Exercise 60

The parity of the permutation is determined by the number of transpositions: the parity is even if the number of transpositions is even, and the parity is odd if the number of transpositions is odd. Thus it is enough to show that any transposition reverses the orientation (because reversing it twice gives the original orientation).

But the matrix of the linear transformation mapping a basis to the same basis but with a single transposition differs from the identity matrix only in that two rows (or two columns) are swapped. But as an elementary row operation, the effect of swapping two rows is to change the sign of the determinant. Since the determinant of the identity matrix is positive, the determinant of the linear transformation must be negative, and hence making a transposition reverses the orientation.

Exercise 61

Let ω be a positively oriented volume form on M. Consider a specific chart (U_α, ϕ_α). In local coordinates on this chart, ω is of the form $f\,dx^1 \wedge \cdots \wedge dx^n$, or equivalently,

$$\omega = \phi_\alpha^{-1}(f\,dx^1 \wedge \cdots \wedge dx^n).$$

Since ω is a volume form, $f \neq 0$ on $\phi_\alpha(U_\alpha)$. But f is continuous and its domain is connected, so either $f > 0$ or $f < 0$. If $f > 0$, then define $\tilde{\phi}_\alpha = \phi_\alpha$, otherwise define $\tilde{\phi}_\alpha$ by

$$\tilde{\phi}_\alpha(x^1, \ldots, x^{n-1}, x^n) = \phi_\alpha(x^1, \ldots, x^{n-1}, -x^n)$$

In terms of this new chart

$$\omega = \phi_\alpha^{-1}(-f\,dx^1 \wedge \cdots \wedge dx^n)$$

and $-f > 0$. So the new atlas with charts $(U_\alpha, \tilde{\phi}_\alpha)$ has the required property: pulling back $dx^1 \wedge \cdots \wedge dx^n$ gives $\frac{1}{g}\omega$ where $g = \pm f \circ \phi_\alpha > 0$.

Exercise 62

Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of charts on M such that the transition functions are orientation-preserving. Let $\psi = \phi_\alpha \circ \phi_\beta^{-1}$ be the transition function from U_β to U_α. The induced map on differential forms can be applied to the canonical volume form on U_α, mapping it to a multiple of the volume form on U_β:

$$\psi^{-1}(dx^1 \wedge \cdots \wedge dx^n) = f_{\alpha\beta} dx^{1\prime} \wedge \cdots dx^{n\prime}$$

Here $f_{\alpha\beta}$ is a smooth function on $U_\alpha \cap U_\beta$, which is also positive everywhere, since ψ is orientation-preserving. Note that $f_{\alpha\alpha} = 1$ and $f_{\alpha\beta} = f_{\beta\alpha}$. If we consider a third chart, U_γ, and use the fact that on $U_\alpha \cap U_\beta \cap U_\gamma$ the transition function from U_β to U_α is the same as going via U_γ, then it follows that $f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} = 1$. These facts will be needed later on.

The idea is now to introduce a positive function g_α on each U_α such that the local volume forms

$$g_\alpha dx^1 \wedge \cdots \wedge dx^n$$

can be patched together to give a volume form on M (which is the goal of the exercise). Since

$$\psi^{-1}(g_\alpha dx^1 \wedge \cdots \wedge dx^n) = f_{\alpha\beta} g_\alpha dx^{1\prime} \wedge \cdots dx^{n\prime}$$

must equal

$$g_\beta dx^{1\prime} \wedge \cdots dx^{n\prime}$$

for the patching to work, the g_α's should satisfy

$$f_{\alpha\beta} g_\alpha = g_\beta$$

i.e.

$$f_{\alpha\beta} = \frac{g_\beta}{g_\alpha}$$

Since all these functions are strictly positive, we can take logarithms. Define $c_\alpha = \log g_\alpha$ and $d_{\alpha\beta} = \log f_{\alpha\beta}$. Then the previous equation becomes

$$d_{\alpha\beta} = c_\beta - c_\alpha$$

To be precise: we obtain the functions $f_{\alpha\beta}$ from the given atlas. Then we obtain $d_{\alpha\beta}$. Below we show how the c_α's are obtained from these, and then finally we apply the exponential function to get the g_α's, which would complete the exercise.

