9 May 2010

Baez + Munian exercise 15

15. Note: this is part of a deep idea which is set to replace the foundations of smooth manifolds; see eg

Baez and Hoffnung, Convenient categories of smooth spaces, arXiv:0807.1704.

"Smooth is as smooth does"

or

"if you smell like you're smooth, then you're smooth!"

Lemma 1: A map $\phi : M \to \mathbb{R}$ is smooth in the new sense (i.e. $f \in C^\infty(\mathbb{R}) \Rightarrow \phi^* f \in C^\infty(M)$ for all $f \in C^\infty(\mathbb{R})$, where $C^\infty(\mathbb{R})$ and $C^\infty(M)$ mean "smooth in the old sense"!)

$$\Longleftrightarrow$$

ϕ is smooth in the old sense (i.e. $\phi \circ \psi_\alpha^{-1}$ is smooth in the really old calculus sense for all charts $\psi_\alpha : U_\alpha \to \mathbb{R}^n$ for M).

Proof

$\Rightarrow$: Suppose $\phi : M \to \mathbb{R}$ is smooth in the new sense. So $\phi^* f \in C^\infty(M)$ for all (smooth in the old sense) $f : \mathbb{R} \to \mathbb{R}$. In other words,

$$(f \circ \phi) \circ \psi_\alpha^{-1} \text{ is smooth map (in really old calculus sense)}$$

Set f = identity function from $\mathbb{R} \to \mathbb{R}$. Then we have

$$(id \circ \phi) \circ \psi_\alpha^{-1} \text{ is smooth in calculus sense for all charts } \psi_\alpha$$

i.e. $\phi \circ \psi_\alpha^{-1}$ is smooth in calculus sense

i.e. ϕ is smooth in old sense.

$\Leftarrow$: Suppose $\phi : M \to \mathbb{R}$ is smooth in old sense.

i.e. $\phi \circ \psi_\alpha^{-1}$ is smooth map (in calculus sense)
from $\mathbb{R}^m$ to $\mathbb{R}$

Let $f \in C^\infty(\mathbb{R})$.
$\uparrow$ old sense

In other words, for each chart (K_α, V_α) of $\mathbb{R}$, we have

$$f \circ K_\alpha^{-1} : V_\alpha \to \mathbb{R}$$

is smooth in calculus sense.

But we can take our chart for $\mathbb{R}$ to be $(id, \mathbb{R})$ itself!

So $f : \mathbb{R} \to \mathbb{R}$

is smooth in calculus sense.

So,

$$\phi^*(f) \circ \psi_\alpha^{-1} = (f \circ \phi) \circ \psi_\alpha^{-1}$$

$$= \underbrace{f}_{\substack{\text{calculus smooth map} \\ \mathbb{R} \to \mathbb{R}}} \circ \underbrace{(\phi \circ \psi_\alpha^{-1})}_{\substack{\text{calculus smooth map} \\ \mathbb{R}^m \to \mathbb{R}}}$$

$\therefore$ $\phi^*(f)$ is a (calculus) smooth map $\mathbb{R} \to \mathbb{R}$

$\therefore$ $\phi^*(f)$ is also smooth in the old sense, i.e. $\phi^*(f) \in C^\infty(\mathbb{R})$

For maps $\gamma : \mathbb{R} \to M$, note that we defined γ to be smooth in the old sense (pg. 29) if for all $f \in C^\infty(M)$, $f \circ \gamma$ is a smooth map $\mathbb{R} \to \mathbb{R}$. In other words, this is precisely "smooth in the new sense" ... there's nothing to check.

More interesting would be to check that this definition of $\gamma : \mathbb{R} \to M$ being "smooth" corresponds to the "<u>local</u> definition", namely that $\phi_\alpha \circ \gamma$ is smooth for all charts $U_\alpha \xrightarrow{\phi_\alpha} \mathbb{R}^m$ of M. See Baez-Hoffnung article.