Gauge fields, knots and gravity - Baez and Muniain

Exercise 1

Note that there is an error in the statement of the problem. It should read $\vec{k} \times \vec{E} = -i\omega \vec{E}$.

Let $\vec{E} = \langle E_1, E_2, E_3 \rangle$, and write

$$\vec{\mathcal{E}}(t,\vec{x}) = \left\langle E_1 e^{-i(\omega t - \vec{k}\cdot\vec{x})}, E_2 e^{-i(\omega t - \vec{k}\cdot\vec{x})}, E_3 e^{-i(\omega t - \vec{k}\cdot\vec{x})} \right\rangle$$

$$= \langle \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \rangle$$

Then $\dfrac{\partial \mathcal{E}_1}{\partial x^1} + \dfrac{\partial \mathcal{E}_2}{\partial x^2} + \dfrac{\partial \mathcal{E}_3}{\partial x^3}$

$$= e^{-i(\omega t - \vec{k}\cdot\vec{x})} \left(k^1 E_1 + k^2 E_2 + k^3 E_3 \right)$$

$$= e^{-i(\omega t - \vec{k}\cdot\vec{x})} \left(\vec{k} \cdot \vec{E} \right)$$

$= 0$, so that $\nabla \cdot \vec{\mathcal{E}} = 0$ as required

$$\nabla \times \vec{\mathcal{E}} = \left\langle \frac{\partial \mathcal{E}_3}{\partial x^2} - \frac{\partial \mathcal{E}_2}{\partial x^3}, \frac{\partial \mathcal{E}_3}{\partial x^1} - \frac{\partial \mathcal{E}_1}{\partial x^3}, \frac{\partial \mathcal{E}_2}{\partial x^1} - \frac{\partial \mathcal{E}_1}{\partial x^2} \right\rangle$$

$$= i e^{-i(\omega t - \vec{k}\cdot\vec{x})} \left\langle k^2 E_3 - k^3 E_2, k^1 E_3 - k^3 E_1, k^1 E_2 - k^2 E_1 \right\rangle$$

$$= i e^{-i(\omega t - \vec{k}\cdot\vec{x})} \vec{k} \times \vec{E}$$

$$= \omega e^{-i(\omega t - \vec{k}\cdot\vec{x})} \vec{E}$$

$$i \frac{\partial \vec{\mathcal{E}}}{\partial t} = i(-i\omega) e^{-i(\omega t - \vec{k}\cdot\vec{x})} \vec{E} = \omega e^{-i(\omega t - \vec{k}\cdot\vec{x})} \vec{E}$$