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Understanding the infinite II: Coalgebra

David Corfield

School of European Culture and Languages, University of Kent, Kent CT2 7NZ, UK

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ABSTRACT

In this paper we give an account of the rise and development of coalgebraic thinking in mathematics and computer science as an illustration of the way mathematical frameworks may be transformed. Originating in a foundational dispute as to the correct way to characterise sets, logicians and computer scientists came to see maximizing and minimizing extremal axiomatisations as a dual pair, each necessary to represent entities of interest. In particular, many important infinitely large entities can be characterised in terms of such axiomatisations. We consider reasons for the delay in arriving at the coalgebraic framework, despite many unrecognised manifestations occurring years earlier, and discuss an apparent asymmetry in the relationship between algebra and coalgebra.

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1. Introduction

There is confusion afoot as to what the practice-oriented philosophy of mathematics is about. What are its practitioners trying to achieve? If they succeed, why should we take these achievements to be philosophical? What *is* clear is that philosophers of this stripe take the *case study*, some description of an episode in the history of mathematics, whether ancient or recent, as a central component of their work. But what then are these case studies for? In that they describe pieces of real mathematics in favourable or unfavourable terms, passing judgment on them, these philosophers seem to be interfering in mathematicians' own affairs. In that they describe mathematical practice detachedly, aren't they merely engaging in an observational practice, like a historian or a sociologist?

Recognising some justice in the charge that we have not made ourselves plain, I would like here to explain more clearly what I take myself to be trying to do. My starting point is the observation that the ways in which mathematical activity is regimented have undergone considerable reforms through the course of the history of the discipline. One cannot do in the twenty-first century what was done in the seventeenth or nineteenth centuries. This is not merely a question of adopting acceptable levels of rigour. It is also a question of how one is living the life of a mathematician, how one is contributing to a living practice of intellectual enquiry which has absorbed many of the finest minds since antiquity, and perhaps staking a claim for one's work to be included within a future history of mathematics. Now, if we take these changes in the way mathematical activity is assessed to be at least partially justifiable, we should be able to understand both the reasons for these changes and how they are good reasons.

A second observation, which lends importance to mathematics as a discipline, is that while there are significant reformations of practice and reformulations of theory, there is also a substantial conservatism. The position Euclidean geometry holds in the body of mathematics may be very different these days from the time of Euclid, for instance, due to Felix Klein, we have had a view of it for well over a century as concerning the properties preserved by a certain group of transformations, but still Pythagoras's theorem holds true within that geometry. The Babylonian recipe for finding the positive root of a type of quadratic equation still works four millennia later, even if it is now a very elementary part of algebra. There is a something of which mathematics is the study and of which we are gaining a clearer understanding. Mathematicians' minds are becoming more adequate to their object. They know better the nature of space and quantity, and the nature of our minds' understanding of space and quantity (see Cartier (2001) and Lawvere (2005)). Mathematics as a discipline thus reveals itself to be a vibrant tradition of intellectual enquiry, where

E-mail address: d.corfield@kent.ac.uk

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radical transformation of conception is possible, but only acceptable once it has passed the severest test of making sense of centuries of earlier work while casting new light into the murky fringes of our current understanding.

As philosophers, in what relation do we stand to this living tradition if neither external observer nor director of operations? An intermediate position for the philosopher is one Collingwood tried to capture in his 1935 letter *The Present Need of a Philosophy*.

> If the philosopher is no pilot, neither is he a mere spectator, watching the ship from his study window. He is one of the crew; but what, as such, is his function? (Collingwood, 1989, p. 167)

His answer is that philosophy may provide 'reasoned conviction' concerning the desirability and feasibility of certain projects. The project which concerns him in the letter, as is fitting for someone writing in the 1930s, is no less than that of bringing about improvements in human relations: "personal and social morality, economic organization, international relationship." The philosopher is to provide reasoned conviction that "progress is possible and that the problems of moral and political life are in principle soluble". I set myself the easier task of providing reasoned conviction that progress in mathematics is possible and desirable. It may seem that this is a modest task in view of the widely acknowledged success of mathematics, but there is much about the nature and value of this success that is not widely understood and as such needs to made explicit.

To address these issues we must steep ourselves in allusive episodes from the history of mathematics. Our case studies must convey aspects of the experience of reasoning like a mathematician. A problem then ensues. If one is to reconstruct the complex thought processes of mathematicians, one must enjoy their work in something approximating the way they enjoy it themselves. Also one must write in a way that makes readers want to catch enough of the drift of the mathematical thought to appreciate what is going on. Since case study material requires a considerable investment of time on the part of the reader to discern its salience, they may well make the economic decision to pass that work by. It is tempting then to write in a way which makes one appear to be a passionate advocate for that thought, acting as though the reader would be a fool to leave it aside. However, as I have said, it is not our primary purpose to act as advocate for a particular theoretical and practical outlook, but rather to make a study of wherein the rationality of the reformation of outlook resides. Our case studies will have to be carefully chosen to allow easiest access to deep waters.

Now the way I have phrased the topics of study as the regulation and reformation of the activity of a community would suggest I take it to be a political matter. I would have no objection to this characterisation. For one thing, it will allow me to borrow from Collingwood again.

> The progress of a people from a primitive to an advanced political condition, therefore, is not the imposition of order on what was once orderless. It is the substitution of one order for another; and (so far is it from being true that evolution introduces heterogeneity and complication into a previously homogeneous world) a civilized political system, like a civilized grammar, is often far less complicated than a barbaric. What, then, is the criterion of political progress? On what principle do people adopt a new political system as better than one already in existence? ... When the political spirit of a society is no longer satisfied with its existing structure, no longer finds that structure to express its own political aspirations, it alters it. (Collingwood, 1989, p. 102)

We should be interested, then, in situations where an intellectual community is no longer satisfied with a conceptual framework, when it finds it insufficient to express its thought.

