Exercise 10

For the one direction, suppose $v = w$. Let $p \in M$. Then for all $f \in C^\infty(M)$ we have:

$$\begin{aligned} v_p(f) &= v(f)(p) \\ &= w(f)(p) \\ &= w_p(f) \end{aligned}$$

This implies that $v_p = w_p$.

For the converse, suppose that $v_p = w_p$ for all $p \in M$. Let $f \in C^\infty(M)$. Then for all $p \in M$ we have:

$$\begin{aligned} v(f)(p) &= v_p(f) \\ &= w_p(f) \\ &= w(f)(p) \end{aligned}$$

Which implies that $v(f) = w(f)$ for all $f \in C^\infty(M)$, hence $v = w$.

Exercise 11

Let $u, v, w \in T_p(M)$, and $\alpha, \beta \in \mathbb{R}$. Then

$u + (v + w) = (u + v) + w$

and

$v + w = w + v$

both follow directly from the definition of addition and $\mathbb{R}$ being a commutative group w.r.t. addition.

The zero vector $0 \in T_p(M)$ is defined by requiring that $0(f) = 0$ for all $f \in T_p(M)$. Then $v + 0 = v$ from the definition and 0 being the identity of the additive group $(\mathbb{R}, +)$. For the additive inversion, define $-v$ by $(-v)(f) = -v(f)$. This satisfies the axiom for being an additive inverse again because of the definition of additive inverse in $(\mathbb{R}, +)$.

For the distributive laws:

$$\begin{aligned} (\alpha(v + w))(f) &= \alpha((v + w)(f)) \\ &= \alpha(v(f) + w(f)) \\ &= \alpha v(f) + \alpha w(f) \\ &= (\alpha v)(f) + (\alpha w)(f) \\ &= (\alpha v + \alpha w)(f) \end{aligned}$$

hence $\alpha(v + w) = \alpha v + \alpha w$. And

$$\begin{aligned}((\alpha+\beta)v)(f) &= (\alpha+\beta)v(f)\\ &= \alpha v(f)+\beta v(f)\\ &= (\alpha v)(f)+(\beta v)(f)\\ &= (\alpha v+\beta v)(f)\end{aligned}$$

hence $(\alpha+\beta)v = \alpha v + \beta v$.

Again, $\alpha(\beta v) = (\alpha\beta)v$ follows from the properties of $\mathbb{R}$. And if we define $1 \in T_p(M)$ by requiring $1(f) = 1$ for all $f \in T_p(M)$, then the final property $1v = v$ follows from:

$$\begin{aligned}(1v)(f) &= 1v(f)\\ &= v(f)\end{aligned}$$

which shows that $T_p(M)$ is a vector space over $\mathbb{R}$.

Exercise 12

Let $f, g \in C^\infty(M)$, and $\alpha \in \mathbb{R}$. Then:

1.

$$\begin{aligned}\gamma'(t)(f+g) &= \frac{d}{dt}((f+g)(\gamma(t))\\ &= \frac{d}{dt}(f(\gamma(t))+g(\gamma(t)))\\ &= \frac{d}{dt}(f(\gamma(t))+\frac{d}{dt}(g(\gamma(t))\\ &= \gamma'(t)(f)+\gamma'(t)(g)\end{aligned}$$

2.

$$\begin{aligned}\gamma'(t)(\alpha f) &= \frac{d}{dt}((\alpha f)\gamma(t))\\ &= \frac{d}{dt}(\alpha f(\gamma(t)))\\ &= \alpha\frac{d}{dt}(f(\gamma(t))\\ &= \alpha\gamma'(t)(f)\end{aligned}$$

3.

$$\begin{aligned}\gamma'(t)(fg) &= \frac{d}{dt}((fg)(\gamma(t)))\\ &= \frac{d}{dt}(f(\gamma(t))g(\gamma(t)))\end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{dt} f(\gamma(t)).g(\gamma(t)) + f(\gamma(t)).\frac{d}{dt} g(\gamma(t)) \\
&= \gamma'(t)(f).g(\gamma(t)) + f(\gamma(t)).\gamma'(t)(g)
\end{aligned}
$$

Exercise 13

Let $t \in \mathbb{R}$. Then

$$
\begin{aligned}
(\phi^* x)(t) &= (x \circ \phi)(t) \\
&= x(\phi(t)) \\
&= x(e^t) \\
&= e^t \\
&= e^x(t)
\end{aligned}
$$

thus $\phi^* x = e^x$.

Exercise 14

If a point $(x, y) \in \mathbb{R}^2$ is rotated counterclockwise around the origin by an angle θ, then the resulting point is $(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$. This can be seen by identifying $\mathbb{R}^2$ with $\mathbb{C}$, and multiplying by $e^{i\theta}$ (thank you wikipedia).

$\phi^* x = x \circ \phi$. This is just the first component of the above vector, which is what we need to show. And similar for $\phi^* y$.

Exercise 15

First consider smooth functions $f : M \to \mathbb{R}$.

For the one direction, let $f : M \to \mathbb{R}$ be any function such that $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth for all α, where $\{(U_\alpha, \phi_\alpha)\}$ is a family of charts on M, in other words assume that f is smooth according to the original definition. Let $g \in C^\infty(\mathbb{R})$. Then because the composition of two smooth functions is again smooth, we have that $g \circ (f \circ \phi_\alpha^{-1}) = (g \circ f) \circ \phi_\alpha^{-1}$is smooth for all α. By the original definition, this implies that $g \circ f$ is smooth. Thus f is smooth according to the new definition.

For the converse, assume that f is smooth according to the new definition. In other words, for any $g \in C^\infty(\mathbb{R})$, we have that $g \circ f \in C^\infty(M)$. Take $g = id_{\mathbb{R}}$, the identity map on $\mathbb{R}$. Then it follows that $f \in C^\infty(M)$, which is the old definition.

Next consider smooth curves, $\gamma : \mathbb{R} \to M$. In this case the two definitions (on pages 29 and 32) are exactly the same.