Baez and Munian

Exercises 7–9

7.

$$
\begin{aligned}
(v+w)(f+g) &= v(f+g) + w(f+g) \\
&= v(f) + v(g) + w(f) + w(g) \\
&= v(f) + w(f) + v(g) + w(g) \\
&= (v+w)(f) + (v+w)(g).
\end{aligned}
$$

$$
\begin{aligned}
(v+w)(\alpha f) &= v(\alpha f) + w(\alpha f) \\
&= \alpha\,(v(f) + w(f)) \\
&= \alpha\,(v+w)(f)
\end{aligned}
$$

$$
\begin{aligned}
(v+w)(fg) &= v(fg) + w(fg) \\
&= v(f)g + fv(g) + w(f)g + fw(g) \\
&= v(f)g + w(f)g + fv(g) + fw(g) \\
&= (v+w)(f)g + f(v+w)(g).
\end{aligned}
$$

$$
\begin{aligned}
(gw)(f+h) &= g\cdot(w(f) + w(h)) \\
&= gw(f) + gw(h) \\
&= (gw)(f) + (gw)(h).
\end{aligned}
$$

$$
\begin{aligned}
(gw)(\alpha f) &= g\cdot w(\alpha f) \\
&= g\cdot \alpha w(f) \\
&= \alpha\,(gw)(f).
\end{aligned}
$$

$$(gw)(f \cdot h) = g[w(f)h + fw(h)]$$
$$= gw(f)h + fgw(h)$$
$$= [gw(f)] \cdot h + f[(gw)(h)]$$

8. $$[f(v+w)](g) = f[(v+w)(g)]$$
$$= f[v(g) + w(g)]$$
$$= fv(g) + fw(g)$$
$$= (fv + fw)(g)$$

$\therefore \; f(v+w) = fv + fw$.

$$[(f+g)v](h) = (f+g)v(h)$$
$$= fv(h) + gv(h)$$
$$= [fv + gv](h)$$

$\therefore \; (f+g)v = fv + gv$

$$[(fg)(v)][h] = fg\,v(h)$$
$$= f \cdot (gv(h))$$
$$= f \cdot [(gv)(h)]$$
$$= [f \cdot (gv)](h)$$

$\therefore \; (fg)(v) = f \cdot (gv)$

$$(1v)(f) = 1 . v(f)$$
$$= v(f)$$

$\therefore \; 1v = v$.

9. We have the projection maps

$$f_\nu : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(x_1, \dots, x_n) \longmapsto x_\nu$$

which are certainly smooth. Hence

$$V^\mu \underbrace{\partial_\mu f_\nu}_{\delta_{\mu\nu}} = 0 \quad \Rightarrow \quad V^\nu = 0 \quad \text{for } \nu = 1, \dots, n.$$