The case material for this paper, chosen to show how mathematical aspirations could find new expression, goes by the name *coalgebra*. This choice has the advantage that, as I shall explain later, an earlier part of the story has already appeared in the practice-oriented literature. First, then, we shall need to understand the nature of coalgebra, which, as its name suggests, is defined by its relation to algebra. Of course, the term 'algebra' does not pick out a static body of theories. Corry's study (2007) amply illustrates this fact for the period from 1890 to 1930, where a huge transformation in what is counted as algebra takes place. So coalgebra is a dual to algebra in a very specific recent sense of the term, one which owes much to a category theoretic outlook. But even if the formulation of coalgebra requires a recent theoretical framework, this does not mean we should neglect possible earlier foreshadowings. So a second point to address in this paper is whether or not coalgebraic ways of thinking are to be glimpsed in the past. Finally, to the extent that we recognise coalgebra as a viable enterprise, we may wonder what would be at stake in considering it to be not the less well-known cousin of algebra, but as of no less importance than algebra itself. We begin with an exposition of basic examples from coalgebra, contrasting them with their dual algebraic constructions.

As we shall see, computer scientists have been drawn to coalgebra partly because of the resources it provides to represent infinite data structures. One aim of this paper is to provide evidence to the reader that conceptual reformulation of such a basic concept as that of *infinity* continues outside of set theory, and its inaccessible and Woodin cardinals. This paper is the second of a pair which treats new ways of thinking about infinite structures as directly concerning the working mathematician and computer scientist. The first of these papers (Corfield, 2011) looks at the robustness of infinite structures defined by universal properties.

2. Coalgebra versus algebra

In recent years we hear powerful claims from the keyboards of theoretical computer scientists that a new world of mathematics is being opened up. Here are two such claims, the first taken from a conference announcement,

> Over the last two decades, coalgebra has developed into a field of its own, presenting a mathematical foundation for various kinds of dynamical systems, infinite data structures, and logics. Coalgebra has an ever growing range of applications in and interactions with other fields such as reactive and interactive system theory, object oriented and concurrent programming, formal system specification, modal logic, dynamical systems, control systems, category theory, algebra, analysis, etc. (CMCS10, 2010)

the second from an as yet unfinished draft of a textbook,

> The area of coalgebras has emerged with a unifying claim. It aims to be the mathematics of computational dynamics. It combines notions and ideas from the mathematical theory of dynamical systems and from the theory of state-based computation. The area of coalgebra is still in its infancy, but promises a perspective on uniting, say, the theory of differential equations with automata and process theory, by providing an appropriate semantical basis with associated logic. (Jacobs, n.d., p. iii)

Interestingly enough, this logic is taken to be that favourite of philosophers—*modal* logic. Coalgebra is seen as the right way of formulating it and extending its scope in a way suitable for computer science.

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> Coalgebras are simple but fundamental mathematical structures that capture the essence of dynamic or evolving systems. The theory of universal coalgebra seeks to provide a general framework for the study of notions related to (possibly infinite) behavior such as invariance, and observational indistinguishability. When it comes to modal logic, an important difference with the algebraic perspective is that coalgebras generalize rather than dualize the model theory of modal logic. Many familiar notions and constructions, such as bisimulations and bounded morphisms, have analogues in other fields, and find their natural place at the level of coalgebra. Perhaps even more important is the realization that one may generalize the concept of modal logic from Kripke frames to arbitrary coalgebras. In fact, the link between (these generalizations of) modal logic and coalgebra is so tight, that one may even claim that modal logic is the natural logic for coalgebras—just like equational logic is that for algebra. (Venema, 2007, p. 332)

To give the definition of a coalgebra we have no choice but to introduce some category theory. Readers with little knowledge of the subject need not be alarmed since its use will be very light. In any case, if they have a serious interest in understanding contemporary mathematics they will have to learn category theory sooner or later. If the coalgebraic community are right in their sense of the importance of their movement, this can only enhance the case of the broader category theoretic community. On the other hand, that it is necessary for coalgebraicists to use category theory may incline us to believe in the novelty of coalgebra, category theory itself arising in the 1940s. Having said this, let's at least admit the possibility that while the rendition of coalgebra by category theory was necessary for it to be presented as a unified field, that fragments of this way of thinking are of considerably older date.

The ingredients of a coalgebra are simple. We need a category, and for our purposes we shall largely consider *Set*, the category whose objects are sets and whose arrows are functions. We also need an *endofunctor*, which is something that will systematically map sets to other sets, so that a function $f : X \to Y$ gets sent to a function between the images of X and Y.

Take the functor which acts on a set by adding to it a single element, or, in other words, which forms the disjoint union of the set with a singleton, let us say $\{*\}$. How this functor acts on functions should be clear. Let us designate this functor $F(X) = 1 + X$. Then $F(f)$ is a function from $1 + X$ to $1 + Y$, which operates as f on X, and which sends $*$ to $*$.

Now another definition: an *algebra* for this functor F is merely a choice of a set X and a function $f : F(X) \equiv 1 + X \to X$. The value of f on $*$ is an element of X, and f also maps the set X to itself. In other words, the choice of f corresponds to the choice of a constant in X and of a unary operation on X. There are clearly very many choices of algebra for F. However, one stands out as special

$$s : 1 + \mathbb{N} \to \mathbb{N}, \quad \text{where } s(*) = 0, \text{ and } s(n) = n + 1.$$

The natural numbers with the zero constant, *0*, and the successor function, *s*, is indeed special. It is the *initial object* in the category of *F*-algebras. This means that for any other *F*-algebra there is a unique algebra morphism to it from $\mathbb{N}$. While this phrasing may be unfamiliar to readers, who may be wondering what has been gained, the property of initiality is a powerful one, having as consequences others which may be much more familiar.

