$$(1v)(h) = 1\,v(h)$$
$$= v(h)$$

so $\quad 1v = v.$

## Exercise 9

Let $f^i$ denote the function on $\mathbb{R}^n$ that projects onto the $i^{th}$ coordinate. That is

$$f^i(x^1, \dots, x^n) = x^i$$

Then $\quad \partial_j(f^i) = \delta^i_j \quad$ where $\quad \delta^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

Thus we have

$$0 = v^\mu \partial_\mu f^i = v^1 \partial_1 f^i + \dots + v^i \partial_i f^i + \dots + v^n \partial_n f^n$$
$$= 0 + \dots + v^i 1 + 0 + \dots + 0$$
$$= v^i 1$$

So $\quad v^\mu = 0 \quad$ for all $\mu$.

## Exercise 10

First, let $v = w$. Thus $v$ and $w$ produce the same functions when acting on functions. That is, for any $f \in C^\infty(M)$,

$v(f) = w(f)$. But these are both functions, and functions are equal iff they agree on every point in their domain.

Thus $v(f)(p) = w(f)(p) \quad \forall p \in M$

$\therefore \; v_p(f) = w_p(f) \quad \forall p \in M.$

Conversely, the collection $\{v_p\}_{p \in M}$ and $\{w_p\}_{p \in M}$ define vector fields $v$ and $w$ respectively via

$v(f)(p) = v_p(f)$ for any $f \in C^\infty(M)$.

and $w(f)(p) = w_p(f)$

Thus $w$ and $v$ are equal.

## Exercise 11

By exercise 7, tangent vectors are closed under addition and scalar multiplication. Associativity and commutativity of addition is immediate since the reals have these properties. Further, the function sending every element of $M$ to $0$ is the identity. Indeed, $v_p(f) + 0_p(f) = v_p(f) + 0$

$$= v_p(f)$$

$$= 0 + v_p(f)$$

$$= 0_p(f) + v_p(f).$$

Next, if $v_p \in T_pM$, we may define another tangent vector $(-v)_p : C^\infty(M) \to \mathbb{R}$ by $(-v)_p(f) = -v_p(f)$.

Then $v_p f + (-v)_p f = v_p f - v_p f = 0 = -v_p f + v_p f$
$= (-v)_p f + v_p f$

The final vector space axioms follow by Exercise 8.

Exercise 12

Let $f, g \in C^\infty(M)$, and $\gamma : \mathbb{R} \to M$ a curve in $M$.

Then $\frac{d}{dt}(f+g)(\gamma(t))$

$$= \frac{d}{dt}\left( f(\gamma(t)) + g(\gamma(t))\right)$$

$$= \frac{d}{dt} f(\gamma(t)) + \frac{d}{dt} g(\gamma(t))$$ since differentiation distributes over addition.

If $\alpha \in \mathbb{R}$, then

$$\frac{d}{dt} \alpha f(\gamma(t)) = \alpha \frac{d}{dt} f(\gamma(t))$$ again

follows trivially by the properties of differentiation.

Lastly, by the product rule,

$$\frac{d}{dt}(fg)\circ(\gamma(t)) =$$

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Then $v_p f + (-v)_p f = v_p f - v_p f = 0 = -v_p f + v_p f$

$$= (-v)_p f + v_p f$$

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follows trivially by the properties of differentiation.

Lastly, by the product rule,

$$\frac{d}{dt}(fg)\circ(\gamma(t)) =$$

$$f \frac{d}{dt} g(\gamma(t)) + g \frac{d}{dt} f(\gamma(t))$$

Exercise 13

$$\phi^* x(t) := (x \circ \phi)(t) = x(\phi(t))$$
$$= x(e^t)$$
$$= e^t$$

Also, $e^{x(t)} = e^t$

Thus $\phi^* x = e^x$

Exercise 14

Let $(x,y) \in \mathbb{R}^2$. Then there exist numbers $r$ and $\alpha$ such that $x = r\cos\alpha$ and $y = r\sin\alpha$.

Hence
$$\phi(x,y) = \phi(r\cos\alpha, r\sin\alpha) = (r\cos(\alpha+\theta), r\sin(\alpha+\theta))$$
$$= (r\cos\alpha\cos\theta - r\sin\alpha\sin\theta, \; r\sin\alpha\cos\theta + r\cos\alpha\sin\theta)$$
$$= (x\cos\theta - y\sin\theta, \; y\cos\theta + x\sin\theta)$$

Thus $\phi^* x = x\cos\theta - y\sin\theta$

and $\phi^* y = x\sin\theta + y\cos\theta$.

Exercise 15

Consider $\mathbb{R}$ a manifold with its global chart.
Let $f: M \to \mathbb{R}$ be smooth in the "new" definition.

Then, for any smooth function on $\mathbb{R}$, the pullback along $f$ is smooth on $M$ using the "old" definition.
In particular, the identity function is smooth on $\mathbb{R}$.
But then $f^* \mathrm{id}_{\mathbb{R}} = f$, so that $f$ is also smooth using the old definition.

Conversely, suppose $f$ is smooth using the old definition.
Thus, for any chart $(U_\alpha, \varphi_\alpha)$ of $M$,
$f \circ \varphi_\alpha^{-1}$ is smooth ~~in~~ in the ordinary calculus definition.

Let $g \in C^\infty(\mathbb{R})$.

~~Then~~ We want $f^* g$ smooth on $M$.

Thus, we want ~~[illegible]~~ $(f^* g) \circ \varphi_\alpha^{-1}$ smooth for any chart $(U_\alpha, \varphi_\alpha)$ of $M$.

But $$(f^* g) \circ \varphi_\alpha^{-1} = (g \circ f) \circ \varphi_\alpha^{-1}$$
$$= g \circ (f \circ \varphi_\alpha^{-1})$$ which is smooth

since it is the composition of two smooth maps.

The argument for $\gamma: \mathbb{R} \to M$ is similar.