Baez + Munian

Exercise 50 , 24/04/2010

50. Uniqueness: Suppose that $F = B + E \wedge dt$ and also that $F = B' + E' \wedge dt$ such that in any chart x^i on S we have

$$B = \tfrac{1}{2} B_{ij}\, dx^i \wedge dx^j \ , \quad E = E_i\, dx^i$$

$$B' = \tfrac{1}{2} B'_{ij}\, dx^i \wedge dx^j \ , \quad E' = E'_i\, dx^i$$

~~But~~ So locally, we can write

$$F = \tfrac{1}{2} B_{ij}\, dx^i \wedge dx^j + E_i\, dx^i \wedge dt = \tfrac{1}{2} B'_{ij}\, dx^i \wedge dx^j + E'_i\, dx^i \wedge dt$$

Now the forms $\{dx^i \wedge dx^j, dx^i \wedge dt\}$ are linearly independent (in fact they span the full 2-form space at each point) so we must have

$$B_{ij} = B'_{ij} \quad \text{and} \quad E_i = E'_i .$$

Existence: Suppose F is a 2-form on $\mathbb{R} \times S$. We must construct a 2-form B on $\mathbb{R} \times S$ and a 1-form E on $\mathbb{R} \times S$ such that $F = B + E \wedge dt$ and such that B and E look locally wrt a chart (x^i, t) on $\mathbb{R} \times S$ like

$$E = E_i\, dx^i \ , \quad B = \tfrac{1}{2} B_{ij}\, dx^i \wedge dx^j .$$

Indeed, in such a chart we can write F as

$$F = \tfrac{1}{2} B_{ij}\, dx^i \wedge dx^j + E_i\, dx^i \wedge dt$$

Since at each point $(t,p) \in \mathbb{R}\times S$, the wedge products

$$\{ (dx^i)_p \wedge (dx^j)_p \, , \, (dx^i)_p \wedge dt \}_{i,j=1 \ldots \dim S}$$

span $\Lambda^2 T^*_{(t,p)}(\mathbb{R}\times S)$. Thus we have locally defined the 2-form B and the 1-form E. And since F patches up to give a globally well-defined 2-form, so must B and E.