Indeed, the ordinary principle of mathematical induction is equivalent to the fact that there is no smaller subalgebra of $\mathbb{N}$, a general property of initial algebras being that injections be isomorphisms. In other words, we cannot carve out a subset of $\mathbb{N}$ with the properties that it contains 0 and for every number its successor, without capturing all of the natural numbers. On the other hand, when we use a recursive definition to define a function we are relying on the unique existence of an arrow out of $\mathbb{N}$. So I can define a function indexed by the natural numbers by specifying an *F*-algebra. There is such an algebra $f : 1 + \mathbb{N} \to \mathbb{N}$, with $f(*) = 1$ and $f(n) = 3n$. The unique arrow to this algebra tells us of the existence of a recursively defined function $g(0) = 1$ and $g(s(n)) = 3g(n)$, that is, $g(n) = 3^n$.

There is always the danger when one reassures readers that a new construction they have learned entails a range of things they already knew, that they will wonder about the point of putting old wine in new bottles. In this case we could point to the resources within this framework of a systematic understanding of induction so that other examples of initial algebras give rise to their own variant kinds of induction and recursion, for example, on lists or trees. We shall focus instead, however, on how formulating things this way has also given rise to a dual theory, that of terminal or final coalgebras, leading to *co*induction and *co*recursion.

So let's now take our first look at the opposite kind of mathematics, *coalgebra*. We shall start with the same functor as above $F(X) = 1 + X$, but this time we look for *coalgebras* of *F*. One of these is a set T with a function g, $g : T \to 1 + T \equiv F(T)$. The arrow has been reversed. When fed an element of T, g either returns with an element of T or with $*$. We can think of this as a partial function from T to itself. Where g is undefined it results in output $*$.

Again there's a special coalgebra for this functor, namely the final or terminal coalgebra. In this case it is the so-called *extended* natural numbers, $\overline{\mathbb{N}} \equiv \mathbb{N} \cup \{\infty\}$. The coalgebra function from $\overline{\mathbb{N}}$ to $1 + \overline{\mathbb{N}}$ is predecessor,

$$pred(n) = \begin{cases} * & \text{if } x = 0 \\ n - 1 & \text{if } x = n > 0 \\ \infty & \text{if } x = \infty. \end{cases}$$

A way of illustrating the semantics of coalgebras is via the notion of a black box. In this case the box has an inaccessible internal state. Every time I press a button, either the box shows a green light or else a red light. If the green light is shown, I can press the button again, the machine whirrs away and again either shows red or green. If it shows red however, the machine has frozen and no further response occurs. The final coalgebra represents all possible distinct behaviours, here the number of times green shows before the machine freezes on red. It is possible that this will never happen, so we include ∞ as a possible behaviour. The predecessor function maps the behaviour of a machine to the same behaviour where the first light is discounted. For finite positive numbers this results simply in reducing the number by one. It is undefined on zero, and it leaves ∞ unchanged.

Perhaps we can begin to glimpse why some computer scientists are so excited about coalgebra, sensing that it will provide them with models for their systems. Think of a computer running an operating system. In certain respects it seems hard to understand it as a Turing machine calculating an output from a completely specified input given to it before it starts functioning. Instead we tell our computers to open new programs and to connect to other machines while performing a range of other functions. Our computer continually adjusts itself to our demands as new instructions are piled on. It could not wait until the input is complete since it could not know when this had happened.

Now we can look for consequences of terminality dual to induction and recursion. Well, we know that every *F*-coalgebra has a unique map to $\overline{\mathbb{N}}$, as a terminal coalgebra, and this allows for *co*recursion. On the other hand, every surjection out of $\overline{\mathbb{N}}$ must be an isomorphism, and this is *co*induction. Let's illustrate these. We can use corecursion to define addition for extended natural numbers by specifying a function to make $\overline{\mathbb{N}} \times \overline{\mathbb{N}}$ into a coalgebra which will force the unique morphism to the final coalgebra to

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be the addition we want. So we define $k : \overline{\mathbb{N}} \times \overline{\mathbb{N}} \to \overline{\mathbb{N}} \times \overline{\mathbb{N}} + 1$, such that

$$k(n,m) = \begin{cases} (pred(n), m) & \text{if } n \neq 0 \\ (n, pred(m)) & \text{if } n = 0,\ m \neq 0 \\ * & \text{if } n = m = 0. \end{cases}$$

The idea to keep in mind is that we are defining as output a pair of extended naturals whose sum is the predecessor of that of the input, unless both inputs are zero in which case the output is $*$. We have then an F-coalgebra, so we know that there must be a coalgebra function from $\overline{\mathbb{N}} \times \overline{\mathbb{N}}$ to $\overline{\mathbb{N}}$. If you follow things through, you will be able to see that this is the desired addition, agreeing with standard addition for finite natural numbers.

Then we can use coinduction to show that this extended addition is commutative. We just need to show a so-called *bisimulation* R for which $(n + m)R(m + n)$. The way to think of a bisimulation is as a relation which pairs together entities which behave in the same way. As we know there can be no surjection out of $\overline{\mathbb{N}}$ which sends distinct elements to the same place—it is as coarse as it can be—any bisimulation must be contained within the identity relation. All we need do then to establish an identity between terms denoting elements of $\overline{\mathbb{N}}$ is to find a relation between them, such that when the coalgebra function is applied the respective outputs are still related. This is possible for sums written in both orders.

