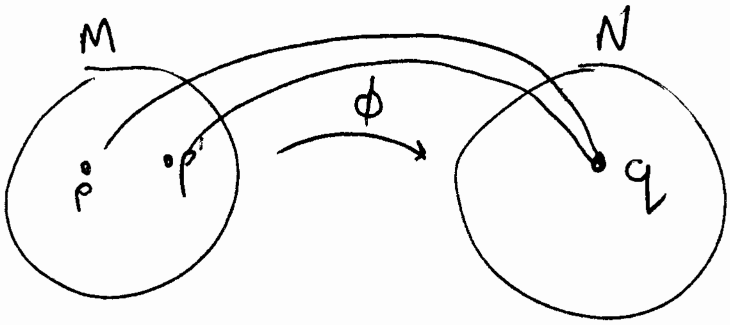
# Baez and Munian

## Exercises 32 – 40



32. **Uniqueness:**




If we define $\phi^*\omega$ by $(\phi^*\omega)_p = \phi^*(\omega_q)$ where $\phi(p)=q$, then this is certainly unique since there are manifestly no choices involved!

We need to check existence, in other words we need to check that

$$\phi^*(\omega) \in \Omega^1(M)$$

i.e. we must check that if $v \in Vect(M)$, then $\phi^*(\omega)(v)$ is really a smooth function on $M$; that is, is the function

$$p \longmapsto \omega(\phi_*(v_p))$$

a smooth function on $M$? We only need to check this locally. If $x^\mu$ is a chart around $p$, and $y^\alpha$ is a chart around $\phi(p)$, then the function above locally computes as

$$x \longmapsto \omega(\phi_*(v(x))) = \omega_\alpha dy^\alpha(\phi_*(v^\mu(x)\partial_\mu$$

$$= \omega_\alpha \, dy^\alpha \left( V^\mu(x) \frac{\partial y^\beta}{\partial x^\mu} \partial_\beta \right)$$ (2)

$$= \omega_\beta(\phi(x)) V^\mu(x) \frac{\partial y^\beta}{\partial x^\mu}$$

which is indeed a smooth function of $x$, since the components $\omega_\beta, V^\mu$ are smooth, and so is the Jacobian $\frac{\partial y^\beta}{\partial x^\mu}$.

33. From naturality, we have

$$\begin{aligned} \phi^*(dx) &= d(\phi^* x) \\ &= d(\sin t) \\ &= \cos t \, dt. \end{aligned}$$

34.

$$\begin{aligned} \phi^*(dx) &= d(\phi^* x) \\ &= d(\cos\theta \cdot u - \sin\theta \cdot v) \\ &= d(\cos\theta \cdot x - \sin\theta \, y) \\ &= \cos\theta \, dx - \sin\theta \, dy. \end{aligned}$$

$\phi(u, v) = (\cos\theta u - \sin\theta v, \sin\theta u + \cos\theta v)$

[change variable names $(u,v) \leftrightarrow (x,y)$!]

$$\begin{aligned} \phi^*(dy) &= d(\phi^* y) \\ &= d(\sin\theta x + \cos\theta y) \\ &= \sin\theta \, dx + \cos\theta \, dy. \end{aligned}$$

35. By definition,

$$(dx^\mu)(\partial_\nu) = \phi^*(\underbrace{dx^\mu}_{\text{on } \mathbb{R}^n})(\phi_*^{-1}(\underbrace{\partial_\nu}_{\text{on } \mathbb{R}^n})) \quad (*)$$

On the other hand, if we think of $dx^\mu$ as "d of the function $x^\mu$", then

$$dx^\mu(\partial_\nu) = \partial_\nu(x^\mu)$$
$$= \delta^\mu_\nu .$$

Let us see that (*) reproduces this:

$$\phi^*(\underbrace{dx^\mu}_{\text{on } \mathbb{R}^n})(\phi_*^{-1}(\underbrace{\partial_\nu}_{\text{on } \mathbb{R}^n}))(p) = \underbrace{dx^\mu}_{\text{on } \mathbb{R}^n}(\phi_*\phi_*^{-1}(\underbrace{\partial_\nu}_{\text{on } \mathbb{R}^n}))(p)$$
$$= \underbrace{dx^\mu}_{\text{on } \mathbb{R}^n}(\underbrace{\partial_\nu}_{\text{on } \mathbb{R}^n})$$
$$= \delta^\mu_\nu .$$