Note that the properties of the $f_{\alpha\beta}$'s given above translate into the following properties for the $g_{\alpha\beta}$'s:

$$\begin{aligned} g_{\alpha\alpha} &= 0 \\ g_{\alpha\beta} &= -g_{\alpha\beta} \\ g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} &= 0 \end{aligned}$$

Let $\lambda_\alpha$be a partition of unity for the atlas on M. Consider the smooth function $d_{\alpha\gamma}\lambda_\gamma$defined on $U_\alpha \cap U_\gamma$. Extend it to a function on U_α by setting it to 0 outside U_γ. Since the support of λ_γ is contained in U_γ, it follows that the extended function is still smooth. (To see this, consider $x \in U_\alpha$. If x is in the support of λ_γ, then it has a neighbourhood inside $U_\gamma$where the function must be smooth. Otherwise it vanishes on a neighbourhood, hence also smooth.)

Now on U_α define

$$c_\alpha = -\sum_\gamma d_{\alpha\gamma}\lambda_\gamma$$

Note that only finitely many of the λ_γ's do not vanish on U_α, so the sum exists.

Then on $U_\alpha \cap U_\beta$:

$$\begin{aligned} c_\beta - c_\alpha &= -\sum_\gamma d_{\beta\gamma}\lambda_\gamma + \sum_\gamma d_{\alpha\gamma}\lambda_\gamma \\ &= \sum_\gamma (d_{\alpha\gamma} - d_{\beta\gamma})\lambda_\gamma \\ &= \sum_\gamma d_{\alpha\beta}\lambda_\gamma \\ &= d_{\alpha\beta} \end{aligned}$$

which is what we needed to show.

Exercise 63

Denote the oriented orthonormal basis of cotangent vectors at $p \in M$ by $\{e^\mu\}$ (i.e. use superscripts). Let $x^1, \ldots, x^n$ be local coordinates on a chart containing p. Define the metric components

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu).$$

Then the volume form associated to the metric is

$$\mathrm{vol} = \sqrt{|\det g_{\mu\nu}|}dx^1 \wedge \cdots \wedge dx^n,$$

and its value at p is

$$\mathrm{vol}_p = \sqrt{|\det(g_{\mu\nu})_p|}(dx^1)_p \wedge \cdots \wedge (dx^n)_p \in \wedge^n T_p^* M.$$

Furthermore,

$$\langle (dx^\mu)_p, (dx^\nu)_p \rangle = (g^{\mu\nu})_p$$

where

$$\det(g_{\mu\nu})_p \cdot \det(g^{\mu\nu})_p = 1.$$

Define the invertible matrix to transform between the two bases of $\wedge^n T_p^* M$:

$$(dx^\mu)_p = T^\mu_\lambda e^\lambda.$$

Using this, the inner product can also be calculated as

$$\begin{aligned} \langle (dx^\mu)_p, (dx^\nu)_p \rangle &= \langle T^\mu_\lambda e^\lambda, T^\nu_\lambda e^\lambda \rangle \\ &= \epsilon(\lambda) T^\mu_\lambda T^\lambda_\nu \end{aligned}$$

where $\epsilon(\lambda) = \langle e^\lambda, e^\lambda \rangle$. Note that if g is actually a Riemannian metric, so that $\epsilon(\lambda) = 1$ for all λ, then this last expression is just the matrix T^μ_λ multiplied by its transpose. In this more general situation, it is of the form $T^\mu_\lambda \cdot (\epsilon(\lambda) T^\mu_\lambda)^T$. Taking the determinant gives

$$\epsilon(\det T^\mu_\lambda)^2$$

where

$$\epsilon = \prod_{\lambda} \epsilon(\lambda) = \pm 1$$

Taking the determinant of the other expression of the inner product yields:

$$\begin{aligned} \det(g^{\mu\nu})_p &= \det(g_{\mu\nu})^{-1} \\ &= \epsilon |\det(g_{\mu\nu})|^{-1} \end{aligned}$$

Setting the two expressions equal to one another yields

$$\det T^{\mu}_{\lambda} = \frac{1}{\sqrt{|\det(g_{\mu\nu})|}}$$

Finally:

$$\begin{aligned} \mathrm{vol}_p &= \sqrt{|\det(g_{\mu\nu})_p|}(dx^1)_p \wedge \cdots \wedge (dx^n)_p \\ &= \sqrt{|\det(g_{\mu\nu})_p|} T^1_{\lambda} e^{\lambda} \wedge \cdots \wedge T^n_{\lambda} e^{\lambda} \\ &= \sqrt{|\det(g_{\mu\nu})_p|} \det T^{\mu}_{\lambda} e^1 \wedge \cdots \wedge e^n \\ &= e^1 \wedge \cdots \wedge e^n \end{aligned}$$

Exercise 64

We can assume $\mu = e^{i_1} \wedge \cdots \wedge e^{i_p}$, since such elements span the space of p-forms, and the result would follow by linearity of the wedge product and bi-linearity of the inner product. Changing the order of the exponents would not cause well-definedness problems due to the $\mathrm{sign}(i_1, \ldots, i_p)$ factor used in the definition.

