Baez and Muniain

Exercises 86 - 89

86. We have defined

$$\int_M \omega := \sum_\alpha \int_{\mathbb{R}^n} (\phi_\alpha^{-1})^*(\rho_\alpha \omega)$$

where (U_α, ϕ_α) is an oriented atlas of M and ρ_α is a partition of unity subordinate to (U_α).

Suppose (V_α, ψ_α) is another oriented atlas of M and χ_α is a partition of unity subordinate to (V_α). Then

$$\sum_\alpha \int_{\mathbb{R}^n} (\phi_\alpha^{-1})^*(\rho_\alpha \omega) = \sum_\alpha \int_{\mathbb{R}^n} (\phi_\alpha^{-1})^*\left(\sum_\beta \chi_\beta \rho_\alpha \omega\right) \quad \left[\text{since } \sum_\beta \chi_\beta = 1\right]$$

$$= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} (\phi_\alpha^{-1})^*(\chi_\beta \rho_\alpha \omega) \quad [\text{integration is linear}]$$

$$= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} (\psi_\beta^{-1})^*(\chi_\beta \rho_\alpha \omega)$$

[since, firstly, supp $\chi_\beta \rho_\alpha \omega$ lies in $U_\alpha \cap V_\beta$, and secondly we have shown in the book that integration over a single oriented chart is independent of the coordinates used]

$$= \sum_\beta \int_{\mathbb{R}^n} (\psi_\beta^{-1})^*\left(\sum_\alpha \rho_\alpha \chi_\beta \omega\right)$$

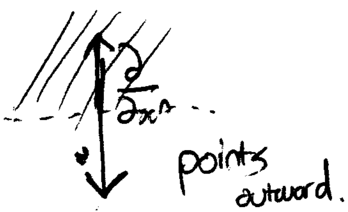
$$= \sum_\beta \int_{\mathbb{R}^n} (\psi_\beta^{-1})^*(\chi_\beta \omega).$$

$\square$

§7. It is clear that ∂D^n, with the charts from exercise 84, in fact equals $S^{n-1} := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 = 1\}$, as a smooth manifold. What we need to check is that their orientations agree. Baez + Muniain were sloppy in their definitions of the orientation on ∂M; here is mine:

$$\boxed{\begin{gathered}\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}}\right) \text{ is an oriented basis for } T_p \partial M \\ \Longleftrightarrow \\ \left(\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}}\right) \text{ is an oriented basis for } T_p M.\end{gathered}}$$

where $\frac{\partial}{\partial x^n}$ points ~~toward~~ outward:



By this definition,

$(V_1, \dots, V_n)$ is an oriented basis for ∂D^n if

$(p, V_1, \dots, V_n)$ is oriented basis for D^n.

Also, the standard definition of the orientation on S^{n-1} is

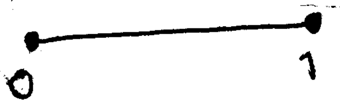
$(V_1, \dots, V_n)$ is an oriented basis for S^{n-1} if

$(p, V_1, \dots, V_n)$ is an oriented basis of $\mathbb{R}^n$.

Since the orientation on D^n is the orientation on the ambient $\mathbb{R}^n$, we see that the orientations on ∂D^n and S^{n-1} agree.

88. $\displaystyle\int_0^1 f'(x)\,dx = \int_{[0,1]} df = \int_{\partial[0,1]} f$ [Stokes]

$= f(1) - f(0)$ [using my defn. of orientation on ∂M above, and the definiti[on] of integration on a zero-[dimensional] oriented manifold, namely sig[ned] addition].



89. In this case, Stokes' theorem would say

note mistake in question

$$\int_0^\infty f'(x)\,dx = \int_{[0,\infty)} df$$

$$= \int_{\partial[0,\infty)} f = -f(0).$$ (using my convention for orientation on ∂M)