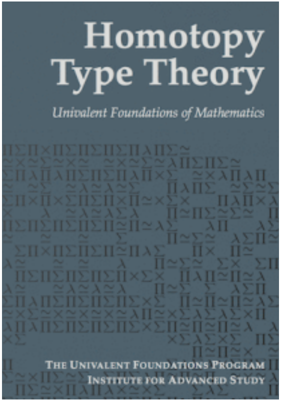
Type Theory and Philosophy Workshop

University of Kent

9 June, 2016

We live in interesting times!



A new foundational language for mathematics has just appeared.

Computational trinitarianism

Constructive logic Programming languages

Category theory

"The central dogma of computational trinitarianism holds that Logic, Languages, and Categories are but three manifestations of one divine notion of computation. There is no preferred route to enlightenment: each aspect provides insights that comprise the experience of computation in our lives." (Robert Harper)

Why I might have predicted the second coming

I came to philosophy on a diet of:

- Imre Lakatos
- Albert Lautman
- Category theory and categorical logic

Albert Lautman

- Proto-category theorist: thematic similarities everywhere.
- Rather than accord logic philosophical priority over other parts of mathematics, we should consider it as any other branch, a place where key ideas recurrently manifest themselves.

My Masters thesis - categorical logic

Question: What should we make of the two kinds of semantics for intuitionistic logic?

- Proof theoretic
- Topological

Categorical logic went some way to explaining this.

Constructive type theory as the 'internal language' of a topos.

Category theory

- Formulated in the 1940s, it looks for common constructions throughout mathematics.
- Entities are gathered together in categories with some relevant kind of mapping between them.
- The nature of an entity in a category is determined by the patterns of arrows in and out of it.
- Some categories are especially 'nice' and support a 'logic' of a certain strength.
- Toposes are extremely nice, and support an (extensional) type theory.

Category theory

- ∞-toposes are needed in modern geometry (Lurie).
- *Homotopy Type Theory* corresponds to their internal language.
- HoTT = Intensional Martin-Löf type theory + Higher inductive types + Univalence axiom

Lawvere on quantifiers

For **H** is a topos (or ∞-topos) $f : X \to Y$ an arrow in **H**, then base change induces between over-toposes:

$$(\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}/X \underset{f_*}{\overset{\overset{f_!}{\rightarrow}}{\overset{f^*}{\leftarrow}} \atop \rightarrow} \mathbf{H}/Y$$

Lawvere on quantifiers

Take a mapping

$$Owner : Dog \to Person,$$

then any property of people can be transported over to a property of dogs, e.g.,

$$Being\ French \mapsto Being\ owned\ by\ a\ French\ person.$$

We shouldn't expect every property of dogs will occur in this fashion.

In other words, we can't necessarily invert this mapping to send, say, 'Pug' to a property of People.

Lawvere on quantifiers

We can try...

Pug $\mapsto$ *Owning some pug* $\mapsto$???

Lawvere on quantifiers

But then

$$Pug \mapsto Owning\ some\ pug \mapsto Owned\ by\ someone\ who\ owns\ a\ pug.$$

However, people may own more than one breed of dog.

Lawvere on quantifiers

How about

$$Pug \mapsto Owning\ only\ pugs \mapsto ???$$

Lawvere on quantifiers

But this leads to

$$\textit{Pug} \mapsto \textit{Owning only pugs} \mapsto \textit{Owned by someone owning only pugs}$$

But again, not all pugs are owned by single breed owners.

Lawvere on quantifiers

In some sense, these are the best approximations to an inverse (left and right adjoints). They correspond to the type theorist's *dependent sum* and *dependent product*.

Were we to take the terminal map so as to group all dogs together ($Dog \to \mathbf{1}$), then the attempts at inverses would send a property such as 'Pug' to familiar things:

'Some dog is a pug' and 'All dogs are pugs'.

What if we take a map *Worlds* $\rightarrow \mathbf{1}$?

We begin to see the modal logician's *possibly* (in some world) and *necessarily* (in all worlds) appear.

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Things work out well if we form the (co)monads of dependent sum (product) followed by base change, so that possibly P and necessarily P are dependent on the type *Worlds*.

Such composites will be adjoint to each other, expressing their 'opposition'.

['Reader monad' and 'write comonad' are other two composites.]

These constructions applied to our pug case are:

$$Pug \mapsto Owning\ some\ pug \mapsto Owned\ by\ someone\ who\ owns\ a\ pug.$$

$$Pug \mapsto Owning\ only\ pugs \mapsto Owned\ by\ someone\ owning\ only\ pugs$$

We have equivalents of

- $P \to \bigcirc P$ and $\bigcirc \bigcirc P \to \bigcirc P$
- $\Box P \to P$ and $\Box P \to \Box\Box P$

Nothing prevents us doing the same with sets rather than propositions. E.g., the possible and necessary ingredients of a beef stew.

