	gauge measure	$\perp$			
	discrete	$\bar{\flat}$	$/$	$\flat$	quantity $\dashv$	$\sharp$	$= \mathrm{loc}_{\neg\neg}$		continuous (intensive/extensive)
				$\vee$	ground	$\vee$			
				$\emptyset$	$\dashv$	$*$			

Internalisation of judgements

Curry's proposal was to take $\bigcirc\phi$ *as the statement "in some stronger (outer) theory,* ϕ *holds". As examples of such nested systems of reasoning (with two levels) he suggested Mathematics as the inner and Physics as the outer system, or Physics as the inner system and Biology as the Outer. In both examples the outer system is more encompassing than the inner system where reasoning follows a more rigid notion of truth and deduction. The modality* $\bigcirc$*, which Curry conceived of as a modality of possibility, is a way of reflecting the relaxed, outer notion of truth within the inner system. (Fairtlough and Mendler, On the Logical Content of Computational Type Theory: A Solution to Curry's Problem)*

Reflections of objects and morphisms across adjunctions



Dan Licata and Felix Wellen, Synthetic Mathematics in Modal Dependent Type Theories.

Licata-Shulman-Riley project

2-Bifibrations

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- Simple: "syntax for multivariable adjunctions"
- Dependent: "syntax for dependently typed multivariable adjunctions".

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There is considerable overlap with Melliès-Zeilberger on type refinement and the unification of intrinsic and extrinsic types. Their "functor as a type refinement system" is the vertical view.

What is an n-theory?

In a syntactic 2-theory with multiple generating types, the objects of the resulting semantic 2-category are not single structured categories, but diagrams of several categories with functors and natural transformations between them. Thus, the corresponding syntactic 1-theories have several "classes" of types, one for each category. These classes of types are generally called "modes", type theory or logic with multiple modes is called "modal", and the functors between these categories are called "modalities". Thus, modal logics are particular 2-theories, to which our framework applies. (Mike Shulman)

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- Applications in computer science and in mathematics are already happening.
- What philosophy will make of it all is much harder to predict.