Exercise 100

We know that

$$\star dz = dx \wedge dy$$

Given that

$$r^2 = x^2 + y^2$$

it follows that

$$2rdr = 2xdx + 2ydy$$

and hence

$$\begin{aligned} 2rdr \wedge d\theta &= \frac{1}{x^2+y^2}(2x^2 dx \wedge dy + 2y^2 dx \wedge dy) \\ &= 2dx \wedge dy \end{aligned}$$

Thus

$$\star dz = rdr \wedge d\theta$$

which implies the desired result.

Exercise 101

Starting with the left side:

$$\begin{aligned} \star d\theta &= \frac{1}{x^2+y^2}(x \star dy + y \star dx) \\ &= \frac{1}{r^2}(xdz \wedge dx - ydy \wedge dx) \end{aligned}$$

and to calculate the right side:

$$\begin{aligned} 2rdz \wedge dr &= dz \wedge (2rdr) \\ &= dz \wedge (2xdx + 2ydy) \\ &= 2xdz \wedge dx - 2ydy \wedge dz \end{aligned}$$

thus the right side is

$$\begin{aligned} \frac{1}{r}dz \wedge dr &= \frac{1}{r^2}(xdz \wedge dx - ydy \wedge dz) \\ &= \star d\theta \end{aligned}$$

Exercise 102

The left side is

$$\begin{aligned} d \star B &= dg(r)d\theta \\ &= \partial_r g(r) dr \wedge d\theta + \partial_z g(r) dz \wedge d\theta \\ &= g'(r) dr \wedge d\theta \end{aligned}$$

since g is independent of z.

The right side is

$$\star j = rf(r)dr \wedge d\theta$$

Thus $d \star B = \star j$ if and only if

$$g'(r) = rf(r)$$

Exercise 103

Firstly, according to the errata, this exercise is not so relevant. But otherwise, the hint describes how to calculate the 1-forms $d\theta_i$, and all that remains is to show that they are closed but not exact.

They are closed because the 1-form $d\theta$ is closed on S^1. They are exact because we can integrate $d\theta_i$ along a loop

$$\gamma_i(s) = (\alpha_1, \ldots, \alpha_{i-1}, s, \alpha_{i+1}, \ldots, \alpha_n)$$

on T^n and the result will not be 0. They are distinct since integrating $d\theta_i$ along the loop γ_j for $j \neq i$ gives 0.

Exercise 104

To show that $dE = 0$ we use only that $e(r)$ is a function of r and not of ϕ or θ. Thus

$$\begin{aligned} dE &= \partial_r e dr \wedge dr + \partial_\phi e d\phi \wedge dr + \partial_\theta e d\theta \wedge dr \\ &= e'(r).0 + 0.d\phi \wedge dr + 0.d\theta \wedge dr \\ &= 0 \end{aligned}$$

To show $d \star E = 0$, we must first determine $\star dr$. First find an orthonormal basis. From the given matrix, it follows that

$$\begin{aligned} \langle dr, dr \rangle &= 1 \\ \langle d\phi, d\phi \rangle &= \frac{1}{f(r)^2} \\ \langle d\theta, d\theta \rangle &= \frac{1}{f(r)^2 \sin^2 \phi} \end{aligned}$$

Hence an orthonormal basis is given by dr, $f(r)d\phi$, and $f(r)\sin\phi d\theta$. And thus

$$\begin{aligned} \star dr &= (f(r)d\phi) \wedge (f(r)\sin\phi d\theta) \\ &= f(r)^2 \sin\phi d\phi \wedge d\theta \\ \star E &= e(r) \star dr \\ &= \frac{q \sin\phi}{4\pi} d\phi \wedge d\theta \end{aligned}$$

and finally

$$
\begin{aligned}
d \star E &= \partial_r \frac{q \sin\phi}{4\pi} dr \wedge d\phi \wedge d\theta \\
&= 0
\end{aligned}
$$

since the coefficient $\frac{q\sin\phi}{4\pi}$ has no dependency on r.