For similar linearity reasons, it is enough to assume $\omega = e^{j_1} \wedge \cdots \wedge e^{j_p}$. So

$$\omega \wedge \star\mu = e^{j_1} \wedge \cdots \wedge e^{j_p} \wedge e^{i_{p+1}} \wedge \cdots \wedge e^{i_n}$$

If $j_k = i_l$ for some k and l, then $\omega \wedge \star\mu = 0$, and also

$$\begin{aligned} \langle \omega, \mu \rangle &= \det \langle e^{j_k}, e^{i_l} \rangle \\ &= 0 \end{aligned}$$

since one of the rows (and one of the columns) contains only 0's. So the property is satisfied in this case, and we can assume that the j_k's and i_l's are all different $(l > p)$. Thus

$$\omega = \pm e^{i_1} \wedge \cdots \wedge e^{i_p}$$

where $\pm$ corresponds to the sign of the permutation mapping $(j_1, \ldots j_p)$ to $(i_1, \ldots, i_p)$. Then

$$\begin{aligned} \omega \wedge \star\mu &= \pm e^{i_1} \wedge \cdots \wedge e^{i_n} \\ &= \pm e^1 \wedge \cdots \wedge e^n \\ &= \pm \mathrm{vol} \end{aligned}$$

where $\pm$ now corresponds to the product of what it was before and the sign of the permutation mapping $(1, \ldots, n)$ to $(i_1, \ldots, i_n)$. The right-hand side of the equation is ± 1, where the sign is the same (since swapping two rows of a matrix changes the sign of the determinant, and the total number of swaps is equal to the sum of the number of transpositions of the two mentioned permutations).

Exercise 65

Let

$$\omega = \omega_x dx + \omega_y dy + \omega_z dz.$$

Then

$$\begin{aligned} d\omega &= (\partial_y \omega_z - \partial_z \omega_y) dy \wedge dz + \\ &\quad (\partial_z \omega_x - \partial_x \omega_z) dz \wedge dx + \\ &\quad (\partial_x \omega_y - \partial_y \omega_x) dx \wedge dy \end{aligned}$$

and finally

$$\begin{aligned} \star d\omega &= (\partial_y \omega_z - \partial_z \omega_y) dx + \\ &\quad (\partial_z \omega_x - \partial_x \omega_z) dy + \\ &\quad (\partial_x \omega_y - \partial_y \omega_x) dz \end{aligned}$$

Exercise 66

$$\begin{aligned} \star\omega &= \omega_x dy \wedge dz + \omega_y dz \wedge dx + \omega_z dx \wedge dy \\ d \star \omega &= (\partial_x \omega_x + \partial_y \omega_y + \partial_z \omega_z) dx \wedge dy \wedge dz \\ \star d \star \omega &= \partial_x \omega_x + \partial_y \omega_y + \partial_z \omega_z \end{aligned}$$

Exercise 67

For the 0-form and the volume form we should get $\star^2 = -1$:

$$\begin{aligned} \star 1 &= dt \wedge dx \wedge dy \wedge dz \\ \star dt \wedge dx \wedge dy \wedge dz &= -1 \end{aligned}$$

For 1-forms and 3-forms we should get $\star^2 = 1$:

$$\begin{aligned}
\star dt &= -dx\wedge dy\wedge dz\\
\star dx\wedge dy\wedge dz &= -dt\\
\star dx &= -dt\wedge dy\wedge dz\\
\star dt\wedge dy\wedge dz &= -dx\\
\star dy &= dt\wedge dx\wedge dz\\
\star dt\wedge dx\wedge dz &= dy\\
\star dz &= -dt\wedge dx\wedge dy\\
\star dt\wedge dx\wedge dy &= -dz
\end{aligned}$$

