Homotopy type theory: A revolution in the foundations of mathematics?

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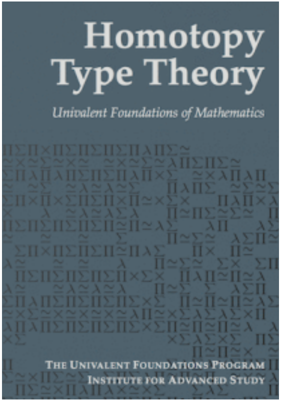
We live in interesting times!



A new foundational language for mathematics has just appeared.

Why I might have predicted the second coming

I came to philosophy on a diet of:

- Imre Lakatos
- Albert Lautman
- Category theory and categorical logic

Albert Lautman

- Proto-category theorist: thematic similarities everywhere.
- Rather than accord logic philosophical priority over other parts of mathematics, we should consider it as any other branch, a place where key ideas recurrently manifest themselves.

My Masters thesis - categorical logic

Question: What should we make of the two kinds of semantics for intuitionistic logic?

- Proof theoretic
- Topological

Categorical logic went some way to explaining this. Topos theory combines the logical and the spatial.

Constructive type theory as 'internal language' of a topos.

In the intervening 24 years...

- However much one speaks of ‘the rich, lived experience of the mathematician’, ‘mathematics as a tradition of intellectual enquiry’, etc., there were always those philosophers content with set theory who claim that nothing goes beyond its bounds.
- Many of the ‘rebels’ turned to category theory (Marquis, McLarty, me), and yet categories can be thought of as sets or classes with a certain structure.
- However much one protests that set theory has lost contact with the conceptual content of mathematics, still it is possible to maintain mathematics to be the accumulation of timeless truths in the framework of first-order logic + axioms of set theory.

Maybe this time

- Where *Homotopy Type Theory* differs is in having logic already contained intrinsically within it.
- Just as 1960s algebraic geometry gave rise to the topos, so 2000s mathematics gave rise to the ∞-topos.
- Homotopy type theory is the internal language of ∞-toposes.

My recent Concepts paper

Two theses:

1. This new foundational language, homotopy type theory, provides an important perspective from which to understand varieties of mathematical concept.
2. Mathematics displays *vertical unity*, so that concepts met in elementary and in research level mathematics are related.

Vertical unity

Vertical unity is a term introduced by Borovik to indicate that the products of mathematical research never completely depart from the kinds of concepts accessible to those with little mathematical training.

> *...'recreational', 'elementary', 'undergraduate' and 'research' mathematics are no more than artificial subdivisions of a single continuous spectrum" (Borovik 2005, p. 1).*

Where we may take justifiable delight in the 'horizontal' unity provided by far-flung analogies between apparently different fields,

> *[t]he vertical unity of mathematics, with many simple ideas and tricks working both at the most elementary and at rather sophisticated levels, is not so frequently discussed – although it appears to be highly relevant to the very essence of mathematics education. (Borovik 2005, p. 10)*

Analogies between logic and arithmetic

If we assign the values 1 to True and 0 to False, then forming the conjunction ("and") of two propositions, the resulting truth value is formed very much as a product of numbers chosen from $\{0, 1\}$ is formed:

- Unless both values are 1, the product will be 0.
- Unless both truth values are True, the truth value of the conjunction will be False.

It is natural then to wonder if the disjunction ("or") of two propositions corresponds to addition. Here things don't appear to work out precisely. In the case of 'True or True', we seem to be dealing with an addition capped at 1.

Implication

- $(A \wedge B) \rightarrow C$ is True if and only if $A \rightarrow (B \rightarrow C)$ is True.
- $c^{(a \times b)} = (c^b)^a$

A proof of an implication is a mapping of proofs. Very much the approach of Martin-Löf and Dummett.

Similarly, these arithmetic quantities measure the cardinalities of sets of functions.

- We can explain this analogy via **Type theory**.
- Basic judgements involve declaring something to be of a certain type, $a : A$, and declaring of two elements of a type that they are equal, $a = b : A$.
- **Intensional** type theories add the twist that an identity is not just a proposition but a type in itself $Id_A(a, b)$.
- Propositions are then taken as a certain kind of type (sometimes called 'mere propositions' when proof irrelevance assumed).

We have a hierarchy of types:

...	...
2	2-groupoid
1	groupoid
0	set
-1	mere proposition
-2	

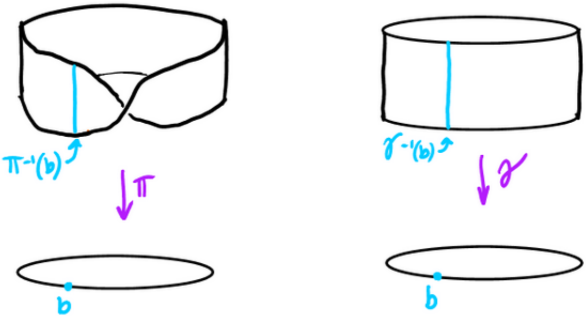
An important part of Martin-Löf type theory is the notion of a *dependent* type, denoted

$$x : A \vdash B(x) : Type.$$

Here the type $B(x)$ *depends* on an element of A, as in

- $Days(m)$ for $m : Month$
- $Players(t)$ for $t : Team$

It's helpful to have in mind the imagery of spaces fibred over other spaces:



Two useful constructions we can apply to these types are *dependent sum* and *dependent product*: total space and sections.