A very important characteristic of initial algebras and terminal coalgebras is that they are fixed points for their functors, that is, isomorphic to their images, and indeed they are least and greatest fixed points respectively. Let's now consider some other algebras and coalgebras. Take the functor $F(X) = 1 + A \times X$, for a fixed set A. Searching for the least fixed point, we put in as little as possible. So we build up in stages from the empty set, taking any existing set of lists to a new set which includes the empty list and all lists generated by adding an element of A to the old lists. All finite lists of A elements are generated this way and nothing else, so these lists form the elements of the initial algebra for F. But a larger set is fixed by F, namely, the set of A-streams, or finite and infinite lists of A elements. This in fact is the final coalgebra.

Again, coalgebras are about observation. We can think of a coalgebra $f : X \to 1 + A \times X$ as observing about an entity whether it contains something A-detectable or not, and if so which element of A it detects. Having observed something it modifies it. The final coalgebra has as elements all possible outcomes of the behaviour you might observe. Do you still have observations to add as list elements? If ever no, we have a finite list. If always yes, we have an infinite list. And there's no other behaviour that can be detected.

Imagine we have a function $d : A \to A$, and we want to extend it to d' which acts on a list of elements of A to give the list of the members' images. Doing things the algebraic-recursive way, we need to define d' on the constructors, that is on the empty list, and on a list σ' which is $a :: \sigma$. First, we want that d' of the empty list be the empty list. Second, we want $d'(a :: \sigma) = d(a) :: d'(\sigma)$. Now extending d to A-streams the coalgebraic-corecursive way, we need to define d' on the destructors, $d'(\sigma) = d(head(\sigma)) :: d'(tail(\sigma))$, if σ is not empty, otherwise $d'(\sigma)$ is empty. In general, if the function to be specified appears in the term defining it, use of the *induction* principle requires that repeated unfolding of the definition makes the arguments smaller and smaller, whereas use of the *coinduction* principle requires that more and more information about the result can be generated.

Let me give a flavour of some of the other regularly used coalgebras.

- $X \to F(X) = D(X)$, the set of probability distributions on X: Markov chain on X.
- $X \to F(X) = P(X)$, the powerset on X: Binary relation on X.
- $X \to F(X) = X^A \times B$: Deterministic automaton.
- $X \to F(X) = P(X^A \times B)$: Nondeterministic automaton.
- $X \to F(X) = A \times X \times X$, for a set of labels A: labelled binary trees.

Coalgebra might be said to lie more on the side of semantics. Coalgebras are models for types of transition systems. We must know the possibilities for movement from one state of the system to another. Here we can see the connection with modal logic. Possible worlds models for a modal theory are based on the idea of a collection of worlds with an accessibility relation defined on it. In other words, possible worlds form a coalgebra for the powerset functor (see Cîrstea et al., 2011).

By contrast there is a syntactical flavour to algebra. I can define a language in terms of basic terms in a set X and constructors, elements of a signature operating on tuples of elements of X to generate new terms. The full language is then closed under term construction. It is a minimal fixed point, containing constructed terms and only these. The language may contain infinitely many terms, and yet each term is constructed from finitely many symbols.

3. Finsler, Aczel and non-well-founded sets

So we have a duality between, on the one hand, the algebraic syntax of terms constructed from primitives and operators, and, on the other, coalgebraic models of systems and the observations we can make on them, which may be thought of as a kind of destruction. This raises the question of why the explicit appearance of coalgebra occurred so late in the day. Mathematics has sought to provide methods for physics in which we model dynamical systems and their observability. Why haven't we seen coalgebra before now? To give an adequate response to this question we need to dive back into the past.

One landmark in the history of coalgebra was the book *Non-well-founded Sets* published by Peter Aczel in 1988. In this book Aczel shows us that it is possible to work with a set theory which allows for sets flouting the Axiom of Foundation. Recall that this axiom states that no set can have an infinitely long membership chain. In other words, there is no infinite path along branches from the root in a set's membership tree. By contrast, non-well-founded sets allow for infinitely extended trees. But we've seen that possibly infinite data structures such as trees are often elements of a terminal coalgebra. And indeed we find this to be the case here. In the most recent formulation of the approach, known as *algebraic set theory* (Awodey, 1998), we take as category, V, the category of all classes. Now upon the category of all classes there acts an endofunctor which maps a class to the class of all subsets of elements of that class. Note that I specified 'subset' rather than 'subclass'. A general rule is that no thingie will be the same 'size' as the collection of its subthingies. But as we're only gathering together subsets from a class it is possible we may have a fixed point, and indeed we do, with two extreme cases corresponding to an initial algebra and a terminal coalgebra. The initial algebra is the class of all well-founded sets; the terminal coalgebra is the class of all non-well-founded sets. We can think of the latter as a system where for any state, that is, set, we can destroy it to yield a set of sets. We can represent the set as an infinite tree—each node represents a set and it has a set of branches emerging from it corresponding to its elements. Now we see the resemblance with the previous section. Well-founded sets don't allow for infinitely deep trees just as the elements of the initial algebra for $X \to A + A \times X \times X$ are *finite* labelled binary trees.

But what is the motivation for this work? Well, Aczel had drawn inspiration from computer science:

> The original stimulus for my own interest in the notion of a non-well-founded set came from a reading of the work of Robin

Milner in connection with his development of a mathematical theory of concurrent processes. (Aczel, 1988, p. xix)

Non-well-founded sets were then used after Aczel published his book in attempts to model situations semantics (Barwise and Moss, 1996), where circular situations can occur, that is, when a situation includes facts about that same situation. Perhaps then we could say that coalgebraic thinking owes its origins wholly to strands within cognitive science. However, Aczel was aware that he was also tapping into a mainstream mathematical way of thinking. Both kinds of extremal reasoning, the algebraic minimal and the coalgebraic maximal, occur in mathematics:

> Thus, in the case of the axiom system for the natural numbers, the extremal axiom is the principle of mathematical induction, which is a minimalisation axiom, as it expresses that no objects can be subtracted from the domain of natural numbers while keeping the truth of the other axioms. The axiom systems for Euclidean Geometry and the real numbers involve on the other hand completeness axioms. These are maximalisation axioms; i.e. they express that the domain of objects cannot be enlarged while preserving the truth of the other axioms. (Aczel, 1988, p. 106)

I shall take up the real numbers in the next section.

