Baez and Munian

Exercises 52 - 56

52. (The vector space V has to be finite here, otherwise this is false. For instance, if V is the vector space of finite sequences of real numbers

$$a = (a_1, a_2, \cdots, a_n, 0, 0, 0, \cdots) ,$$

Then we have a natural metric,

$$(a,b) = \sum_i a_i b_i \qquad \text{(the sum is finite)}$$

But clearly the functional which sends

$$a \mapsto \sum_i a_i \qquad \text{(the sum is finite)}$$

lives in V^*, but it isn't of the form $(b, \cdot)$ for some b

The map $v \mapsto g(v, \cdot)$ is one-to-one, since its kern is zero (if $g(v,w) = 0$ for all w then $v = 0$ by defn of nondeg
Hence we have a one-to-one map from a finite dimensional vector space to itself, so it is also surjective.

53. We have

$$\begin{aligned} g(v^\mu e_\mu, e_\nu) &= v^\mu g(e_\mu, e_\nu) \\ &= v^\mu g_{\mu\nu} \end{aligned}$$

While

$$V_\alpha f^\alpha(e_\nu) = g_{\beta\alpha} V^\beta \underbrace{f^\alpha(e_\nu)}_{\delta^\alpha_\nu}$$

$$= g_{\beta\nu} V^\beta$$

or
in other words $= V^\mu g_{\mu\nu}$

So $g(V^\mu e_\mu, \cdot) = V_\nu f^\nu$ since they have the same action on all vector fields.

54. We must show that if we convert $\omega^\nu e_\nu$ into a one-form using the metric, then we recover ω. Indeed, from exercise 53, we know that if we convert $\omega^\nu e_\nu$ into a 1-form, it will have componets

a W, not an ω $\longrightarrow$ $W_\nu = g_{\mu\nu}\,\omega^\mu$

$$= \underbrace{g_{\mu\nu}\, g^{\alpha\mu}}\,\omega_\alpha$$

$$= \delta^\alpha_\nu$$

$$= \omega_\nu$$

In other words, we do indeed recover ω.

55. Obvious.

56. $g^\mu_\nu = g^{\mu\alpha} g_{\alpha\nu} = \left([g]^{-1}[g]\right)_{\mu,\nu} = \delta^\mu_\nu$.