Exercise 2

First, assume f is continuous according to the topological definition. Let $\vec{x} \in \mathbb{R}^n$, and let $\varepsilon > 0$ be a real number. Consider the ε-neighbourhood $N_\varepsilon(f(\vec{x}))$ around $f(\vec{x})$. By the continuity of f, $f^{-1}\left(N_\varepsilon(f(\vec{x}))\right)$ is open in $\mathbb{R}^n$. This open set contains a neighbourhood about each of its points. Thus there exists a $\delta > 0$ such that

$N_\delta(\vec{x}) \subseteq f^{-1}\left(N_\varepsilon(f(\vec{x}))\right)$. That is,

$f\left(N_\delta(\vec{x})\right) \subseteq N_\varepsilon\left(f(\vec{x})\right)$. This δ thus satisfies the condition, and f is continuous at $\vec{x}$ in the "ordinary" sense. Since $\vec{x}$ was arbitrarily chosen, f is continuous on $\mathbb{R}^n$ via the ε-δ definition.

Conversely, suppose that f is continuous via the epsilon-delta definition on $\mathbb{R}^n$. Let $V \subseteq \mathbb{R}^m$ be open. Let $U = f^{-1}(V)$. By hypothesis f is continuous at every $\vec{x} \in U$. Thus there is a $\delta_{\vec{x}} > 0$ such that

$f\left(N_{\delta_{\vec{x}}}(\vec{x})\right) \subseteq V$. Thus $N_{\delta_{\vec{x}}}(\vec{x}) \subseteq U$, so that

$$U = \bigcup_{\vec{x} \in U} N_{\delta_{\vec{x}}}(\vec{x}).$$

Thus U is open.

Exercise 3

Firstly, S^n inherits the Hausdorff property from $\mathbb{R}^{n+1}$.

Define the following sets:

$$U^j_+ := \{ \vec{x} \in S^n \mid x^j > 0 \}$$

and

$$U^j_- := \{ \vec{x} \in S^n \mid x^j < 0 \}.$$

The collection $\{ U^j_{+/-} \}_j$ thus forms an open cover of S^n.

Define the chart maps $\phi^j_{+/-} : U^j_{+/-} \to \mathbb{R}^n$ by

$$\phi^j_{+/-}(\vec{x}) = (x^1, \dots, \hat{x}^j, \dots, x^{n+1})$$

where the "^" means that the j^{th} coordinate is omitted.

Since this is a projection map, it is continuous.

The inverse map $\left(\phi^j_{+/-}\right)^{-1} : \mathbb{R}^n \to U^j_{+/-}$ is given by

$$\left(\phi^j_{+/-}\right)^{-1}(x^1, \dots, x^n) = \left(x^1, \dots, x^{j-1}, \pm\sqrt{1-(x^1)^2-\dots-(x^n)^2}, x^j, \dots, x^n\right)$$

This map is algebraic in every component, and so is also continuous (indeed, it is differentiable).

Thus $\phi^j_{+/-}$ is a homeomorphism onto the open disk in $\mathbb{R}^n$.

Finally, since the transition maps, being a composition of an algebraic function and a projection, it is itself also algebraic in each component. Thus it is smooth. Thus S^n is an n-dimensional manifold.

Exercise 4

If $\mathcal{A} = \{(U_i, \phi_i)\}_i$ is an atlas for M, then

$\mathcal{A}' = \{(U \cap U_i, \phi_i')\}_i$ is clearly an atlas for U,

where $\phi_i' = \phi_i|_{U \cap U_i}$

Since $U \cap U_i$ is open, $\phi_i' \circ (\phi_j')^{-1}$ will be smooth as well for any charts $(U \cap U_i, \phi_i')$ and $(U \cap U_j, \phi_j')$ in $\mathcal{A}'$. Hence U is also a manifold.

Exercise 5

If $\mathcal{A} = \{(U_i, \phi_i)\}_i$ is an atlas for X and

$\mathcal{B} = \{(V_j, \psi_j)\}_j$ is an atlas for Y, let

$\mathcal{C} = \{(U_i \times V_j, \phi_i \times \psi_j)\}_{i \times j}$. ~~be an~~ We show that $\mathcal{C}$ is a valid atlas for $X \times Y$.