Aczel also observes that debates around the Axiom of Foundation go back to the Foundational Crisis. When he proposes his own anti-foundation axiom, AFA, he discusses it in relation to other such axioms, including FAFA (for Paul **F**insler) and SAFA (for Dana **S**cott). He notes that

> Although Fraenkel's idea of a minimalising extremal axiom for set theory failed to give rise to a categorical axiom system it led eventually to the formulation of FA. It is in (Finsler, 1926) that we see a formulation of an axiom system for set theory using an extremal axiom of the dual character of a maximalising axiom. This also fails to be a categorical axiom system having similar difficulties to Fraenkel's extremal axiom. Finsler appears to have been unresponsive to the criticisms of his idea. (Aczel, 1988, p. 107)

He also poses the problem of accounting for the fact that Finsler's ideas were neglected for so long.

> It is surprising that it has taken over 50 years for this "success" to come about, whereas Fraenkel only had to wait a handful of years. (Aczel, 1988, p. 107)

Let's look a little further into this matter. Paul Finsler is the mathematician known perhaps best for his generalised geometry, but he also worked on foundational questions concerning sets. His influence on twentieth century set theory, at least before Aczel, has been rather limited however due to the perception that his theory had been shown to be inconsistent. This turns out to be a mistake as we shall see. This story won't be new to anyone who has read *Revolutions in Mathematics* (Gillies, 1992), because contained in this volume is a chapter by Herbert Breger on Finsler's set theory, 'A Restoration that Failed' (Breger, 1992).

In this chapter, Breger is illustrating what he sees as the shape of a revolution. The revolutionary in his case study, however, is not Finsler but Hilbert. For Breger, it is Hilbert who introduces a new style of axiomatisation, which he and his followers have the task of passing off as always having been there. Finsler, he argues, continues in what he takes to be an older tradition, and indeed attempts to restore it. For Finsler, the axiomatiser is trying to capture some existing system by specifying features of the field sufficiently fully that the maximally large system which meets the specifications is the desired one. The system of all sets was certainly something to be captured in this way. So, rather than worry about the set theoretic paradoxes by controlling their construction, building them bottom up from the empty set and various constructions while maintaining consistency, instead Finsler defines the system of sets to be that collection which is maximal with respect to certain properties.

Breger tells us the story of how by 1928 Finsler's approach was thoroughly misunderstood, allowing Reinhold Baer to dismiss Finsler's set theory as inconsistent in a four page paper. Baer showed that any system satisfying Finsler's axioms can be extended, hence that there is no maximal such system. But in doing so he was using different rules of set formation from Finsler. Baer's strategy was to consider a system of sets, Σ, satisfying Finsler's axioms, then adjoin to this system the set N consisting of every set A of Σ which is not an element of itself. According to Baer the new system is a larger system satisfying Finsler's axioms. But why should N be considered a set?

> To Zermelo, Fraenkel, Baer, and others, the idea of definite property is inseparably connected with the notion of set, the underlying philosophy being that well-formed definitions create the object. (Breger, 1992, p. 261)

Finsler did not accept this rule and was left to face a lonely battle. We hear about him still writing in 1969, the year before his death, on how the continuum hypothesis remains open in his set theory, in what he calls 'classical mathematics'. That so many decades had to pass before what was valuable in Finsler's work could be appreciated by someone else may cause us to reflect on the lack of flexibility of thought that may afflict a discipline. On the other hand, perhaps we could say that a formal apparatus for dealing with maximal axioms had first to become available.

Something in Baer's argument may remind the reader of the argument against omnipotence. If God is all powerful, why can't he make something so vast that he is unable to move it? The objector here imagines that merely writing down a well-formed power property entails that a maximally powerful being would have to possess it. The Finslerian version would look to describe a being with logically consistent properties which is maximally powerful. Just because you can set down in words a power which is inconsistent with other powers, here the power to make something too large for you to move, does not mean that a maximally powerful being should possess it. It may happen however that there is no way to form a unique maximal system, just as not every functor gives rise to a terminal coalgebra.

This metaphysical aside resonates with Breger's observation that

> To the revolutionary, the most striking difference between Zermelo's and Finsler's axioms is the certain ontological flavour of Finsler's axioms. To the conservative, the philosophical background of Zermelo's axioms is the implicit assumption that sets do not exist unless they can be derived from given sets by axiomatically fixed rules. Axiom 3 [the axiom of completeness] is of particular interest. It is the analogue of Hilbert's somewhat problematic axiom of completeness for geometry. Weyl and Fraenkel purposefully took the contrary into consideration, namely an axiom of restriction postulating the minimal system which fulfils the other axioms. Weyl's and Fraenkel's axiom is obviously motivated by the revolutionary idea that axioms and definitions create objects, and that sets which are too big should not be brought into existence, whereas Finsler's axiom of completeness is motivated by the conservative idea that big sets exist anyway, so set theory should investigate them. (Breger, 1992, pp. 258–259)

Let us remind ourselves of Hilbert's completeness axiom:

> *V.2: Line completeness. To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that*

> *the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.*

If it strikes the reader as odd that Hilbert should be both instrumental in the axiomatisation of set theory in the minimal fashion, and, as the axiomatiser of Euclidean geometry, ready to use maximal axioms, then it should be observed that one can perfectly well define maximal axioms in a minimally defined set theory. As for why he felt obliged to define his foundational system in minimal terms, perhaps this reflects the burning issue of the consistency of mathematics *in toto*. Breger writes

> The different philosophical backgrounds imply different consequences for the consistency topic. The consistency of arithmetic and Euclidean geometry had not been a problem as long as the Platonistic interpretation of the objects had been self-evident. (Breger, 1992, p. 259)

But while this interpretation satisfied Finsler, who never saw the need for a minimal-axiomatisation of set theory, Hilbert was sufficiently concerned by what mathematicians of the turn of the nineteenth century had discovered about potential inconsistency that he wanted to provide such an axiomatisation.