Exercise 105

Define $\phi(r)$ as

$$\int_0^r \frac{-q}{4\pi f(s)^2} ds - \frac{q}{4\pi f(0)^2}$$

if $r \geq 0$, and

$$\int_r^0 \frac{q}{4\pi f(s)^2} ds + \frac{q}{4\pi f(0)^2}$$

if $r < 0$. Then $d\phi = -E$.

Exercise 106

Assume that $|r|$ is large enough so that

$$E = \frac{qdr}{r\pi r^2}$$

Then

$$
\begin{aligned}
\star E &= \frac{q}{4\pi r^2} r^2 \sin\phi d\theta \wedge d\phi \\
&= \frac{q \sin\theta}{4\pi} d\theta \wedge d\phi
\end{aligned}
$$

So

$$
\begin{aligned}
\int_{S^2} \star E &= \frac{q}{4\pi} \int_0^{\pi} \int_0^{2\pi} \sin\phi d\theta d\phi \\
&= \frac{q}{2} \int_0^{2\pi} \sin\phi d\theta d\phi \\
&= \frac{q}{2} (-\cos\phi)|_0^{\pi} \\
&= q
\end{aligned}
$$

Note: on p144 where spherical coordinates is first mentioned, $\sin\phi$ is used instead of $\sin\theta$. So I'm staying with $\sin\phi$ and interpreting ϕ as the angle from the vertical downward as seen from the side, and θ as the horizontal angle as seen from above.

Exercise 107

Given the sphere, with either $r > 0$ or $r < 0$, we want to calculate the integral of the normal component of the electric field over the sphere. To have the same perspective from both sides of the wormhole, the normal component should point outward.

The integral that was calculated in the previous exercise assumes a volume form ω on S^2 such that $dr \wedge \omega$ is the volume form on the whole space. Under the correspondence between the volume form and the normal to the surface, this means that in both cases we were integrating the normal component pointing in the direction of increasing r. For S^2 with $r > 0$ this is indeed the normal component pointing outwards, but for S^2 with $r < 0$ this is the component pointing inwards. Thus in the latter case we should have used a volume form ω such that $-dr \wedge \omega$ is the volume form on the whole space. This amounts to taking the negative of the answer, hence $-q$.

Exercise 108

In higher dimensions, say dimension n, if E is still a 1-form, then $\star E$ is an $(n-1)$-form, so it must be integrated over an $n-1$ dimensional surface. Given that E is chosen to make $\star E$ closed, it implies the space must have H^{n-1} non-zero for an $n-1$ dimensional surface S with $\int_S \star E \neq 0$ to exist.

Exercise 109

$d\omega$ is the sum of three terms. Due to symmetry between the variables, we only need to calculate the first to know how the others will look:

$$\partial_x \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Thus the sum over the three terms is

$$\begin{aligned} d\omega &= \frac{3(x^2+y^2+z^2)-3x^2-3y^2-3z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} \\ &= 0 \end{aligned}$$

Exercise 110

The required $(n-1)$-form is

$$\omega = \frac{x^1 dx^2 \wedge \cdots \wedge dx^n + x^2 dx^3 \wedge \cdots \wedge dx^n \wedge dx^1 + \cdots + x^n dx^1 \wedge \cdots \wedge dx^{n-1}}{((x^1)^2+\cdots+(x^n)^2)^{\frac{n}{2}}}$$

The proof that this form is closed is exactly as in the previous exercise (I know). And it can be shown to be non-exact in a similar way (I hope).

Exercise 111

We showed in exercise 104 that

$$\star dr = f(r)^2 \sin\phi d\theta \wedge d\phi$$

(well, we had $d\phi \wedge d\theta$ instead of $d\theta \wedge d\phi$, but it's not so important here.)

This implies that

$$B = \frac{m \sin\phi}{4\pi} d\theta \wedge d\phi$$

The result

$$\int_{S^2} B = m$$

then follows exactly as in exercise 106 (just with m instead of q).