Exercise 80

E is closed:

$$\begin{aligned} dE &= \frac{1}{x^2+y^2}(dx \wedge dy - dy \wedge dx) \\ &= 0 \end{aligned}$$

To calculate the integrals, note that the path lies on a 1-dimensional space which can be reparametrised with

$$\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned}$$

The 1-form then becomes

$$\frac{\cos\theta\cos\theta - \sin\theta(-\sin\theta)}{\cos^2\theta + \sin^2\theta} = 1$$

and the integrals are:

$$\begin{aligned} \int_{\gamma_0} E &= \int_{\pi}^{0} 1 d\theta \\ &= -\pi \end{aligned}$$

and

$$\begin{aligned} \int_{\gamma_1} E &= \int_{-\pi}^{0} 1 d\theta \\ &= \pi \end{aligned}$$

Exercise 81

Let γ_0 and $\gamma_1$be two paths from p to q. Define the homotopy γ as

$$\gamma(s,t) = (1-s)\gamma_0(t) + s\gamma_1(t)$$

Then

$$\gamma(0,t) = \gamma_0(t)$$

and

$$\gamma(1,t) = \gamma_1(t)$$

Also

$$\gamma(s,0) = (1-s)p + sp = p$$

and similarly $\gamma(s,1) = q$. Finally, γ is smooth.

Exercise 82

Suppose the 1-form E is exact. Then there is a function ϕ on M such that $E = d\phi$. Let γ be any loop on M. Denote $\gamma(0) = \gamma(1)$ by $p \in M$. Then

$$\begin{aligned}\int_\gamma E &= \int_\gamma d\phi \\ &= \phi(\gamma(1)) - \phi(\gamma(0)) \\ &= \phi(p) - \phi(p) \\ &= 0\end{aligned}$$

Conversely, suppose $\int_\gamma E = 0$ for all loops γ on M. This implies that

$$\int_{\gamma_0} E = \int_{\gamma_1} E$$

for any two paths γ_0 andγ_1 both starting at $p \in M$ and ending at $q \in M$:

$$\int_{\gamma_1} E - \int_{\gamma_0} E = \int_{\gamma_0\gamma_1^{-1}} E = 0$$

since $\gamma_0\gamma_1^{-1}$ is a loop on M.

But this means that we can follow the exact same steps as on p109 to construct a function ϕ on M such that $E = -d\phi$, showing that E is exact.

Exercise 83

Parametrize S^1 by $\theta \in [0, 1] \subset \mathbb{R}$ such that $\theta(0) = \theta(1)$. Choose any two charts, say $V_1 = (-0.1, 0.6)$ and $V_2 = (0.4, 1.1)$. Then construct a 1-form on S^1 which is $d\theta$ on either of the charts.

If $\{U_i\}$ is an atlas of charts on M, then $\{V_1 \times U_i\} \cup \{V_2 \times U_i\}$ is an atlas for $S^1 \times M$ and on each chart, $d\theta$ is still a 1-form. Integrating it around the loop $S^1 \times \{m\}$ gives a non-zero value.

Exercise 84

Consider the open subset $U_1^+ \subset D^n$, where

$$U_1^+ = \{x \in D^n : x_1 > 0\}$$

Let $\phi : U_1^+ \to \mathbb{R}^n$ map $(x_1, \ldots, x_n)$ to

$$\frac{\|x\|}{x_1}(x_1, \ldots x_n)$$

The image of ϕ is

$$\{x \in \mathbb{R}^n : 0 < x_1 \leq 1\}$$

with points with norm 1 mapping to points with $x_1 = 1$. We can compose with another mapping from $\mathbb{R}^n$ to itself that maps x_1 to $(1 - x_1)$. This gives a chart which maps U_1^+ to $\mathbf{H}^n$ with the boundary points being exactly the ones with norm 1.

Finally, charts of the form $U_i^+$and U_i^- cover D^n, hence forming an atlas.