A striking lesson we learn from this episode is how what appear to be irreconcilable *metaphysical* differences end up becoming reconcilable *mathematical* differences: Finsler's sets as the terminal coalgebra and Hilbert and followers' sets as the initial algebra for an endofunctor on the category of classes. A similar kind of metaphysical deflation occurred in the case of quaternions. Recall the intense Kantian and Coleridgean metaphysical motivation behind William Hamilton's drive to reformulate mathematical physics in quaternionic terms. The quaternions today have their place in the family of normed division algebras, and find applied usage in computer graphics applications, but without bearing the huge responsibility with which Hamilton wanted to burden them. Similarly we can happily live with the coexistence of minimal-algebraic and maximal-coalgebraic thinking in contemporary mathematics.

4. Coalgebraic reals and analysis

Any time there is a representation of a mathematical entity as composed of elements which are completed infinities of some kind, this may be a candidate for coalgebraic reformulation. If that entity is also maximal in some sense, it is very likely. Now we have already seen Aczel point out the real numbers as a maximally complete kind of system, and of course real numbers may have infinitely long expansions, but it took until the late 1990s for someone to put the reals in a coalgebraic setting. Then Peter Freyd discovered that the real interval $[0, 1]$ is indeed a coalgebra and announced it in a note to a web-based discussion group (Freyd, 1999). Let C be the category of bipointed sets, by which we are to understand sets X equipped with two distinct basepoints x_-, x_+. You can wedge together any two bipointed sets X and Y to form a new bipointed set $X \vee Y$. In the act of wedging we are to identify x_+ and y_-, and take as the two named points of the new set x_- and y_+. Then this operation is an endofunctor $W : X \mapsto X \vee X$ of C. Moreover, we have that

- The initial W-algebra is the set of dyadic rationals in $[0, 1]$, with 0 and 1 as basepoints.
- The terminal W-coalgebra is the set $[0, 1]$, again with the endpoints as basepoints.

Here *dyadic* rational means a rational whose denominator is a power of 2. In other words the initial algebra consists of all numbers in $[0, 1]$ with finite binary (as opposed to decimal) expansions, $0 \cdot b_1 b_2 \cdots b_n$, whereas the terminal coalgebra consists of all numbers in $[0, 1]$ with any (possibly infinite) binary expansions, $0 \cdot b_1 b_2 \cdots$, where certain identifications take place, for example, so that $0 \cdot 1111 \cdots = 1$.

As for the whole set of real numbers, no-one has yet described this in coalgebraic terms. Freyd, however, has developed an 'algebraic real analysis' in a paper of that name (Freyd, 2008), and claims there that $[0, 1]$ is the more fundamental object.

> For reasons that can easily be considered abstruse we were led to the belief that the closed interval–not the entire real line–is the basic structure of interest. (Freyd, 2008, p. 215)

Freyd goes on to develop a real analysis using this coalgebraic characterisation of the reals. It is interesting to read Freyd describe how what he "needed was someone to kick me into coalgebra mode". This idea that coalgebra requires a shift in thinking is echoed by Bart Jacobs:

> The author's experiences in lecturing about coalgebras is that the material in itself is usually not seen as difficult, but that it takes a subtle change of view (with respect to the traditional algebraic approach) to be able to appreciate, recognise and apply the coalgebraic notions and techniques. (Jacobs, n.d., p. 2)

The 'kick' Freyd received came from a paper by Pavlović and Pratt (1999)—'On Coalgebra of Real Numbers'. The first of these authors had also worked on casting the calculus in coalgebraic form. Let's see now how we might have foreseen coalgebra cropping up in this basic field of mathematics. In a paper with Escardó, entitled 'Calculus in Coinductive Form', the authors write:

> Coinduction is often seen as a way of implementing infinite objects. Since real numbers are typical infinite objects, it may not come as a surprise that calculus, when presented in a suitable way, is permeated by coinductive reasoning. What *is* surprising is that mathematical techniques, recently developed in the context of computer science, seem to be shedding a new light on some basic methods of calculus.
>
> We introduce a coinductive formalization of elementary calculus that can be used as a tool for symbolic computation, and geared towards computer algebra and theorem proving. So far, we have covered parts of ordinary differential and difference equations, Taylor series, Laplace transform and the basics of the operator calculus. (Pavlović and Escardó, 1998, p. 408)

The trick here is to see Taylor series for real functions as streams of coefficients. For example,

$$exp(x) = 1 + x + x^2/2! + x^3/3! + \cdots$$

we represent as the stream $[1, 1, 1, 1, \ldots]$. Now the destructors for a stream correspond to

$$head(\sigma) = f(0), \quad tail(\sigma) = f'.$$

Then $f = exp$ is the unique function for which $f(0) = 1$, $f' = f$, as can be seen by forming the stream σ for which $head(\sigma) = 1$, $tail(\sigma) = \sigma$. $exp = 1 :: exp$. We are then shown a wide variety of coalgebraic constructions in elementary calculus.