Finally for 2-forms we should get $\star^2 = -1$:

$$\begin{aligned}
\star dt\wedge dx &= -dy\wedge dz\\
\star dy\wedge dz &= dt\wedge dx\\
\star dt\wedge dy &= dx\wedge dz\\
\star dx\wedge dz &= -dt\wedge dy\\
\star dt\wedge dz &= -dx\wedge dy\\
\star dx\wedge dy &= dt\wedge dz
\end{aligned}$$

Exercise 68

Applying the definition on p89 twice gives

$$\begin{aligned}
\star\star(e^{i_1}\wedge\cdots\wedge e^{i_p}) &= \star(\pm e^{i_{p+1}}\wedge\cdots\wedge e^{i_n})\\
&= \pm e^{i_1}\wedge\cdots\wedge e^{i_p}
\end{aligned}$$

The sign is given by

$$\operatorname{sign}(i_1,\ldots,i_n)\operatorname{sign}(i_{p+1},\ldots,i_n,i_1,\ldots,i_p)\epsilon(i_1)\cdots\epsilon(i_p)\epsilon(i_{p+1})\cdots\epsilon(i_n)$$

For the part with the ϵ's, the sign is $(-1)^s$. Permuting $(i_{p+1},\ldots,i_n,i_1,\ldots,i_p)$ to get $(i_1,\ldots,i_n)$ requires $p(n-p)$ transpositions, so the contribution to the sign is $(-1)^{p(n-p)}$. The remaining permutation from $(i_1,\ldots,i_n)$ is performed twice, so if it is negative, it cancels. Putting the two parts together gives the required formula.

Exercise 69

Assume that the coefficient in the final equation is $\frac{1}{p!(n-p)!}$ instead of $\frac{1}{p!}$.

Note that

$$\begin{aligned}
\epsilon^{i_1\ldots i_p}{}_{j_1\cdots j_{n-p}} &= g^{i_1k_1}g^{i_2k_2}\ldots g^{i_pk_p}\epsilon_{k_1k_2\ldots k_pj_1\ldots j_{n-p}}\\
&= \epsilon(i_1)\epsilon(i_2)\cdots\epsilon(i_p)\epsilon_{i_1\ldots i_pj_1\ldots j_{n-p}}\\
&= \begin{cases}\epsilon(i_1)\cdots\epsilon(i_p)\operatorname{sign}(i_1,\ldots,i_p,j_1,\ldots,j_{n-p}) & \text{if } \{j_1,\ldots,j_{n-p}\}=\{i_{p+1},\ldots,i_n\}\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

First consider the case where ω only consists of one term. Let $\{j'_1, \ldots, j'_{n-p}\}$ be the indices distinct from $\{i_1, \ldots, i_p\}$ in a specific, fixed order. Then

$$\begin{aligned}
\star\omega &= \frac{1}{p!}\omega_{i_1 \ldots i_p} \star (e^{i_1} \wedge \cdots \wedge e^{i_p}) \\
&= \frac{1}{p!}\omega_{i_1 \ldots i_p}\epsilon(i_1)\cdots\epsilon(i_p)\mathrm{sign}(i_1, \ldots, i_p, j'_1, \ldots, j'_{n-p})\, e^{j'_1} \wedge \cdots \wedge e^{j'_{n-p}} \\
&= \frac{1}{p!(n-p)!}\omega_{i_1 \ldots i_p}\epsilon^{i_1 \ldots i_p}{}_{j_1 \ldots j_{n-p}} e^{j_1} \wedge \cdots \wedge e^{j_{n-p}}
\end{aligned}$$

In the last line, the fixed set of indices j'_k is replaced by a summation over all possible sets of indices j_k. There are only $(n-p)!$ cases where the coefficient ϵ is not 0, and in all these cases

$$\epsilon^{i_1 \ldots i_p}{}_{j_1 \ldots j_{n-p}} e^{j_1} \wedge \cdots \wedge e^{j_{n-p}} = \epsilon(i_1)\cdots\epsilon(i_p)\mathrm{sign}(i_1, \ldots, i_p, j'_1, \ldots, j'_{n-p})\, e^{j'_1} \wedge \cdots \wedge e^{j'_{n-p}}$$

Next consider the case where ω consists of several terms, but all with the same set of indices $\{i_1, \ldots, i_p\}$, just with different permutations applied. Then each term will produce an expression as above with fixed i_k's, and adding them up will yield the same expression where we sum over the i_k's.

