Baez and Munian

Exercise 3, 2 May 2010

3. We'll use the "hemisphere" charts. Set

$$U_{i,\epsilon} = S^n \cap \mathbb{R}^{i,n+1}_{\epsilon} = \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} : \epsilon x_i > 0\}$$

$\epsilon = \pm 1$

so $U_{i,\epsilon}$ is open by definition of the subspace topology, since $\mathbb{R}^{i,n}_{\epsilon}$ is open in $\mathbb{R}^{n+1}$. Our charts are

$(i=1\dots n+1)$
$$\phi_{i,\epsilon} : U_{i,\epsilon} \to \mathbb{R}^n$$
$$(x_1,\dots,x_{n+1}) \mapsto (x_1,\dots,\hat{x}_i,\dots,x_n).$$
leave out the i^{th} one.

Note that $\phi_{i,\epsilon}$ is continuous since it's the restriction of the projection map $\pi_i : \mathbb{R}^{n+1} \to \mathbb{R}^n$ to S^n, and by definition of the subspace topology any "restricted function" must be continuous (Let $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion map; then if V is open in $\mathbb{R}^n$, we must have $(\pi \circ i)^{-1}(V) = \underbrace{\pi^{-1}(V)}_{\text{open in } \mathbb{R}^{n+1}} \cap S^n$ so $\pi \circ i$ is continuous).

The image of $\phi_{i,\epsilon}$ is the open ball B of radius 1 in $\mathbb{R}^n$. Its inverse is

$$\phi^{-1}_{i,\epsilon} : B \to U_{i,\epsilon}$$
$$(t_1,\dots,t_n) \mapsto \left(t_1,\dots,t_{i-1}, \epsilon\sqrt{1-\sum_{k=1}^{n} t_k^2}, t_{i+1},\dots,t_n\right)$$

The charts $U_{i,\epsilon}$ cover S^n (every point must have some nonzero coordinate). The map $\phi^{-1}_{i,\epsilon}$ is clearly continuous (since it is continuous as a map into $\mathbb{R}^{n+1}$, and the subspace topology is precisely the topology making these kinds of maps continuous).

We must check whether the transition functions are smooth:

$$\phi_{i,\epsilon} \circ \phi_{j,\delta}^{-1} : \underset{(\subseteq \mathbb{R}^n)}{\overset{\text{an open subset of}}{B}} \longrightarrow \underset{(\subseteq \mathbb{R}^n)}{\overset{\text{an open subset of}}{B}}$$

$$(t_1, \cdots, t_n) \overset{\phi_{j,\delta}^{-1}}{\longmapsto} \left(t_1, \cdots, t_{j-1}, \delta\sqrt{1-\sum_{k=1}^{n} t_k^2}, t_j, \cdots, t_n\right)$$

$$\overset{\phi_{i,\epsilon}}{\longmapsto} \left(t_1, \cdots, t_{j-1}, \delta\sqrt{1-\sum_{k=1}^{n} t_k^2}, t_j, \cdots, \hat{t}_i, \cdots, t_n\right)$$

So, at least where it is defined, the transition function

$$\phi_{i,\epsilon}\, \phi_{j,\delta}^{-1}$$

takes the form "leave out the i^{th} coordinate and insert a square root function at position j"

which is a smooth map. So S^n is a manifold.