2. Say that f is topologically continuous if pullbacks of open sets are open, and ε, δ–continuous if it satisfies the usual ε, δ–condition. For $x \in \mathbb{R}^d$ and $r > 0$, let $B(x,r) := \{y \in \mathbb{R}^d : ||y - x|| < r\}$ denote the open ball of radius r centered at x. The ε, δ –definition of continuity of $f : \mathbb{R}^n \to \mathbb{R}^m$ can be phrased as

$$\forall x \in \mathbb{R}^n \; \forall \varepsilon > 0 \; \exists \delta > 0 \left[B(x,\delta) \subseteq f^{-1}B(f(x),\varepsilon) \right] \qquad (*)$$

Suppose first that $f : \mathbb{R}^n \to \mathbb{R}^m$ is ε, δ–continuous, and that $U \subseteq \mathbb{R}^m$ is open, and that $x \in f^{-1}U$. Since U is open and $f(x) \in U$, there is $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq U$, and thus by $(*)$ a $\delta > 0$ such that

$$x \in B(x,\delta) \subseteq f^{-1}B(f(x),\varepsilon) \subseteq f^{-1}U$$

from which it is clear that $f^{-1}[U]$ is open.

Next, suppose that f is topologically continuous, that $x \in \mathbb{R}^n$, and that $\varepsilon > 0$. Since $f^{-1}B(f(x),\varepsilon)$ is open and $x \in f^{-1}B(f(x),\varepsilon)$, there is $\delta > 0$ such that $B(x,\delta) \subseteq f^{-1}B(f(x),\varepsilon)$, from which it follows that f is ε, δ–continuous (via $(*)$).

3. For $i = 1, \ldots, n+1$, let $V_i^\pm := \{\mathbf{x} \in \mathbb{R}^{n+1} : \pm x_i > 0\}$, and let $U_i\pm := V_i^\pm \cap S^n$, so that each $U_i^\pm$ is open in S^n w.r.t. the induced topology. It is clear that the hemispheres $U_i^\pm$ cover S^n. Define maps

$$\phi_i^\pm : U_i^\pm \to \mathbb{R}^n : \mathbf{x} \mapsto \underbrace{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})}_{x_i \text{ omitted}}$$

so that each $\phi_i^\pm$ is essentially a projection onto the n–dimensional open unit disc $D^n := \{\mathbf{u} \in \mathbb{R}^n : ||\mathbf{u}|| < 1\}$ in the coordinate plane $x_i = 0$.

Now, e.g., for $i < j$ we have $\phi_i^+ \circ (\phi_j^\pm)^{-1} : D^n \to D^n$ by

$$\phi_i^+ \circ (\phi_j^\pm)^{-1}(u_1, \ldots, u_n) = \underbrace{(u_1, \ldots, u_{i-1}, u_{i+1}, u_{j-1}, \pm(1 - \sum_{j=1}^n u_j^2)^{\frac{1}{2}}, u_j, u_{j+1}, \ldots, u_n)}_{u_i \text{ omitted, } \pm(1-\sum_{j=1}^n u_j^2)^{\frac{1}{2}} \text{ inserted}}$$

(or something like that) which is clearly C^∞. We do not have to consider the case were $i = j$ as $U_i^+ \cap U_i^- = \emptyset$.

(One can do this with just two charts, though obviously not fewer, as S^n is compact, but no open subset of $\mathbb{R}^{n+1}$ is compact.)

4. If $\varphi_i : U_i \to \mathbb{R}^n$ are charts for a manifold M, and $U \subseteq M$ is open, then the restrictions $\varphi_i | U \cap U_i$ are charts for U.

5. Given an n–dimensional manifold M, as witnessed by charts $\phi_i : U_i \to V_i \subseteq \mathbb{R}^n, i \in I$, and an n'–dimensional manifold M', as witnessed by charts $\phi'_{i'} : U'_{i'} \to V'_{i'} \subseteq \mathbb{R}^{n'}, i' \in I'$, the product mappings

$$\phi_i \times \phi'_{i'} : U_i \times U'_{i'} \to V_i \times V'_{i'} \subseteq \mathbb{R}^{n+n'} \qquad (i, i') \in I \times I'$$

show that $M \times M'$ is an $(n+n')$–dimensional manifold. Indeed,

$$(\phi_i \times \phi'_{i'}) \circ (\phi_j \times \phi'_{j'})^{-1} = (\phi_i \circ \phi_j^{-1}) \times (\phi'_{i'} \circ \phi'^{-1}_{j'}) : \phi_j[U_i \cap U_j] \times \phi'_{j'}[U'_{i'} \times U'_{j'}] \to \mathbb{R}^{n+n'}$$

is a product of differentiable maps, and hence differentiable.

6. If Φ, Φ' are atlases for two disjoint $n--$dimensional manifolds M, M', the union $\Phi \cup \Phi'$ is clearly an atlas for the union $M \cup M'$.

7. By definition vector field on a manifold M is a linear map $C^\infty(M) \to C^\infty(M)$ satisfying the Leibniz rule. It is obvious that if $v, w \in \mathrm{Vect}(M)$ and $h \in C^\infty(M)$, then $v + w, hv$ are linear. To verify the Leibniz rule, just observe that

$$\begin{aligned}(v+w)(f \cdot g) &= v(f \cdot g) + w(f \cdot g) = v(f) \cdot g + f \cdot v(g) + w(f) \cdot g + f \cdot w(g) \\ &= [v(f) + w(f)] \cdot g + f \cdot [v(g) + w(g)] = (v+w)(f) \cdot g + f \cdot (v+w)(g)\end{aligned}$$

and that

$$(hv)(f \cdot g) = h[v(f \cdot g)] = h[v(f) \cdot g + f \cdot v(g)] = (hv)(f) \cdot g + f \cdot (hv)(g)$$

8. Also easy.

9. Clearly for each $i \leq n$, the projection $\pi^i : \mathbb{R}^n \to \mathbb{R} : (x^1, \ldots, x^n) \mapsto x^i$ has $\pi^i \in C^\infty(\mathbb{R}^n)$, and $\partial_j \pi^i = \delta^i_j$. So if $v^j \partial_j = 0$, then $v^i = v^j \partial_j \pi^i = 0$ for each $\leq n$. Thus $\{\partial_i : i \leq n\}$ is a set of linearly independent elements of $\mathrm{Vect}(\mathbb{R}^n)$.

   [To see that this set is also a basis, and not just linearly independent, note that, for fixed x_0, a first–order Taylor expansion yields

$$f(x) = f(x_0) + \partial_i f(x_0) \cdot (x^i - x^i_0) + \varepsilon_{x_0}(x) \cdot ||x - x_0|| \quad \text{where } \varepsilon_{x_0}(x) \to 0 \text{ as } x \to x_0$$

   Now as $v(c) = 0$ when c is constant (because $v(c) = cv(1) = cv(1 \cdot 1) = 2cv(1) = 2v(c)$), we obtain

$$\begin{aligned}v(f)(x_0) &= v(\pi^i)(x_0)\partial_f(x_0) + v(\varepsilon_{x_0})(x_0) \cdot ||x_0 - x_0|| + \varepsilon_{x_0}(x_0) \cdot v(||\mathrm{id} - x_0||)(x_0) \\ &= v(\pi^i)(x_0)\partial_i f(x_0)\end{aligned}$$

   Hence

$$v = v(\pi^i)\partial_i$$

   expresses v as a linear combination of the basis vectors ∂_i.]