Exercise 90

Let $\{(U_\alpha, \phi_\alpha)\}$ be a collection of charts such that the open sets U_α cover S and such that

$$S \cap U_\alpha = \phi_\alpha^{-1}(\mathbb{R}^k)$$

holds for each α. (This is possible given the definition of a sub-manifold in the text.)

Let $V_\alpha = S \cap U_\alpha$, and let $\psi_\alpha = \phi_\alpha|_{V_\alpha}$. Then the V_α's are open subsets of S, and the functions $\psi_\beta \circ \psi_\alpha^{-1}$ go from $\mathbb{R}^k$ to itself. Hence $\{(V_\alpha, \psi_\alpha)\}$ forms an atlas for S.

Exercise 91

As a topological space, it is compact since it is closed (being the pre-image of the closed set $\{1\}$ under the norm function) and bounded (all points have norm less than 2).

To show that it's a submanifold, the same function as used in exercise 84 will work:

Define

$$U_1^+ = \{x \in \mathbb{R}^n : x_1 > 0\}$$

and define the map

$$\phi_1^+ : U_1^+ \to \mathbb{R}^n$$

by mapping $(x_1, \ldots, x_n)$ to

$$\frac{\|x\|}{x_1}(x_1, \ldots, x_n)$$

(U_1^+, ϕ_1^+) is a chart on $\mathbb{R}^n$ that maps $S^{n-1} \cap U_1^+$ bijectively to the hyperplane

$$\{x \in \mathbb{R}^n : x_n = 1\}$$

which is $\mathbb{R}^{n-1}$. The collection of open sets $U_i^\pm$ (with U_i^-defined as expected) cover S^{n-1}.

Exercise 92

Let $V \subset M$ be an open subset, and let $\{(U_\alpha, \phi_\alpha)\}$be an atlas for M. Then the family of open sets $U_\alpha \cap V$ together with the charts $\phi_\alpha|_V$ forms a suitable family to meet the requirements of the definition of a submanifold.

Exercise 93

This is exactly the same as exercise 90 except that $\mathbb{R}^k$ should be replaced everywhere by ($\mathbb{R}^k$ or $\mathbf{H}^k$), roughly speaking.

Exercise 94

Again, this is almost the same as exercise 91, using the same map.

Exercise 95

Let

$$\omega = \omega_x dx + \omega_y dy$$

be an arbitrary 1-form on S. Then

$$d\omega = \partial_x \omega_y - \partial_y \omega_x dx \wedge dy$$

so Stokes' theorem,

$$\int_S d\omega = \int_{\partial S} \omega$$

becomes

$$\int_S \partial_x \omega_y - \partial_y \omega_x dx dy = \int_{\partial S} (\omega_x dx + \omega_y dy)$$

Exercise 96

Let $S \in \mathbb{R}^3$ be a 2-dimensional compact orientable submanifold with boundary. Choose an atlas for S as a submanifold. It should be possible to subdivide S into a finite number of smaller 2-dimensional compact orientable submanifolds with boundary, such the S is the union of these smaller parts, and the intersection of any two parts is either empty, or a 1-dimensional manifold.

Then the sum of integrating over all the boundaries of these smaller parts is the same as integrating over the boundary of S since opposite integrals cancel out. Also, the sum of integrating over the surfaces of all the smaller parts yields the same answer as integrating over the whole of S. It follows that we only need to prove the given statement for one of the smaller parts, i.e. we can assume S is such a smaller part.

Then we can choose coordinates such that S lies in the plane $z = 0$. Let

$$\omega = \omega_x dx + \omega_y dy$$

be a 1-form on S. Then

$$d\omega = (\partial_x \omega_y - \partial_y \omega_x) dx \wedge dy$$

and Stokes' theorom implies that

$$\int_S (\partial_x \omega_y - \partial_y \omega_x) dx \wedge dy = \int_{\partial S} \omega$$

Now let $F = (F_x, F_y, F_z)$ be a vector field (on an open subset of $\mathbb{R}^3$ containing S). According to the usual Stokes' theorem,

$$\int_S (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot dr$$

Let

$$\omega = F_x dx + F_y dy + F_z dz$$

be a 1-form defined using the components of F. Given the orientation of S, the normal dS always points in the z-direction. Thus

$$(\nabla \times F) \cdot dS$$

corresponds to the z-component of $d\omega$, which is

$$\partial_x F_y - \partial_y F_x$$

Thus the left-hand sides of the classic version and the more general version of Stokes' theorem, agree. For the right-hand side, note that dr will always be orthogonal to the z-direction, so the z-component of F can be ignored here as well. The right-hand sides then also agree.

Exercise 97

In this case, let

$$\omega = \omega_x dy \wedge dz + \omega_y dz \wedge dx + \omega_z dx \wedge dy$$

be the 2-form corresponding to the vector field $F = (\omega_x, \omega_y, \omega_z)$. Then integrating the divergence of F over the volume is clearly the same as integrating

$$d\omega = (\partial_x \omega_x + \partial_y \omega_y + \partial_z \omega_z) dx \wedge dy \wedge dz$$

over the volume. To show that integrating the normal of F over the surface (the boundary of the volume) is the same as integrating ω over the surface, choose local coordinates such that the surface lies in the plane $z = 0$. Then the normal component of F is just ω_z, and restricted to the surface, the 2-form ω becomes $\omega_z dx \wedge dy$, and integrating the two gives the same result.

Exercise 98

Let ϕ be a map from M to N, and let ω be a p-form on N.

Suppose ω is closed, i.e. $d\omega = 0$. Then

$$\begin{aligned} d(\phi^*\omega) &= \phi^*(d\omega) \\ &= \phi^*(0) \\ &= 0 \end{aligned}$$

so $\phi^*\omega$ is also closed.

Suppose that ω is exact, i.e. $\omega = d\nu$, where ν is a $p-1$-form on N. Then

$$\begin{aligned} \phi^*(\omega) &= \phi^*(d\nu) \\ &= d\phi^*(\nu) \end{aligned}$$

so $\phi^*\omega$ is also exact.

Exercise 99

Firstly, the linear map on p-forms from $\Omega^p(M')$ to $\Omega^p(M)$ has been defined earlier. The previous exercise shows that if we restrict this map to the closed forms $Z^p(M')$, then the image lies in $Z^p(M)$. So we get a map from $Z^p(M')$ to $Z^p(M)$. The previous exercise also shows that the kernel of this map contains the exact forms $B^p(M')$, hence it induces a map on the equivalence classes, from $H^p(M')$ to $H^p(M)$, as desired.

For the second part, showing that the map on the cohomology groups commutes with composition of maps follows from the fact that the pull-back map on p-forms commutes with composition of maps between manifolds.