The authors remark,

> In essence, calculus is coinductive programming. It consists mostly of using final fixpoints and constructing various transforms between them. When applying standard methods for solving differential equations, we are actually using coinduction without realising it! (Pavlović and Escardó, 1998, pp. 8–9)

Again we run up the question of why it took so long for the coalgebraic-coinductive perspective to emerge. Pavlović and Pratt ask

> Why would so foundational a principle wait for the late 20th century to be discovered? ... the idea was put forward that coinduction is new only by name, while it had actually been around for a long time, concealed within the infinitistic methods of mathematical analysis. Roughly, the idea is that
>
> $$\frac{\textit{induction}}{\textit{arithmetic}} \approx \frac{\textit{coinduction}}{\textit{analysis}}.$$
>
> The basic passage to infinity in elementary calculus is coinductive, dual to the inductive passage to infinity in elementary arithmetic. (Pavlović and Pratt, 2002, p. 106)

So if analysis is well-described coalgebraically, we have evidence for the claim that coalgebra has been around for a very long time, but unrecognised as such. I would argue that what was necessary for its recognition was for category theory to become a readily available language. Now that it is available we find coalgebraic work on self-similarity by a category theorist showing that any given compact metrizable space has a terminal-coalgebra characterization (Leinster, 2007). Then we find that symbolic dynamics is based on the existence of a cofree coalgebra in the category of complete metric spaces (Rutten, 2000, Section 18). And there are many more episodes in the coalgebraic characterisation of mathematics, see (Turi and Rutten, 1998). But if coalgebra is rightly regarded as being pervasive throughout mathematics, ought we to view it as an equal partner to algebra, or perhaps just an offshoot? Let us now address this question.

5. The status of coalgebra

The reader should by now have a sense of why the 'co' in 'coalgebra', but perhaps is wondering why the 'algebra'. What does a map from $FX \to X$ have to do with anything we meet in algebra? We saw some basic examples of F-algebras, such as the natural numbers, but we don't usually consider them as 'algebraic'. When we think of algebra, we think of groups, rings, fields, modules. So let's take one of the simplest algebraic structures, the monoid, to show the connection.

A monoid is a set with a binary operation, m, and an identity element, e, satisfying basic equations expressing the identity properties of e and the associativity of m. We can now see how one of these can be considered as an algebra for a functor. Consider the functor, F, which sends a set to the set of finite strings of its elements. For example, $F(\{x, y\})$ has members such as *yyxxx* and *xxxyyxxxxx*, and the empty string e. Then an ordinary monoid is an algebra for this functor, that is, a map $f : FX \to X$ for some set X. If you like, f carries out the multiplication operating in the monoid. For example, $F(\{x, y\})$ is itself an F-algebra, indeed a free one. There is a map $F(F(\{x, y\})) \to F(\{x, y\})$, which acts on strings of strings to give strings. For example,

$$(xyx)(xxyy)(yx) \mapsto xyxxxyyyx.$$

Any monoid is in the same way an F-algebra. Monoids are algebras, then, in the sense of this paper.

Where then are the comonoids? These should be a special case of F-coalgebras. But now something about the category of sets comes into play, and this may account for the delay in hitting on the idea of coalgebra. Comonoids in *Sets* and categories of that type (cartesian monoidal category) are as boring as they can be. On each set we can place precisely one comonoid structure—our choices are also forced upon us. The coidentity operation sends elements to $*$, and the comultiplication acts as duplication. The equivalent of the property in a monoid that multiplication by the identity element leaves an element unchanged, is that when you duplicate an element, then erase one copy, you're back to where you started.

So we have to look elsewhere for interesting comonoids, away from cartesian monoidal categories. Now one of the most frequently encountered non-cartesian categories, where product is not simply set product, is $Vect_F$, the category of vector spaces over a field F and linear mappings. In this category for every map resembling a multiplication on a vector space $A \otimes A \to A$, there is a comultiplication on the dual space $A^* \to A^* \otimes A^*$. Unsurprisingly then, interesting comonoids are to be found in $Vect_F$. But now we encounter terminological confusion—a comonoid within *Vect* is called a 'coalgebra', to mirror the fact that a monoid in *Vect* is called an 'algebra'. Indeed, this is the environment in which the term 'coalgebra' first arose. It is unfortunate that these names have been given to very specific examples of a much more general concept. But we learn an important lesson that to find interesting objects with structure which is dual to the frequently encountered algebraic structures, we need to distance ourselves from *Set* and *Set*-like categories. Category theory again does well explaining global features of the collection of a certain kind of entity.

What if we took the category which is the opposite of *Set* itself, that is, we keep the objects the same but reverse the arrows, so that an arrow from $A \to B$ corresponds to a function from B to A? In turns out that this category is equivalent to *CABA*, the category of complete atomic Boolean algebras, a result which is a part of Stone duality (Johnstone, 1982). Now a comonoid in *CABA* is a monoid in *Set*, and *vice versa*. So now we find the monoids in *CABA* to be very limited. Each such Boolean algebra supports precisely one monoid structure, while it may support plenty of different comonoids. Had we taken *CABA* to be the natural category for our work, we might see coalgebra as the more basic. But normally we define Boolean algebras in terms of sets with various structure and properties. Viewing matters in terms of the whole category of sets, should make us wonder whether there is something about the asymmetry of *Set* which inclines us to use it.

So now we may be left wondering whether it is possible that we should have followed Finsler in opting for a more coalgebraic approach to mathematics. Let us review the situation at which we have arrived. It seems clear that *explicitly* coalgebraic mathematics is still a fringe activity, most frequently seen in computer science. But we can react to this state of affairs in very different ways.

1. It is not a distinction worth making—a coalgebra for (C, F) is an algebra for (C^{op}, F^{op}).
2. It is a distinction worth making, but there's plenty of coalgebraic thinking going on—it's just not flagged as such.
3. Coalgebra is a small industry providing a few tools for specific situations, largely in computer science, but with occasional uses in topology, etc.

