Baez and Munian

Exercises 46 – 49

46. We can write ω and μ as a linear combination of wedge products of p and q vectors respectively:

$$\omega = \omega_1 V_{11} \wedge \cdots \wedge V_{1p} + \omega_2 V_{21} \wedge \cdots \wedge V_{2p} + \cdots + \omega_n V_{n1} \wedge \cdots \wedge V_{np}$$

$$\mu = \mu_1 U_{11} \wedge \cdots \wedge U_{1q} + \mu_2 U_{21} \wedge \cdots \wedge U_{2q} + \cdots + \mu_m U_{m1} \wedge \cdots \wedge U_{mq}$$

So $\omega \wedge \mu$ will consist of linear combinations of terms of the form

$$\underbrace{V \wedge \cdots \wedge V}_{p} \wedge \underbrace{U \wedge \cdots \wedge U}_{q}$$

To put all the "U"'s in front costs $(-1)^{pq}$ minus signs (a factor of $(-1)^p$ for each U).

47. We only need to define ϕ^* on wedge products (we will then extend to linear combinations of wedge products by linearity). So we set

$$\phi^*(\omega_1 \wedge \cdots \wedge \omega_p) := \phi^*(\omega_1) \wedge \cdots \wedge \phi^*(\omega_p)$$

Clearly this definition satisfies the properties listed in in the exercise. This proves <u>existence</u>. To prove <u>uniqueness</u>, observe that ϕ^* must behave as above on wedge products, since that is demanded by the requirement $\phi^*(\omega \wedge \mu) = \phi^*\omega \wedge \phi^*\mu$. But then the properties

$$\phi^*(\omega+\mu) = \phi^*\omega + \phi^*\mu \quad \text{and} \quad \phi^*(\alpha\omega) = \alpha\,\phi^*(\omega)$$

determines ϕ^* uniquely on <u>all</u> forms.

48. Firstly, note that

$$P^*(dx^\mu) = -dx^\mu$$

since by definition, $P^*(dx^\mu)\left(\frac{\partial}{\partial x^\nu}\right) = dx^\mu\left(P_*\left(\frac{\partial}{\partial x^\nu}\right)\right) = dx^\mu\left(-\frac{\partial}{\partial x^\nu}\right)$

$$= -\delta^\mu_\nu .$$

Hence we have

$$P^*(\omega_\mu dx^\mu) = \omega_\mu P^*(dx^\mu) \qquad \text{[from exercise 47]}$$

$$= -\omega_\mu dx^\mu .$$

so P changes the sign of 1-forms. Similarly,

$$P^*\left(\tfrac{1}{2}\omega_{\mu\nu}\, dx^\mu \wedge dx^\nu\right) = \tfrac{1}{2}\omega_{\mu\nu}\, P^*(dx^\mu) \wedge P^*(dx^\nu)$$

[exercise 47]

$$= \tfrac{1}{2}\omega_{\mu\nu}\,(-dx^\mu)\wedge(-dx^\nu)$$

$$= \tfrac{1}{2}\omega_{\mu\nu}\, dx^\mu \wedge dx^\nu .$$

So parity P does <u>not</u> change the sign of a 2-form. (In physics terminology, 2-forms are hence "pseudovectors", eg. the magnetic field).

49. $$d(\omega_\mu dx^\mu) = d\omega_\mu \wedge dx^\mu + \omega_\mu \, ddx^\mu$$

[Leibniz]

$$= \partial_\nu \omega_\mu \, dx^\nu \wedge dx^\mu + 0 \, .$$

[differential of function, and $d^2 = 0$]