Exercise 85

In chapter 3, tangent vectors at $p \in M$ were defined as functions from $C^\infty(M)$ to $\mathbb{R}$ satisfying three properties. Let $C_p^\infty(M)$ be the germ of smooth functions at p, and consider the surjective mapping from $C^\infty(M)$ to $C_p^\infty(M)$. The kernel consists of functions defined on a neighbourhood of p that vanish on this neighbourhood. From the second property ($v_p(\alpha f) = \alpha v_p(f)$) it follows that tangent vectors vanish on this kernel. Hence we can consider them as functions on $C_p^\infty(M)$, and could also have defined them like that.

Now let p be on the boundary of M. We can assume $p \in \mathbf{H}^n$. Let $\mathbf{H}_\epsilon^n$ be the manifold without boundary defined like $\mathbf{H}^n$ but with $x^n > -\epsilon$ instead of $x^n \geq 0$. Then the tangent space defined at $p \in \mathbf{H}_\epsilon^n$ using $C_p^\infty(\mathbf{H}_\epsilon^n)$ is the usual n-dimensional vector space. And the tangent space at $p \in \mathbf{H}^n$ defined using $C_p^\infty(\mathbf{H}^n)$ is the space that we are interested in (for which we must show that the dimension is also n).

But again there is a surjective map from $C_p^\infty(\mathbf{H}_\epsilon^n)$ to $C_p^\infty(\mathbf{H}^n)$. The kernel consists of smooth functions defined on a neighbourhood of $p \in \mathbb{R}^n$ that vanish for $x^n \geq 0$. Such functions do not necessarily vanish on a neighbourhood of p, but the tangent vector does vanish on such functions. This is essentially because at least one side of any small straight line through p must lie in $\mathbf{H}^n$.

Therefore this last map induces a map between the two tangent spaces which is an isomorphism.

Exercise 86

(I googled this and found p226 of 'Introduction to manifolds' on Google books which gave most of the proof.)

Let $\{U_\alpha\}$ be the original atlas with $\{f_\alpha\}$ the corresponding partition of unity. Let $\{V_\beta\}$ be another atlas where all the charts have the same orientation as in the original atlas, with $\{g_\beta\}$ a subordinate (i.e. support$(g_\beta) \subset V_\beta$) partition of unity. Then

$$\begin{aligned}
\sum_\alpha \int_{U_\alpha} f_\alpha \omega &= \sum_\alpha \int_{U_\alpha} f_\alpha (\sum_\beta g_\beta \omega) \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha} f_\alpha g_\beta \omega \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} f_\alpha g_\beta \omega
\end{aligned}$$

By symmetry,

$$\sum_\beta \int_{V_\beta} g_\beta \omega$$

is equal to the same expression, and so the two different ways of defining the integral $\int_M \omega$ give the same value. Interchanging the summation and integration signs above is allowed due to the second property of partitions of unity, namely that for any point $p \in M$ there is a neighbourhood where only finitely many of the partition functions f_α do not vanish.

I think the orientations of the charts being the same is used implicitely above when writing integrals without coordinates.

Exercise 87

Using the same charts as defined in exercise 84, we saw that ∂D^n is precisely

$$\{x \in D^n : \|x\| = 1\}$$

This is the same set of points as can be used to define S^{n-1}.

Exercise 88

Let x be the implied local coordinate on M. Then using Stokes theorem with $\omega = f$, a function, gives the required result. The different signs follow from the induced orientation on the boundary points, though I don't know how to make this precise.

Perhaps by considering two maps: one from a neighbourhood of 0 to $0 \in H^1$, the other from 1 to $0 \in H^1$. In the latter case the map would be orientation-reversing. This should imply that the signs of $f(0)$ and $f(1)$ are different, but does not show why $f(1)$ is the positive one.

Exercise 89

Let

$$f(x) = e^x.$$

Then

$$\int_0^\infty f'(x)dx \quad = \quad \int_0^\infty e^x dx$$

is not defined, or is ∞. But if we had applied Stoke's theorem, it should be equal to

$$-f(0) = -1.$$