We are less likely to be inclined to believe (1) if we think there is something which breaks the symmetry in mathematics and leads us to favour working in certain kinds of category. The category of sets possesses a great number of properties, some pairs of which, such as completeness and cocompleteness, are dual, but many of which are not. If these latter unmatched properties are valued over their duals then we have grounds to distinguish algebra from coalgebra. On the other hand, a category such as *Rel*, with objects sets and arrows binary relations between domain and codomain, is self-dual. In any case we may need to use algebraic and coalgebraic reasoning within the same setting, requiring us to distinguish them. Positions (2) and (3) are now in direct competition. I hope I have given sufficient evidence for position (2), having shown that a considerable amount of mathematics is readily interpretable as coalgebraic.

6. Conclusion

In this paper I have told a part of the story of how some of the component pieces of the emerging field of coalgebra have come together. We saw that an important impetus was provided by computer scientists needing to capture a certain kind of data structure, the identity of whose composition is revealed by an unfolding or destruction. The maximal fixed point for this process of unfolding was seen to be an entity of great significance—it measures the collection of all observable behaviours. It was important then to understand how this picture was the dual of an already understood one: the minimal fixed point under a process of construction. An instantiation of these dual pictures was found to have been at play in the foundational debate of the 1920s between Finsler and Hilbert's followers, but without the coalgebraic framework it was not possible to recognise the duality relating the two sides of the debate. It took well over half a century for this to become clear. Seen from the modern perspective, we no longer need choose between the passionately held metaphysical beliefs fuelling the debate. Rather, we can take both sides to have had mathematisable ideas which can both be accommodated. So where, as Breger tells us, van der Waerden could take the Finsler–Baer axiomatisation dispute to be a metaphysical one, as when he told Finsler,

> If sets are pre-existing objects, then you are right; but if sets are made by human beings, then Baer is right. (Breger, 1992, p. 262),

both approaches could be pursued in computer science. There the relevant distinction concerns data which is guaranteed to be fully unfoldable by a program and *co*data which can be operated on as and when it unfolds, even if this process is potentially unending. It is interesting to see once more how metaphysical ideas can mutate into practical effects which may be given a very different interpretation. Given the coalgebraic understanding of modal logic alluded to earlier, a not unrelated case of metaphysical mutation concerns the centuries of philosophical thought devoted to the modalities which had yielded by the late twentieth century a range of modal logics, many used in computer science, including one of which is used very effectively as a practical tool in model checking.

A fuller account of coalgebra would talk about coalgebras in the narrow sense, that is, coalgebras as structures on vector spaces or modules, encountered frequently in representation theory and mathematical physics (Brouder, 2005), and of the somewhat related use of coalgebras to capture decomposition in combinatorics (Joni and Rota, 1979). Unlike in situations where products are determined by their underlying sets, here there are rich opportunities to study dualised algebraic operations, and indeed to study the interplay of algebraic and coalgebraic operations, or in other words the interplay between composition and decomposition, or construction and destruction. This occurs in the widely studied concepts of Hopf algebras and quantum groups. If we take the Hopf algebra *motif* and instantiate it in the category of sets, the coalgebraic part 'collapses' to give that workhorse of mathematics—the group. We can see the group construct, therefore, as a projection of a self-dual construct into the non-self-dual world of sets.

Neither the theory of codata, nor the theory of Hopf algebras force us out of the setting of set theory, if we do not wish to leave. We can, if we choose, find models in ordinary set theory. But it is worth reflecting how issues of duality throw into relief the choice that is made when opting for sets, and whether it is optimal. We seem to have a predisposition to want to represent our experience of the world as arrows from sets to sets, according to the arrow of time where an effect determines its cause, and multiple causes may bring about the same effect. However, insights from many areas suggest that we might choose better in some situations. For example, Baez (2006) explains how away from the world of medium sized objects, general relativity and quantum field theory may find better homes in categories which are not set-like, where there exists a duality lacking in the set environment.

Mathematics is a fascinating *mangle of practice*, to use a term of Pickering (1995). Ingredients of different passages and styles of thought fall into the mix, their origins often unrespected, to be worked over, blended and then fed on for more mangling. It is not for us as philosophers to dictate how the story of coalgebra might go from here, which isn't to say that the telling of the story in a certain way could not resound with someone and affect the way the story continues. What we can comment on is what we take to be most important features of the way stories unfold, and about how we take the process to be going well or badly. Where the willingness to deal with the practical demands of defining certain data types with mathematics still considered by some to be too abstract reflects well on the computer scientists concerned, the missed encounter between Baer and Finsler reflects a lack of openness which delayed the progress of the coalgebraic approach. But this is not so surprising. In *The Idea of History* Collingwood remarks that

> ...nothing is harder than for a given generation in a changing society, which is living in a new way of its own, to enter sympathetically into the life of the last. It sees that life as a mere incomprehensible spectacle, and seems driven to escape from sympathy with it by a kind of instinctive effort to free itself from parental influences and bring about the change on which it is blindly resolved. (Collingwood, 1946, p. 326)

At some stage, however, the effort of coming to terms with the past must be made,

> ...progress is not the replacement of the bad by the good, but of the good by the better. In order to conceive a change as a progress, then, the person who has made it must think of what he has abolished as good, and good in certain definite ways. This, he can only do on condition of his knowing what the old way of life was like, that is, having historical knowledge of his society's past while he is actually living in the present he is creating: for historical knowledge is simply the re-enactment of past experiences in the mind of the present thinker. Only thus can the two ways of life be held together in the same mind for a comparison of their merits, so that a person choosing one and rejecting the other can know what he has gained and what he has lost, and decide that he has chosen the better. In short: the revolutionary can only regard his revolution as a progress in so far as he is also an historian, genuinely re-enacting in his own historical thought the life he nevertheless rejects. (Collingwood, 1946, p. 326)

With the contemporary understanding of the duality between algebra and coalgebra in place, we are now in a position to recognise what was good in Finsler's thinking and in that of Hilbert and his followers, and how it has been possible to improve on both.

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