Baez and Munian

Exercises 69 - 71

69. If

$$\omega = \frac{1}{p!}\sum_{i_1,\dots,i_p} \omega_{i_1\cdots i_p}\, e^{i_1}\wedge\cdots\wedge e^{i_p}$$

then we know from the explicit ~~formula~~ that

$$*\omega = \frac{1}{p!}\sum_{i_1,\dots,i_p} \omega_{i_1\cdots i_p}\, \epsilon(i_1)\cdots\epsilon(i_p)\, \mathrm{sign}(i_1,\dots,i_p,i_{p+1},\dots,i_n)\, e^{i_{p+1}}\wedge\cdots\wedge e^{i_n}$$

Here, $(i_{p+1},\dots,i_n)$ was a particular (any choice will do) choice of ordering for the complement variables $\{1,\dots,n\} - \{i_1,\dots,i_n\}$. So we can replace the sum above by

$$*\omega = \frac{1}{p!}\frac{1}{(n-p)!}\sum_{i_1,\dots,i_n} \omega_{i_1\cdots i_p}\, \epsilon(i_1)\cdots\epsilon(i_p)\, \mathrm{sign}(i_1,\dots,i_n)\, e^{i_{p+1}}\wedge\cdots\wedge e^{i_n} \quad \cdots (A)$$

where we must divide by $(n-p)!$. ~~Thus~~

On the other hand, we are trying to show that

$$*\omega \overset{?}{=} \frac{1}{(n-p)!}\frac{1}{p!}\sum_{j_1,\dots,j_{n-p}} \epsilon^{i_1\cdots i_p}{}_{j_1\cdots j_{n-p}}\, \omega_{i_1\cdots i_p}\, e^{j_1}\wedge\cdots\wedge e^{j_{n-p}} \quad \cdots (B)$$

Now,

$$\epsilon^{i_1\cdots i_p}{}_{j_1\cdots j_{n-p}} = \epsilon(i_1)\cdots\epsilon(i_p)\, \mathrm{sign}(i_1,\dots,i_p,j_1,\dots,j_{n-p})$$

where we are intending that $\mathrm{sign}(i_1,\cdots,i_p,j_1,\cdots,j_{n-p}) := 0$ if $(i_1,\cdots,i_p,j_1,\cdots,j_{n-p})$ is not a permutation. With this understanding, we can rewrite (B) as

$$(B) = \frac{1}{(n-p)!}\frac{1}{p!}\sum_{i_1,\cdots,i_n} \epsilon(i_1)\cdots\epsilon(i_p)\,\mathrm{sign}(i_1,\cdots,i_p,i_{p+1},\cdots,i_n)\,\omega_{i_1\cdots i_p}\, e^{i_{p+1}}\wedge\cdots\wedge e^{i_n}$$

So that we see that (A) = (B).

70. We know from exercise 66 that $*_s d_s *_s E$ calculates the divergence of the vector field $\vec{E}$ corresponding to the one-form E, so we get out $\nabla\cdot\vec{E} = \rho$.

Also, we know that $*_s B$ is the 1-form corresponding to the vector field $\vec{B}$, and also that $*_s d_s$ essentially calculates the curl on 1-forms, so

$$-\partial_t E + *_s d_s *_s B$$

corresponds to

$$-\partial_t \vec{E} + \nabla\times\vec{B}.$$

71. Firstly, if $F = E\wedge dt + B$, then

$$*\, E\wedge dt = *_s E$$

since $*\, dx\wedge dt = dy\wedge dz$, $*\, dy\wedge dt = dz\wedge dx$ and $*\, dz\wedge dt = dx\wedge dy$.

Also,

$$*B = -*_s B \wedge dt$$

since $\quad *dx = -dy \wedge dz \wedge dt \quad$, etc.

Thus $\quad *F = *_s E - *_s B \wedge dt.$

Taking "d" of this equation, we find that

$$d*_s E = \underbrace{dt \wedge \partial_t *_s E}_{} + d_s *_s E$$

$$= \partial_t *_s E \wedge dt \quad (\text{since } *_s E \text{ is a 2-form})$$

$$= *_s \partial_t E \wedge dt \quad (*_s \text{ is linear})$$

and also that

$$d*_s B \wedge dt = d_s *_s B \wedge dt$$

since the dt part of the "d" operation must vanish. Hence

we arrive at

$$d*F = *_s \partial_t E \wedge dt + d_s *_s E - d_s *_s B \wedge dt.$$

To take the $*$ of the RHS, we must calculate

$$*\underbrace{*_s \partial_t E}_{\substack{\text{a 2-form containing} \\ dx\wedge dy \text{ terms etc.}}} \wedge dt = -\partial_t E$$

since $\quad * dx \wedge dy \wedge dt = -dz \quad$, etc.

Also,

$$*\underbrace{d_s *_s E}_{\substack{\text{a 3-form} \\ \text{on space}}} = -*_s d_s *_s E \wedge dt$$

since $\quad * dx \wedge dy \wedge dz = -dt.$

Finally,

$$\underbrace{* d_s *_s B}_{\text{a 2-form on space}} \wedge dt = - *_s d_s *_s B$$

since $\quad * dx \wedge dy \wedge dt = -dz$, etc.

Thus we arrive at

$$*d*F = \underbrace{-\partial_t E}_{\text{terms in } dx, dy, dz} - \underbrace{(*_s d_s *_s E) \wedge dt}_{\text{a } dt \text{ term}} + \underbrace{*_s d_s *_s B}_{\text{terms in } dx, dy, dz}$$

which we set equal to $\mathring{j} - \rho\, dt$, thus arriving at

$$-\partial_t E + *_s d_s *_s B = \mathring{j}$$
$$*_s d_s *_s E = \rho \, .$$