In the definition of $\mathcal{C}$, $U_i \times V_j$ is the ordinary cartesian product. By $\phi_i \times \psi_j$ we denote the map from $X \times Y \to \mathbb{R}^{m+n}$ defined by

$\phi_i \times \psi_j\,(\vec{x},\vec{y}) = (\phi_i(\vec{x}), \psi_j(\vec{y}))$ where $\vec{x} \in \mathbb{R}^m$ and $\vec{y} \in \mathbb{R}^n$.

The charts certainly form an open cover for $X \times Y$. It remains therefore to check that the transition maps are smooth.

But $(\phi_i \times \psi_j) \circ (\phi_k \times \psi_\ell)^{-1} (\vec{x},\vec{y})$

$= (\phi_i \circ \phi_k^{-1}(\vec{x}), \psi_j \circ \psi_\ell^{-1}(\vec{y}))$ which is smooth since the component functions are smooth by definition. Thus $X \times Y$ is an $(m+n)$-dimensional manifold.

Exercise 6

As before, let $A = \{(U_i, \phi_i)\}_i$ be an atlas for X and $B = \{(V_j, \psi_j)\}_j$ be an atlas for Y.

Then $A \cup B$ is trivially an atlas for $X \cup Y$ since the collection of U_i's and V_j's cover X and Y respectively, and since $U_i \cap V_j = \phi$ for any such charts, the transition functions only exist on X or Y seperately, hence being smooth by definition. Thus $X \cup Y$ is an n-dimensional manifold

Exercise 7

We check that $v+w$ satisfies the three defining properties of a vector field.

Let $f, g \in C^{\infty}(M)$, and α a scalar.

Then
$$
\begin{aligned}
(v+w)(f+g) &= v(f+g) + w(f+g) \\
&= v(f) + v(g) + w(f) + w(g) \\
&= v(f) + w(f) + v(g) + w(g) \\
&= (v+w)(f) + (v+w)(g)
\end{aligned}
$$

Further,
$$
\begin{aligned}
(v+w)(\alpha f) &= v(\alpha f) + w(\alpha f) \\
&= \alpha v(f) + \alpha w(f) \\
&= \alpha \big(v(f) + w(f) \big) \\
&= \alpha \big((v+w)(f) \big).
\end{aligned}
$$

Lastly,
$$
\begin{aligned}
(v+w)(fg) &= v(fg) + w(fg) \\
&= f v(g) + v(f) g + f w(g) + w(f) g. \\
&= f \big(v(g) + w(g) \big) + \big(v(f) + w(f) \big) g \\
&= f (v+w)(g) + (v+w)(f) g
\end{aligned}
$$

Hence $v+w \in \mathrm{Vect}(M)$.

Similarly, we check that gv satisfies the conditions.

Let $f, h \in C^{\infty}(M)$, and let α be a scalar.

$$
\begin{aligned}
(gv)(f+h) = g v(f+h) &= g \big(v(f) + v(h) \big) \\
&= g v(f) + g v(h) \\
&= (gv)(f) + (gv)(h)
\end{aligned}
$$

$$
\begin{aligned}
(gv)(\alpha f) &= g\, v(\alpha f) \\
&= g\,(\alpha\, v(f)) \\
&= \alpha\,(g\, v(f)) \\
&= \alpha\,(gv)(f)
\end{aligned}
$$

$$
\begin{aligned}
(gv)(fh) &= g\, v(fh) \\
&= g\,\big(v(f)h + f\,v(h)\big) \\
&= g\, v(f) h + gf\, v(h) \\
&= (gv)(f)h + f(gv)(h)
\end{aligned}
$$

Thus $(gv) \in \mathrm{Vect}(M)$.

Exercise 8

Let $h \in C^\infty(M)$

Then
$$
\begin{aligned}
f(v+w)(h) &= f\big(v(h) + w(h)\big) \\
&= f\,v(h) + f\,w(h) \\
&= (fv)(h) + (fw)(h)
\end{aligned}
$$

so that $f(v+w) = fv + fw$.

$$
\begin{aligned}
(f+g)v(h) &= f\,v(h) + g\,v(h) \\
&= (fv)(h) + (gv)(h)
\end{aligned}
$$

so that $(f+g)v = fv + gv$

$$
\begin{aligned}
f(gv)(h) &= f\big(g\,v(h)\big) \\
&= (fg)\,v(h) \\
&= \big((fg)v\big)(h)
\end{aligned}
$$

so that $f(gv) = (fg)v$.