In general we can think of this **dependent sum** as sitting ‘fibred’ above the base type A, as one might imagine the collection of league players lined up in fibres above their team name.

Likewise an element of the **dependent product** is a choice of a player from each team, such as $Captain(t)$.

Dependent sum	Dependent product
$\sum_{x:A} B(x)$ is the collection of pairs (a, b) with $a : A$ and $b : B(a)$	$\prod_{x:A} B(x)$, is the collection of functions, f, such that $f(a) : B(a)$
When A is a set and $B(x)$ is a constant set B: The product of the sets.	When A is a set and $B(x)$ is a constant set B: The set of functions from A to B.
When A is a proposition and $B(x)$ is a constant proposition, B: The conjunction of A and B.	When A is a proposition and $B(x)$ is a constant proposition, B: The implication $A \to B$.

- Consider the case where A is a set, and $B(a)$ is a proposition for each a in A.
- Perhaps A is the set of animals, and $B(a)$ states that a particular animal, a, breathes.
- Then an element of the dependent sum is an element a of A and a proof of $B(a)$, so something witnessing a breathing animal.
- Meanwhile an element of the dependent product is a mapping from each $a : A$ to a proof of $B(a)$.
- There will only be such a mapping if $B(a)$ is true for each a.
- If this were the case, we would have a proof of the universal statement 'for all x in A, $B(x)$', in our example, 'All animals breathe.'

- Returning to the dependent sum, this is almost expressing the existential quantifier 'there exists x in A such that $B(x)$', except that it's gathering all such a for which $B(a)$ holds, or, in our case, gathering all breathing animals.
- As we have seen before in the capped addition of the Boolean truth values in a disjunction, to treat this dependent sum as a proposition, there needs to be a 'truncation' from set to proposition, so that we ask merely whether this set is inhabited, in our case 'Does there exist a breathing animal?'
- This extra step should be expected as existential quantification resembles forming a long disjunction.
- That we don't need to adapt for universal quantification tallies with the straightforward form of the product of Boolean values.

The bottom line is that homotopy type theory for the lower levels of the hierarchy encapsulates:

- Propositional logic
- (Typed) predicate logic
- Structural set theory

It is a structural theory *par excellence*. It seems impossible to say anything more by speaking of 'the structure of A' or 'places in the structure of A'.

Constructivity

- Unless otherwise specified, HoTT adopts a constructive outlook.
- But there's no difficulty in adding in classical principles if these are required.
- However, then one loses computational benefits, interpretation in wider range of settings, etc.

Variants

- Plain HoTT
- Cohesive HoTT
- Directed HoTT
- Linear HoTT

What to do then?

- Give a historical/relativised a priori/Dynamics of Reason style account of what's happening?
- Redo analytic philosophy (language, metaphysics, etc.) with a much better formalism.

Given a plain type, A, we can turn any type C into one dependent on A by formulating $x : A \vdash (A \times C)(x) := C$. If A and C are sets, think of lining up a copy of C over every element of A, the product of the two sets projecting down to the first of them, $A \times C \to A$.

Now, we can think of approximating an inverse to this process, which would need to send A-dependent types to plain types. Such approximations, or adjoints do indeed exist. Left adjoint to this mapping is dependent sum, and right adjoint is dependent product. The fact that these are adjoints may be rendered as follows, for B a type depending on A:

- $Hom_{\mathcal{C}}(\sum_{x:A} B, C) \cong Hom_{\mathcal{C}/A}(B, A \times C)$
- $Hom_{\mathcal{C}}(C, \prod_{x:A} B) \cong Hom_{\mathcal{C}/A}(A \times C, B)$

Were these A, B and C sets, then their cardinalities would satisfy:

- $c^{\sum_i b_i} = \prod_i c^{b_i}$
- $(\prod_i b_i)^c = \prod_i (b_i)^c$

So a pupil being taught that, say, $3^5 \times 7^5 = 21^5$ is being exposed to the shadow of an instance of an important adjunction, which in turn, as with all of the discussion above of dependent sums and products, works for types up and down the hierarchy of n-types.

- In homotopy type theory, the way to express that a set, A, is acted on by a group, G, is to write it as a dependent type $* : BG \vdash A(*) : Type$.
- The type BG is a version of the group G, but where we're taking it to be a single object with looping arrows labelled by each group element.
- Applying the dependent sum construction from the last section, we find that $\sum_{*:BG} A(*)$ is a type with structure that of the action groupoid, which should indicate to us that the concept is a fundamental one.
- Similarly we can form the dependent product, which in this case is composed of the 'fixed points' of the action, those elements of the set left unchanged by all elements of the group.