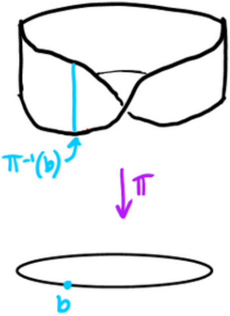
n-type hierarchy

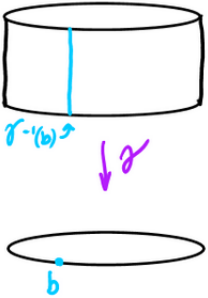
Intensional type theory allows for more interesting identity structures on types:

...	...
2	2-groupoid
1	groupoid
0	set
-1	mere proposition
-2	

Forming identity types, $id_A(a, b)$, lowers the level.

With dependent types it's helpful to have in mind the imagery of spaces fibred over other spaces:





Two useful constructions we can apply to these types are *dependent sum* and *dependent product*: total space and sections.

In general we can think of this **dependent sum** as sitting ‘fibred’ above the base type A, as one might imagine the collection of league players lined up in fibres above their team name.

Likewise an element of the **dependent product** is a choice of a player from each team, such as $Captain(t)$.

Context

Ranta explains how a narrative builds up a context, e.g.,

> *A women walked down a street. She laughed. Next to her was a small child. It was her son...*

A context, Γ, will correspond to an object, $[\Gamma]$, and reasoning in that context corresponds to working in the slice over $[\Gamma]$.

Then, as with 'Worlds', earlier:

- Possibly P: In some way of extending the context, P holds.
- Necessarily P: In every way of extending the context, P holds.

Building up a context, there are dependencies of types and terms on earlier types and terms.

Counterfactuals will require stripping back a context to one compatible with the antecedent:

- Had I taken an aspirin earlier, I wouldn't have a headache now.
- Had Julius Caesar been a general in the Korean War, he would have used nuclear weapons.

'The' introduction

We can't just use 'the' without proper warrant from earlier in the context.

Only if it has been established that a type A is contractible, may we form 'the A':

$$A : Type, p : IsContr(A) \vdash the(A, p) : A$$

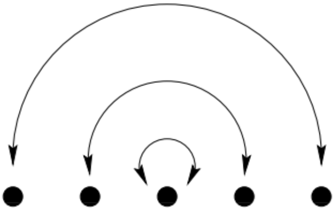
This works for the mathematicians' 'generalized the', as in 'the product of two sets'.

A context of symmetries

Since types need not just be sets, we should see what happens when we work in a simple non-set context such as $* : \mathbf{B}G$, for a group G.

A type in this context is something equipped with an action by G.

For example, a set of 5 objects acted on by the group of order 2:



If we don't know the position of another person, we shouldn't say 'the one on the left'.

Then for the unique map $\mathbf{B}G \to \mathbf{1}$

- Dependent sum is the quotient (action groupoid, orbits);
- Dependent product is the fixed points of the action.

We can speak of 'the end position' and 'the middle object'.

Consider with Black (1952), a universe empty apart from two identical spheres. If I cannot describe a differentiating property, how many spheres are there: 0, 1 or 2?

Variants of HoTT

- Plain HoTT
- Cohesive HoTT
- Directed HoTT
- Linear HoTT (as in the ‘linear’ of ‘linear logic’)

‘Modalities’ can be introduced to allow for spaces to be constructed by gluing together their parts.

Physics and HoTT

First, pre-quantum geometry ... is naturally axiomatized in cohesive homotopy type theory; second, quantization (geometric quantization and path integral quantization, in fact we find a subtle mix of both) is naturally axiomatized in linear homotopy-type theory.

In fact we find that linear homotopy type theory provides an improved quantum logic that, contrary to the common perception of traditional quantum logic, indeed serves as a powerful tool for reasoning about what is just as commonly perceived as the more subtle aspects of quantum theory, such as the path integral, quantum anomalies, holography, motivic structure. (Urs Schreiber)

Graeme Segal on *The Ubiquity of Homotopy*

Much of mathematics is about discovering robust kinds of structure which organize and illuminate large areas of the subject. Perhaps the most basic organizing concept of our thought is space. It leads us to the homotopy category, which captures many of our geometric intuitions but also arises unexpectedly in contexts far from ordinary spaces. Still more is this true of the 'stable homotopy' category, which sits midway between geometry and algebra.

The theme of my lectures is the strangeness and the ubiquity of the homotopy and stable homotopy categories, and how they give us new ideas of what a space is, and why manifolds and spaces with algebraic structure play such a special role.