Finally, including more terms in ω with different sets of indices, and applying the $\star$ gives a completely different set of terms, so the result still holds.

Exercise 70

For the first equation,

$$\nabla \cdot \overrightarrow{E} = \rho$$

is the same as

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = \rho$$

But if

$$E = E_x dx \wedge E_y dy \wedge E_z dz$$

then

$$\begin{aligned}
\star_S E &= E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy \\
d_S \star_S E &= (\partial_x E_x + \partial_y E_y + \partial_z E_z) dx \wedge dy \wedge dz \\
\star_S d_S \star_S E &= \partial_x E_x + \partial_y E_y + \partial_z E_z
\end{aligned}$$

so the two formulations are the same.

For the second equation, the coefficients for the current on the right-hand side already correspond, so we just check the left-hand side. Using the older formulation,

$$\begin{aligned}
\nabla \times \overrightarrow{B} &= (\partial_y B_z - \partial_z B_y, \partial_z B_x - \partial_x B_z, \partial_x B_y - \partial_y B_x) \\
\frac{\partial \overrightarrow{E}}{\partial t} &= (\partial_t E_x, \partial_t E_y, \partial_t E_z)
\end{aligned}$$

and using the new formulation,

$$
\begin{aligned}
\partial_t E &= \partial_t E_x dx + \partial_t E_y dy + \partial_t E_z dz \\
B &= B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \\
\star_S B &= B_x dx + B_y dy + B_z dz \\
d_S \star_S B &= (\partial_y B_z - \partial_z B_y) dy \wedge dz + \\
&= (\partial_z B_x - \partial_x B_z) dz \wedge dx + \\
&= (\partial_x B_y - \partial_y B_x) dx \wedge dy \\
\star_S d_S \star_S B &= (\partial_y B_z - \partial_z B_y) dx + (\partial_z B_x - \partial_x B_z) dy + (\partial_x B_y - \partial_y B_x) dz
\end{aligned}
$$

so again the coefficients agree.

Exercise 71

First check that

$$\star B = (-\star_S B) \wedge dt$$

This can be checked for each component individually. For instance, if $B = dx \wedge dy$, then

$$
\begin{aligned}
\star(dx \wedge dy) &= -dz \wedge dt \\
\star_S dx \wedge dy &= dz
\end{aligned}
$$

and similarly for the other. Next check that

$$\star(E \wedge dt) = \star_S E$$

in a similar way, using for instance that

$$\star(dx \wedge dt) = dy \wedge dz = \star_S dx$$

Using these, it follows that if

$$F = B + E \wedge dt$$

then

$$
\begin{aligned}
\star F &= \star B + \star(E \wedge dt) \\
&= (-\star_S B) \wedge dt + \star_S E
\end{aligned}
$$

Next apply d:

$$
\begin{aligned}
d(-\star_S B \wedge dt) &= -d_S \star_S B \wedge dt - \partial_t \star_S B \wedge dt \wedge dt \\
&= -d_S \star_S B \wedge dt
\end{aligned}
$$

$$
\begin{aligned}
d(\star_S E) &= d_S \star_S E + \partial_t \star_S E \wedge dt \\
&= d_S \star_S E + \star_S \partial_t E \wedge dt
\end{aligned}
$$

Combining these gives the required expression for $d \star F$:

$$d \star F = \star_S \partial_t E \wedge dt + d_S \star_S E - d_S \star_S B \wedge dt$$

For finding $\star d \star F$, we can again check the individual terms. If $E = E_x dx$, then

$$\begin{aligned} \star(\star_S \partial_t E \wedge dt) &= \star(\star_S \partial_t E_x dx \wedge dt) \\ &= \star(\partial_t E_x dy \wedge dz \wedge dt) \\ &= -\partial_t E_x dx \\ &= -\partial_t E \end{aligned}$$

and this can be checked for general E as well.

For the second term,

$$d_S \star_S E = k\, dx \wedge dy \wedge dz$$

where k is a function. So

$$\begin{aligned} \star d_S \star_S E &= k\, dt \\ &= k \wedge dt \\ &= \star_S(k\, dx \wedge dy \wedge dz) \wedge dt \\ &= \star_S d_S \star_S E \wedge dt \end{aligned}$$

and the third term can be checked similarly. This gives the expression for $\star d \star F$. The last step follows directly as described there.