36. Firstly, we know that $dx'^\nu$ is a linear combination of the $dx^\lambda$:

$$dx'^\nu = T^\nu_\lambda \, dx^\lambda$$

Apply both sides to $\partial_\mu$:

$$\frac{\partial x'^\nu}{\partial x^\mu} = T^\nu_\mu .$$

Thus

$$dx'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} dx^\mu .$$

Now, if $\omega_\mu dx^\mu = \omega'_\nu dx'^\nu$, then we can use the above relation with the roles of $x$ and $x'$ swapped around, to get

$$\omega_\mu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu = \omega'_\nu dx'^\nu$$

i.e. $$\omega'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} \omega_\mu .$$

" the derivative of the old variables with respect to the new variables".

37. Evaluate both sides on $\partial_\mu$:

$$\begin{aligned}
\phi^*(dx'^\nu)(\partial_\mu) &= dx'^\nu(\phi_* \partial_\mu) && \text{(defn)} \\
&= dx'^\nu\left(\frac{\partial x'^\lambda}{\partial x^\mu} \partial'_\lambda\right) && \text{(prev. paragraph)} \\
&= \frac{\partial x'^\lambda}{\partial x^\mu} \underbrace{dx'^\nu(\partial'_\lambda)}_{\delta^\nu_\lambda} \\
&= \frac{\partial x'^\nu}{\partial x^\mu}
\end{aligned}$$

while $$\frac{\partial x'^\nu}{\partial x^\alpha} \underbrace{dx^\alpha(\partial_\mu)}_{\delta^\alpha_\mu} = \frac{\partial x'^\nu}{\partial x^\mu} .$$

38. Clear: $T_p$ must transform the basis $\partial_\nu|_p$ of $T_pM$ into some other basis, i.e. the matrix $T^\nu_\mu(p)$ must be invertible.

39. Uniqueness: Suppose $g^\mu$ are another ~~basis~~ set of 1-forms satisfying

$$g^\mu(e_\nu) = \delta^\mu_\nu .$$

Then clearly $g^\mu = f^\mu$, since a functional is determined by its action on a basis.

Existence: Clearly the formula

$$f^\mu = (T^{-1})^\mu_{\ \nu}\, dx^\nu$$

does the job, since

$$\begin{aligned} T^{-1\mu}_{\ \ \nu}\, dx^\nu(e_\lambda) &= T^{-1\mu}_{\ \ \nu}\, dx^\nu(T_\lambda^{\ \alpha}\partial_\alpha) \\ &= T^{-1\mu}_{\ \ \nu}\, T_\lambda^{\ \nu} \\ &= \delta^\mu_\lambda \end{aligned}$$

as it should.

40. The fact that $f'^\mu = (T^{-1})^\mu_{\ \nu} f^\nu$ was essentially proved above.

If $V = V^\mu e_\mu = V'^\mu e'_\mu$, then we have

$$V^\mu e_\mu = V'^\mu T_\mu^{\ \nu} e_\nu$$

so taking $f^\lambda$ of both sides gives

$$V^\lambda = V'^\mu T_\mu^{\ \lambda}$$

i.e. $$V'^\mu = (T^{-1})^\mu_{\ \nu} V^\nu$$

Similarly if $\omega = \omega_\mu f^\mu = \omega'_\mu f'^\mu$ then

$$\omega_\mu f^\mu = \omega'_\mu (T^{-1})^\mu_{\ \nu} f^\nu$$

so applying both sides to $e_\lambda$ gives

$$\omega_\lambda = \omega'_\mu (T^{-1})^\mu_{\ \lambda} \quad \text{i.e.} \quad \omega'_\mu = T_\mu^{\ \nu}\omega_\nu$$