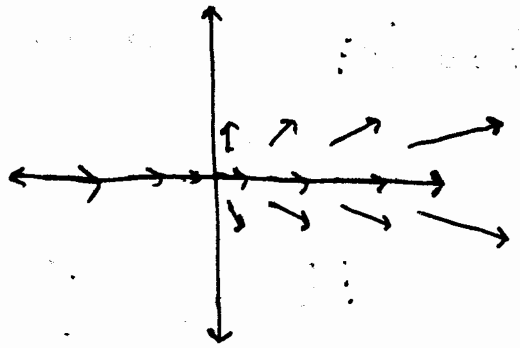
Baez and Munian

Exercises 20 – 21

20.



Want:

$$(\dot{x}(t), \dot{y}(t)) = (x^2, y)$$

$$\dot{x} = x^2, \quad \dot{y} = y$$

general soln: $x = \dfrac{1}{A-t}, \quad y = Be^t$

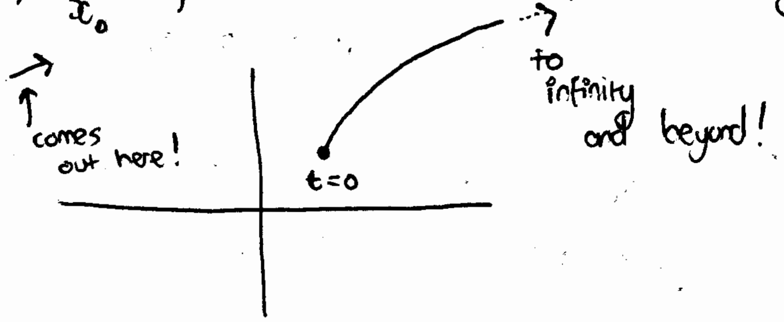
Would like $x(0) = y(0) = 0$

$$\Rightarrow A = \frac{1}{x_0}, \ B = y_0$$

So the curve starting at (x_0, y_0) at time $t=0$ is

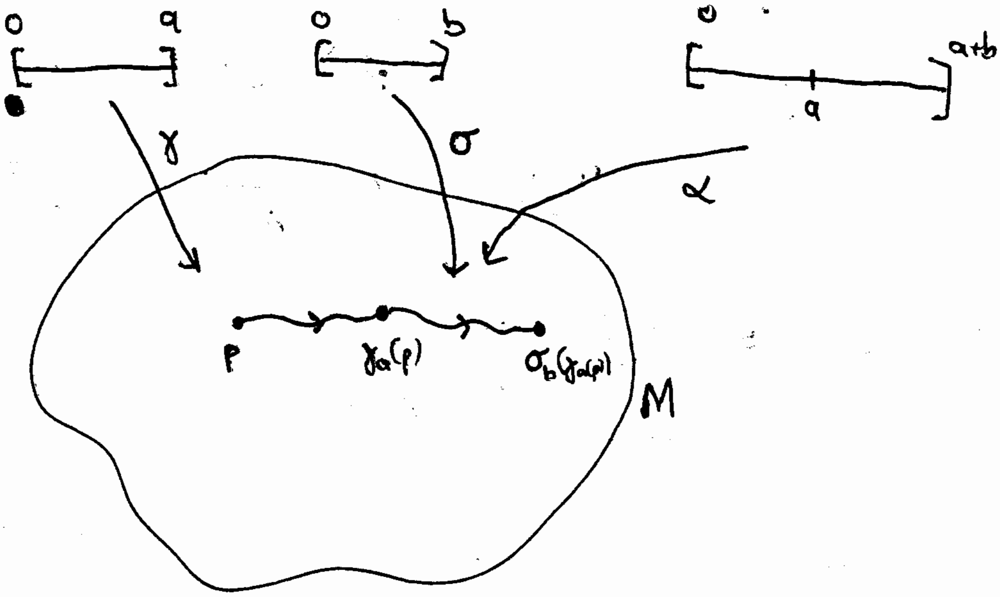
$$\gamma(t) = \left(\frac{1}{\frac{1}{x_0} - t}, y_0 e^t \right)$$

When $t \to \dfrac{1}{x_0}$, the x-coordinate escapes to infinity!



21. The fact that $\phi_0(p) = p$ is due to our convention about integral curves through p starting at p at $t=0$ (pg. 34).

To see that $(\phi_b \circ \phi_a)(p) = \phi_{b+a}(p)$, note that there are 3 integral curves at stake (actually 4):



$\sigma : \mathbb{R} \to$

$\phi_b(\phi_a(p))$ = the value at $t=b$ of the unique curve propagating according to the vector field, with $\sigma(0) = \gamma_a(p)$.

(where $\gamma_a(p)$ = the value at $t=a$ of the unique integral curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$)

Claim that $\sigma(t) = \alpha(t+a)$, where $\alpha : \mathbb{R} \to M$ is the unique integral curve with $\alpha(0) = p$.

We just need to check:

$$\sigma(0) = \gamma_a(p) \checkmark$$
$$= \alpha(a) \quad \text{(by uniqueness of soln.)}$$

And, $\sigma'(t) = V_{\sigma(t)}$,

while

if we set $\beta(t) = \alpha(t+a)$, then

Here we have used a very basic version of the chain rule.

$$\beta'(t) = \alpha'\big|_{t+a} \circ \left(\text{the derivative at } t \text{ of the map } \underbrace{t \mapsto t+a}_{=1} \right)$$

$$= \alpha'\big|_{t+a}$$

$$= \alpha'(t+a) \qquad [\text{notation}]$$

$$= V_{\alpha(t+a)}$$

$$= V_{\beta(t)}$$

i.e. $\sigma'(t) = \beta'(t)$, with some initial condition

$$\Rightarrow \sigma = \beta.$$